MATH 217 PRACTICE FINAL SOLUTIONS

1. $y' + \frac{1}{x} y = x$

$$g(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$xy' + y = x^2$$

$$(xy)' = x^2$$

$$xy = \frac{x^3}{3} + C$$

$$y = \frac{x^2}{3} + \frac{C}{x}$$

$$1 = y(1) = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}.$$

Sol. to i.v.p. is $y = \frac{x^2}{3} + \frac{2/3}{x}$.

2. $(xy - x^3)\,dx + x^2\,dy = 0$. Not exact

$$g(x) = \frac{x - 2x}{x^2} = \frac{-x}{x^2} = -\frac{1}{x}$$

$$\mu(x) = e^{\int g(x)\,dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}.$$

This is an integrating factor because

$$\frac{1}{x}(xy - x^3)\,dx + \frac{1}{x} x^2\,dy = 0$$

$$(y - x^2)\,dx + x\,dy = 0 \text{ is exact.}$$

3. $xy'' - y' = x$

$y' = p,\ y'' = p'$

So $xp' - p = x$

$$p' - \frac{1}{x}p = 1$$

$$g(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

$$\frac{1}{x}p' - \frac{1}{x^2}p = \frac{1}{x}$$

$$\left(\frac{1}{x}p\right)' = \frac{1}{x}$$

$$\frac{1}{x}p = \ln x + C$$

$$p = x\ln x + Cx$$

$$y' = x\ln x + Cx$$

$$y = \int x\ln x + Cx$$

$$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}dx + C\frac{x^2}{2}$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{Cx^2}{2} + D$$

$$= \frac{x^2}{2}\ln x + Ex^2 + D$$

4. $y'' + 5y' + 6y = 0,\ y(0) = 1,\ y'(0) = 1.$

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0$$

$$r = -2, -3$$

So $y = Ae^{-2x} + Be^{-3x}$,

$1 = y(0) = A + B \Rightarrow 2 = 2A + 2B$

$1 = y'(0) = -2A - 3B \quad 1 = -2A - 3B$

$3 = -B \Rightarrow B = -3$

$A = 4$

So $y = 4e^{-2x} + (-3)e^{-3x}$ is solution of i.v.p.

5. $xy'' - y' = 0$

$y'' - \frac{1}{x}y' = 0$

$v = \int \frac{1}{(x^2)^2} e^{-\int -\frac{1}{x}dx} dx = \int \frac{1}{x^4} \cdot x \, dx = \int x^{-3} dx$

$= -\frac{1}{2}x^{-2}$.

Thus $y_2 = vy_1 = -\frac{1}{2}x^{-2} \cdot x^2 = -\frac{1}{2}$.

General solution is

$y = Ax^2 + B \cdot (-\frac{1}{2}) = Ax^2 + C$.

6. $y'' - 3y' + 2y = x \quad (*)$

$y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r = 2, 1$

$y = Ae^{2x} + Be^{x}$ is solution of homogeneous

Guess $y_p(x) = \alpha x^2 + \beta x + \gamma$ is solution of $(*)$.

So $y_p' = 2\alpha x + \beta$

$y_p'' = 2\alpha$

$$2\alpha - 3(2\alpha x + \beta) + 2(\alpha x^2 + \beta x + \gamma) = x$$

$$2\alpha x^2 + (-6\alpha + 2\beta)x + (2\alpha - 3\beta + 2\gamma) = x$$

$2\alpha = 0 \Rightarrow \alpha = 0$

$2\alpha - 3\beta + 2\gamma = 0 \Rightarrow -3\beta + 2\gamma = 0$

$-6\alpha + 2\beta = 1 \Rightarrow \quad 2\beta = 1 \Rightarrow \beta = \frac{1}{2}$

So $\gamma = \frac{3}{4}$.

$$y_p(x) = \frac{1}{2}x + \frac{3}{4}$$

So general solution is $y = \left(\frac{1}{2}x + \frac{3}{4}\right) + Ae^{2x} + Be^{x}$.

7. $y' = y$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$\sum_{j=1}^{\infty} j a_j x^{j-1} = \sum_{j=0}^{\infty} a_j x^j$$

$$\sum_{j=0}^{\infty} (j+1) a_{j+1} x^j - \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=0}^{\infty} \left[(j+1) a_{j+1} - a_j\right] x^j = 0.$$

$$(j+1)a_{j+1} - a_j = 0, \quad j \geq 0$$

$$a_{j+1} = \frac{a_j}{j+1}, \quad j \geq 0$$

$$a_0 = C$$

$$a_1 = \frac{a_0}{1} = C$$

$$a_2 = \frac{a_1}{2} = \frac{C}{2}$$

$$a_3 = \frac{a_2}{3} = \frac{C}{3 \cdot 2}$$

$$a_4 = \frac{a_3}{4} = \frac{C}{4 \cdot 3 \cdot 2}$$

$$\vdots$$

$$a_j = \frac{C}{j!}$$

Solution is

$$y = \sum_{j=0}^{\infty} \frac{C}{j!} x^j = C \sum_{j=0}^{\infty} \frac{x^j}{j!} = Ce^x.$$

8. $f(x) = x$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x\,dx = 0$$

In fact f is odd so all cosine terms are 0.

$$b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin jx\,dx = \frac{2}{\pi} \int_0^{\pi} x \sin jx\,dx$$

$$= \frac{2}{\pi} \left[x\left(\frac{-\cos jx}{j} \right) \right]_0^{\pi} + \int_0^{\pi} \frac{\cos jx}{j}\,dx$$

$$= \frac{2}{\pi} \left[\frac{\pi\left(-(-1)^j\right)}{j} - 0 + \left[\frac{\sin jx}{j^2}\right]_0^{\pi} \right]$$

$$= \frac{2(-1)^{j+1}}{j} - 0 + (0-0) = \frac{2(-1)^{j+1}}{j}.$$

So Fourier series is

$$x = f(x) = \sum_{j=1}^{\infty} \frac{2(-1)^{j+1}}{j} \sin jx.$$

9. $y'' + 3y' - 4y = 2t$, $y(0) = 0$, $y'(0) = 0$.

$z(p) = p^2 + 3p - 4$.

$$h(t) = L^{-1}\left(\frac{1}{p^2+3p-4}\right) = L^{-1}\left(\frac{1}{(p+4)(p-1)}\right)$$

$$= L^{-1}\left(\frac{-1/5}{p+4} + \frac{1/5}{p-1}\right)$$

$$= -\frac{1}{5}e^{-4t} + \frac{1}{5}e^{t}.$$

Hence

$$y(t) = \int_0^t \left(-\frac{1}{5}e^{-4(t-\tau)} + \frac{1}{5}e^{(t-\tau)}\right) 2\tau \, d\tau$$

$$= -\frac{1}{5}e^{-4t}\int_0^t 2\tau e^{4\tau} + \frac{1}{5}e^{t}\int_0^t 2\tau e^{-\tau}\, d\tau$$

$$= -\frac{1}{5}e^{-4t}\left[\left[2\tau\frac{e^{4\tau}}{4}\right]_0^t - \int_0^t \frac{2}{4}e^{4\tau}\, d\tau\right]$$

$$+ \frac{e^t}{5}\left[2\tau\left(-e^{-\tau}\right)\Big|_0^t + \int_0^t 2e^{-\tau}\, d\tau\right]$$

$$= -\frac{1}{5}e^{-4t}\left[\frac{2te^{4t}}{4} - \frac{1}{8}e^{4\tau}\Big]_0^t\right]$$

$$+ \frac{e^t}{5}\left[-2te^{-t} + (-2e^{-t}) + 2\right]$$

$$= -\frac{1}{10}t + \frac{1}{40} - \frac{1}{40}e^{-4t} - \frac{2}{5}t + -\frac{2}{5} + \frac{2}{5}e^{t}$$

$$= -\frac{1}{2}t - \frac{3}{8} - \frac{1}{40}e^{-4t} + \frac{2}{5}e^{t}.$$

10. Only $\lambda > 0$ makes sense. Then the solutions of the ODE are

$$y = A\sin\sqrt{\lambda}\,x + B\cos\sqrt{\lambda}\,x$$

But $y(0) = 0 \Rightarrow B = 0$.

So $y = A\sin\sqrt{\lambda}\,x$. And $y\left(\frac{\pi}{2}\right) = 0 \Rightarrow \frac{\pi}{2}\sqrt{\lambda} = n\pi$

$$\text{so } \sqrt{\lambda} = 2n \Rightarrow \lambda = 4n^2$$

Thus $y_n(x) = \sin 2nx$.

11. $f(\theta) = \begin{cases} 2 & \text{if } 0 \le \theta \le \pi \\ 1 & \text{if } -\pi \le \theta < 0. \end{cases}$

So

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{\pi}\int_{-\pi}^{0} 1\,d\theta + \frac{1}{\pi}\int_{0}^{\pi} 2\,d\theta$$

$$= +1 + 2 = 3.$$

For $j \ge 1$,

$$a_j = \frac{1}{\pi}\int_{-\pi}^{0} 1\cos j\theta\,d\theta =$$

$$+ \frac{1}{\pi}\int_{0}^{\pi} 2\cos j\theta\,d\theta$$

$$= \frac{1}{\pi}\frac{\sin j\theta}{j}\Big]_{-\pi}^{0} + \frac{1}{\pi}\frac{2\sin j\theta}{j}\Big]_{0}^{\pi} = 0.$$

For $j \geq 1$,

$$b_j = \frac{1}{\pi}\int_{-\pi}^{0} 1 \sin j\theta \, d\theta + \frac{1}{\pi}\int_{0}^{\pi} 2 \sin j\theta \, d\theta$$

$$= \frac{1}{\pi}\left(\frac{-\cos j\theta}{j}\right)\Bigg]_{-\pi}^{0} + \frac{1}{\pi}\left(-\frac{2\cos j\theta}{j}\right)\Bigg]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{-1}{j} + \frac{(-1)^j}{j}\right) + \frac{1}{\pi}\left(\frac{-2(-1)^j}{j} + \frac{2}{j}\right)$$

$$= \frac{1}{\pi j} + \frac{1}{\pi j}\left((-1)^j - 2(-1)^j\right)$$

$$= \frac{1}{\pi j}\left(1 + (-1)^{j+1}\right).$$

So if $j = 2k$ is even then $b_j = 0$

if $j = 2k-1$ is odd then $b_j = \frac{2}{\pi j}$,

Hence

$$f(\theta) = \frac{3}{2} + \frac{2}{\pi}\left(\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots\right)$$

The solution of the Dirichlet problem is

$$w(r,\theta) = \frac{3}{2} + \frac{2}{\pi}\left(r\sin\theta + \frac{r^3\sin 3\theta}{3} + \frac{r^5 \sin 5\theta}{5} + \cdots\right).$$

12. $z(x,0) = -\sin x + 6\sin x\cos x = -\sin x + 3\sin 2x.$

Hence

$$z(x,t) = -e^{-t^2}\sin x + 3e^{-4t^2}\sin 2x,$$