First Order Predicate Logic

Expanding upon Propositional Logic

CIS 32

Functionalia

Project 1 Deliverables

Demos?

HW 3 is out on the webpage.

Today:

- Proof Systems
- Propositional Logic Examples
- Predicate Logic

Review

Greek Letters

ϕ	phi
Φ	capital Phi
ψ	psi
π	pi
τ	tau
χ	chi

Logical Inference

$$\text{KB} \vdash_i \phi$$

“ is derived from KB by “ *or* “*i* derives from KB”

KB is a haystack, ϕ is a needle.

Entailment is the needle being in the haystack; ***Inference*** is finding it.

Sound Inference Algorithms derive only entailed sentences.

Unsound Inference Algorithms makes things up along the way (finding non-existent needles).

Complete Inference Algorithms can derive any sentence that is entailed.

Finite KB, it’s simple: systematic examination is complete.

Infinite KB, it’s a bit more problematic.

- **Definition**: (*Soundness*) A proof system $\vdash$ is said to be *sound* with respect to semantics $\models$ iff

$$\phi_1, \ldots, \phi_n \vdash \phi$$

implies

$$\phi_1, \ldots, \phi_n \models \phi.$$

- **Definition**: (*Completeness*) A proof system $\vdash$ is said to be *complete* with respect to semantics $\models$ iff

$$\phi_1, \ldots, \phi_n \models \phi$$

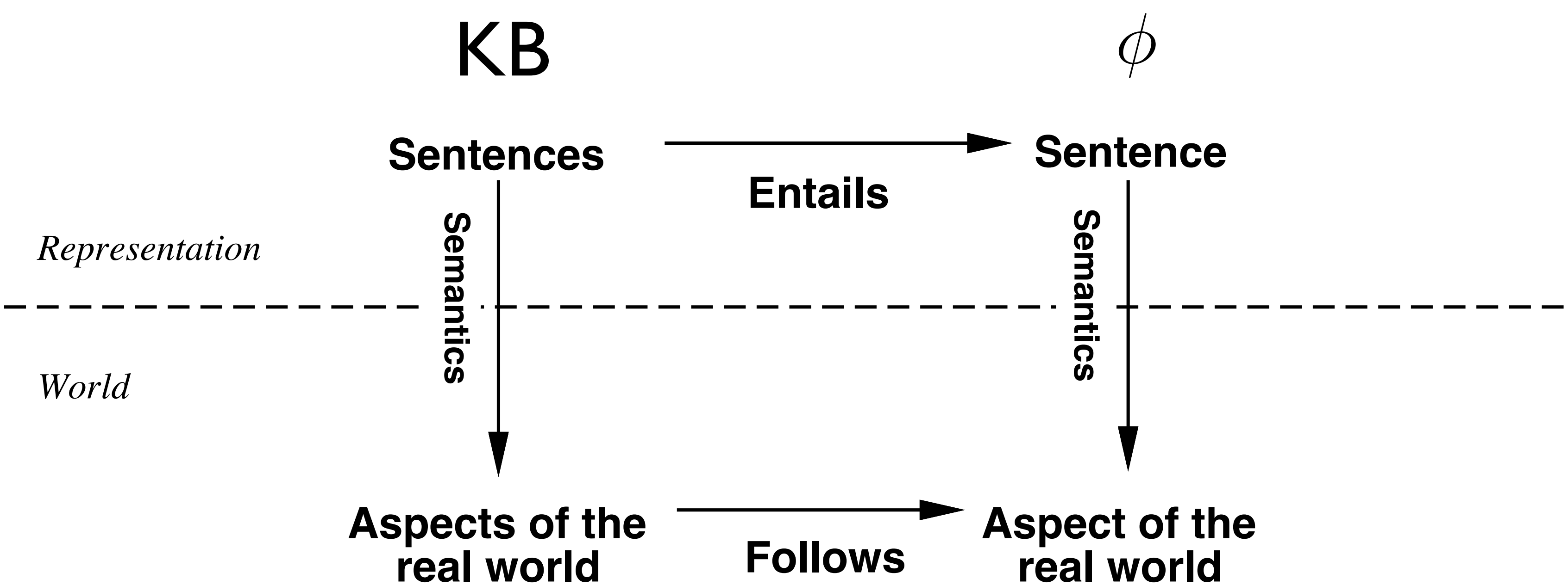
implies

$$\phi_1, \ldots, \phi_n \vdash \phi.$$

Entailment

If the Knowledge Base (KB) is true in the real world, then any sentence ϕ derived from KB by a sound inference procedure $\vdash$ is also true in the real world.

$$\text{KB} \vdash \phi$$



A Proof System

- There are many proof systems for propositional logic; we shall look at a simple one.
- First, we have an unusual rule that allows us to introduce any tautology.

$$\frac{}{\vdash \phi}\ \mathsf{TAUT} \qquad \text{if } \phi \text{ is a tautology}$$

- Because a tautology is true there is no problem bringing it into the proof.

Eliminating Connectives

- Next, rules for *eliminating* connectives.

AND Elimination

$$\frac{\vdash \phi \wedge \psi}{\vdash \phi; \vdash \psi} \wedge\text{-E}$$

OR Elimination

$$\frac{\begin{array}{l}\vdash \phi_1 \vee \cdots \vee \phi_n; \\ \phi_1 \vdash \phi; \\ \quad \cdots; \\ \phi_n \vdash \phi\end{array}}{\vdash \phi} \vee\text{-E}$$

Modus Ponens

- An alternative $\vee$ elimination rule is:

$$\frac{\begin{array}{l}\vdash \phi \vee \psi; \\ \vdash \phi \Rightarrow \chi; \\ \vdash \psi \Rightarrow \chi\end{array}}{\vdash \chi} \quad \vee\text{-E}$$

- Next, a rule called *modus ponens*, which lets us eliminate $\Rightarrow$.

$$\frac{\vdash \phi \Rightarrow \psi; \vdash \phi}{\vdash \psi} \quad \Rightarrow\text{-E}$$

Introducing Connectives

- Next, rules for *introducing* connectives.

$$\frac{\vdash \phi_1; \cdots ; \vdash \phi_n}{\vdash \phi_1 \wedge \cdots \wedge \phi_n} \wedge\text{-I}$$

$$\frac{\vdash \phi_1; \cdots ; \phi_n}{\vdash \phi_1 \vee \cdots \vee \phi_n} \vee\text{-I}$$

- We have a rule called the deduction theorem. This rule says that if we can prove ψ from ϕ, then we can prove that $\phi \Rightarrow \psi$.

$$\frac{\phi \vdash \psi}{\vdash \phi \Rightarrow \psi} \Rightarrow\text{-I}$$

- There are a whole range of other rules, which we shall not list here.

Proof Examples

- In this section, we give some examples of proofs in the propositional calculus.

- Example 1:
$p \wedge q \vdash q \wedge p$

1.	$p \wedge q$	Given
2.	p	From 1 using $\wedge$-E
3.	q	1,$\wedge$-E
4.	$q \wedge p$	2, 3, $\wedge$-I

- Example 2:

$p \wedge q \vdash p \vee q$

1.	$p \wedge q$	Given
2.	p	1, $\wedge$-E
3.	$p \vee q$	2, $\vee$-I

- Example 3:
  $p \wedge q, p \Rightarrow r \vdash r$

1.	$p \wedge q$	Given
2.	p	1, $\wedge$-E
3.	$p \Rightarrow r$	Given
4.	r	2, 3, $\Rightarrow$-E

- Example 4:

$p \Rightarrow q, q \Rightarrow r \vdash p \Rightarrow r$

1.	$p \Rightarrow q$	Given	
2.	$q \Rightarrow r$	Given	
3.	p	Ass	|
4.	q	1, 3, $\Rightarrow$-E	|
5.	r	2, 4, $\Rightarrow$-E	|
6.	$p \Rightarrow r$	3, 5, $\Rightarrow$-I	

- Example 5:
  $(p \land q) \Rightarrow r \vdash p \Rightarrow (q \Rightarrow r)$

1.	$(p \land q) \Rightarrow r$	Given	
2.	p	Ass	$\mid$
3.	q	Ass	$\mid\mid$
4.	$p \land q$	2, 3, $\land$-I	$\mid\mid$
5.	r	1, 4, $\Rightarrow$-I	$\mid\mid$
6.	$q \Rightarrow r$	3–5, $\Rightarrow$-I	$\mid$
7.	$p \Rightarrow (q \Rightarrow r)$	2–6, $\Rightarrow$-I	

- Example 6:

  $p \Rightarrow (q \Rightarrow r) \vdash (p \land q) \Rightarrow r$

1.	$p \Rightarrow (q \Rightarrow r)$	Given	
2.	$p \land q$	Ass	|
3.	p	2, $\land$-E	|
4.	q	2, $\land$-E	|
5.	$q \Rightarrow r$	1, 3, $\Rightarrow$-E	|
6.	r	4, 5, $\Rightarrow$-E	|
7.	$(p \land q) \Rightarrow r$	2–6, $\Rightarrow$-I	

- Example 7:
  $p \Rightarrow q, \neg q \vdash \neg p$

1.	$p \Rightarrow q$	Given
2.	$\neg q$	Given
3.	p	Ass
4.	q	1, 3, $\Rightarrow$-E
5.	$q \wedge \neg q$	2, 4, $\wedge$-I
6.	$\neg p$	3, 5, $\neg$-I

- Example 8:

  $p \Rightarrow q \vdash \neg(p \wedge \neg q)$

1.	$p \Rightarrow q$	Given	
2.	$p \wedge \neg q$	Ass	|
3.	p	2, $\wedge$-E	|
4.	$\neg q$	2, $\wedge$-E	|
5.	q	1, 3, $\Rightarrow$-E	|
6.	$q \wedge \neg q$	4, 5, $\wedge$-I	|
7.	$\neg(p \wedge \neg q)$	6, $\neg$-I	

- Example 9:

Jim will party all night and pass AI? That must be wrong. If he works hard he won't have time to party. If he doesn't work hard he's not going to pass AI.

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p	Jim will party all night
q	Jim will pass AI
r	Jim works hard

Formalisation of argument:

- Example 9:

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Let:

p Jim will party all night
q Jim will pass AI
r Jim works hard

Formalisation of argument:

$$\vdash$$

- Example 9:

Jim will party all night and pass AI? That must be wrong. If he works hard he won't have time to party. If he doesn't work hard he's not going to pass AI.

Let:

p	Jim will party all night
q	Jim will pass AI
r	Jim works hard

Formalisation of argument:

$$r \Rightarrow \neg p, \neg r \Rightarrow \neg q \vdash \neg(p \wedge q)$$

1.	$r \Rightarrow \neg p$	Given	
2.	$\neg r \Rightarrow \neg q$	Given	
3.	$p \wedge q$	Ass	$\vert$
4.	r	Ass	$\vert\vert$
5.	$\neg p$	1, 4, $\Rightarrow$-E	$\vert\vert$
6.	p	3, $\wedge$-I	$\vert\vert$
7.	$p \wedge \neg p$	5, 6, $\wedge$-I	$\vert\vert$
8.	$\neg r$	4, 7, $\neg$-I	$\vert$
9.	$\neg q$	2, 9, $\Rightarrow$-E	$\vert$
10.	q	3, $\wedge$-E	$\vert$
11.	$q \wedge \neg q$	9, 10, $\wedge$-I	$\vert$
12.	$\neg(p \wedge q)$	3, 11, $\neg$-I	

Predicate Logic

- *First-order predicate logic*
- More expressive than propositional logic.
- Consider the following argument:
  - *all monitors are ready;*
  - *X12 is a monitor;*
  - *therefore X12 is ready.*
- Sense of this argument *cannot* be captured in propositional logic.
- Propositional logic is too *coarse grained* to allow us to represent and reason about this kind of statement.

Syntax

- We shall now introduce a generalization of propositional logic called first-order logic (FOL). This new logic affords us much greater expressive power.

- **Definition**: The alphabet of FOPL contains:

   1. a set of ***constants***;

   2. a set of ***variables***;

   3. a set of ***function symbols***;

   4. a set of ***predicates symbols***;

   5. the ***connectives*** $\vee$, $\neg$;

   6. the ***quantifiers*** $\forall$, $\exists$, $\exists_1$;

   7. the punctuation symbols), (.

Terms : Constants

- The basic components of FOL are called *terms*.
- Essentially, a term is an object that denotes some object other than $\top$ or $\perp$.
- The simplest kind of term is a ***constant***.
- A value such as 8 is a constant.
- The ***denotation*** of this term is the number 8.
- Note that a constant and the number it denotes are different!
- Aliens don't write "8" for the number 8, and nor did the Romans.

Terms : Variables

- The second simplest kind of term is a *variable*.
- A variable can stand for anything in the *domain of discourse*.
- The domain of discourse (usually abbreviated to domain) is the set of all objects under consideration.
- Sometimes, we assume the set contains "everything".
- Sometimes, we explicitly *give* the set, and *state* what variables/constants can stand for.

Terms : Functions

- We can now introduce a more complex class of terms — *functions*.
- The idea of functional terms in logic is similar to the idea of a function in programming: recall that in programming, a function is a procedure that takes some arguments, and *returns a value*.

In C:

```
T myfunction( T1 a1, ..., Tn an ) {
   ...
}
```

this function takes *n* arguments; the first is of type **`T1`**, the second is of type **`T2`**, and so on. The function returns a value of type **`T`**.

- In FOL, we have a set of *function symbols*; each symbol corresponds to a particular function. (It denotes some function.)

Function: arity

- Each function symbol is associated with a number called its *arity*. This is just the number of arguments it takes.

- A *functional term* is built up by *applying* a function symbol to the appropriate number of terms.

- Formally …

Definition: Let *f* be an arbitrary function symbol of arity *n*. Also, let $\tau_1, \ldots, \tau_n$ be terms. Then

$$f(\tau_1, \ldots, \tau_n)$$

is a functional term.

Function : *arity*

- All this sounds complicated, but isn't. Consider a function *plus*, which takes just two arguments, each of which is a number, and returns the first number added to the second.

Then:

– *plus*(2, 3) is an acceptable functional term;

– *plus*(0, 1) is acceptable;

– *plus*(*plus*(1, 2), 4) is acceptable;

– *plus*(*plus*(*plus*(0, 1), 2), 4) is acceptable;

Functions

- In maths, we have many functions; the obvious ones are

$$+ \quad - \quad / \quad * \quad \sqrt{} \quad \sin \quad \cos \quad \ldots$$

- The fact that we write

$$2 + 3$$

instead of something like

$$\text{plus}(2, 3)$$

is just convention, and is not relevant from the point of view of logic; all these are functions in exactly the way we have defined.

Function

• Using functions, constants, and variables, we can build up *expressions*, e.g.:

$$(x + 3) * \sin 90$$

(which might just as well be written

$$\mathit{times}(\mathit{plus}(x, 3), \mathit{sin}(90))$$

for all it matters.)

Predicates

• In addition to having terms, FOL has *relational operators*, which capture *relationships* between objects.

• The language of FOL contains *predicate symbols*.

• These symbols stand for *relationships between objects*.

• Each predicate symbol has an associated *arity* (number of arguments).

• **Definition**: Let *P* be a predicate symbol of arity *n*, and $\tau_1, \ldots, \tau_n$ are terms.

Then

$$P(\tau_1, \ldots, \tau_n)$$

is a predicate, which will either be $\top$ or $\perp$ under some interpretation.

- **EXAMPLE**. Let *gt* be a predicate symbol with the intended interpretation '*greater than*'. It takes two arguments, each of which is a natural number.

Then:

  – *gt*(4, 3) is a predicate, which evaluates to $\top$;

  – *gt*(3, 4) is a predicate, which evaluates to $\bot$.

- The following are standard mathematical predicate symbols:

$$> \; < \; = \; \neq \; \geq \; \leq \; \ldots$$

- The fact that we are normally write $x > y$ instead of $gt(x, y)$ is just convention.

- We can build up more complex predicates using the connectives of propositional logic:

$$(2 > 3) \wedge (6 = 7) \vee (\sqrt{4} = 2)$$

- So a predicate just expresses a relationship between some values.
- What happens if a predicate contains *variables*: can we tell if it is true or false?

  Not usually; we need to know an *interpretation* for the variables.

- A predicate that contains no variables is a proposition.

Properties

- Predicates of *arity* 1 are called properties.
- **EXAMPLE**. The following are properties:

$$Man(x)$$

$$Mortal(x)$$

$$Malfunctioning(x).$$

- We interpret $P(x)$ as saying x is in the set P.
- Predicate that have arity 0 (i.e., take no arguments) are called *primitive propositions*.

These are identical to the primitive propositions we saw in propositional logic.

Quantifiers

- We now come to the central part of first order logic: *quantification*.
- Consider trying to represent the following statements:
  - *all men have a mother ;*
  - *every positive integer has a prime factor.*
- We can't represent these using the apparatus we've got so far; we need *quantifiers*.

Quantifiers

- We use three quantifers:

  $\forall$ — the *universal quantifier* ;

  - is read 'for all...'

  $\exists$ — the *existential quantifier* ;

  - is read 'there exists...'

  $\exists_1$ — the unique quantifier ;

  - is read 'there exists a unique...'

- The simplest form of quantified formula is as follows:

  quantifier variable $\cdot$ *predicate*

where

- *quantifier* is one of $\forall$, $\exists$, $\exists_1$;
- *variable* is a variable;
- and *predicate* is a predicate.

Examples

- $\forall x \cdot \mathit{Man}(x) \Rightarrow \mathit{Mortal}(x)$

  'For all *x*, if *x* is a man, then *x* is mortal.'

  (i.e. all men are mortal)

- $\forall x \cdot \mathit{Man}(x) \Rightarrow \exists_1 y \cdot \mathit{Woman}(y) \wedge \mathit{MotherOf}(x, y)$

  'For all *x*, if *x* is a man, then there exists exactly one *y* such that *y* is a woman and the mother of *x* is *y*.'

  (i.e., every man has exactly one mother).

Examples

- $\exists m \cdot Monitor(m) \land MonitorState(m, ready)$

  'There exists a monitor that is in a ready state.'

- $\forall r \cdot Reactor(r) \Rightarrow \exists_1 t \cdot (100 \leq t \leq 1000) \land temp(r) = t$

  'Every reactor will have a temperature in the range 100 to 1000.'

- $\exists n \cdot posInt(n) \wedge n = (n * n)$

  "Some positive integer is equal to its own square."

- $\exists c \cdot EUcountry(c) \wedge Borders(c, Albania)$

  "Some EU country borders Albania."

- $\forall m, n \cdot Person(m) \wedge Person(n) \Rightarrow \neg Superior(m, n)$

  "No person is superior to another."

- $\forall m \cdot Person(m) \Rightarrow \neg \exists n \cdot Person(n) \wedge Superior(m, n)$

  (same as previous)

Domains & Interpretations

- Suppose we have a formula $\forall x \cdot P(x)$.

  What does x *range* over?

  Physical objects, numbers, people, times, . . . ?

- Depends on the *domain* that we intend.
- Often, we *name* a domain to make our intended interpretation clear.

Example of Domains

- Suppose our intended interpretation is the positive integers.
Suppose $>, +, *, \ldots$ have the usual mathematical interpretation.

- Is this formula satisfiable under this interpretation?

$$\exists n \cdot n = (n * n)$$

- Now suppose that our domain is all living people,
and that $*$ means "is the child of".

- Is the formula satisfiable under this interpretation?

Conjunctions

• Note that universal quantification is similar to *conjunction*.

Suppose the domain is the numbers {2, 4, 6}. Then

$$\forall n \cdot Even(n)$$

is the same as

$$Even(2) \wedge Even(4) \wedge Even(6).$$

• Existential quantification is similar to disjunction. Thus with the same domain,

$$\exists n \cdot Even(n)$$

is the same as

$$Even(2) \vee Even(4) \vee Even(6)$$

- The universal and existential quantifiers are in fact *duals* of each other:

$$\forall x \cdot P(x) \Leftrightarrow \neg \exists x \cdot \neg P(x)$$

Saying that everything has some property is the same as saying that there is nothing that does not have the property.

$$\exists x \cdot P(x) \Leftrightarrow \neg \forall x \cdot \neg P(x)$$

Saying that there is something that has the property is the same as saying that its not the case that everything doesn't have the property.

Validity

- In propositional logic, we saw that some formulae were tautologies — they had the property of being true under all interpretations.
- We also saw that there was a procedure which could be used to tell whether any formula was a tautology — this procedure was the truth-table method.
- A formula of FOL that is true under all interpretations is said to be *valid*.
- So in theory we could check for validity by writing down all the possible interpretations and looking to see whether the formula is true or not.

Decidability and Undecidability

- Unfortuately in general we can't use this method.
- Consider the formula:

$$\forall n \cdot Even(n) \Rightarrow \neg Odd(n)$$

- There are an infinite number of interpretations.
- Is there any other procedure that we can use, that will be guaranteed to tell us, in a finite amount of time, whether a FOL formula is, or is not, valid?
- The answer is *no*.
- FOL is for this reason said to be *undecidable*.

Proof in FOL

- Proof in FOL is similar to propositional logic (PL); we just need an extra set of rules, to deal with the quantifiers.
- FOL inherits all the rules of PL.
- To understand FOL proof rules, need to understand *substitution*.
- The most obvious rule, for ∀-E.

Tells us that if everything in the domain has some property, then we can infer that any particular individual has the property.

$$\frac{\vdash \forall x \cdot \phi(x);}{\vdash \phi(a)}\ \forall\text{-E}\qquad \text{for any } a \text{ in the domain}$$

Going from *general* to *specific*

Example 1

Let's use ∀-E to get the Socrates example out of the way.

$$Man(s); \forall x \cdot Man(x) \Rightarrow Mortal(x)$$
$$\vdash Mortal(s)$$

1.	$Man(s)$	Given
2.	$\forall x \cdot Man(x) \Rightarrow Mortal(x)$	Given
3.	$Man(s) \Rightarrow Mortal(s)$	2, ∀-E
4.	$Mortal(s)$	1, 3, ⇒-E

- Existential Introduction Rule 1 (∃-I(1)).
- We can also go from the general to the slightly less specific!

$$\frac{\vdash \forall x \cdot \phi(x);}{\vdash \exists x \cdot \phi(x)} \; \exists\text{-I}(1) \qquad \text{if domain not empty}$$

Note the *side condition*.

The ∃ quantifier *asserts the existence* of at least one object.

The ∀ quantifier does not.

- Existential Introduction Rule 2 (∃-I(2)).

- We can also go from the very specific to less specific.

$$\frac{\vdash \phi(a);}{\vdash \exists x \cdot \phi(x)}\ \exists\text{-I}(2)$$

- In other words once we have a concrete example, we can infer there exists something with the property of that example.

• We often informally make use of arguments along the lines…

1. We know somebody is the murderer.

2. Call this person *a*.

3. …

(Here, *a* is called a *Skolem constant*.)

• We have a rule which allows this, but we have to be careful how we use it!

$$\frac{\vdash \exists x \cdot \phi(x);}{\vdash \phi(a)} \; \exists\text{-E} \quad a \text{ doesn't occur elsewhere}$$

- Here is an invalid use of this rule:

1.	$\exists x \cdot Boring(x)$	Given
2.	$Lecture(AI)$	Given
3.	$Boring(AI)$	1, $\exists$-E

- (The conclusion may be true, the argument isn't sound.)

Example 2

1. Everybody is either happy or rich.
2. Simon is not rich.
3. Therefore, Simon is happy.

Predicates:

- $H(x)$ means x is happy;
- $R(x)$ means x is rich.

• *Formalisation*:

$\forall x\ \ H(x) \vee R(x); \neg R(Simon) \vdash H(Simon)$

Proof

1.	$\forall x.H(x) \vee R(x)$	Given
2.	$\neg R(Simon)$	Given
3.	$H(Simon) \vee R(Simon)$	1, $\forall$-E
4.	$\neg H(Simon) \Rightarrow R(Simon)$	3, defn $\Rightarrow$
5.	$\neg H(Simon)$	Assumption
6.	$R(Simon)$	4, 5, $\Rightarrow$-E
7.	$R(Simon) \wedge \neg R(Simon)$	2, 6, $\wedge$-I
8.	$\neg\neg H(Simon)$	5, 7, $\neg$-I
9.	$H(Simon) \Leftrightarrow \neg\neg H(Simon)$	PL axiom
10.	$(H(Simon) \Rightarrow \neg\neg H(Simon))$ $\wedge(\neg\neg H(Simon) \Rightarrow H(Simon))$	9, defn $\Leftrightarrow$
11.	$\neg\neg H(Simon) \Rightarrow H(Simon)$	10,$\wedge$-E
12.	$H(Simon)$	8, 11, $\Rightarrow$-E

Summary

- This lecture looked at predicate (or first order) logic.
- Predicate logic is a generalization of propositional logic.
- The generalization requires the use of quantifiers, and these need special rules for handling them when doing inference.
- We looked at how the proof rules for propositional logic need to be extended to handle quantifiers.