# Structural Theorems for Symbolic Summation

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**Abstract** Starting with Karr's structural theorem for summation —the discrete version of Liouville's structural theorem for integration— we work out crucial properties of the underlying difference fields. This leads to new and constructive structural theorems for symbolic summation. E.g., these results can be applied for harmonic sums which arise frequently in particle physics.

**Keywords** structural theorems · symbolic summation · difference fields · symbolic integration · Liouville's theorem

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## 1 Introduction

In [21, 22] M. Karr developed a summation algorithm in which indefinite nested sums and products can be simplified. More precisely, such expressions are rephrased in a $\Pi\Sigma$-field $\mathbb{F}$, a very general class of difference fields[1], and first order linear difference equations defined over $\mathbb{F}$ are solved by Karr's algorithm. In this way, one can decide constructively, if a given indefinite sum or product with a summand or multiplicand $f$ from $\mathbb{F}$ can be expressed in terms of $\mathbb{F}$. For instance, given $\mathbb{F} = \mathbb{Q}(k)(S_1(k), S_2(k), S_3(k))$ where $S_r(k) = \sum_{i=1}^{k} \frac{1}{i^r}$ denotes the generalized harmonic numbers of order $r \geq 1$ and given

$$f(k) = \frac{\left(S_2(k)(k+1)^2 + 1\right) S_3(k) + S_1(k)((k+1)S_3(k) - S_2(k))}{S_3(k)\left(S_3(k)(k+1)^3 + 1\right)} \in \mathbb{F},$$

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[1] Throughout this article all fields contain the rational numbers $\mathbb{Q}$ as subfield.

Karr's algorithm decides constructively if there is an antidifference $g \in \mathbb{F}$ for $f$, i.e.,

$$g(k+1) - g(k) = f(k); \tag{1}$$

In our concrete example, the algorithm produces the solution $g(k) = \frac{S_1(k)S_2(k)}{S_3(k)}$. Then summing the telescoping equation (1) over $k$ leads to the simplification

$$\sum_{i=1}^{k} f(i) = \frac{S_2(k)(k+1)^2 + S_1(k)\left(S_2(k)(k+1)^3 + k + 1\right) + 1}{S_3(k)(k+1)^3 + 1} - 1 \in \mathbb{F}.$$

Karr's algorithm can be considered as the discrete analogue of Risch's algorithm [36,37] for indefinite integration. Here the essential building blocks of exponentials and logarithms can be expressed in terms of an elementary differential field $\mathbb{F}$, and Risch's algorithm can decide constructively, if for a given $f \in \mathbb{F}$ there exists an antiderivative $g \in \mathbb{F}$, i.e.,

$$D(g) = f; \tag{2}$$

here $D$ denotes the differential operator acting on the elements of $\mathbb{F}$. In this regard, Liouville's theorem of integration, see, e.g., [28,31,38], plays an important role. In a nutshell, it states that for integration with elementary functions it suffices to restrict to logarithmic extensions, i.e., one can neglect exponential and algebraic function extensions; for an explicit formulation we refer to Section 2.1. In particular, Risch's algorithm provides a constructive version of Liouville's theorem: his algorithm finds such an extension in terms of logarithms for a given input integral, or it outputs that there does not exist such an extension in which the integral is expressible.

Inspired by Rosenlicht's algebraic proof [38] of Liouville's theorem, Karr could derive a structural theorem for symbolic summation [21,22]. To be more precise, he refined his $\Pi\Sigma$-difference field theory to the so-called reduced and normalized $\Pi\Sigma$-fields in which a discrete version of Liouville's theorem is applicable. For instance, given $\mathbb{F}$ from above and given $f(k) \in \mathbb{Q}(k)$, any solution $g(k) \in \mathbb{F}$ of (1) has the form

$$g(k) = w(k) + c_1 S_1(k) + c_2 S_2(k) + c_3 S_3(k) \tag{3}$$

for some $w(k) \in \mathbb{Q}(k)$ and $c_1, c_2, c_3 \in \mathbb{Q}$.

In previous work [42–44,23,48,24,45,25] we incorporated and generalized, e.g., the ($q$–)hypergeometric algorithms presented in [2,18,54,34,32,35,33,6, 20,4], the summation of ($q$–)harmonic sums [10,51,29,11,1] arising, e.g, in particle physics, and parts of the holonomic approach [53,52,15,14] in Karr's unified framework of $\Pi\Sigma$-difference fields [21]. Here we restricted ourself to $\Pi\Sigma^*$-extensions and $\Pi\Sigma^*$-fields being slightly less general than Karr's $\Pi\Sigma$-fields, but covering all sums and products treated explicitly by Karr's work.

In this article we turn Karr's theorem to constructive and refined versions. Based on the algorithm given in [40] we show that any $\Pi\Sigma^*$-field can be transformed to a reduced $\Pi\Sigma^*$-field in which Karr's structural theorem can be applied; see Theorem 23. In addition, we complement Karr's structural results by taking into account the nesting depth of the recursively defined $\Pi\Sigma^*$-extensions. We show how any $\Pi\Sigma^*$-field can be transformed to a completely reduced ordered $\Pi\Sigma^*$-field in which one can bound the nesting depth of an indefinite nested sum; see Theorem 39. Finally, we relate these results with the difference field theory of depth-optimal $\Pi\Sigma^*$-fields that have been introduced recently [41,47,49]. Comparing Karr's approach and depth-optimal $\Pi\Sigma^*$-extensions we obtain additional insight in $\Pi\Sigma$-field theory (see Theorems 19,30,54), and we derive new structural theorems that are independent of the order in which the generating elements are adjoined; see Theorems 48, 50.

We stress that the suggested results and the underlying algorithms implemented in the summation package Sigma [46] play an important role in the simplification of d'Alembertian solutions [30,3,39], a subclass of Liouvillian solutions [19] of a given recurrence relation. In this regard, special emphasis is put on the simplification of harmonic sum expressions that arise frequently in particle physics; we refer to [7–9] for typical examples in the frame of difference fields.

The general structure of this article is as follows. In Section 2 we state Liouville's structural theorem, and we relate it to Karr's results in terms of reduced $\Pi\Sigma^*$-fields. In Section 3 we work out the crucial properties of reduced $\Pi\Sigma^*$-extensions, and in Section 4 we show that any $\Pi\Sigma^*$-field can be transformed algorithmically to a reduced $\Pi\Sigma^*$-field. In Section 5 reduced extensions are refined to completely reduced extensions. In Section 6 we focus on structural theorems that bound the nesting depth of a telescoping solution; it turns out that this is only possible if the reduced extensions are built up in a particular ordered way. Finally, in Section 7 we relate depth-optimal $\Pi\Sigma^*$-extensions to reduced and completely reduced $\Pi\Sigma^*$-extensions. We present structural theorems that are independent of the order of the explicitly given tower of extensions.

## 2 Liouville's and Karr's structural theorems

We start with a short outline of Liouville's theorem for differential fields and relate it to Karr's achievements for the discrete analogue of difference fields.

### 2.1 An outline of Liouville's theorem

Let $(\mathbb{F}, D)$ be a *differential field*, i.e., $\mathbb{F}$ is a field with a function $D\colon \mathbb{F} \to \mathbb{F}$ such that $D(a+b) = D(a) + D(b)$ and $D(a\,b) = D(a)b + aD(b)$ for all $a, b \in \mathbb{F}$; $D$ is also called *differential operator*. The set of constants is defined by $\text{const}_D\mathbb{F} = \{c \in \mathbb{F} | D(c) = 0\}$; note that $\text{const}_D\mathbb{F}$ (also called constant field) forms a subfield of $\mathbb{F}$ which contains $\mathbb{Q}$. A differential field $(\mathbb{E}, \tilde{D})$ is called a *differential field extension* of a differential field $(\mathbb{F}, D)$ if $\mathbb{F}$ is a subfield of $\mathbb{E}$ and $\tilde{D}(a) = D(a)$ for all $a \in \mathbb{F}$; subsequently, we do not distinguish anymore between $D$ and $\tilde{D}$. Finally, a differential field extension $(\mathbb{F}(t), D)$ of $(\mathbb{F}, D)$ is called *elementary*, see, e.g., [12, Def. 5.1.3] if $t$ is algebraic over $\mathbb{F}$ or if $t$ is transcendental over $\mathbb{F}$ and

(1) $D(t) = D(b)/b$ for some $b \in \mathbb{F}^*$ (a *logarithm*)
(2) $D(t)/t = D(b)$ for some $b \in \mathbb{F}$ (an *exponential*).

In addition, an extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is called *elementary*, if it is a tower of elementary extensions. Then Liouvillian's theorem reads as follows.

**Theorem 1** *(Liouville's theorem) Let $(\mathbb{E}, D)$ be an elementary extension of $(\mathbb{F}, D)$ with $\mathrm{const}_D\mathbb{E} = \mathrm{const}_D\mathbb{F}$, and let $f \in \mathbb{F}$. If there is a $g \in \mathbb{E}$ with (2), then there are $w \in \mathbb{F}$, $c_1, \dots, c_n \in \mathrm{const}_D\mathbb{F}$ and $f_1 \dots, f_n \in \mathbb{F}^*$ such that*

$$f = D(w) + \sum_{i=1}^{n} c_i \frac{D(f_i)}{f_i}.$$

In other words, it suffices to search for a solution $g$ with (2) in logarithmic extensions, and one can neglect algebraic or exponential extensions.

*Remark* **2** Liouville's theorem has been observed already by Laplace [27, p.7] — but the first precise formulation together with a proof based on analytic arguments has been given by Liouville [28]. In particular, the first algebraic proof in terms of differential fields has been provided by [31]; a complete proof dealing also with algebraic extensions has been accomplished by Rosenlicht [38]. For an extensive list of literature and generalizations/refinements, like, e.g.,[50], we refer to [12].

A constructive version of Liouville's theorem was given by Risch in [36, 37]. For instance, let $(\mathbb{F}, D)$ be a differential field with $\mathbb{K} = \mathrm{const}_D\mathbb{F}$ given by a tower of elementary transcendental extensions over the differential field $(\mathbb{K}(x), D)$ with $D(x) = 1$. Then Risch's algorithm can decide in a finite number of steps, if for a given $f \in \mathbb{F}$ there exists a tower of elementary transcendental extensions $(\mathbb{F}(t_1)\dots(t_e), D)$ of $(\mathbb{F}, D)$ in which we have $g$ with (2); in particular, if such an extension exists, it computes such $w$, $f_i$ and $c_i$ as given in Theorem 1. For a detailed description of this algorithm see [12].

### 2.2 Karr's summation theorems

M. Karr [21,22] developed a theory of $\Pi\Sigma$-difference fields which can be considered as the discrete version of elementary transcendental extensions (whose constant fields remain unchanged). In this context we need the following definitions. Let $(\mathbb{F}, \sigma)$ be a *difference field*, i.e., $\mathbb{F}$ is a field and $\sigma\colon \mathbb{F} \to \mathbb{F}$ is a field automorphism, and define the set of constants by $\mathrm{const}_\sigma\mathbb{F} := \{c \in \mathbb{F} | \sigma(c) = c\}$; as in the differential case, $\mathrm{const}_\sigma\mathbb{F}$ forms a subfield of $\mathbb{F}$ which contains $\mathbb{Q}$; $\mathrm{const}_\sigma\mathbb{F}$ is also called the *constant field* of $(\mathbb{F}, \sigma)$. In such a difference field we define the *forward difference operator* as follows: for $a \in \mathbb{F}$,

$$\Delta(a) := \sigma(a) - a.$$

A difference field $(\mathbb{E}, \tilde{\sigma})$ is a *difference field extension* of a difference field $(\mathbb{F}, \sigma)$ if $\mathbb{F}$ is a subfield of $\mathbb{E}$ and $\tilde{\sigma}(a) = \sigma(a)$ for all $a \in \mathbb{F}$; subsequently, we do not distinguish between $\sigma$ and $\tilde{\sigma}$ anymore.

In the following we introduce $\Pi\Sigma^*$-extensions being slightly less general than Karr's $\Pi\Sigma$-fields [21], but covering all sums and products treated explicitly by Karr's work. A difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is a *$\Pi\Sigma^*$-extension* if $t$ is transcendental over $\mathbb{F}$, $\text{const}_\sigma\mathbb{F}(t) = \text{const}_\sigma\mathbb{F}$ and one of the following holds:

(1) $\Delta(t) = b$ for some $b \in \mathbb{F}^*$ (a *$\Sigma^*$-extension*)
(2) $\sigma(t)/t = b$ for some $b \in \mathbb{F}^*$ (a *$\Pi$-extension*).

$(\mathbb{F}(t_1)\dots(t_e), \sigma)$ is a *$\Pi\Sigma^*$-extension* (resp. a $\Sigma^*$-extension or a $\Pi$-extension) of $(\mathbb{F}, \sigma)$ if it is a tower of such extensions (this implies that $\text{const}_\sigma\mathbb{F}(t_1)\dots(t_e) = \text{const}_\sigma\mathbb{F}$). A difference field $(\mathbb{K}(t_1)\dots(t_e), \sigma)$ is a *$\Pi\Sigma^*$-field over* $\mathbb{K}$ if it is a $\Pi\Sigma^*$-extension of $(\mathbb{K}, \sigma)$ and $\text{const}_\sigma\mathbb{K} = \mathbb{K}$.
If it is clear from the context, we identify a $\Pi\Sigma^*$-extension with the explicitly given generating element $t_i$ of the corresponding field extension; see, e.g., Definition 7.

*Example* **3** We rephrase $\mathbb{Q}(k)(S_1(k), S_2(k), S_3(k))$ from Section 1 in terms of a $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ as follows. Consider the difference field $(\mathbb{Q}, \sigma)$ with $\sigma(q) = q$ for all $q \in \mathbb{Q}$, i.e., $\text{const}_\sigma\mathbb{Q} = \mathbb{Q}$. Now take the rational function field $\mathbb{Q}(k)$ and extend the field automorphism $\sigma$ to $\sigma\colon \mathbb{Q}(k) \to \mathbb{Q}(k)$ by $\sigma(k) = k + 1$; note that $\sigma$ is uniquely determined in this way. Since $\text{const}_\sigma\mathbb{Q}(k) = \text{const}_\sigma\mathbb{Q} = \mathbb{Q}$, $(\mathbb{Q}(k), \sigma)$ forms a $\Sigma^*$-extension of $(\mathbb{Q}, \sigma)$. Next, we represent the harmonic numbers $S_1(k)$ with the shift behavior $S_1(k+1) = S_1(k) + \frac{1}{k+1}$ as follows. Define (uniquely) the difference field extension $(\mathbb{Q}(k)(s_1), \sigma)$ of $(\mathbb{Q}(k), \sigma)$ such that $s_1$ is transcendental and $\sigma(s_1) = s_1 + \frac{1}{k+1}$. Again, since $\text{const}_\sigma\mathbb{Q}(k)(s_1) = \mathbb{Q}$, this forms a $\Sigma^*$-extension. In this way, $S_1(k)$ is rephrased by the variable $s_1$, and the shift operator in $k$ acting on $S_1(k)$ is modeled by the field automorphism $\sigma$. Repeating this approach, we obtain the $\Sigma^*$-extension $(\mathbb{F}, \sigma)$ of $(\mathbb{Q}, \sigma)$ with the rational function field $\mathbb{F} = \mathbb{Q}(k)(s_1)(s_2)(s_3)$ and with the field automorphism $\sigma\colon \mathbb{F} \to \mathbb{F}$ uniquely defined by

$$\sigma(k) = k+1, \quad \sigma(s_1) = s_1 + \tfrac{1}{k+1}, \quad \sigma(s_2) = s_2 + \tfrac{1}{(k+1)^2}, \quad \sigma(s_3) = s_3 + \tfrac{1}{(k+1)^3}; \tag{4}$$

since $\text{const}_\sigma\mathbb{F} = \mathbb{Q}$, $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{Q}$. In particular, the sums $S_1(k), S_2(k), S_3(k)$ and the shift operator acting on them are modeled by the variables $s_1, s_2, s_3$ and the field automorphism $\sigma$.

*Remark* **4** Note that, e.g., $\log(x)$ with $D\log(x) = \frac{1}{x}$ and the harmonic numbers $S_1(k) = \sum_{i=1}^{k} \frac{1}{i}$ with $\Delta(S_1(k)) = \frac{1}{k+1}$ are closely related; in particular $\lim_{k\to\infty}(H_k - log(k)) = \gamma$ where $\gamma = 0.5772...$ denotes Euler's constant. Similarities between elementary unimonomial extensions and $\Pi\Sigma^*$-extensions in the algebraic setting of difference/differential fields are worked out, e.g., in [13].

As it turns out, the discrete version of Liouville's structural theorem in the context of $\Pi\Sigma^*$-extensions can be stated in the following surprisingly simple form: a sum of $f \in \mathbb{F}$ is either expressible in $\mathbb{F}$ or it can be represented by one $\Sigma^*$-extension; in particular, one can neglect $\Pi$-extensions. This follows by the following result.

**Theorem 5** *([21]) Let $(\mathbb{F}(t), \sigma)$ be an extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = t + f$ for some $f \in \mathbb{F}$. Then this is a $\Sigma^*$-extension iff there is no $g \in \mathbb{F}$ such that $\sigma(g) = g + f$.*

Namely, let $(\mathbb{F}, \sigma)$ be a difference field with $f \in \mathbb{F}$. Then either there exists a solution[2] $g \in \mathbb{F}$ of the telescoping equation

$$\Delta(g) = f; \tag{5}$$

or if not, there is the $\Sigma^*$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(t) = t + f$ by Theorem 5, i.e., $t$ forms a solution of (5).
Similar to Risch, Karr developed an algorithm in [21] which makes these observations constructive. Given a $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}$ and given $f \in \mathbb{F}$, decide in finite number of steps if there is a $g \in \mathbb{F}$ s.t. (5) holds; if yes, output such a $g$.

In a nutshell, a sum $S(k) = \sum_{i=1}^{k} F(i)$ can be modeled in Karr's framework as follows. First, construct a difference field $(\mathbb{F}, \sigma)$ in which one represents the shifted summand $F(i + 1)$ by an explicitly given $f \in \mathbb{F}$. Then either one finds $g \in \mathbb{F}$ such that (5) holds and $S(k)$ can be represented by $g + c$ for some $c \in \text{const}_\sigma\mathbb{F}$; or one constructs the $\Sigma^*$-extension $(\mathbb{F}(s), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(s) = s + f$ and one can model $S(k)$ by $s$.

In all our examples we will stick to harmonic sums which are defined as follows [10,51]: for positive integers $m_1, \dots, m_r \in \mathbb{N} \setminus \{0\}$,

$$S_{m_1,\dots,m_r}(k) = \sum_{i_1=1}^{k} \frac{1}{i_1^{m_1}} \sum_{i_2=1}^{i_1} \frac{1}{i_2^{m_2}} \cdots \sum_{i_r=1}^{i_{r-1}} \frac{1}{i_r^{m_r}};$$

in addition, truncated Euler sums [17] of the form $\sum_{i=1}^{k} \frac{S_{m_1}(i) \dots S_{m_r}(i)}{i^u}$ for some $u \in \mathbb{N} \setminus \{0\}$ will arise.

*Example* **6** We start with the $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ from Example 3 in which we model $S_1(k), S_2(k), S_3(k)$. In order to represent in addition the harmonic sum $S_{1,3}(k) = \sum_{i=1}^{k} \frac{S_3(i)}{i}$ with the shift behavior $S_{1,3}(k + 1) = S_{1,3}(k) + \frac{S_3(k+1)}{k+1}$ we proceed as follows. Take $f = \frac{\sigma(s_3)}{k+1} \in \mathbb{F}$. Using, e.g., Karr's algorithm, or the simplified version [44] implemented in the summation package Sigma, one can check that there is no $g \in \mathbb{F}$ such that (5) holds. Hence we can construct the $\Sigma^*$-extension $(\mathbb{F}(s_{1,3}), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(s_{1,3}) = s_{1,3} + f$. In this way, the harmonic sum $S_{1,3}(k)$ is represented by $s_{1,3}$ where the shift operator acting on $S_{1,3}(k)$ is reflected by the field automorphism $\sigma$ acting on $s_{1,3}$. Completely analogously, we can represent the harmonic sum $S_{6,1,3}(k)$

[2] Note that the telescoping problem (1) is rephrased in the algebraic setting of difference fields.

by $s_{6,1,3}$ and the truncated Euler sum $\sum_{i=1}^{k} \frac{S_2(i)S_3(i)}{i}$ by $x$. In summary, we construct the difference field extension $(\mathbb{F}(s_{1,3})(x)(s_{6,1,3}), \sigma)$ of $(\mathbb{F}, \sigma)$ with the rational function field $\mathbb{F}(s_{1,3})(x)(s_{6,1,3})$ and with

$$\sigma(s_{1,3}) = s_{1,3} + \frac{\sigma(s_3)}{k+1},\ \sigma(x) = x + \frac{\sigma(s_2)\,\sigma(s_3)}{k+1},\ \sigma(s_{6,1,3}) = s_{6,1,3} + \frac{\sigma(s_{1,3})}{(k+1)^6}; \tag{6}$$

in particular, we can check algorithmically that this extension forms a tower of $\Sigma^*$-extensions by verifying iteratively the non-existence of solutions of the corresponding telescoping problems. Note also that one can verify by the same mechanism that the base field $(\mathbb{F}, \sigma)$ constructed in Ex. 3 forms a $\Pi\Sigma^*$-field over $\mathbb{Q}$.

We remark that Karr's framework covers also $q$–analogues of harmonic sums [5, 16, 11] or generalized harmonic sums [29]; for a package which combines the ideas of [10, 51, 29] with the difference field approach see [1].

### 2.3 Karr's structural theorem

In [21, 22] Karr arrives at the following conclusion: one can predict the structure of a solution $g$ for (5) in a refined version of $\Pi\Sigma$-fields; see [22, page 314]. For $\Pi\Sigma^*$-extensions this refinement reads as follows.

*Definition* **7** A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is called *reduced over* $\mathbb{F}$, or in short *reduced*, if for any $\Sigma^*$-extension $t_i$ $(1 \le i \le e)$ with $f := \Delta(t_i) \in \mathbb{F}(t_1)\dots(t_{i-1}) \setminus \mathbb{F}$ the following property holds: there do not exist a $g \in \mathbb{F}(t_1)\dots(t_{i-1})$ and an $f' \in \mathbb{F}$ such that

$$\Delta(g) + f' = f. \tag{7}$$

The following special case is immediate.

**Lemma 8** *Let $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t_i) - t_i \in \mathbb{F}$ or $\sigma(t_i)/t_i \in \mathbb{F}$ for $1 \le i \le e$. Then this extension is reduced.*

In Section 4 we provide an algorithmic approach which enables one to check whether a $\Pi\Sigma^*$-extension is reduced. In particular, if this is not the case, this machinery automatically transforms the given extension to an isomorphic difference field which is built by a tower of reduced $\Pi\Sigma^*$-extensions; see Theorem 23. In other words, one can always apply the following structural theorem (in a given reduced $\Pi\Sigma^*$-extension or in an isomorphic extension which is reduced).

**Theorem 9** *(Karr's structural theorem) Let $(\mathbb{E}, \sigma)$ be a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ and $\sigma(t_i) = a_i\, t_i + f_i$ (where either $a_i = 1$ or $f_i = 0$), and define*[3]

$$S := \{1 \le i \le e | \Delta(t_i) \in \mathbb{F}\}; \tag{8}$$

[3] Note that $S$ consists of exactly those $i$ such that $t_i$ is a $\Sigma^*$-extension, and $f_i = \Delta(t_i) \in \mathbb{F}$.

*let $f \in \mathbb{F}$. If there is a $g \in \mathbb{E}$ with (5), there are $w \in \mathbb{F}$ and $c_i \in \text{const}_\sigma\mathbb{F}$ s.t.*

$$f = \Delta(w) + \sum_{i \in S} c_i f_i; \tag{9}$$

*in particular, for any such $g$ there is some $c \in \text{const}_\sigma\mathbb{F}$ such that*

$$g = c + w + \sum_{i \in S} c_i t_i. \tag{10}$$

For a proof in the context of $\Pi\Sigma$-fields we refer the reader to [22, Result, page 315], and for the corresponding proof for reduced $\Pi\Sigma^*$-extensions as given in Theorem 9 we refer the reader to [39, Thm 4.2.1]; the proofs follow Rosenlicht's proof strategy [38] of Liouville's theorem.

We emphasize that Karr's result exceeds Liouville's theorem in the following sense: given a reduced $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ and given $f \in \mathbb{F}$ one can forecast to a certain extent how the solution $g \in \mathbb{E}$ is composed; for a typical application see, e.g., page 15.

*Example* **10** Consider the $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ over $\mathbb{Q}$ with $\mathbb{F} = \mathbb{Q}(k)(s_1)(s_2)(s_3)$ and (4). Note that $(\mathbb{F}, \sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{Q}(k), \sigma)$ by Lemma 8. Hence by Theorem 9 any solution $g \in \mathbb{F}$ of (5) for a given $f \in \mathbb{Q}(k)$ is of the form

$$g = w + c_1\, s_1 + c_2\, s_2 + c_3\, s_3 \qquad \text{for some } w \in \mathbb{Q}(k) \text{ and } c_1, c_2, c_3 \in \mathbb{Q}; \tag{11}$$

for a precise formulation of how (3) and (11) are related, we refer the reader to [48, 49]

*Example* **11** Start with the $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ over $\mathbb{Q}$ from Example 10, and consider the $\Sigma^*$-extension $(\mathbb{F}(s_{1,3})(x)(s_{6,1,3}), \sigma)$ of $(\mathbb{F}, \sigma)$ with (6) from Example 6; later we can check that this extension is reduced over $\mathbb{F}$; see Example 29. Hence for any $g \in \mathbb{F}(s_{1,3})(x)(s_{6,1,3})$ with (5) for some $f \in \mathbb{F}$ it follows that

$$g = w + c_1\, s_{1,3} + c_2\, x \qquad \text{for some } c_1, c_2 \in \mathbb{Q} \text{ and } w \in \mathbb{F}. \tag{12}$$

*Example* **12** Again, start with the $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ over $\mathbb{Q}$ from Example 10, and consider the $\Sigma^*$-extension $(\mathbb{F}(s_{1,3})(x)(s_{2,1,3}), \sigma)$ of $(\mathbb{F}, \sigma)$ with

$$\sigma(s_{1,3}) = s_{1,3} + \frac{\sigma(s_3)}{k+1}, \quad \sigma(x) = x + \frac{\sigma(s_2)\,\sigma(s_3)}{k+1}, \quad s_{2,1,3} = s_{2,1,3} + \frac{\sigma(s_{1,3})}{(k+1)^2}.$$

In this instance, the extension is not reduced. E.g., for $f = \frac{(k+1)^5+1}{(k+1)^6}$ there is

$$g = -s_3^2 + 2x + s_1 - 2 s_{1,3} s_2 + 2\, s_{2,1,3} \tag{13}$$

s.t. (5) holds: if this extension were reduced, $g$ should be free of $s_{2,1,3}$ and $g$ should contain $s_{1,3}$ only in the form $c\, s_{1,3}$ for some $c \in \mathbb{Q}$ by Theorem 9.

*Remark* **13** Reinterpreting the variables in $f$ and $g$ of the previous example as harmonic sums and summing (1) over $k$ lead to the following identity: for $k \geq 0$,

$$\sum_{i=1}^{k} \frac{i^5+1}{i^6} = -S_3(k)^2 + 2\sum_{i=1}^{k} \frac{S_2(i)S_3(i)}{i} + S_1(k) - 2S_{1,3}(k)S_2(k) + 2\,S_{2,1,3}(k).$$

Obviously, the obtained right hand side is more complicated (i.e., consists of sums with higher nesting depth) than the given left hand side. In Sections 6 and 7 we work out in details why this is the case in general; for our particular case see Ex. 35.

### 2.4 A simple structure theorem for $\Pi\Sigma^*$-extensions

We conclude this section with the following simple "structural theorem" which is valid for any $\Pi\Sigma^*$-extension. Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with the rational function field $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ and $\sigma(t_i) = a_i\,t_i$ or $\sigma(t_i) = t_i + a_i$ for $1 \leq i \leq e$; let $f \in \mathbb{E}$. We say that one of the generating elements $t_i$ of the rational function field extension does not occur in $f$ if $f \in \mathbb{F}(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_e)$; the latter field is considered as a subfield of $\mathbb{E}$. Now we define the *set of leaf extensions which do not occur in* $f$ by

$$\mathrm{Leaves}_{\mathbb{F}\leq\mathbb{E}}(f) := \{t_i | t_i \text{ does not occur in } f \text{ and } a_{i+1}, \dots, a_e\},$$

and we define the *set of inner node extensions or extensions that occur in* $f$ by

$$\mathrm{InnerNodes}_{\mathbb{F}\leq\mathbb{E}}(f) := \{t_1, \dots, t_e\} \setminus \mathrm{Leaves}_{\mathbb{F}\leq\mathbb{E}}(f);$$

those extensions which are $\Sigma^*$-extensions are denoted by

$$\Sigma^*\text{–}\mathrm{Leaves}_{\mathbb{F}\leq\mathbb{E}}(f) := \{t \in \mathrm{Leaves}_{\mathbb{F}\leq\mathbb{E}}(f) | t \text{ is a } \Sigma^*\text{-extension}\}.$$

We denote all $\Sigma^*$-extensions being leave extensions by $\Sigma^*\text{–}\mathrm{Leaves}_{\mathbb{F}\leq\mathbb{E}} := \Sigma^*\text{–}\mathrm{Leaves}_{\mathbb{F}\leq\mathbb{E}}(1)$.

At this point the following remark is in place. If there is a permutation $\tau \in S_e$ such that $a_{\tau(i)} \in \mathbb{F}(t_{\tau(1)})\dots(t_{\tau(i-1)})$ for all $i$ with $1 \leq i \leq e$, then $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(e)}), \sigma)$ forms again a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$. In particular, one can reorder the $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $f \in \mathbb{E}$ to $(\mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_s), \sigma)$ such that

$$\mathrm{InnerNodes}_{\mathbb{F}\leq\mathbb{E}}(f) = \{x_1, \dots, x_r\} \tag{14}$$

and $\mathrm{Leaves}_{\mathbb{F}\leq\mathbb{E}}(f) = \{y_1, \dots, y_s\}$; note that $\frac{\sigma(y_i)}{y_i} \in \mathbb{F}(x_1)\dots(x_r)$ or $\sigma(y_i) - y_i \in \mathbb{F}(x_1)\dots(x_r)$ for $1 \leq i \leq s$. Hence $(\mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_s), \sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ by Lemma 8. Thus we can apply Theorem 9, and we arrive at the following result.

**Theorem 14** *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $f\in\mathbb{E}$ and define $\{x_1,\dots,x_r\}$ by* (14). *If there is a $g\in\mathbb{E}$ such that* (5) *holds, then*

$$g=\sum_{a\in\Sigma^*\text{–Leaves}_{\mathbb{F}\leq\mathbb{E}}(f)} c_a a+w \quad \textit{for some } c_a\in\text{const}_\sigma\mathbb{F} \textit{ and } w\in\mathbb{F}(x_1,\dots,x_r).$$

*Example* **15** Consider the $\Pi\Sigma^*$-field $(\mathbb{E},\sigma)$ over $\mathbb{Q}$ from Example 12 with $\mathbb{E}=\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{2,1,3})$, and have a look at the solution (13) of (5) for $f=\frac{(k+1)^5+1}{(k+1)^6}$. Then, as predicted in Theorem 14, the solution (13) is given by a linear combination over $\mathbb{Q}$ in terms of the variables $\Sigma^*\text{–Leaves}_{\mathbb{Q}\leq\mathbb{E}}(f)=\{s_1,x,s_{2,1,3}\}$ plus one expression from $\mathbb{Q}(k,s_2,s_3,s_{1,3})$.

Combining Theorem 19 with Theorem 14 we arrive at

**Theorem 16** *(A refinement of Karr's structural theorem) Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$, let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ and let $f\in\mathbb{F}$. Define $S=\{1\leq i\leq e|\Delta(t_i)\in\mathbb{F}\}=\{i_1,\dots,i_u\}$ and consider the $\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{H}=\mathbb{F}(t_{i_1})\dots(t_{i_u})$; define $\{x_1,\dots,x_r\}:=\text{InnerNodes}_{\mathbb{G}\leq\mathbb{H}}(f)$. If there is a $g\in\mathbb{F}(t_1,\dots,t_e)$ such that* (5) *holds, then*[4]

$$g=\sum_{a\in\Sigma^*\text{–Leaves}_{\mathbb{G}\leq\mathbb{H}}(f)} c_a a+w \textit{ for some } c_a\in\text{const}_\sigma\mathbb{G} \textit{ and } w\in\mathbb{G}(x_1,\dots,x_r).$$

*Example* **17** Take the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{6,1,3}),\sigma)$ over $\mathbb{Q}$ from Example 11, and let

$$f=\frac{k^3+3k^2+3k-s_2-(k+1)(k(k+2)(s_2-4)+s_2-5)s_3+5}{(k+1)^4}.$$

We apply Theorem 16 by choosing $\mathbb{G}=\mathbb{Q}$, $\mathbb{F}=\mathbb{Q}(k)(s_1)(s_2)(s_3)$, and $\mathbb{E}=\mathbb{F}(s_{1,3})(x)(s_{6,1,3})$, namely $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$ with $f\in\mathbb{F}$, and $(\mathbb{E},\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. In this instance, we find $S=\{s_{1,3},x\}$, and we get $\Sigma^*\text{–Leaves}_{\mathbb{Q}\leq\mathbb{F}(s_{1,3})(x)}(f)=\{s_1,s_{1,3},x\}$ and $\text{InnerNodes}_{\mathbb{Q}\leq\mathbb{F}(s_{1,3})(x)}(f)=\{k,s_2,s_3\}$. Hence, for any $g\in\mathbb{F}(s_{1,3})(x)(s_{6,1,3})$ with (5) it follows that $g=w+c_1\,s_1+c_2\,s_{1,3}+c_3\,x$ for some $c_1,c_2,c_3\in\mathbb{Q}$ and $w\in\mathbb{Q}(k,s_2,s_3)$. Note that our prediction refines the version given in (12). Indeed, we find $g=s_3^2+s_1+4s_{1,3}-x$.

## 3 Equivalent characterizations of reduced $\Pi\Sigma^*$-extensions

We work out alternative characterizations of whether a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ is reduced. Here we need the following lemma.

[4] Note that $S\subseteq\Sigma^*\text{–Leaves}_{\mathbb{G}\leq\mathbb{H}}(f)$, i.e., Theorem 16 refines Theorem 19.

**Lemma 18** *Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Sigma^*$*-extension of* $(\mathbb{F},\sigma)$ *with* $\sigma(t)=t+f$ *and* $\mathbb{K}:=\mathrm{const}_\sigma\mathbb{F}$*, and let* $f'\in\mathbb{F}$*. Then there are* $c\in\mathbb{K}$ *and* $g\in\mathbb{F}$ *such that*

$$\Delta(g)+c\,f'=f \tag{15}$$

*iff there is a* $\Sigma^*$*-extension* $(\mathbb{F}(s),\sigma)$ *of* $(\mathbb{F},\sigma)$ *with* $\sigma(s)=s+f'$ *in which we find* $h\in\mathbb{F}(s)$ *such that* $\Delta(h)=f$*.*

*Proof* Suppose that there are a $g\in\mathbb{F}$ and $c\in\mathbb{K}$ such that (15) holds, and assume in addition that there is a $g'\in\mathbb{F}$ such that $\Delta(g')=f'$. Then $\Delta(q)=f$ with $q:=g+c\,g'\in\mathbb{F}$, a contradiction with the fact that $(\mathbb{F}(t),\sigma)$ is a $\Sigma^*$-extension of $(\mathbb{F},\sigma)$ by Theorem 5. Hence $(\mathbb{F}(s),\sigma)$ is a $\Sigma^*$-extension of $(\mathbb{F},\sigma)$ where $s$ satisfies $\Delta(s)=f'$ by Thm. 5. Besides this, for $h:=g+c\,s$ we have $\Delta(h)=\Delta(g)+c\,f'=f$.
Conversely, suppose that there is a $\Sigma^*$-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s)=s+f'$ together with a $h\in\mathbb{F}(s)$ such that $\Delta(h)=f$. Since $(\mathbb{F}(s),\sigma)$ is a reduced $\Sigma^*$-extension of $(\mathbb{F},\sigma)$ by Lemma 8, we can apply Theorem 9, and it follows that $g=c\,s+w$ for some $w\in\mathbb{F}$ and $c\in\mathbb{K}$. Thus, $f=\Delta(g)=\Delta(w)+c\,f'$.

**Theorem 19** *Let* $(\mathbb{E},\sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F},\sigma)$ *with* $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ *and define* $S$ *as in* (8)*. Then the following statements are equivalent.*

*(1) This extension is reduced.*
*(2) For any* $g\in\mathbb{E}$ *with* $\Delta(g)\in\mathbb{F}$ *we have* (10) *for some* $c_i\in\mathrm{const}_\sigma\mathbb{F}$ *and* $w\in\mathbb{F}$*.*
*(3) For any* $\Sigma^*$*-extension* $t_i$ *with* $f:=\Delta(t_i)$ *and* $i\notin S$ *the following property holds: There does not exist a* $\Sigma^*$*-extension* $(\mathbb{F}(t_1)\dots(t_{i-1})(s),\sigma)$ *of* $(\mathbb{F}(t_1)\dots(t_{i-1}),\sigma)$ *with* $\Delta(s)\in\mathbb{F}$ *in which we have* $g$ *with* (5)*.*

*Proof* **(1)** $\Rightarrow$ **(2)** follows by Theorem 9. Now suppose that $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is not a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. Then there is an $i$ with $1\leq i\leq e$ such that $f:=\Delta(t_i)\in\mathbb{F}(t_1,\dots,t_{i-1})\setminus\mathbb{F}$ and (7) for some $f'\in\mathbb{F}$ and $g\in\mathbb{F}(t_1)\dots(t_{i-1})$. Hence, we obtain $\Delta(g')=f'\in\mathbb{F}$ with $g':=t_i-g$. Since $f=\Delta(t_i)\notin\mathbb{F}$, $i\notin S$, and thus **(2)** does not hold. This proves the equivalence of **(1)** and **(2)**. Equivalence **(1)** $\Leftrightarrow$ **(3)** is an immediate consequence of Lemma 18.

*Example* **20** Take the $\Pi\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{2,1,3}),\sigma)$ of $(\mathbb{Q}(k)(s_1)(s_2)(s_3),\sigma)$ from Example 12 which is not reduced. Theorem 19 explains why we can find, e.g., $f=\frac{(k+1)^5+1}{(k+1)^6}$ with (13) such that (5) holds. Equivalently, we can take the $\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s),\sigma)$ of $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x),\sigma)$ with $\sigma(s)=s+f$ such that we get $\Delta(h)=f$ with $h=\frac{1}{2}(s+s_3^2-2x-s_1+2s_{1,3}s_2)$.

In summary, it is precisely the property of being reduced which guarantees that the conclusion of Theorem 9 holds (equivalence (1)$\Leftrightarrow$(2) of Theorem 19). In particular, Theorem 19 relates reduced $\Pi\Sigma^*$-extensions to certain refined $\Sigma^*$-extensions (equivalence (1)$\Leftrightarrow$(3)). This observation will be crucial to connect reduced $\Pi\Sigma^*$-extensions to depth-optimal $\Pi\Sigma^*$-extensions; see Section 7.

## 4 Constructive aspects of reduced $\Pi\Sigma^*$-extensions

In [21] it has been outlined that any $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ can be transformed in principle to a reduced version. Subsequently, we make this more precise in terms of difference field isomorphisms, and we show how such a transformation can be carried out algorithmically. As a consequence, one can always apply Karr's structural theorem 9 constructively in the given extension or in the corresponding transformed one.

$\tau\colon \mathbb{F} \to \mathbb{F}'$ is called a *$\sigma$-isomorphism* (resp. *$\sigma$-monomorphism*) between two difference fields $(\mathbb{F},\sigma)$ and $(\mathbb{F}',\sigma')$ if $\tau$ is a field isomorphism (resp. field monomorphism) and $\tau(\sigma(f)) = \sigma'(\tau(f))$ for all $f \in \mathbb{F}$. In particular, let $(\mathbb{E},\sigma)$ and $(\mathbb{E}',\sigma')$ be difference field extensions of $(\mathbb{F},\sigma)$. Then a $\sigma$-isomorphism (resp. $\sigma$-monomorphism) $\tau\colon \mathbb{E} \to \mathbb{E}'$ is a an *$\mathbb{F}$-isomorphism* (resp. *$\mathbb{F}$-monomorphism*) if $\tau(a) = a$ for all $a \in \mathbb{F}$. We start with the following two lemmas.

**Lemma 21** *Let $(\mathbb{F}(t),\sigma)$ be a $\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = t + f$, and let $f' \in \mathbb{F}$ and $g \in \mathbb{F}$ such that (7) holds. Then for any $c \in \mathrm{const}_\sigma\mathbb{F} \setminus \{0\}$ there is a $\Sigma^*$-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s) = s + c\,f'$ together with an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{F}(t) \to \mathbb{F}(s)$ with $\tau(t) = \frac{s}{c} + g$.*

*Proof* By Lemma 18 there is the $\Sigma^*$-ext. $(\mathbb{F}(x),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(x) = x+f'$. Let $c \in \mathrm{const}_\sigma\mathbb{F} \setminus \{0\}$. By Theorem 5 there is no $h \in \mathbb{F}$ such that $\Delta(h) = f'$. Consequently, there is no $h \in \mathbb{F}$ such that $\Delta(h) = c\,f'$, and thus there is the $\Sigma^*$-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s) = s + c\,f'$. Take the field isomorphism $\tau\colon \mathbb{F}(t) \to \mathbb{F}(s)$ with $\tau(h) = h$ for all $h \in \mathbb{F}$ and $\tau(t) = \frac{s}{c} + g$. By $\tau(\sigma(t)) = \tau(t+f) = \tau(t)+f = \frac{s}{c}+g+f = \frac{s}{c}+f'+\sigma(g) = \sigma(\frac{s}{c}+g) = \sigma(\tau(t))$ it follows that $\tau$ is an $\mathbb{F}$-isomorphism.

**Lemma 22** *[[47, Prop. 18]] Let $(\mathbb{F},\sigma)$, $(\mathbb{F}',\sigma')$ be difference fields with a $\sigma$-isomorphism $\tau\colon \mathbb{F} \to \mathbb{F}'$; let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-ext. of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\,t + \beta$. Then there is a $\Pi\Sigma^*$-extension $(\mathbb{F}'(t'),\sigma)$ of $(\mathbb{F}',\sigma)$ with $\sigma(t') = \tau(\alpha)t' + \tau(\beta)$ together with an $\sigma$-isomorphism $\tau'\colon \mathbb{F}(t) \to \mathbb{F}'(t')$ such that $\tau'|_{\mathbb{F}} = \tau$ and $\tau'(t) = t'$.*

By iterative applications of Lemmas 21 and 22 each $\Pi\Sigma^*$-extension can be transformed to an isomorphic reduced $\Pi\Sigma^*$-extension; see Theorem 23. In particular, this construction can be given explicitly if one can solve the following problem.

| |
|---|
| Problem RS (*R*educed *S*ummation): **Given** a $\Pi\Sigma^*$-extension $(\mathbb{D},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{D} = \mathbb{F}(t_1)\dots(t_e)$, and given $f \in \mathbb{F}$; **find** $g \in \mathbb{D}$ and $f' \in \mathbb{F}(t_1)\dots(t_i)$ as in (7) such that $i$ with $0 \le i \le e$ is minimal. |

In the following we call a difference field $(\mathbb{F},\sigma)$ *RS-computable*, if one can solve problem RS for any $\Pi\Sigma^*$-extension $(\mathbb{D},\sigma)$ of $(\mathbb{F},\sigma)$ and for any $f \in \mathbb{F}$.

**Theorem 23** *For any $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ there is a reduced $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{H} \to \mathbb{E}$. Such a $\Pi\Sigma^*$-ext. $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and $\tau$ can be given explicitly, if $(\mathbb{F},\sigma)$ is RS-computable.*

**Algorithm 1** $\texttt{ToReducedField}(\mathbb{F}(t_1)\dots(t_e), k)$

In: A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i) = a_i\,t_i + b_i$ for $1 \leq i \leq e$; $(\mathbb{F},\sigma)$ is RS-computable.

Out: A reduced $\Pi\Sigma^*$-extension $(\mathbb{F}(x_1)\dots(x_e),\sigma)$ of $(\mathbb{F},\sigma)$, and an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{F}(t_1)\dots(t_e) \to \mathbb{F}(x_1)\dots(x_e)$.

```
Let τ: F → F be the identity map.
FOR i = 1 to e DO
    Set a := τ(a_i); f := τ(b_i); h := x_i.
    IF t_i is a Σ*-extension (a_i = a = 1) THEN
        Let f' ∈ F(x_1)...(x_j) \ F(x_1)...(x_{j-1}) and g ∈ F(x_1)...(x_{i-1})
        be the result of problem RS for f and D = F(x_1)...(x_{i-1}).
        IF j = 0, THEN Set f := f'; h := x_i + g FI
    FI
    Construct the ΠΣ*-extension (F(x_1)...(x_i), σ) of (F(x_1)...(x_{i-1}), σ) with
    σ(x_i) = a x_i + f; extend τ: F(t_1)...(t_{i-1}) → F(x_1)...(x_{i-1}) to the
    F-isomorphism τ: F(t_1)...(t_i) → F(x_1)...(x_i) by τ(t_i) = h. OD
RETURN ((F(x_1)...(x_e), σ), τ).
```

*Proof* The induction base is trivial. Suppose that we are given a $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{H} := \mathbb{F}(x_1)\dots(x_e)$ and a reduced $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ together with an $\mathbb{F}$-isomorphism $\tau\colon \mathbb{H} \to \mathbb{E}$. Now consider the $\Pi\Sigma^*$-extension $(\mathbb{H}(x),\sigma)$ of $(\mathbb{H},\sigma)$ with $\sigma(x) = \alpha\,x + \beta$, and take the $\Pi\Sigma^*$-extension $(\mathbb{E}(t),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(t) = \tau(\alpha)\,t + \tau(\beta)$ by Lemma 22; in particular, we can take the $\mathbb{F}$-isomorphism $\tau'\colon \mathbb{H}(x) \to \mathbb{E}(t)$ with $\tau'(x) = t$ and $\tau'(h) = \tau(h)$ for all $h \in \mathbb{H}$. If $(\mathbb{E}(t),\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$, we are done. If not, $\alpha = 1$, and for $f := \tau(\beta) \in \mathbb{E}$ there are $g \in \mathbb{E}$ and $f' \in \mathbb{F}$ such that (7) holds. Note: if $(\mathbb{F},\sigma)$ is RS-computable, we can solve problem RS, and we get such $f'$ and $g$ explicitly. Then by Lemma 21 there is a $\Sigma^*$-extension $(\mathbb{E}(t'),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(t') = t' + f'$ together with an $\mathbb{F}$-isomorphism $\tau''\colon \mathbb{E}(t) \to \mathbb{E}(t')$ with $\tau''(t) = t' + g$ and $\tau''(h) = \tau'(h)$ for all $h \in \mathbb{E}$. Since $(\mathbb{E},\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ by assumption and since $f' = \Delta(t') \in \mathbb{F}$, $(\mathbb{E}(t'),\sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. Moreover, $\rho := \tau'' \circ \tau'$ is an $\mathbb{F}$-isomorphism from $\mathbb{H}(x)$ to $\mathbb{E}(t')$. In particular, if $\tau\colon \mathbb{H} \to \mathbb{E}$ and $g$ are given explicitly, also $\rho\colon \mathbb{H}(x) \to \mathbb{E}(t')$ can be given explicitly with $\rho(x) = t' + g$ and $\rho(h) = \tau(h)$ for all $h \in \mathbb{H}$.

As a consequence, we obtain Alg. 1; the correctness follows by the proof of Theorem 23. From the point of view of application we rely on the following algorithm [40, Algorithm 1]. Namely, due to its generic specification, e.g., the following classes of difference fields $(\mathbb{F},\sigma)$ are RS-computable, i.e., Algorithm 1 can be executed in the summation package Sigma [46]: $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-field or it is a $\Pi\Sigma^*$-extension over a free difference field [23] or over a difference field containing radicals [24], like $\sqrt{k}$.

*Example* **24** Consider the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_{1,1})(s_{1,1,1}),\sigma)$ over $\mathbb{Q}$ with

$$k = k+1,\ \sigma(s_1) = s_1 + \tfrac{1}{k+1},\ \sigma(s_{1,1}) = s_{1,1} + \tfrac{\sigma(s_1)}{k+1},\ \sigma(s_{1,1,1}) = s_{1,1,1} + \tfrac{\sigma(s_{1,1})}{k+1}. \tag{16}$$

By Thm. 19 the extension is not reduced: we find, e.g., for $f = \frac{1}{(k+1)^3}$ the solution

$$g = s_1^3 - 3s_{1,1}\, s_1 + 3s_{1,1,1} \tag{17}$$

of (5). Subsequently, we transform this extension to a reduced one.

(1) We start with the $\Pi\Sigma^*$-field $(\mathbb{Q}(k), \sigma)$ over $\mathbb{Q}$ with $\sigma(k) = k + 1$ and take the $\mathbb{Q}$-isomorphism $\tau\colon \mathbb{Q}(k) \to \mathbb{Q}(k)$ with $\tau(f) = f$ for all $f \in \mathbb{Q}(k)$.

(2) Now we apply our algorithm for problem RS with $f = \frac{1}{k+1}$: since we do not find $f' \in \mathbb{Q}$ and $g \in \mathbb{Q}(k)$ (by executing the implementation of Sigma), it follows that $(\mathbb{Q}(k)(s_1), \sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{Q}(k), \sigma)$. Hence we keep $(\mathbb{Q}(k)(s_1), \sigma)$ and extend the $\mathbb{Q}$-isomorphism from the field $\mathbb{Q}(k)$ to $\tau\colon \mathbb{Q}(k)(s_1) \to \mathbb{Q}(k)(s_1)$ with $\tau(s_1) = s_1$, i.e., $\tau(h) = h$ for all $h \in \mathbb{Q}(k)(s_1)$.

(3) We apply our algorithm for problem RS to $f = \frac{\sigma(s_1)}{k+1}$ (with $\mathbb{D} = \mathbb{Q}(k)(s_1)$) and find $f' = \frac{1}{2(k+1)^2}$ and $g = \frac{1}{2}s_1^2$. Following the proof of Theorem 23 we could construct the $\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s), \sigma)$ of $(\mathbb{Q}(k)(s_1), \sigma)$ with $\sigma(s) = s + \frac{1}{2(k+1)^2}$ which leads to the solution $g' = \frac{1}{2}s_1^2 + s$ for $\Delta(g') = f$. But, to match the harmonic numbers of second order, we normalize the extension to the $\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2), \sigma)$ of $(\mathbb{Q}(k)(s_1), \sigma)$ with $\sigma(s_2) = s_2 + \frac{1}{(k+1)^2}$, and we obtain the solution $g' = \frac{1}{2}(s_1^2 + s_2)$ for $\Delta(g') = f$. To be more precise, we apply Lemma 21 with $c = 2$; as a consequence, we can extend the isomorphism $\tau$ to $\tau\colon \mathbb{Q}(k)(s_1)(s_{1,1}) \to \mathbb{Q}(k)(s_1)(s_2)$ with

$$\tau(s_{1,1}) = \frac{1}{2}\left(s_1^2 + s_2\right). \tag{18}$$

By construction $(\mathbb{Q}(k)(s_1)(s_2), \sigma)$ is a reduced extension of $(\mathbb{Q}(k), \sigma)$.

(4) Finally, we solve RS for $f = \tau(\frac{\sigma(s_{1,1})}{k+1})$ (with $\mathbb{D} = \mathbb{Q}(k)(s_1)(s_2)$) and find $f' = \frac{1}{3(k+1)^3}$ and $g = \frac{1}{6}\left(s_1^3 + 3s_2 s_1\right)$. Hence we can define the $\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3), \sigma)$ of $(\mathbb{Q}(k)(s_1)(s_2), \sigma)$ with $\sigma(s_3) = s_3 + \frac{1}{(k+1)^3}$ (note again that we normalized the extension by pulling out the constant $1/3$); by construction $(\mathbb{Q}(k)(s_1)(s_2)(s_3), \sigma)$ is a reduced extension of $(\mathbb{Q}(k), \sigma)$. In addition, we can extend our $\mathbb{Q}$-isomorphism to $\tau\colon \mathbb{Q}(k)(s_1)(s_{1,1})(s_{1,1,1}) \to \mathbb{Q}(k)(s_1)(s_2)(s_3)$ with

$$\tau(s_{1,1,1}) = \frac{1}{6}\left(s_1^3 + 3s_2 s_1 + 2s_3\right). \tag{19}$$

Since $h = s_3$ is a solution of $\Delta(h) = \frac{1}{(k+1)^3}$, $\tau^{-1}(h)$ (which is nothing else than (17)) is a solution of $\Delta(\tau^{-1}(h)) = \tau^{-1}(\frac{1}{(k+1)^3}) = \frac{1}{(k+1)^3}$.

*Remark* **25** Reinterpreting $s_1, s_{1,1}, s_{1,1,1}$ in Ex. 24 as harmonic sums leads to the following identities which are reflected by (18) and (19): for $k \in \mathbb{N}$,

$$S_{1,1}(k) = \frac{1}{2}\left(S_1(k)^2 + S_2(k)\right),$$
$$S_{1,1,1}(k) = \frac{1}{6}\left(S_1(k)^3 + 3S_2(k)S_1(k) + 2S_3(k)\right);$$

these occur, e.g., in [10] or in [16, Cor. 3] combined with [26, Prop. 2.1].

We remark that any $\mathbb{F}$-isomorphism is of this shape due to the following lemma; note that the product case is analogous, see [43, Prop. 4.4 and 4.8].

**Lemma 26** *Let $(\mathbb{F}(t),\sigma)$ and $(\mathbb{F}(s),\sigma)$ be $\Sigma^*$-extensions of $(\mathbb{F},\sigma)$ with $\sigma(t)=t+f$ and $\sigma(s)=s+f'$, and let $\mathbb{K}:=\mathrm{const}_\sigma\mathbb{F}$. If $\tau\colon\mathbb{F}(t)\to\mathbb{F}(s)$ is an $\mathbb{F}$-isomorphism, there are $g\in\mathbb{F}$ and $c\in\mathbb{K}^*$ as in (15) such that $\tau(t)=c\,s+g$.*

*Proof* Note: $\Delta(\tau(t))=\tau(\Delta(t))=\tau(f)=f$. By Thm. 9 $\tau(t)=c\,s+g$ for some $g\in\mathbb{F}$ and $c\in\mathbb{K}$, and thus (15).

**Application:** Suppose we are given a $\Pi\Sigma^*$-extension $(\mathbb{F}(x_1)\dots(x_e),\sigma)$ of a $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$, and one has to compute solutions $g\in\mathbb{F}(x_1)\dots(x_e)$ of (5) for various instances of $f\in\mathbb{F}$. Then the following strategy is straightforward. Compute once and for all a reduced $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ together with an $\mathbb{F}$-isomorphism $\tau\colon\mathbb{F}(x_1)\dots(x_e)\to\mathbb{F}(t_1)\dots(t_e)$; define $S$ as in (8) and set $f_i:=\Delta(t_i)\in\mathbb{F}$ for $i\in S$. Then for each summand $f\in\mathbb{F}$ we can apply Theorem 9 as follows: it suffices to look for $c_i$ with $i\in S$ and $w\in\mathbb{F}$ such that

$$\Delta(w)=f-\sum_{i\in S}c_i\,f_i;$$

note that this problem (among others) can be solved with Karr's algorithm [21] or our simplified version [44]. Then given such a solution, one gets the solution (10) for (5). Hence with $g':=\tau^{-1}(g)\in\mathbb{F}(x_1)\dots(x_r)$ we get the required solution $\Delta(g')=f$, since $\Delta(g')=\Delta(\tau^{-1}(g))=\tau^{-1}(\Delta(g))=\tau^{-1}(f)=f$.

## 5 Completely reduced $\Pi\Sigma^*$-extensions

We refine reduced $\Pi\Sigma^*$-extension to completely reduced $\Pi\Sigma^*$-extensions.

*Definition* **27** A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is called *completely reduced over* $\mathbb{F}$ or in short *completely reduced* if for any $\Sigma^*$-extension $t_i$ $(1\leq i\leq e)$ with $f:=\Delta(t_i)$ and $r$ with $f\in\mathbb{F}(t_1)\dots(t_r)\setminus\mathbb{F}(t_1)\dots(t_{r-1})$ the following property holds: there are no $g\in\mathbb{F}(t_1)\dots(t_{i-1})$ and $f'\in\mathbb{F}(t_1)\dots(f_{r-1})$ such that (7) holds.

The proof of the following theorem is analogous to the proof of Theorem 23. The resulting algorithm is just Alg. 1: the only difference is that one always executes line (6) independently of whether $j$ is 0 or not.

**Theorem 28** *For any $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ there is a completely reduced $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and an $\mathbb{F}$-isomorphism $\tau\colon\mathbb{H}\to\mathbb{E}$. Such a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ and $\tau$ can be given explicitly, if $(\mathbb{F},\sigma)$ is RS-computable.*

*Example* **29** **(1)** In Example 24 we transformed step by step the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_{1,1})(s_{1,1,1}),\sigma)$ with the automorphism (16) to an isomorphic $\Pi\Sigma^*$-field given by $(\mathbb{Q}(k)(s_1)(s_2)(s_3),\sigma)$ with (4). Since in each step we solved

problem RS, the resulting extension is completely reduced.
**(2)** Take the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{6,1,3}), \sigma)$ with (4) and (6). Solving problem RS (with $\mathbb{F} = \mathbb{Q}$) for each extension shows that the $\Pi\Sigma^*$-field is a completely reduced extension of $(\mathbb{Q}, \sigma)$.

Theorem 19 can be carried over to completely reduced extensions as follows.

**Theorem 30** *Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{E} := \mathbb{F}(t_1) \dots (t_e)$. Then the following statements are equivalent.*

*(1) This extension is completely reduced.*
*(2) For any $i, j$ with $1 \leq i \leq j \leq e$, $(\mathbb{F}(t_1) \dots (t_j), \sigma)$ is a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}(t_1) \dots (t_i), \sigma)$.*
*(3) For any $j$ $(1 \leq j \leq e)$ with*

$$S = S(j) = \{i | j \leq i \leq e \text{ and } \Delta(t_i) \in \mathbb{F}(t_1) \dots (t_{j-1})\} \tag{20}$$

*and for any $g \in \mathbb{E}$ with $\Delta(g) \in \mathbb{F}(t_1) \dots (t_{j-1})$ we have (10) for some $c_i \in \mathrm{const}_\sigma \mathbb{F}$ and $w \in \mathbb{F}(t_1) \dots (t_{j-1})$.*
*(4) For any $\Sigma^*$-extension $t_i$ $(1 \leq i \leq e)$ with $f := \Delta(t_i)$ and $r$ such that $f \in \mathbb{F}(t_1) \dots (t_r) \setminus \mathbb{F}(t_1) \dots (t_{r-1})$ the following holds: There is no $\Sigma^*$-ext. $(\mathbb{F}(t_1) \dots (t_{i-1})(s), \sigma)$ of $(\mathbb{F}(t_1) \dots (t_{i-1}), \sigma)$ with $\Delta(s) \in \mathbb{F}(t_1) \dots (t_{r-1})$ in which we have $g$ with (5).*

*Proof* This extension is not completely reduced if and only if there is a $j$, $1 \leq j \leq e$, such that for $f := \Delta(t_j)$ with $f \in \mathbb{F}(t_1) \dots (t_r) \setminus \mathbb{F}(t_1) \dots (t_{r-1})$ for some $r$ $(1 \leq r \leq j)$ we have the following property: there are $f' \in \mathbb{F}(t_1) \dots (t_{r-1})$ and $g \in \mathbb{F}(t_1) \dots (t_{j-1})$ with (7). But this is equivalent to the fact that there are $r, j$ with $1 \leq r \leq j \leq e$ such that $(\mathbb{F}(t_1) \dots (t_j), \sigma)$ is not a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}(t_1) \dots (t_r), \sigma)$. Hence **(1)** is equivalent to **(2)**. The other equivalences are an immediate consequence of Theorem 19.

We emphasize the equivalence (1)$\Leftrightarrow$(3) of Theorem 30: For any $f \in \mathbb{E}$ we can apply Theorem 9. Namely, let $j$ be minimal such that $f \in \mathbb{F}(t_1) \dots (t_j)$ and define $S = S(j)$ by (20). Then for any solution $g \in \mathbb{E}$ of (5) we have (10) for some $w \in \mathbb{F}(t_1) \dots (t_{j-1})$ and $c_i \in \mathrm{const}_\sigma \mathbb{F}$.

## 6 The depth and reordering of completely reduced $\Pi\Sigma^*$-extensions

As indicated in the introduction, reducing the nesting depth of a given indefinite sum expression, like, e.g., d'Alembertian solutions [30,3,39] of a linear recurrence, is an important issue in the context of $\Pi\Sigma^*$-fields. In order to measure the nesting depth, we introduce the following depth function [47].

Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with the field $\mathbb{E} := \mathbb{F}(t_1) \dots (t_e)$ and with $\sigma(t_i) = a_i\, t_i$ or $\sigma(t_i) = t_i + a_i$ for $1 \leq i \leq e$. The *depth function for elements of* $\mathbb{E}$ *over* $\mathbb{F}$, $\delta_\mathbb{F} : \mathbb{E} \to \mathbb{N}$, is defined as follows.

(1) For any $g \in \mathbb{F}$, $\delta_\mathbb{F}(g) := 0$.

(2) If $\delta_{\mathbb{F}}$ is defined for $(\mathbb{F}(t_1)\dots(t_{i-1}),\sigma)$ with $i>1$, we define $\delta_{\mathbb{F}}(t_i):=\delta_{\mathbb{F}}(a_i)+1$; for $g=\frac{g_1}{g_2}\in\mathbb{F}(t_1)\dots(t_i)$, with $g_1,g_2\in\mathbb{F}[t_1,\dots,t_i]$ coprime, we define

$$\delta_{\mathbb{F}}(g):=\max(\{\delta_{\mathbb{F}}(t_j)|1\leq j\leq i \text{ and } t_j \text{ occurs in } g_1 \text{ or } g_2\}\cup\{0\}).$$

The *extension depth* of a $\Pi\Sigma^*$-extension $(\mathbb{E}(x_1)\dots(x_r),\sigma)$ of $(\mathbb{E},\sigma)$ is defined by $\max(0,\delta_{\mathbb{F}}(x_1),\dots,\delta_{\mathbb{F}}(x_r))$.

*Example* **31** In the $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ with the rational function field $\mathbb{F}=\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{6,1,3})$ and with $\sigma$ defined by (4) and (6) we have $\delta_{\mathbb{Q}}(k)=1$ and

$$\delta_{\mathbb{Q}}(s_1)=\delta_{\mathbb{Q}}(s_2)=\delta_{\mathbb{Q}}(s_3)=2,\ \delta_{\mathbb{Q}}(s_{1,3})=\delta_{\mathbb{Q}}(x)=3,\ \delta_{\mathbb{Q}}(s_{6,1,3})=4.$$

The extension depth of the $\Pi\Sigma^*$-extension $(\mathbb{F},\sigma)$ of $(\mathbb{Q},\sigma)$ is 4.

If one wants to simplify the nesting depth of sums in a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$, the following property is crucial: for any $f,g\in\mathbb{E}$ with (5) we have

$$\delta_{\mathbb{F}}(f)\leq\delta_{\mathbb{F}}(g)\leq\delta_{\mathbb{F}}(f)+1; \tag{21}$$

in other words, if we find a sum representation $g$ for a summand $f$ with (5), the depth of $g$ should be bounded by (21).

Subsequently, we show that property (21) is closely related to reduced and completely reduced $\Pi\Sigma^*$-extensions. For this task we assume that the $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=a_i\,t_i+f_i$ for all $i$ with $1\leq i\leq e$ is $\mathbb{F}$-*ordered*, i.e., the extensions are built in the order of their depths:

$$\delta_{\mathbb{F}}(t_1)\leq\delta_{\mathbb{F}}(t_2)\leq\dots\leq\delta_{\mathbb{F}}(t_e); \tag{22}$$

we remark that any $\Pi\Sigma^*$-extension can be reordered in this form.

**Theorem 32** *Let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with the tower of $\Pi\Sigma^*$-extensions*

$$\mathbb{F}=\mathbb{F}_0\leq\mathbb{F}_1\leq\dots\leq\mathbb{F}_d=\mathbb{E}, \tag{23}$$

*such that for $1\leq i\leq d$ the following holds: $\mathbb{F}_i=\mathbb{F}_{i-1}(x_1^{(i)})\dots(x_{e_i}^{(i)})$ is a $\Pi\Sigma^*$-extension of $\mathbb{F}_{i-1}$ with $e_i>0$ and $\delta_{\mathbb{F}}(x_j^{(i)})=i$ for all $1\leq j\leq e_i$. Then the following two statements are equivalent:*

*(1) For $0\leq i\leq j\leq d$, the $\Pi\Sigma^*$-extension $(\mathbb{F}_j,\sigma)$ of $(\mathbb{F}_i,\sigma)$ is reduced.*
*(2) For any $f,g\in\mathbb{E}$ as in (5) we have (21).*

*Proof* Let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ as claimed above such that statement **(1)** holds. Let $f\in\mathbb{E}$ with $j:=\delta_{\mathbb{F}}(f)$ and $g\in\mathbb{E}$ with (5). If $j=d$, (21) clearly holds. Otherwise, let $j<d$. Since the extension $(\mathbb{E},\sigma)$ of $(\mathbb{F}_j,\sigma)$ is reduced, we can apply Theorem 9 and it follows that $g=\sum_{i=1}^{e_{j+1}}c_ix_i^{(j+1)}+w$ where $w\in\mathbb{F}_j$ and $c_i\in\text{const}_\sigma\mathbb{F}$. Since $\delta_{\mathbb{F}}(g)\leq j+1$, statement **(2)** holds.

Conversely, let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ such that statement **(1)** does not hold. Then there are $l,r\geq 1$ such that $(\mathbb{F}_r,\sigma)$ is not a reduced $\Pi\Sigma^*$-extension of $(\mathbb{F}_l,\sigma)$. In particular, there is a $\Sigma^*$-extension $x_u^{(v)}$ for some $l<v\leq r$ and $1\leq u\leq e_v$ with $f:=\Delta(x_u^{(v)})\notin\mathbb{F}_l$ s.t. the following property holds: there are $f'\in\mathbb{F}_l$ and $g\in\mathbb{F}_{v-1}(x_1^{(v)})\dots(x_{u-1}^{(v)})$ with (7). Note that $\delta_{\mathbb{F}}(f')<\delta_{\mathbb{F}}(f)$. Hence for $h:=x_u^{(l)}-g$, $\delta_{\mathbb{F}}(h)=\delta_{\mathbb{F}}(x_u^{(l)})>\delta_{\mathbb{F}}(f)>\delta_{\mathbb{F}}(f')$ and $\Delta(h)=f-\Delta(g)=f'$. Thus, $\delta_{\mathbb{F}}(h)>\delta_{\mathbb{F}}(f')+1$, and **(2)** does not hold.

$\mathbb{F}$-ordered completely reduced $\Pi\Sigma^*$-extensions are covered by $\mathbb{F}$-ordered $\Pi\Sigma^*$-extensions of the form (23) for which statement (2) of Theorem 32 holds. Hence we get

**Corollary 33** *Let $(\mathbb{E},\sigma)$ be an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. If the extension is completely reduced, then for any $f,g\in\mathbb{E}$ with* (5) *we have* (21).

*Example* **34** As pointed out in Ex. 29.2 the $\mathbb{Q}$-ordered $\Pi\Sigma^*$-extension $(\mathbb{G},\sigma)$ of $(\mathbb{Q},\sigma)$ with $\mathbb{G}=\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{6,1,3})$ and with (4) and (6) is completely reduced. Thus Cor. 33 is applicable: for any $f,g\in\mathbb{G}$ with (5) we have (21). E.g., if $\delta_{\mathbb{F}}(f)=2$, i.e., $f\in\mathbb{Q}(k)(s_1)(s_2)(s_3)$, we have (12). If $\delta_{\mathbb{F}}(f)=1$, i.e., $f\in\mathbb{Q}(k)$, we have (11).

*Example* **35** The $\Pi\Sigma^*$-field from Example 12 is not reduced. Hence, as predicted in Theorem 32 we could find $f$ and $g$ in this field with (5) such that $\delta_{\mathbb{F}}(g)>\delta_{\mathbb{F}}(f)+1$.

In order to exploit Corollary 33 in full generality, it is necessary to transform a $\Pi\Sigma^*$-extension to an $\mathbb{F}$-ordered completely reduced extension. It turns out that this task is not straightforward[5]. We start with the following illustrative example.

*Example* **36** Given $(\mathbb{G},\sigma)$ as in Example 34, we consider the $\Sigma^*$-extension $(\mathbb{G}(s_{2,1,3}),\sigma)$ of $(\mathbb{G},\sigma)$ with $\sigma(s_{2,1,3})=s_{2,1,3}+\frac{\sigma(s_{1,3})}{(k+1)^2}$. Subsequently, we try to transform this extension such that it is again a $\mathbb{Q}$-ordered and completely reduced extension of $(\mathbb{Q},\sigma)$. First, we verify that $s_{2,1,3}$ is not a completely reduced extension: by solving problem RS ($\mathbb{D}=\mathbb{G}(s_{2,1,3})$, $\mathbb{F}=\mathbb{Q}$ and $f=\frac{\sigma(s_{1,3})}{(k+1)^2}$) we arrive at $f'=\frac{1}{2(k+1)^6}$ and $g=\frac{1}{2}\left(s_3^2-2x+2s_{1,3}\,s_2\right)$. Hence we can construct the $\Sigma^*$-extension $(\mathbb{G}(s_6),\sigma)$ of $(\mathbb{G},\sigma)$ with $\sigma(s_6)=s_6+\frac{1}{(k+1)^6}$. In particular, we get

$$\Delta(\frac{1}{2}\left(s_3^2-2x+2s_{1,3}\,s_2+s_6\right))=\frac{\sigma(s_{1,3})}{(k+1)^2}; \tag{24}$$

note that we applied Lemma 21 (in particular, we pulled out the constant $1/2$ by choosing $c=2$ in the lemma). Next, we rearrange the $\Pi\Sigma^*$-field $(\mathbb{G}(s_6),\sigma)$

[5] In Section 7.1 we shall propose another solution by embedding a $\Pi\Sigma^*$-extension into a depth-optimal $\Pi\Sigma^*$-extension; see also Ex. 44 which is related to Ex. 36.

and obtain the $\mathbb{Q}$-ordered $\Pi\Sigma^*$-ext. $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x)(s_{6,1,3}),\sigma)$ of $(\mathbb{Q},\sigma)$. In addition, we find the $\mathbb{Q}$-isomorphism

$$\rho\colon \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{6,1,3})(s_{2,1,3}) \to \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x)(s_{6,1,3})$$

by keeping all variables fixed except

$$\rho(s_{2,1,3}) = \frac{1}{2}\left(s_3^2 - 2x + 2s_{1,3}\, s_2 + s_6\right). \tag{25}$$

Due to this change, we have to check if the extensions $s_{1,3}, x, s_{6,1,3}$ on top of $s_6$ are still completely reduced. Solving the corresponding problems RS shows that $s_{1,3}$ and $x$ are completely reduced, but $s_{6,1,3}$ is not completely reduced. Namely solving problem RS for $\mathbb{D} = \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)$, $\mathbb{F} = \mathbb{Q}$ and $f = \frac{\sigma(s_{1,3})}{(k+1)^6}$, we get $g = s_{1,3}s_6$ and $f' = -\frac{\sigma(s_3)\left(\sigma(s_6)(k+1)^6-1\right)}{(k+1)^7}$. Applying Lemma 21 with $c = -1$, we find the completely reduced $\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x)(y),\sigma)$ of $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x),\sigma)$ with $\sigma(y) = y + \frac{\sigma(s_3)\left(\sigma(s_6)(k+1)^6-1\right)}{(k+1)^7}$) such that

$$\Delta(s_{1,3}s_6 - y) = \frac{\sigma(s_{1,3})}{(k+1)^6}. \tag{26}$$

In particular, we get the $\mathbb{Q}$-isomorphism

$$\mu\colon \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x)(s_{6,1,3}) \to \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x)(y)$$

by keeping all variables fixed except

$$\mu(s_{6,1,3}) = s_{1,3}s_6 - y. \tag{27}$$

To sum up, we managed to transform the $\Pi\Sigma^*$-field $(\mathbb{G},\sigma)$ to the $\mathbb{Q}$-ordered and completely reduced $\Pi\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x)(y),\sigma)$ of $(\mathbb{Q},\sigma)$ with

$$\begin{aligned}
&\sigma(k) = k+1, && \sigma(s_1) = s_1 + \tfrac{1}{k+1}, && \sigma(s_2) = s_2 + \tfrac{1}{(k+1)^2},\\
&\sigma(s_3) = s_3 + \tfrac{1}{(k+1)^3}, && \sigma(s_6) = s_6 + \tfrac{1}{(k+1)^6}, && \sigma(s_{1,3}) = s_{1,3} + \tfrac{\sigma(s_3)}{k+1},\\
&\sigma(x) = x + \tfrac{\sigma(s_2)\,\sigma(s_3)}{k+1}, && \sigma(y) = y + \tfrac{\sigma(s_3)\left(\sigma(s_6)(k+1)^6-1\right)}{(k+1)^7}
\end{aligned} \tag{28}$$

together with the $\mathbb{Q}$-isomorphism $\tau := \mu \circ \rho$ with

$$\tau\colon \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{6,1,3})(s_{2,1,3}) \to \mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x)(y); \tag{29}$$

here all variables are fixed except

$$\tau(s_{2,1,3}) = \frac{1}{2}\left(s_3^2 - 2x + 2s_{1,3}\, s_2 + s_6\right) \quad \text{and} \quad \tau(s_{6,1,3}) = s_{1,3}s_6 - y. \tag{30}$$

*Remark* **37** Reinterpreting the variables of the previous example as indefinite sums yields the following identities (which are reflected by (30)): for all $k \in \mathbb{N}$,

$$S_{2,1,3}(k) = \frac{1}{2}S_3(k)^2 - \sum_{i=1}^{k}\frac{S_2(i)S_3(i)}{i} + S_{1,3}(k)\,S_2(k) + \frac{1}{2}S_6(k),$$
$$S_{6,1,3}(k) = S_{1,3}(k)S_6(k) - \sum_{i=1}^{k}\frac{S_3(i)\left(S_6(i)i^6 - 1\right)}{i^7}. \tag{31}$$

Subsequently, we will make this transformation more precise. In order to deal with $\Pi$-extensions (see case 2 in the proof of Thm. 39), we need the following lemma.

**Lemma 38** *Let* $(\mathbb{E},\sigma)$ *with* $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F},\sigma)$*, let* $f \in \mathbb{E}$*, and let* $(\mathbb{E}(x),\sigma)$ *be a* $\Pi$*-extension of* $(\mathbb{E},\sigma)$ *with* $\frac{\sigma(x)}{x} \in \mathbb{F}$*. If there are* $f' \in \mathbb{F}(x)$ *and* $g \in \mathbb{E}(x)$ *s.t. (7) holds, there are* $\phi' \in \mathbb{F}$ *and* $\gamma \in \mathbb{E}$ *s.t.* $\Delta(\gamma) + \phi' = f$*.*

*Proof* Let $f \in \mathbb{E}$, $g \in \mathbb{E}(x)$ and $f' \in \mathbb{F}(x)$ as claimed above. For convenience, denote by $\mathbb{E}(x)^{(\text{prop})}$ (resp. by $\mathbb{F}(x)^{(\text{prop})}$) all proper rational functions from $\mathbb{E}(x)$ (resp. from $\mathbb{F}(x)$), i.e., for each element the degree of the numerator (w.r.t. $x$) is smaller than the degree of the denominator. By polynomial division we can write $g = p_1 + q_1$ and $f' = p_2 + q_2$ such that $p_1 \in \mathbb{E}[x]$, $q_1 \in \mathbb{E}(x)^{(\text{prop})}$ and $p_2 \in \mathbb{F}[x]$, $q_2 \in \mathbb{F}(x)^{(\text{prop})}$. Since $\frac{\sigma(x)}{x} \in \mathbb{F}$, it is immediate that $\sigma(p_1) \in \mathbb{E}[x]$, and consequently, $\Delta(p_1) \in \mathbb{E}[x]$. Moreover, since $\sigma(q_1) \in \mathbb{E}(x)^{(\text{prop})}$ (the degrees of polynomials in $x$ do not change under the action of $\sigma$), $\Delta(q_1) \in \mathbb{E}(x)^{(\text{prop})}$. Analogously, $\Delta(p_2) \in \mathbb{F}[x]$ and $\Delta(q_2) \in \mathbb{F}(x)^{(\text{prop})}$. Since $\mathbb{E}(x) = \mathbb{E}[x] \oplus \mathbb{E}(x)^{(\text{prop})}$ forms a direct sum (as vector spaces over $\mathbb{E}$) and $\mathbb{F} < \mathbb{E}$, (7) implies $\Delta(p_1) + p_2 = f$ and $\Delta(q_1) + q_2 = 0$. Consider now $p_1, p_2, f$ as polynomials in $\mathbb{E}[x]$, and let $\gamma, \phi' \in \mathbb{E}$ be the constant terms of $p_1, p_2$, respectively; note that $f \in \mathbb{E}$. Then by coefficient comparison in $\Delta(p_1) + p_2 - f = 0$, $\Delta(\gamma) + \phi - f = 0$; this completes the lemma.

**Theorem 39** *For any* $\Pi\Sigma^*$*-extension* $(\mathbb{E},\sigma)$ *of* $(\mathbb{F},\sigma)$ *there is a completely reduced* $\mathbb{F}$*-ordered* $\Pi\Sigma^*$*-extension* $(\mathbb{E}',\sigma)$ *of* $(\mathbb{F},\sigma)$ *together with an* $\mathbb{F}$*-isomorphism* $\tau\colon \mathbb{E} \to \mathbb{E}'$*; in particular,*

$$\delta_{\mathbb{F}}(\tau(h)) \leq \delta_{\mathbb{F}}(h) \tag{32}$$

*for all* $h \in \mathbb{H}$*. Such* $(\mathbb{E}',\sigma)$ *and* $\tau$ *can be given explicitly, if* $(\mathbb{F},\sigma)$ *is RS-computable.*

*Proof* Let $(\mathbb{E},\sigma)$ with $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma^*$-ext. of $(\mathbb{F},\sigma)$. We show the theorem by induction on the depth. If $\delta_{\mathbb{F}}(t_1) = \dots \delta_{\mathbb{F}}(t_e) = 1$, the claim follows by Lemma 8. Now suppose that we have shown the assumption for any extension whose extension depth is $\leq d+1$ and $r \geq 0$ or less extensions have depth $d+1$. Subsequently, assume that the $\Pi\Sigma^*$-ext. $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with extension

depth $d+1$ has exactly $r+1$ extensions with depth $d+1$. W.l.o.g. assume that this extension is $\mathbb{F}$-ordered, and thus $\delta_{\mathbb{F}}(t_e)=d+1$. By our assumption we get an $\mathbb{F}$-ordered completely reduced $\Pi\Sigma^*$-ext. $(\mathbb{G},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{G}:=\mathbb{F}(x_1)\dots(x_{e-1})$ together with an $\mathbb{F}$-isomorphism $\tau\colon\mathbb{F}(t_1)\dots(t_{e-1})\to\mathbb{G}$ such that (32) holds for all $h\in\mathbb{F}(t_1)\dots(t_{e-1})$.
For the $\Pi\Sigma^*$-extension $t_e$ on top assume that $\sigma(t_e)=\alpha\,t_e+\beta$ (either $\alpha=1$ or $\beta=0$), and define $a=\tau(\alpha)$ and $f=\tau(\beta)$ (i.e., either $a=1$ or $f=0$). Finally, take the $\Pi\Sigma^*$-extension $(\mathbb{G}(x_e),\sigma)$ of $(\mathbb{G},\sigma)$ with $\sigma(x_e)=a\,x_e+f$, and extend the $\mathbb{F}$-isomorphism $\tau$ with $\tau(t_e)=x_e$; this is possible by Lemma 22. Note that $\delta_{\mathbb{F}}(x_e)=\max(\delta_{\mathbb{F}}(a),\delta_{\mathbb{F}}(f))+1\leq\max(\delta_{\mathbb{F}}(\alpha),\delta_{\mathbb{F}}(\beta))+1=\delta_{\mathbb{F}}(t_e)$ by (32); hence, (32) for all $h\in\mathbb{F}(x_1)\dots(x_e)$.
**Case 1:** $x_e$ is a $\Pi$-ext., i.e., $f=0$. **Case 1.1:** If $\delta_{\mathbb{F}}(x_e)=d+1$, $(\mathbb{G}(x_e),\sigma)$ is an $\mathbb{F}$-ordered completely reduced $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$, and we are done.
**Case 1.2:** Otherwise bring it to an $\mathbb{F}$-ordered form: for some $l$ with $0\leq l<e$, we obtain[6] $(\mathbb{F}(x_1)\dots(x_l)(x_e)(x_{l+1})\dots(x_{e-1}),\sigma)$. Suppose that one of the extensions $x_i$ with $i>l$ is not completely reduced; let $j$ be minimal s.t. $h:=\Delta(x_i)\in\mathbb{F}(x_1)\dots(x_l)(x_e)(x_{l+1})\dots(x_j)$. Then there are $g\in\mathbb{F}(x_1)\dots(x_{i-1})(x_e)$ and $h'\in\mathbb{F}(x_1)\dots(x_{j-1})(x_e)$ s.t. $\Delta(g)+h'=h$. Hence by Lemma 38 we find such $h'$, $g$ which are free of $x_e$, and thus $(\mathbb{F}(x_1)\dots(x_j),\sigma)$ is not a completely reduced extension of $(\mathbb{F},\sigma)$; a contradiction. This completes this part of the proof.
**Case 2:** $x_e$ is a $\Sigma^*$-extension, i.e., $a=1$. Let $j$ $(j<e)$ be minimal such that there are $f'\in\mathbb{F}(x_1)\dots(x_j)$ and $g\in\mathbb{G}$ as in (7). Since $\delta_{\mathbb{F}}(x_1)\leq\dots\leq\delta_{\mathbb{F}}(x_{e-1})$, $\delta_{\mathbb{F}}(f')\leq\delta_{\mathbb{F}}(f)$. Note: If $(\mathbb{F},\sigma)$ is RS-computable, such $f'$ and $g$ can be computed explicitly. By Lemma 21 there is a $\Sigma^*$-extension $(\mathbb{G}(s),\sigma)$ of $(\mathbb{G},\sigma)$ and $\sigma(s)=s+f'$; in particular, there is the $\mathbb{F}$-isomorphism $\rho\colon\mathbb{G}(x_e)\to\mathbb{G}(s)$ with $\rho(h)=h$ for all $h\in\mathbb{G}$ and $\rho(x_e)=s+g$. Next, we show that $\delta_{\mathbb{F}}(\rho(x_e))\leq\delta_{\mathbb{F}}(x_e)$. Note that $\Delta(s+g)=\Delta(x_e)=f$, and hence $\Delta(g)=f-f'$. Since $(\mathbb{G},\sigma)$ is an $\mathbb{F}$-ordered completely reduced extension of $(\mathbb{F},\sigma)$ and $f,f',g\in\mathbb{G}$ it follows that $\delta_{\mathbb{F}}(g)\leq\delta_{\mathbb{F}}(f-f')+1$ by Cor. 33. Consequently $\delta_{\mathbb{F}}(\tau(x_e))=\delta_{\mathbb{F}}(s+g)\leq\max(\delta_{\mathbb{F}}(s),\delta_{\mathbb{F}}(g))=\max(\delta_{\mathbb{F}}(f')+1,\delta_{\mathbb{F}}(g))$, and hence with $\delta_{\mathbb{F}}(f')\leq\delta_{\mathbb{F}}(f)$ it follows that $\delta_{\mathbb{F}}(\tau(x_e))\leq\delta_{\mathbb{F}}(f)+1=\delta_{\mathbb{F}}(x_e)$. With $\delta_{\mathbb{F}}(\rho(h))\leq\delta_{\mathbb{F}}(h)$ for all $h\in\mathbb{G}$ (see above), we get $\delta_{\mathbb{F}}(\rho(h))\leq\delta_{\mathbb{F}}(h)$ for all $h\in\mathbb{G}(x_e)$. Observe that $(\mathbb{G}(s),\sigma)$ is a completely reduced $\Pi\Sigma^*$-ext. of $(\mathbb{F},\sigma)$ by construction ($f',g$ solve problem RS for $f$ in $(\mathbb{G},\sigma)$).
**Case 2.1:** If $\delta_{\mathbb{F}}(f')=\delta_{\mathbb{F}}(f)=d$, then $(\mathbb{G}(s),\sigma)$ is also an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. Finally, with $\tau'=\rho\circ\tau$ we get an $\mathbb{F}$-isomorphism from $\mathbb{E}$ to $\mathbb{G}(s)$ such that $\delta_{\mathbb{F}}(\tau'(h))\leq\delta_{\mathbb{F}}(h)$ for all $h\in\mathbb{E}$; this completes this part of the induction.
**Case 2.2:** If $\delta_{\mathbb{F}}(f')<\delta_{\mathbb{F}}(f)$, rearrange the extension $(\mathbb{G}(s),\sigma)$ to an $\mathbb{F}$-ordered $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{H}=\mathbb{F}(x_1)\dots\dots(x_l)(s)(x_{l+1})\dots(x_{e-1})$ for some $l>j$ (see again footnote 6). Note that in this case the number of extensions with depth $d+1$ have been reduced at least by 1. Consequently,

[6] Note that the extensions below of $x_{l+1}$ are $\mathbb{F}$-ordered and completely reduced; this fact will be exploited in Alg. 2.

**Algorithm 2** $\texttt{ToCompleteReducedOrderedField}((\mathbb{E},\sigma),k)$

In: A $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ s.t. $(\mathbb{F},\sigma)$ is RS-computable; $k\in\mathbb{N}$ s.t. $(\mathbb{F}(t_1)\dots(t_k),\sigma)$ is an $\mathbb{F}$-ordered completely reduced extension of $(\mathbb{F},\sigma)$.
Out: An $\mathbb{F}$-ordered completely reduced $\Pi\Sigma^*$-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{F},\sigma)$ together with an $\mathbb{F}$-isomorphism $\tau\colon\mathbb{E}\to\mathbb{E}'$.

```
IF k ≥ e, THEN RETURN ((E, σ), id_E) FI
((F(x_1)...(x_{e-1}), σ), τ):=ToCompleteReducedOrderedField((F(t_1)...(t_{e-1}), σ), k);
IF t_e is a Π-extension, i.e., a := τ(σ(t_e)/t_e) ∈ F(x_1)...(x_{e-1}) THEN
    Take the Π-ext. (F(x_1)...(x_e), σ) of (F(x_1)...(x_{e-1}), σ) with σ(x_e) = a x_e; bring
    it to an F-ordered form with E' := F(x_1)...(x_l)(x_e)(x_{l+1})...(x_{e-1}) for some l
    with 0 ≤ l < e. Take the F-isomorphism τ': F(t_1)...(t_e) → E' with τ'(h) = τ(h)
    for all h ∈ F(t_1)...(t_{e-1}) and τ'(t_e) = x_e. RETURN ((E', σ), τ').
FI
Let f' ∈ F(x_1)...(x_j) \ F(x_1)...(x_{j-1}) and g ∈ F(x_1)...(x_{e-1}) be the result of
  problem RS for f := τ(Δ(t_e)) and D = F(x_1)...(x_{e-1}).
Define the Σ*-extension (H, σ) of (F(x_1)...(x_{e-1}), σ) with H := F(x_1)...(x_{e-1})(s)
  and σ(s) = s + f' together with the F-isomorphism ρ: F(x_1)...(x_e) → H with ρ(h) =
  h for all h ∈ F(x_1)...(x_{e-1}) and ρ(x_e) = s + g.
IF δ_F(f') = δ_F(f) THEN RETURN ((H, σ), ρ ∘ τ) FI
Bring (H, σ) to an F-ordered ext. (H', σ) with H' = F(x_1)...(x_l)(s)(x_{l+1})...(x_{e-1})
  for some l > j. As pointed out in Footnote 6 we can execute
  ((E', σ), μ):=ToCompleteReducedOrderedField((H', σ), l + 1).
RETURN ((E', σ), μ ∘ ρ ∘ τ).
```

we can apply our induction assumption: we transform $(\mathbb{H},\sigma)$ to an $\mathbb{F}$-ordered completely reduced extension $(\mathbb{E}',\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E}'=\mathbb{F}(x_1')\dots(x_e')$ together with an $\mathbb{F}$-isomorphism $\mu\colon\mathbb{H}\to\mathbb{E}'$ such that $\delta_{\mathbb{F}}(\mu(h))\leq\delta_{\mathbb{F}}(h)$ for all $h\in\mathbb{H}$. Hence with $\tau':=\mu\circ\rho\circ\tau$ we get an $\mathbb{F}$-isomorphism $\tau'\colon\mathbb{E}\to\mathbb{E}'$ with $\delta_{\mathbb{F}}(\tau'(h))\leq\delta_{\mathbb{F}}(h)$ for all $h\in\mathbb{E}$. This finishes the induction step.

Extracting the reduction steps of the inductive proof of Theorem 39 and taking into account Footnote 6 lead to Algorithm 2. For instance, in Example 36 the algorithm is carried out for the input $((\mathbb{F}(s_{2,1,3}),\sigma),7)$. In general, given a $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ one computes with Alg. 2 and the input $((\mathbb{E},\sigma),1)$ an isomorphic $\mathbb{F}$-ordered completely reduced extension.

*Remark* **40** Note that we could proceed differently. **Step 1:** Bring a $\Pi\Sigma^*$-extension to the form (23) such that statement (2) in Theorem 32 holds; then we are already in the position to exploit property (1) given in Theorem 32. **Step 2:** Now the computation of an $\mathbb{F}$-ordered completely reduced extension is immediate: just apply the underlying algorithm of Theorem 28 (it is easy to see that the depth of the extensions cannot be reduced further, and hence the output is an $\mathbb{F}$-ordered completely reduced extension). However, in order to perform step 1, our arguments lead to the same algorithm as given in Algorithm 2; only subproblem RS can be slightly modified/simplified. Since we could not see that these modifications lead to any substantial improvement, we just presented Algorithm 2, and we set aside a detailed presentation of the variation sketched in this remark.

## 7 Depth-optimal $\Pi\Sigma^*$-extensions and refined structural Theorems

In [41] $\Pi\Sigma^*$-extensions have been elaborated to depth-optimal $\Pi\Sigma^*$-extensions. As it turns out, such extensions are closely related to reduced and completely reduced $\Pi\Sigma^*$-extensions. But, there are also major differences: depth-optimal $\Pi\Sigma^*$-extensions satisfy additional properties that are highly relevant in the field of symbolic summation; see [47,49]. Subsequently, we present in detail how the derived properties of reduced and completely reduced $\Pi\Sigma^*$-extensions can be carried over to depth-optimal $\Pi\Sigma^*$-extensions. Besides this, we work out their crucial differences in the context of symbolic summation. As a spin off we obtain refined structural theorems that are preferable, e.g., to Theorems 9 and 16.

In the context of reduced $\Pi\Sigma^*$-extensions depth-optimal $\Pi\Sigma^*$-extensions can be introduced as follows. Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(x_1)\dots(x_l)$. Then by Theorem 19 there is the following alternative characterization for a reduced $\Pi\Sigma^*$-extension $(\mathbb{E}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{E},\sigma)$: for any $\Sigma^*$-extension $t_i$ with $f:=\Delta(t_i)\in\mathbb{E}$ $(1\leq i\leq e)$ there is no $\Sigma^*$-extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ in which we have $g\in\mathbb{E}(s)$ with (5). Now suppose in addition the following ordering:

$$\max(\delta_{\mathbb{F}}(x_1),\dots,\delta_{\mathbb{F}}(x_l))+1=\delta_{\mathbb{F}}(t_1)=\delta_{\mathbb{F}}(t_2)=\dots=\delta_{\mathbb{F}}(t_e).$$

Then the above statement can be rephrased as follows. For any $\Sigma^*$-extension $f:=\Delta(t_i)$ there does not exist a single-nested $\Sigma^*$-extension $\mathbb{E}(s)$ with $\delta_{\mathbb{F}}(s)\leq\delta_{\mathbb{F}}(f)$ which provides us with a solution $g\in\mathbb{E}(s)$ for (5).

Essentially, depth-optimal $\Pi\Sigma^*$-extension follow up this construction with the constraint that there does not exist a *tower* of $\Sigma^*$-extensions, say $\mathbb{S}=\mathbb{E}(s_1)\dots(s_r)$ with $\delta_{\mathbb{F}}(s_i)\leq\delta_{\mathbb{F}}(f)$ for $1\leq i\leq r$, which provides us with a solution $g\in\mathbb{S}$ for (5). To be more precise, we introduce depth-optimal $\Pi\Sigma^*$-extensions as follows; see [47].

*Definition* **41** Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. A difference field extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(s)=s+f$ is called *depth-optimal* $\Sigma^*$-extension, in short $\Sigma^\delta$-extension, if there is no $\Sigma^*$-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ with extension[7] depth $\leq\delta_{\mathbb{F}}(f)$ in which there is a $g\in\mathbb{S}$ such that (5) holds. A $\Pi\Sigma^*$-extension $(\mathbb{E}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{E},\sigma)$ is depth-optimal, in short a $\Pi\Sigma^\delta$-extension, if all $\Sigma^*$-extensions[8] are depth-optimal. A *depth-optimal* $\Pi\Sigma^*$*-field* (in short a $\Pi\Sigma^\delta$-field) over $\mathbb{F}$ is a $\Pi\Sigma^*$-field over $\mathbb{F}$ which consists of $\Pi$- and $\Sigma^\delta$-extensions.

Then $\Pi\Sigma^\delta$-extensions can be related to reduced extensions as follows.

**Lemma 42** *Let* $(\mathbb{E},\sigma)$ *be an* $\mathbb{F}$*-ordered* $\Pi\Sigma^\delta$*-extension of* $(\mathbb{F},\sigma)$ *with* (23) *s.t. for* $1\leq i\leq d$ *we have that* $\mathbb{F}_i=\mathbb{F}_{i-1}(x_1^{(i)})\dots(x_{e_i}^{(i)})$ *is a* $\Pi\Sigma^*$*-extension of*

[7] Note that $\Pi\Sigma^\delta$-extensions are defined relatively to the ground field $(\mathbb{F},\sigma)$ over which the depth-function $\delta_{\mathbb{F}}$ is defined. Throughout this section we assume that this ground field is $\mathbb{F}$.

[8] In addition, note that $\Sigma^\delta$-extensions belong to the class of $\Sigma^*$-extensions by Theorem 5.

$\mathbb{F}_{i-1}$ *with* $e_i > 0$ *and* $\delta_{\mathbb{F}}(x_j^{(i)}) = i$ *for all* $1 \leq j \leq e_i$. *Then for* $0 \leq i \leq j \leq d$, *the* $\Pi\Sigma^\delta$*-extension* $(\mathbb{F}_j, \sigma)$ *of* $(\mathbb{F}_i, \sigma)$ *is reduced.*

*Proof* Suppose that the lemma holds with depth $d \geq 0$ and consider a $\Pi\Sigma^\delta$-extension $(\mathbb{F}_{d+1}, \sigma)$ of $(\mathbb{F}_d, \sigma)$ with $\mathbb{F}_{d+1} = \mathbb{F}_d(t_1)\dots(t_e)$ and $\delta_{\mathbb{F}}(t_i) = d + 1$ for $1 \leq i \leq e$. Clearly, $(\mathbb{F}_{d+1}, \sigma)$ is a reduced extension of $(\mathbb{F}_d, \sigma)$ by Lemma 8. For any $j$ $(1 \leq j \leq e)$ with $f_j := \Delta(t_j) \in \mathbb{F}_d$ and for any $r$ $(0 \leq r < d)$ we conclude as follows. Since $t_j$ is a $\Sigma^\delta$-ext., there is no $\Sigma^*$-ext. $(\mathbb{F}_d(t_1)\dots(t_{j-1})(s), \sigma)$ of $(\mathbb{F}_d(t_1)\dots(t_{j-1}), \sigma)$ with $\Delta(s) \in \mathbb{F}_r$ s.t. $\Delta(g) = f_j$ for some $g \in \mathbb{F}_d(t_1)\dots(t_{j-1})(s)$. By the equivalence (1)$\Leftrightarrow$(3) of Thm. 19, $(\mathbb{F}_d(t_1)\dots(t_e), \sigma)$ is a reduced extension of $(\mathbb{F}_r, \sigma)$. This completes the induction step.

### 7.1 Embeddings of $\Pi\Sigma^*$-extensions into $\Pi\Sigma^\delta$-extensions

Similar to reduced and completely reduced $\Pi\Sigma^*$-extensions, we can apply Lemmata 21 and 22 iteratively in order to translate a $\Pi\Sigma^*$-extension into a $\Pi\Sigma^\delta$-extension. In particular, this construction can be given explicitly, if one can solve the following problem.

| |
|---|
| Problem DOS (*D*epth *O*ptimal *S*ummation): **Given** a $\Pi\Sigma^\delta$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$, and given $f \in \mathbb{E}$; **find**, if possible, a $\Sigma^\delta$-extension $(\mathbb{E}(x_1)\dots(x_r), \sigma)$ of $(\mathbb{E}, \sigma)$ with extension depth$\leq \delta_{\mathbb{F}}(f)$ together with a $g \in \mathbb{E}(x_1)\dots(x_r)$ for (5). |

Namely, assume that the difference field $(\mathbb{F}, \sigma)$ is *DOS-computable*, i.e., for any $\Pi\Sigma^\delta$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ and any $f \in \mathbb{E}$ one can solve problem DOS algorithmically. E.g., due to [47, Algorithm 1] implemented in Sigma any $\Pi\Sigma^*$-field is DOS-computable. In fact, a difference field is DOS-computable if and only if it is RS-computable; for further difference field examples see page 13.

Then the embedding mechanism works as follows. Suppose we are given a $\Pi\Sigma^*$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ which we managed to embed into a $\Pi\Sigma^\delta$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\tau\colon \mathbb{H} \to \mathbb{E}$. Now consider the $\Sigma^*$-extension $(\mathbb{H}(t), \sigma)$ of $(\mathbb{H}, \sigma)$ with $\sigma(t) = t + f$. Then one can either find a $\Sigma^\delta$-extension $(\mathbb{E}', \sigma)$ of $(\mathbb{E}, \sigma)$ with $g \in \mathbb{E}'$ such that $\Delta(g) = \tau(f)$ (by solving problem DOS). In this case, one can embed $(\mathbb{H}(t), \sigma)$ into $(\mathbb{E}', \sigma)$ by extending the $\mathbb{F}$-monomorphism $\tau$ to $\tau\colon \mathbb{H}(t) \to \mathbb{E}'$ with $\tau(t) = g$; the correctness follow by $\sigma(\tau(t)) = \sigma(g) = g + \tau(f) = \tau(t + f) = \tau(\sigma(t))$.
Otherwise, if there does not exist such a solution, we can adjoin the $\Sigma^\delta$-extension $(\mathbb{E}(s), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(s) = s + \tau(f)$ and we can extend the $\mathbb{F}$-monomorphism $\tau$ to $\tau\colon \mathbb{H}(t) \to \mathbb{E}(s)$ by $\tau(t) = s$. Similarly, one can treat a $\Pi$-extension $\sigma(t) = a\,t$ for some $a \in \mathbb{H}^*$; see [47, Result 5] for further details. Summarizing, we arrive at

**Theorem 43** *For any* $\Pi\Sigma^*$*-extension* $(\mathbb{E}, \sigma)$ *of* $(\mathbb{F}, \sigma)$ *there is a* $\Pi\Sigma^\delta$*-extension* $(\mathbb{E}', \sigma)$ *of* $(\mathbb{F}, \sigma)$ *and an* $\mathbb{F}$*-monomorphism* $\tau\colon \mathbb{E} \to \mathbb{E}'$. *Such* $(\mathbb{E}', \sigma)$ *and* $\tau$ *can be constructed explicitly if* $(\mathbb{F}, \sigma)$ *is DOS-computable.*

*Example* **44** We embed the $\Pi\Sigma^*$-field $\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{6,1,3})(s_{2,1,3})$ with (4), (6) and $\sigma(s_{2,1,3}) = s_{2,1,3} + \frac{\sigma(s_{1,3})}{(k+1)^2}$ from Example 36 into a depth-optimal $\Pi\Sigma^*$-field. It is easy to see that $(\mathbb{F},\sigma)$ with $\mathbb{F} = \mathbb{Q}(k)(s_1)(s_2)(s_3)$ is already a $\Pi\Sigma^\delta$-field; see also [47, Prop. 17]. We continue as follows.

(1) We apply our algorithms implemented in Sigma and verify that there is no $\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth$\leq 2$ in which we find $g \in \mathbb{E}$ with $\Delta(g) = \frac{\sigma(s_3)}{k+1}$. Hence the $\Sigma^*$-extension $(\mathbb{F}(s_{1,3}),\sigma)$ of $(\mathbb{F},\sigma)$ is depth-optimal.
(2) Similarly, we verify that $(\mathbb{F}(s_{1,3})(x),\sigma)$ is a $\Sigma^\delta$-extension of $(\mathbb{F}(s_{1,3}),\sigma)$.
(3) Now, we check the extension $s_{6,1,3}$ by looking at problem DOS with $f = \frac{\sigma(s_{1,3})}{(k+1)^6}$: we find the $\Sigma^\delta$-extension $(\mathbb{F}(s_{1,3})(x)(s_6)(y),\sigma)$ of $(\mathbb{F}(s_{1,3}),\sigma)$ with
$$\sigma(s_6) = s_6 + \frac{1}{(k+1)^6} \quad \text{and} \quad \sigma(y) = y + \frac{\sigma(s_3)\left(\sigma(s_6)(k+1)^6+1\right)}{(k+1)^7}$$
and $\delta_{\mathbb{F}}(s_6), \delta_{\mathbb{F}}(y) \leq 3$ such that (26) holds; the $\mathbb{Q}$-monomorphism $\mu$ from $\mathbb{F}(s_{1,3})(x)(s_{6,1,3})$ to $\mathbb{F}(s_{1,3})(x)(s_6)(y)$ is defined by $\mu(h) = h$ for all $h \in \mathbb{F}(s_{1,3})(x)$ and (27).
(4) We treat $s_{2,1,3}$ by solving problem DOS for $f = \frac{\sigma(s_{1,3})}{(k+1)^2}$. This time no extension is needed, since we find (24); we can extend the $\mathbb{Q}$-monomorphism as in (25).

Summarizing, we arrive at the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_6)(y),\sigma)$ over $\mathbb{Q}$ with (28) together with the $\mathbb{Q}$-isomorphism (29) given by (30).

Usually, one obtains difference field monomorphisms where the transcendental degree of the embedding extension is larger than the embedded extension. For instance, in step 3 of Ex. 44 we embedded a $\mathbb{Q}$-ordered completely reduced extension with degree 7 into a depth-optimal extension with degree 8.

*Remark* **45** Note that in Ex. 44 we rediscovered identity (31): we simplified the sum $S_{6,1,3}(k)$ of depth 4 to a sum expression with depth 3 by introducing the tower of sum extensions $S_6(k)$ and $\sum_{i=1}^k i^{-7} S_3(i)\left(S_6(i)i^6 - 1\right)$.
In a nutshell, in ordered completely reduced $\Pi\Sigma^*$-fields, like for instance $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_{6,1,3}),\sigma)$ from the Ex. 34 and 44, one might fail to produce sum representations with smallest possible depth. But, transformations of $\Pi\Sigma^*$-fields to $\Pi\Sigma^\delta$-fields lead always to sum representations with optimal nesting depth; a detailed proof of this observation is carried out in [49].

## 7.2 Structural theorems

Comparing reduced and completely reduced $\Pi\Sigma^*$-extensions with depth-optimal $\Pi\Sigma^*$-extensions, the following theorem[9] summarizes one of the decisive differences.

[9] The proof of Thm. 46 relies on additional properties of $\Pi\Sigma^\delta$-extensions elaborated in [47].

**Theorem 46** *([47, Result 2]) Let* $(\mathbb{E}, \sigma)$ *be a* $\Pi\Sigma^\delta$*-extension of* $(\mathbb{F}, \sigma)$*. Any possible reordering (as a* $\Pi\Sigma^*$*-extension) is again a* $\Pi\Sigma^\delta$*-extension.*

Namely, if one adjoins a $\Pi\Sigma^\delta$-extension $t$ on top of a $\Pi\Sigma^\delta$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ and if one reorganizes, e.g., this extension to an $\mathbb{F}$-ordered version, then this $\mathbb{F}$-ordered extension is again depth-optimal. This flexibility is completely different to reduced and completely reduced $\Pi\Sigma^*$-extensions: as worked out in Algorithm 2 and illustrated in Example 36, one has to reorganize the whole difference field in order to get back an $\mathbb{F}$-ordered completely reduced $\Pi\Sigma^*$-extension.

*Example* **47** The $\Pi\Sigma^\delta$-extension $(\mathbb{Q}(k)(s_1)(s_3)(s_{1,3})(x)(s_6)(y)(s_2), \sigma)$ of the constant field $(\mathbb{Q}, \sigma)$ with (28) (see Example 44) can be rearranged, e.g., to the $\mathbb{Q}$-ordered $\Pi\Sigma^*$-extension $(\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_{1,3})(x)(s_6)(y), \sigma)$ of $(\mathbb{Q}, \sigma)$, which we constructed already in Example 36. Then due to Theorem 46 this extension is again a $\Pi\Sigma^\delta$-extension.

As an immediate consequence, we end up at structural properties which do not depend on the order of the extensions; compare, e.g., Corollary 33.

**Theorem 48** *($\Pi\Sigma^\delta$-structural theorem) Let* $(\mathbb{E}, \sigma)$ *be a* $\Pi\Sigma^\delta$*-ext. of* $(\mathbb{F}, \sigma)$*. Then for any* $f, g \in \mathbb{E}$ *with* (5) *we have* (21)*. In particular, if* $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ *and*

$$S = \{1 \leq i \leq e | \delta_{\mathbb{F}}(t_i) = \delta_{\mathbb{F}}(f) + 1 \textit{ and } t_i \textit{ is a } \Sigma^*\textit{-extension}\},$$

*then* (10) *for some* $c, c_i \in \mathbb{K}$ *and* $w \in \mathbb{E}$ *with* $\delta_{\mathbb{F}}(w) \leq \delta_{\mathbb{F}}(f)$*.*

*Proof* By Theorem 46 we can bring the $\Pi\Sigma^\delta$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ to an $\mathbb{F}$-ordered extension as in (23). By Lemma 42 the $\Pi\Sigma^*$-extension $(\mathbb{F}_j, \sigma)$ of $(\mathbb{F}_i, \sigma)$ is reduced for any $0 \leq i \leq j \leq d$. Hence by Theorem 32 the first part follows. The second part follows by Theorem 9.

*Example* **49** Take the depth-optimal $\Pi\Sigma^*$-field $(\mathbb{E}, \sigma)$ with the rational function field $\mathbb{E} = \mathbb{Q}(k)(s_1)(s_3)(s_{1,3})(x)(s_6)(y)(s_2)$ and (28), and let $f \in \mathbb{E}$ with $\delta_{\mathbb{Q}}(f) = 2$. Then for any $g \in \mathbb{E}$ with (5) we have

$$g = w + c_1\, s_{1,3} + c_2\, x + c_3\, y \quad \text{for some } w \in \mathbb{Q}(k, s_1, s_2, s_3, s_6) \text{ and } c_1, c_2, c_3 \in \mathbb{Q}.$$

Combining this result with Theorem 14 we end up at the following refinement.

**Theorem 50** *(Refined* $\Pi\Sigma^\delta$*-structural theorem) Let* $(\mathbb{E}, \sigma)$ *be a* $\Pi\Sigma^\delta$*-ext. of* $(\mathbb{F}, \sigma)$ *with* $f \in \mathbb{E}$*; suppose*[10] *that* $\mathbb{E} = \mathbb{F}(s_1)\dots(s_u)(t_1)\dots(t_e)$ *such that* $\delta_{\mathbb{F}}(s_i) \leq \delta_{\mathbb{F}}(f) + 1$ *for all* $1 \leq i \leq u$ *and such that* $\delta_{\mathbb{F}}(t_i) > \delta_{\mathbb{F}}(f) + 1$ *for all* $1 \leq i \leq e$*; let* $\{x_1, \dots, x_r\} = \mathrm{InnerNodes}_{\mathbb{F} \leq \mathbb{F}(s_1)\dots(s_u)}(f)$*. If there is a* $g \in \mathbb{E}$ *with* (5)*, then*

$$g = \sum_{a \in \Sigma^*\text{–}\mathrm{Leaves}_{\mathbb{F} \leq \mathbb{F}(s_1)\dots(s_u)}(f)} c_a a + w \textit{ for some } c_a \in \mathrm{const}_\sigma \mathbb{F} \textit{ and } w \in \mathbb{F}(x_1, \dots, x_r).$$

[10] W.l.o.g. any extension can be brought to this form by Theorem 46.

*Example* **51** Take again the $\Pi\Sigma^{\delta}$-field $(\mathbb{E},\sigma)$ as in Example 49, and take on top the $\Sigma^{\delta}$-extension $(\mathbb{E}(t),\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(t)=t+\frac{\sigma(s_{1,3})\sigma(s_2)}{k+1}$; let

$$f=\frac{k^2\left(k^2(s_2+k(s_3+k(s_3(s_2+2s_6+3)+1)))-1\right)-2s_3}{k^7}$$

with $\delta_{\mathbb{Q}}(f)=2$. We apply Theorem 50 by choosing $\mathbb{F}=\mathbb{Q}$ ($(\mathbb{F},\sigma)$ is trivially a $\Pi\Sigma^{\delta}$-extension of $(\mathbb{F},\sigma)$). Following our theorem we reorder the $\Pi\Sigma^{\delta}$-field to the $\mathbb{Q}$-ordered $\Pi\Sigma^{\delta}$-field $(\mathbb{D}(t),\sigma)$ with $\mathbb{D}=\mathbb{Q}(k)(s_1)(s_2)(s_3)(s_6)(s_{1,3})(x)(y)$. In this instance, $\mathrm{InnerNodes}_{\mathbb{Q}\leq\mathbb{D}}(f)=\{k,s_2,s_3,s_6\}$ and $\Sigma^*$–$\mathrm{Leaves}_{\mathbb{F}\leq\mathbb{D}}(f)=\{s_1,s_{1,3},x,y\}$. Hence for any $g\in\mathbb{E}(t)$ such that (5) holds, we have

$$g=c_1\,s_1+c_2\,s_{1,3}+c_3\,x+c_4\,y+w$$

for some $c_1,c_2,c_3,c_4\in\mathbb{Q}$ and $w\in\mathbb{Q}(k,s_2,s_3,s_6)$; note that we could exclude $t$ from $g$. Indeed, we find

$$g=s_1+3s_{1,3}+x+2\,y+\frac{s_2s_3k^7-(s_3(s_2+2s_6+3)+1)k^6-s_3k^5-s_2k^4+k^2+2s_3}{k^7}.$$

Note that these results lead to fine-tuned telescoping algorithms that enable one to handle efficiently a tower of up to 100 $\Sigma^{\delta}$-extensions in the summation package Sigma; for an example from particle physics see [9]. Besides this, we emphasize

**Theorem 52** *([47, Result 6]) Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^{\delta}$-ext. of $(\mathbb{F},\sigma)$; let $f\in\mathbb{E}$. If there is a $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with $g\in\mathbb{H}$ s.t.* (5) *holds, then there is a $\Sigma^{\delta}$-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{F},\sigma)$ with a solution $g'\in\mathbb{E}'$ of $\Delta(g')=f$ such that $\delta_{\mathbb{F}}(g')\leq\delta_{\mathbb{F}}(g)$.*

In short, $\Pi$-extensions are not needed to find a telescoping solution with optimal depth. This result is connected to Liouville's theorem 1 where exponential extensions can be excluded if one looks for a solution of the integration problem.

Finally, we work out alternative characterizations as given in Theorems 19 and 30 for reduced and completely reduced $\Pi\Sigma^*$-extensions. Here we need

**Lemma 53** *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $f\in\mathbb{E}$. If there is a $\Sigma^*$-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ with extension depth$\leq d$ such that there is a $g\in\mathbb{S}\setminus\mathbb{E}$ with* (5)*, then there is a $\Sigma^*$-extension $(\mathbb{S}'(s),\sigma)$ of $(\mathbb{E},\sigma)$ with extension depth$\leq d$ and with $\Sigma^*$–$\mathrm{Leaves}_{\mathbb{E}\leq\mathbb{S}'(s)}=\{s\}$ such that there is a $w\in\mathbb{S}'$ with $\Delta(s+w)=f$.*

*Proof* We construct the desired extension from the given extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$. Note that we cannot find a $g'\in\mathbb{E}$ such that $\Delta(g')=f$; otherwise $\Delta(g-g')=0$, and hence with $g-g'\notin\mathbb{E}$ the constants are extended – a contradiction to the assumption that we adjoined only $\Sigma^*$-extensions. Let $\Sigma^*$–$\mathrm{Leaves}_{\mathbb{E}\leq\mathbb{S}}=\{s_1,\dots,s_r\}$. Then we can reorder the difference field $(\mathbb{S},\sigma)$ to $(\mathbb{E}(x_1)\dots(x_l)(s_1)\dots(s_r),\sigma)$ such that this is a $\Sigma^*$-extension of $(\mathbb{E},\sigma)$. W.l.o.g. we may assume that $g\notin\mathbb{E}(x_1)\dots(x_l)$: otherwise, we neglect the leaf extensions $s_1, \dots, s_r$ and repeat the construction from above. If $r = 1$, we are done. Otherwise, we continue as follows. Since $\Delta(s_i) \in \mathbb{E}(x_1)\dots(x_l)$ for $1 \leq i \leq r$ (the $x_i$ are leaf extensions), $(\mathbb{E}(x_1)\dots(x_l)(s_1)\dots(s_r), \sigma)$ is a reduced $\Sigma^*$-extension of $(\mathbb{E}(x_1)\dots(x_l), \sigma)$ by Lemma 8. Applying Theorem 9 it follows that $g = w + \sum_{i=1}^{r} c_i s_i$ for $c_i \in \text{const}_\sigma\mathbb{F}$ and $w \in \mathbb{E}(x_1)\dots(x_l)$; w.l.o.g. we may assume that $c_r \neq 0$, otherwise we reorder the extensions $s_i$ accordingly. Define $\phi := \sum_{i=1}^{r} c_i \Delta(s_i) \in \mathbb{E}(x_1)\dots(x_l)$; note that $\delta_{\mathbb{F}}(\phi) < d$. Then observe that there is no $\gamma \in \mathbb{E}(x_1)\dots(x_l)$ such that $\Delta(\gamma) = \phi$. Otherwise, for $h := (\gamma - \sum_{i=1}^{r-1} c_i s_i)/c_r \in \mathbb{E}(x_1)\dots(x_l)(s_1)\dots(s_{r-1})$ we get $\Delta(h) = \Delta(s_r)$, and thus $s_r$ is not a $\Sigma^*$-extension by Theorem 5; a contradiction. Consequently, we can apply Theorem 5 and construct the $\Sigma^*$-extension $(\mathbb{E}(x_1)\dots(x_l)(s), \sigma)$ of $(\mathbb{E}(x_1)\dots(x_l), \sigma)$ with $\sigma(s) = s + \phi$ and $\delta_{\mathbb{F}}(s) \leq d$. Note that for $q := w + s \in \mathbb{E}(s_1)\dots(s_k)(s)$ we have $\Delta(q) = \Delta(g) = f$. If $\Sigma^*\text{–Leaves}_{\mathbb{E} \leq \mathbb{E}(x_1)\dots(x_l)(s)}$ contains only $s$, we are done. Otherwise we repeat the construction from above. Since in each such step at least one extension is eliminated, this construction will lead to the desired result.

**Theorem 54** *Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$. Then the following statements are equivalent:*

*(1) This extension is depth-optimal.*
*(2) For any $\Sigma^*$-extension $t_i$ ($1 \leq i \leq e$) with $f := \Delta(t_i) \in \mathbb{F}(t_1)\dots(t_{i-1})$ there does not exist a $\Pi\Sigma^*$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}(t_1)\dots(t_{i-1}), \sigma)$ with extension depth $\leq \delta_{\mathbb{F}}(f)$ in which we find $g \in \mathbb{H}$ such that (5) holds.*
*(3) For any $\Sigma^*$-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{E}, \sigma)$ with extension depth $\mathfrak{d}$ the following holds:*

$$\forall f, g \in \mathbb{S} : \ \Delta g = f \ \wedge \ \delta_{\mathbb{F}}(f) \geq \mathfrak{d} \ \Rightarrow \ \delta_{\mathbb{F}}(g) \leq \delta_{\mathbb{F}}(f) + 1. \qquad (33)$$

*Proof* **(1)⇔(2)** follows by Theorem 52. We show the implication **(1)⇒(3)**. Consider a $\Sigma^*$-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{E}, \sigma)$ with $\mathbb{S} = \mathbb{E}(s_1)\dots(s_r)$ such that $\delta_{\mathbb{F}}(s_i) \leq \mathfrak{d}$ for $1 \leq i \leq r$; let $f, g \in \mathbb{S}$ with (5) and $\delta_{\mathbb{F}}(f) \geq \mathfrak{d}$. By Theorem 46 we may suppose that the $\Pi\Sigma^\delta$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ is ordered with $\mathbb{E} = \mathbb{H}(t_1)\dots(t_e)$ where $\delta_{\mathbb{F}}(\mathbb{H}) = \mathfrak{d}$ and $\mathfrak{d} < \delta_{\mathbb{F}}(t_1) \leq \dots \leq \delta_{\mathbb{F}}(t_e)$; note that $f \in \mathbb{H}$. If $e = 0$, nothing has to be shown. Otherwise, by reordering we get the $\Pi\Sigma^*$-extension $(\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_e), \sigma)$ of $(\mathbb{H}, \sigma)$. Now suppose that a $\Sigma^*$-extension $t_l$ for some $1 \leq l \leq e$ is not depth-optimal; set $\phi := \Delta(t_l)$. Then there is a $\Sigma^*$-extension $(\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_{l-1})(x_1)\dots(x_u), \sigma)$ of $\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_{l-1})$ with $\delta_{\mathbb{F}}(x_i) \leq \delta_{\mathbb{F}}(\phi)$ for $1 \leq i \leq u$ and $\gamma \in \mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_{l-1})(x_1)\dots(x_u)$ such that $\Delta(\gamma) = \phi$. Since $\delta_{\mathbb{F}}(t_1) > \mathfrak{d}$, we have $\delta_{\mathbb{F}}(\phi) \geq \mathfrak{d}$, and thus $(\mathbb{H}(t_1)\dots(t_{l-1})(s_1)\dots(s_r)(x_1)\dots(x_u), \sigma)$ is a $\Sigma^*$-extension of $(\mathbb{H}(t_1)\dots(t_{l-1}), \sigma)$ with extension depth $\leq \delta_{\mathbb{F}}(\phi)$. Hence $(\mathbb{H}(t_1)\dots(t_l), \sigma)$ is not a $\Sigma^\delta$-extension of $(\mathbb{H}(t_1)\dots(t_{l-1}), \sigma)$, a contradiction. We conclude that $(\mathbb{H}(s_1)\dots(s_r)(t_1)\dots(t_e), \sigma)$ is a $\Pi\Sigma^\delta$-extension of $(\mathbb{H}(s_1)\dots(s_r), \sigma)$. Moreover, it is a reduced extension of $(\mathbb{H}, \sigma)$ by Lemma 42. Hence by Thm. 9, $g$ depends only on those $t_i$ with $\Delta(t_i) \in \mathbb{H}$, i.e., $\delta_{\mathbb{F}}(t_i) \leq \mathfrak{d}+1$. Thus $\delta_{\mathbb{F}}(g) \leq \mathfrak{d} + 1$.

Finally, we show the implication **(3)⇒(1)**. Suppose that the $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ is not depth-optimal. We may suppose that $\mathbb{E}$ is ordered, i.e., $\delta_{\mathbb{F}}(t_i)\leq\delta_{\mathbb{F}}(t_{i+1})$ for all $i$. Then there is a $\Sigma^*$-extension $t_u$ with $f:=\Delta(t_u)$ and $\mathfrak{d}:=\delta_{\mathbb{F}}(f)$ with the following property: there is a $\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_r),\sigma)$ of $(\mathbb{F}(t_1)\dots(t_{u-1}),\sigma)$ with $\delta_{\mathbb{F}}(s_i)\leq\mathfrak{d}$ and $f_i:=\Delta(s_i)$ for all $i$ s.t. there is a $g\in\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_r)$ with (5); w.l.o.g. we may assume that $\delta_{\mathbb{F}}(s_1)\leq\dots\leq\delta_{\mathbb{F}}(s_r)$. Suppose we can adjoin all $s_i$ as a tower of $\Sigma^*$-extensions to $\mathbb{F}(t_1)\dots(t_u)$: by reordering we get the $\Sigma^*$-ext. $(\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_r)(t_u),\sigma)$ of $(\mathbb{F},\sigma)$; since $g\in\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_r)$ with (5), $t_u$ is not a $\Sigma^*$-extension by Theorem 5; a contradiction. Consequently there is a $j$ $(1\leq j\leq r)$ s.t. we can construct the $\Sigma^*$-extension $(\mathbb{E}(s_1)\dots(s_{j-1}),\sigma)$ of $(\mathbb{E},\sigma)$ with $f_i=\Delta(s_i)$ for all $i$ with $1\leq i<j$, but we fail to construct the $\Sigma^*$-extension $s_j$ with $f_j=\Delta(s_j)$ on top. By Lemma 53 we can assume that there is only one leaf extension on top; hence $\delta_{\mathbb{F}}(s_1)\leq\dots\leq\delta_{\mathbb{F}}(s_{j-2})<\delta_{\mathbb{F}}(s_{j-1})\leq\mathfrak{d}$. By the choice of $j$ it follows with Thm. 5 that there is a $g'\in\mathbb{E}(s_1)\dots(s_{j-1})$ such that $\Delta(g')=f_j$. Since $(\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_j),\sigma)$ is a $\Sigma^*$-extension of $(\mathbb{F}(t_1)\dots(t_{u-1}),\sigma)$, $g'\notin\mathbb{F}(t_1)\dots(t_{u-1})(s_1)\dots(s_{j-1})$, i.e., $g'$ depends on a $t_\lambda$ with $\lambda\geq u$. Thus, $\delta_{\mathbb{F}}(g')\geq\delta_{\mathbb{F}}(t_\lambda)\geq\delta_{\mathbb{F}}(t_u)=\delta_{\mathbb{F}}(f)+1>\mathfrak{d}\geq\delta_{\mathbb{F}}(s_j)=\delta_{\mathbb{F}}(f_j)+1$. Hence, (33) does not hold.

To sum up, the structural properties given in Theorems 48 and 50 are valid, even if one adjoins $\Sigma^*$-extensions (up to a certain depth) which are not depth-optimal (see equivalence (1)⇒(3) of Theorem 54). Conversely, it is precisely property (3) of Theorem 54 that characterizes $\Pi\Sigma^\delta$-extensions, and that illuminates the difference to reduced and completely reduced extensions (compare Theorems 19 and 30).

## 8 Conclusion

Starting with Karr's structural theorem, we obtained various refined versions for reduced, completely reduced and depth-optimal $\Pi\Sigma^*$-extensions. In particular we worked out one essential draw back of Karr's version of reduced $\Pi\Sigma^*$-extensions if one wants to reduce, e.g., the nesting depth of sum expressions: his optimality depends on the order how the elements are adjoined in the field. In particular, if one reorders the tower of extensions w.r.t. the nesting depth given by the shift-operator, Karr's structural theorem usually cannot be applied: only if the difference field is reorganized by expensive transformations, one gets back a reduced $\Pi\Sigma^*$-extension of the desired ordered shape; compare Theorem 39. In contrast to that, in the recently defined depth-optimal $\Pi\Sigma^*$-fields any possible reordering (as a $\Pi\Sigma^*$-field) gives again a depth-optimal $\Pi\Sigma^*$-field. As a consequence we could show structural properties that are independent of the extension order.

We emphasize that the presented theorems for telescoping (1) can be immediately carried over to Zeilberger's creative telescoping paradigm [54] used for definite summation; for more details in the setting of $\Pi\Sigma^*$-fields we refer to [46]. More generally, we obtain structural results for parameterized telescoping. For illustrative purposes we rephrase Theorems 9 and 56 explicitly.

**Theorem 55** *(Karr's structural theorem for parameterized telescoping) Let* $(\mathbb{E},\sigma)$ *be a reduced* $\Pi\Sigma^*$*-extension of* $(\mathbb{F},\sigma)$ *with* $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ *and* $\sigma(t_i)=a_i\,t_i+f_i$ *(where either* $a_i=1$ *or* $f_i=0$*), and define* $S$ *by* (8)*; let* $\phi_1,\dots,\phi_n\in\mathbb{F}$*. If there are* $\kappa_1,\dots,\kappa_n\in\text{const}_\sigma\mathbb{F}$ *and* $g\in\mathbb{E}$ *such that the parameterized telescoping equation*

$$\Delta(g)=\kappa_1\,\phi_1+\dots+\kappa_n\,\phi_n \tag{34}$$

*holds, then there are* $w\in\mathbb{F}$ *and* $c_i\in\text{const}_\sigma\mathbb{F}$ *such that* (9) *holds; in particular, for any such* $g$ *there is some* $c\in\text{const}_\sigma\mathbb{F}$ *such that* (10) *holds.*

**Theorem 56** *(*$\Pi\Sigma^\delta$*-structural thm. for parameterized telescoping) Let* $(\mathbb{E},\sigma)$ *be a* $\Pi\Sigma^\delta$*-ext. of* $(\mathbb{F},\sigma)$*; let* $\phi_1,\dots,\phi_n$ *with* $d:=\max(\delta_{\mathbb{F}}(\phi_1),\dots,\delta_{\mathbb{F}}(\phi_n))$*. Then for* $g\in\mathbb{E}$ *and* $\kappa_1,\dots,\kappa_n\in\text{const}_\sigma\mathbb{F}$ *with* (34) *we have* $\delta_{\mathbb{F}}(g)\leq d+1$*. In particular, if* $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ *and*

$$S=\{1\leq i\leq e|\delta_{\mathbb{F}}(t_i)=d+1 \textit{ and } t_i \textit{ is a } \Sigma^*\textit{-extension}\},$$

*then we have* (10) *for some* $c,c_i\in\mathbb{K}$ *and* $w\in\mathbb{E}$ *with* $\delta_{\mathbb{F}}(w)\leq d$*.*

By concluding, we remark once more that Karr's structural theorem in [21, 22] (Theorem 9) is closely related to Liouville's theorem (Theorem 1) and Rosenlicht's algebraic proof [38] in the language of differential fields. A natural question is how our new results can be carried over to the differential field case. A positive answer should throw new light on the differential theory of elementary extensions.

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## References


1. Ablinger, J.: A computer algebra toolbox for harmonic sums related to particle physics. Diploma thesis, RISC, Johannes Kepler University (2009)
2. Abramov, S.: On the summation of rational functions. Zh. vychisl. mat. Fiz. **11**, 1071–1074 (1971)
3. Abramov, S., Petkovšek, M.: D'Alembertian solutions of linear differential and difference equations. In: J. von zur Gathen (ed.) Proc. ISSAC'94, pp. 169–174. ACM Press (1994)
4. Abramov, S., Petkovšek, M.: Rational normal forms and minimal decompositions of hypergeometric terms. J. Symbolic Comput. **33**(5), 521–543 (2002)
5. Andrews, G., Uchimura, K.: Identities in combinatorics IV: Differentiation and harmonic numbers. Util. Math. **28**, 265–269 (1985)
6. Bauer, A., Petkovšek, M.: Multibasic and mixed hypergeometric Gosper-type algorithms. J. Symbolic Comput. **28**(4–5), 711–736 (1999)
7. Bierenbaum, I., Blümlein, J., Klein, S., Schneider, C.: Difference equations in massive higher order calculations. In: Proc. ACAT 2007, vol. PoS(ACAT)082 (2007). [arXiv:hep-ph/0707.4659]

8. Bierenbaum, I., Blümlein, J., Klein, S., Schneider, C.: Two–Loop Massive Operator Matrix Elements for Unpolarized Heavy Flavor Production to $O(\epsilon)$. Nucl.Phys. B **803**(1-2), 1–41 (2008). [arXiv:hep-ph/0803.0273]
9. Blümlein, J., Kauers, M., Klein, S., Schneider, C.: Determining the closed forms of the $O(a_s^3)$ anomalous dimensions and Wilson coefficients from Mellin moments by means of computer algebra. Computer Physics Communications p. To appear (2009). [arXiv:hep-ph/0902.4091]
10. Blümlein, J., Kurth, S.: Harmonic sums and Mellin transforms up to two-loop order. Phys. Rev. **D60** (1999)
11. Bradley, D.: Duality for finite multiple harmonic $q$-series. Discrete Math. **300**(1–3), 44–56 (2005)
12. Bronstein, M.: Symbolic Integration I, Transcendental functions. Springer, Berlin-Heidelberg (1997)
13. Bronstein, M.: On solutions of linear ordinary difference equations in their coefficient field. J. Symbolic Comput. **29**(6), 841–877 (2000)
14. Chyzak, F.: An extension of Zeilberger's fast algorithm to general holonomic functions. Discrete Math. **217**, 115–134 (2000)
15. Chyzak, F., Salvy, B.: Non-commutative elimination in ore algebras proves multivariate identities. J. Symbolic Comput. **26**(2), 187–227 (1998)
16. Dilcher, K.: Some $q$–series identities related to divisor functions. Discrete Math. **145**(1-3), 83–93 (1995)
17. Flajolet, P., Salvy, B.: Euler sums and contour integral representations. Experimental Math. **7**(1), 15–35 (1998)
18. Gosper, R.: Decision procedures for indefinite hypergeometric summation. Proc. Nat. Acad. Sci. U.S.A. **75**, 40–42 (1978)
19. Hendriks, P., Singer, M.: Solving difference equations in finite terms. J. Symbolic Comput. **27**(3), 239–259 (1999)
20. Hoeij, M.: Finite singularities and hypergeometric solutions of linear recurrence equations. J. Pure Appl. Algebra **139**(1-3), 109–131 (1999)
21. Karr, M.: Summation in finite terms. J. ACM **28**, 305–350 (1981)
22. Karr, M.: Theory of summation in finite terms. J. Symbolic Comput. **1**, 303–315 (1985)
23. Kauers, M., Schneider, C.: Indefinite summation with unspecified summands. Discrete Math. **306**(17), 2021–2140 (2006)
24. Kauers, M., Schneider, C.: Symbolic summation with radical expressions. In: C. Brown (ed.) Proc. ISSAC'07, pp. 219–226 (2007)
25. Kauers, M., Schneider, C.: Automated proofs for some Stirling number identities. Electron. J. Combin. **15** (2008). ♯R2
26. Kirschenhofer, P.: A note on alternating sums. Electron. J. Combin. **3**(2) (1996)
27. Laplace, P.: Théorie analytique des probabilités. Vol. I. 2nd edition, $V^e$ Courcier, Paris (1814)
28. Liouville, J.: Mémoire sur l'intégration d'une classe de fonctions transcendantes. J. Reine Angew. Math. **13**, 93–118 (1835)
29. Moch, S., Uwer, P., Weinzierl, S.: Nested sums, expansion of transcendental functions, and multiscale multiloop integrals. J. Math. Phys. **6**, 3363–3386 (2002)
30. Nörlund, N.: Differenzenrechnung. Springer, Berlin (1924)
31. Ostrowski, A.: Sur l'intégrabilité élémentaire de quelques classes d'expressions. Comment. Math. Helv. **18**, 283–308 (1946)
32. Paule, P.: Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)
33. Paule, P., Riese, A.: A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated aproach to $q$-hypergeometric telescoping. In: M. Ismail, M. Rahman (eds.) Special Functions, q-Series and Related Topics, vol. 14, pp. 179–210. Fields Institute Toronto, AMS (1997)
34. Petkovšek, M.: Hypergeometric solutions of linear recurrences with polynomial coefficients. J. Symbolic Comput. **14**(2-3), 243–264 (1992)
35. Petkovšek, M., Wilf, H.S., Zeilberger, D.: $A = B$. A. K. Peters, Wellesley, MA (1996)
36. Risch, R.: The problem of integration in finite terms. Trans. Amer. Math. Soc. **139**, 167–189 (1969)

37. Risch, R.: The solution to the problem of integration in finite terms. Bull. Amer. Math. Soc. **76**, 605–608 (1970)
38. Rosenlicht, M.: Liouville's theorem on functions with elementary integrals. Pacific J. Math. **24**, 153–161 (1968)
39. Schneider, C.: Symbolic summation in difference fields. Ph.D. thesis, RISC-Linz, J. Kepler University, Linz (2001)
40. Schneider, C.: Symbolic summation with single-nested sum extensions. In: J. Gutierrez (ed.) Proc. ISSAC'04, pp. 282–289. ACM Press (2004)
41. Schneider, C.: Finding telescopers with minimal depth for indefinite nested sum and product expressions. In: M. Kauers (ed.) Proc. ISSAC'05, pp. 285–292. ACM (2005)
42. Schneider, C.: A new Sigma approach to multi-summation. Adv. in Appl. Math. Special Issue Dedicated to Dr. David P. Robbins **34**(4), 740–767 (2005)
43. Schneider, C.: Product representations in $\Pi\Sigma$-fields. Ann. Comb. **9**(1), 75–99 (2005)
44. Schneider, C.: Solving parameterized linear difference equations in terms of indefinite nested sums and products. J. Differ. Equations Appl. **11**(9), 799–821 (2005)
45. Schneider, C.: Simplifying sums in $\Pi\Sigma$-extensions. J. Algebra Appl. **6**(3), 415–441 (2007)
46. Schneider, C.: Symbolic summation assists combinatorics. Sém. Lothar. Combin. **56**, 1–36 (2007). Article B56b
47. Schneider, C.: A refined difference field theory for symbolic summation. J. Symbolic Comput. **43**(9), 611–644 (2008). [arXiv:0808.2543v1]
48. Schneider, C.: Parameterized telescoping proves algebraic independence of sums. Ann. Comb. p. To appear (2009). [arXiv:0808.2596]
49. Schneider, C.: A symbolic summation approach to find optimal nested sum representations. In: Proceedings of the Conference on Motives, Quantum Field Theory, and Pseudodifferential Operators, To appear in the Mathematics Clay Proceedings (2009)
50. Singer, M.F., Saunders, B.D., Caviness, B.F.: An extension of Liouville's theorem on integration in finite terms. SIAM J. Comput. **14**(4), 966–990 (1985)
51. Vermaseren, J.: Harmonic sums, Mellin transforms and integrals. Int. J. Mod. Phys. **A14**, 2037–2976 (1999)
52. Wegschaider, K.: Computer generated proofs of binomial multi-sum identities. Diploma thesis, RISC Linz, Johannes Kepler University (1997)
53. Zeilberger, D.: A holonomic systems approach to special functions identities. J. Comput. Appl. Math. **32**, 321–368 (1990)
54. Zeilberger, D.: The method of creative telescoping. J. Symbolic Comput. **11**, 195–204 (1991)