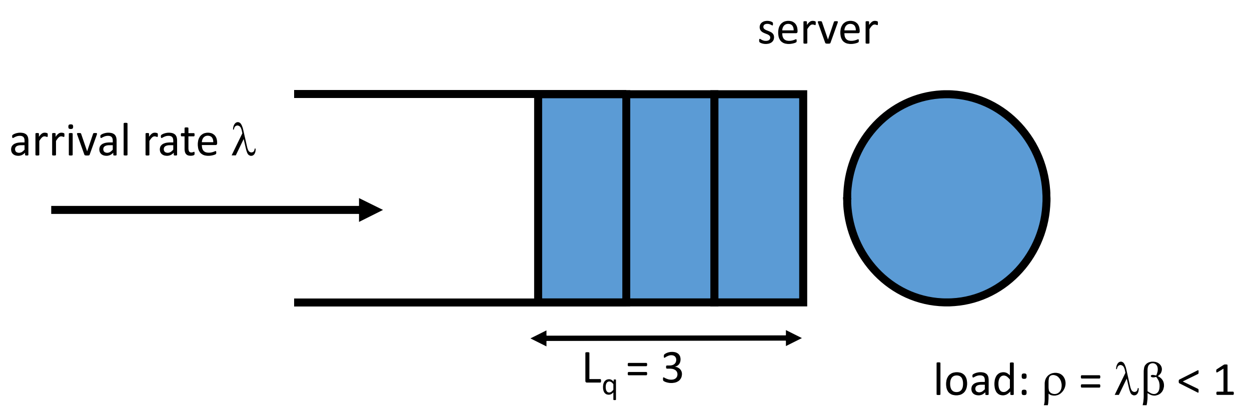
Algorithmic Methods in Queueing Theory (AIQT)



Thanks to: Ahmad Al Hanbali

Rob van der Mei
CWI and VU University Amsterdam

Contact: mei@cwi.nl, www.few.vu.nl/~mei

Time schedule

Course	Teacher	Topics	Date
1	Rob	Methods for equilibrium distributions for Markov chains	25/01/21
2	Rob	Markov processes and transient analysis	01/02/21
3	Rob	M/M/1-type models and matrix-geometric method	08/02/21
4	Rob	Buffer occupancy method	15/02/21
5	Rob	Descendant set approach	22/02/21

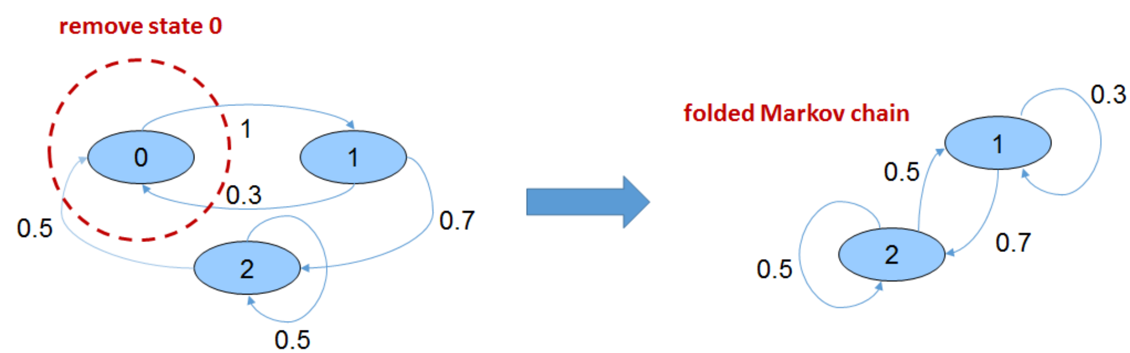
Wrap up of last two courses

- **Calculating equilbrium distributions for discrete-time Markov chains**

- **Direct methods**
  1. Gaussian elimination – numerically unstable
  2. GTH method (reduction of Markov chains via folding)

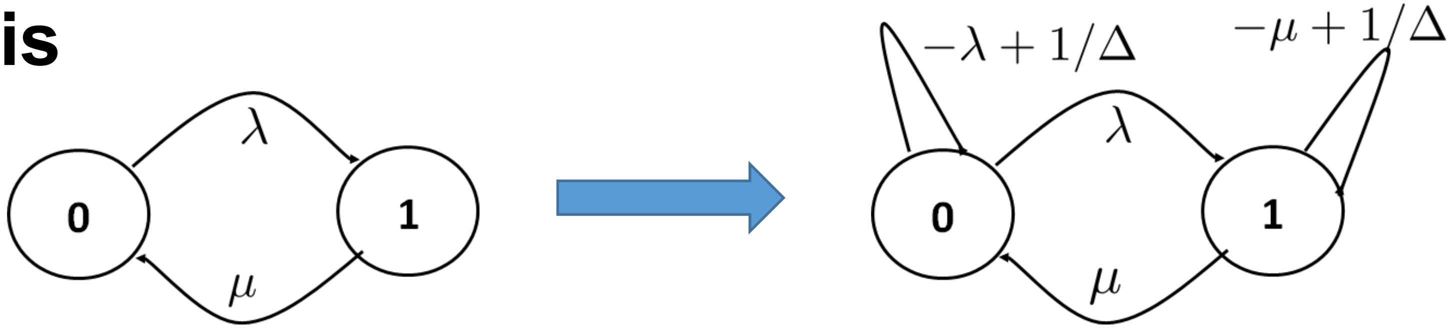
- Spectral decomposition theorem for matrix power Q^n
- Maximum eigenvalue (spectral radius) and second largest eigenvalues (sub-radius)

- **Indirect methods**
  1. Matrix powers
  2. Power method
  3. Gauss-Seidel



Wrap up of last week

- **Continuous-time Markov chains and transient analysis**



- **Continuous-time Markov chains**
  1. Generator matrix Q
  2. Uniformization
  3. Relation CTMC with discrete-time MC at jump moments

- **Transient analysis**
  1. Kolmogorov equations
  2. Differential equation
  3. Expression $P(t) = e^{Qt} = X \operatorname{diag}\left(e^{\lambda_i t}\right) Y^T = \sum_{i=0}^{N} e^{\lambda_i t} x_i y_i^T$
  4. Mean occupancy time over (0,T)

Lecture 7:

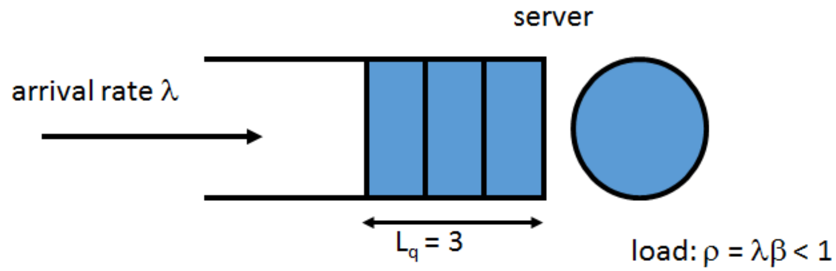
Algorithmic methods for M/M/1-type models

Lecture 7 overview

- This Lecture deals with continuous time Markov chains with **infinite state space** as opposed to finite space Markov chains in Lectures 5 and 6

- **Objective**:
  To find the equilibrium distribution of the Markov chain

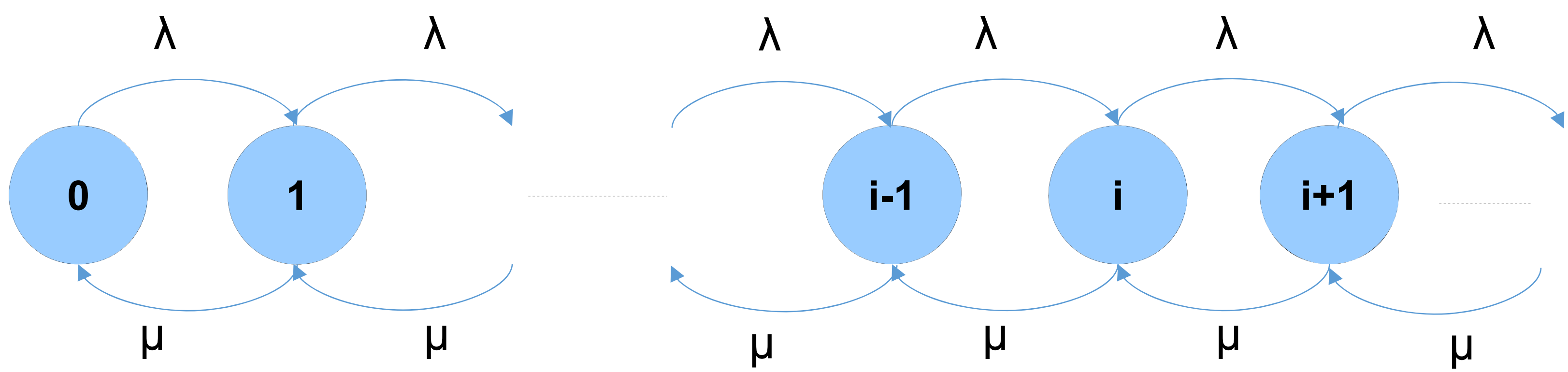
Background (1): M/M/1 queue



- Customers arrive according to **Poisson process** of rate λ, i.e., the inter-arrival times are iid exponential random variables (rv) with rate λ
- Customers' **service times** are iid **exponential** rv with mean $1/\mu$
- Inter-arrival times and service times are independent
- Service discipline can be First-In-First-Out (FIFO), Last-In-First-Out (LIFO), or Processor Sharing (PS)

Under above assumptions, $\{N(t), t \geq 0\}$, the number of customers in M/M/1 queue at time t is continuous-time infinite space Markov chain

Background (2): M/M/1 queue



Let p_i denote equilibrium probability of state i, then

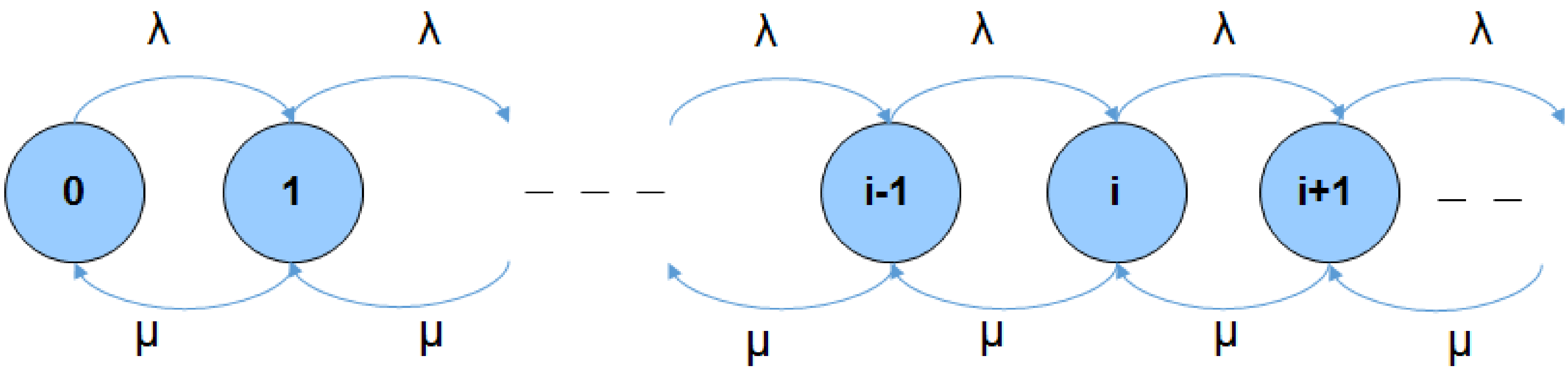
- $-\lambda p_0 + \mu p_1 = 0,$
- $\lambda p_{i-1} - (\lambda + \mu) p_i + \mu p_{i+1} = 0, \; i = 1,2, \ldots$

Solving equilibrium equations for $\rho = \lambda/\mu < 1$ (with $\sum_{i \geq 0} p_i = 1$) gives:

$$p_i = p_{i-1}\rho, \; p_0 = 1 - \rho \Rightarrow \boxed{p_i = (1-\rho)\rho^i, i \geq 0,}$$

This means N is **geometrically** distributed with parameter ρ

Background (3): M/M/1 queue



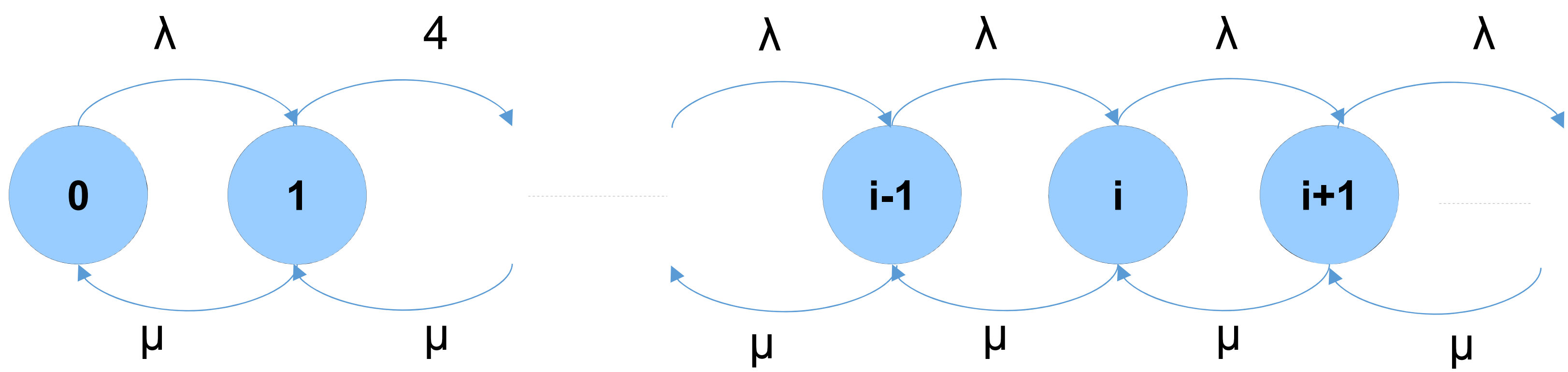
Solution: $p_i = (1-\rho)\rho^i, i \geq 0$

Feature: jumps only to neighbouring states

Idea of generalization to "M/M/1-type" queues:

1. State i replaced by **set of states** (called level i)
2. Load ρ replaced by **rate matrix R**

Background (4): M/M/1 queue



Solution: $p_i = (1-\rho)\rho^i, i \geq 0$

Tail probabilities:

conditioning w.r.t. number of customers upon arrival + PASTA

$$P(W > t) = \sum_{i=0}^{\infty}(1-\rho)\rho^i \sum_{j=0}^{i-1}\frac{(\mu t)^j}{j!}e^{-\mu t} = \sum_{j=0}^{\infty}\frac{(\mu t)^j}{j!}e^{-\mu t}\sum_{i=j+1}^{\infty}(1-\rho)\rho^i$$

$$= \sum_{j=0}^{\infty}\frac{(\mu t)^j}{j!}e^{-\mu t}\rho^{j+1} = \rho e^{-\mu(1-\rho)t}, \qquad t \geq 0.$$

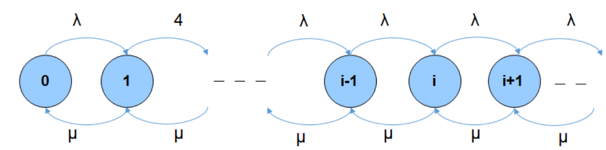
given i customers upon arrival, waiting time Erlang-i distributed with mean i/μ

if Poisson process with rate μ, # arrivals in (0;t) is Poisson with mean μt

Prob { time until i-th arrival > t) = Prob { # of arrivals in (0;t) } < i

Background (5): M/M/1 queue

- **Stability condition:** expected queue length is finite, load ρ:= λ/μ < 1. This can be interpreted as drift to the right is smaller than drift to left
- In stable case, ρ is probability the M/M/1 system is non-empty, i.e. , $P(N=0)=1-\rho$
- Q, the **generator** of M/M/1 queue, is a **tri-diagonal** matrix, and has the form



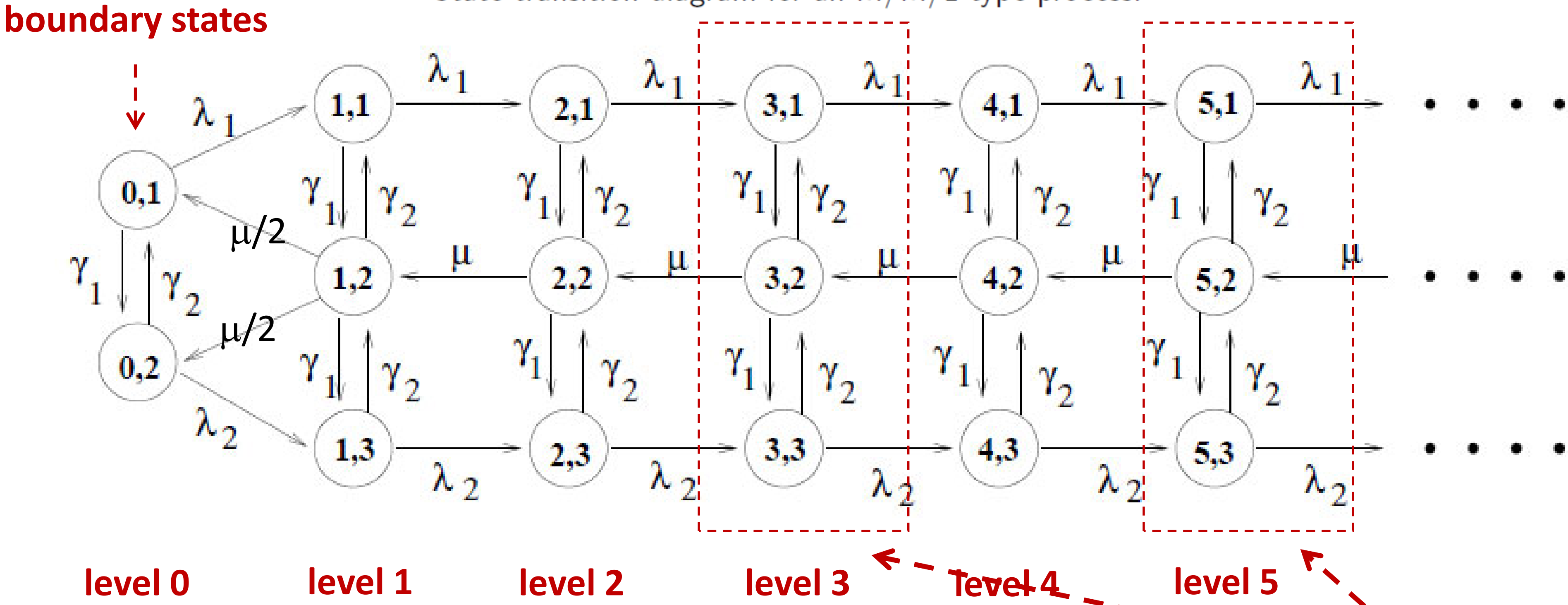
$$Q=\begin{pmatrix} -\lambda & \lambda & 0 & \cdots \\ \mu & -\lambda-\mu & \lambda & \ddots \\ 0 & \mu & -\lambda-\mu & \ddots \\ \vdots & \ddots & \mu & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Quasi Birth Death (QBD) process

- A two-dimensional irreducible continuous time Markov process with states (i, j), where $i = 0, \ldots, \infty$ and $j = 0, \ldots, m-1$
- Subset of state space with common i entry is called **level** i $(i > 0)$ and denoted $l(i) = \{(i,0), (i,1), \ldots, (i, m-1)\}$. $l(0) = \{(0,0), (0,1), \ldots, (i, n-1)\}$. This means state space is $\cup_{i \geq 0} l(i)$
- Transition rate from (i, j) to (i', j') is equal to zero for $|i - i'| \geq 2$
- For $n > 1$, transition rate between states in $l(i)$ and between states in $l(i)$ and $l(i \pm 1)$ is **independent of n**

Example of QBD process

State transition diagram for an M/M/1-type process.



same transitions for all levels >1

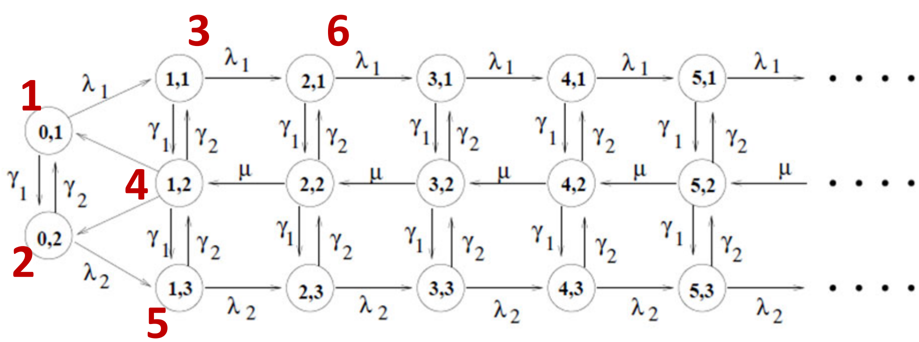
<u>Block tri-diagonal structure</u>: transitions are permitted

- between states at the same level (diagonal blocks)
- to states in the next highest level (super-diagonal blocks)
- and to states in the adjacent lower level (sub-triangular blocks)

Example of QBD process

Order the states **lexicographically**, i.e., $(0,0), \dots, (0, n-1), (1,0), \dots, (1, m-1), (2,0), \dots, (2, m-1), \dots$, the *generator* of the QBD has the following form:

lexicographical ordering

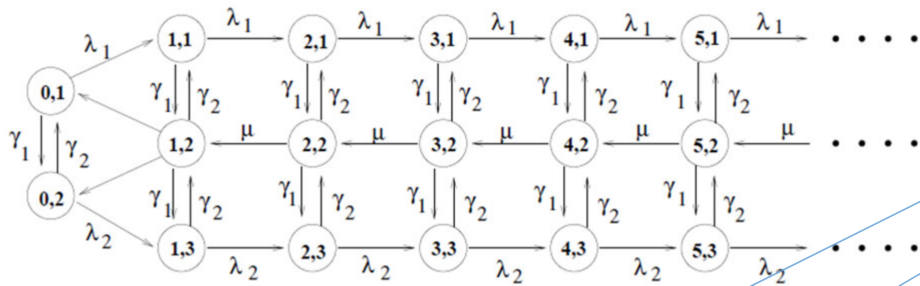


$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Note that row sums are 0:

$(B_{00} + B_{01})e = 0$, $(B_{10} + A_1 + A_2)e = 0$, and $(A_0 + A_1 + A_2)e = 0$

Example of QBD process



$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

n = 2

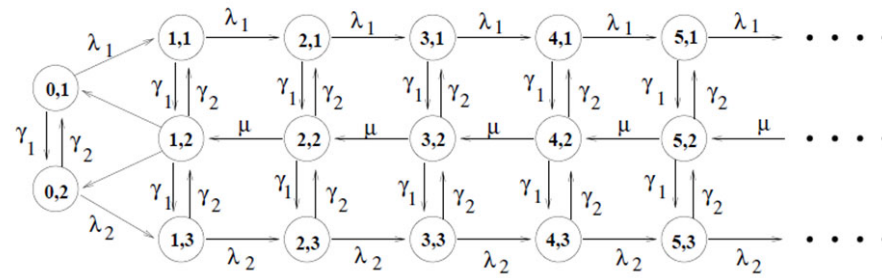
m = 3

	n		m			m			m			m			m				
n	$*$	γ_1	λ_1																level 0
	γ_2	$*$			λ_2														
m			$*$	γ_1		λ_1													level 1
	$\mu/2$	$\mu/2$	γ_2	$*$	γ_1														
				γ_2	$*$			λ_2											
m						$*$	γ_1		λ_1										level 2
				μ		γ_2	$*$	γ_1											
							γ_2	$*$			λ_2								
m									$*$	γ_1		λ_1							level 3
							μ		γ_2	$*$	γ_1								
										γ_2	$*$			λ_2					
m												$*$	γ_1		λ_1				level 4
										μ		γ_2	$*$	γ_1					
													γ_2	$*$			λ_2		
													$\ddots$		$\ddots$			$\ddots$	
	level 0		level 1			level 2			level 3			level 4			level 5				

Example of QBD process

Generator matrix

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$



Block matrices

departures: i → i-1 **arrivals: i → i+1**

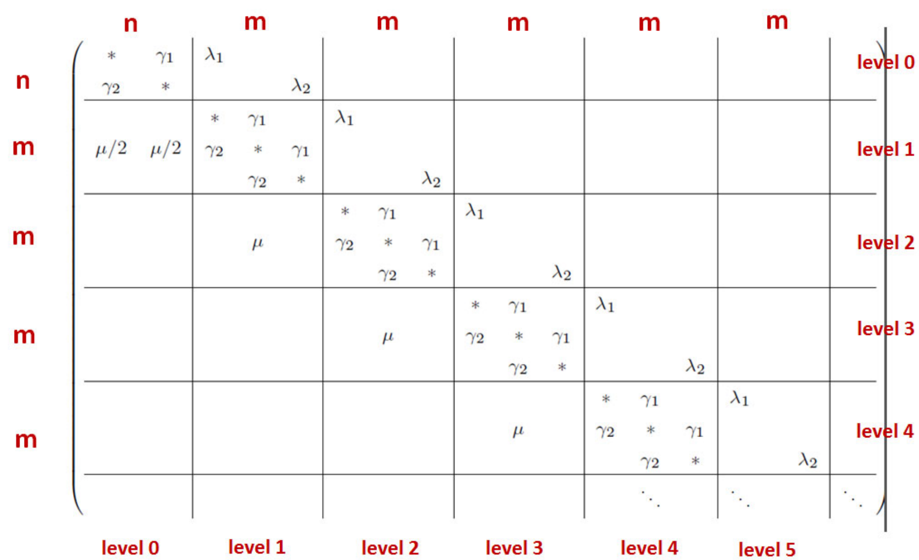
$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -(\gamma_1+\lambda_1) & \gamma_1 & 0 \\ \gamma_2 & -(\mu+\gamma_1+\gamma_2) & \gamma_1 \\ 0 & \gamma_2 & -(\gamma_2+\lambda_2) \end{pmatrix}$$

generator of transitions within level

$$B_{00} = \begin{pmatrix} -(\gamma_1+\lambda_1) & \gamma_1 \\ \gamma_2 & -(\gamma_2+\lambda_2) \end{pmatrix},$$

$$B_{01} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad B_{10} = \begin{pmatrix} 0 & 0 \\ \mu/2 & \mu/2 \\ 0 & 0 \end{pmatrix}.$$



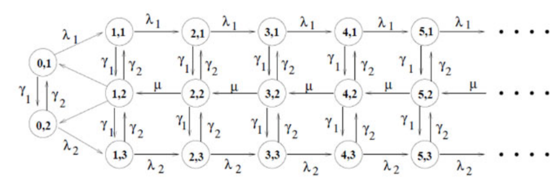
Stability: Neuts' Drift Theorem

Theorem: The QBD is **ergodic** (i.e., mean recurrence time of the states is finite) iff

$$\pi A_2 e < \pi A_0 e \text{ (*mean drift condition*)}$$

where e is the column vector of ones and π is the equilibrium distribution of the *irreducible* Markov chain with generator $A = A_0 + A_1 + A_2$,

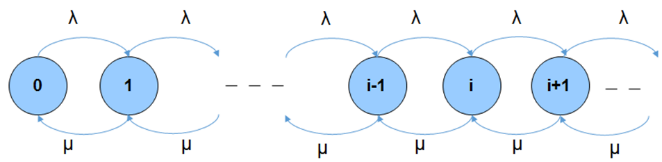
$$\pi A = 0, \qquad \pi e = 1$$

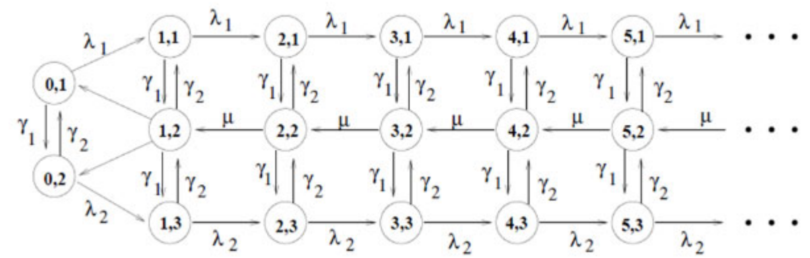


Interpretation: $\pi A_2 e$ is mean drift from level i to $i+1$. $\pi A_0 e$ is mean drift from level i to $i-1$ (Neuts' drift condition)

Generator A describes the behavior of QBD **within level**

Interpretation of Neuts' Drift Theorem





$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \quad A_1 = \begin{pmatrix} -(\gamma_1+\lambda_1) & \gamma_1 & 0 \\ \gamma_2 & -(\mu+\gamma_1+\gamma_2) & \gamma_1 \\ 0 & \gamma_2 & -(\gamma_2+\lambda_2) \end{pmatrix}$$

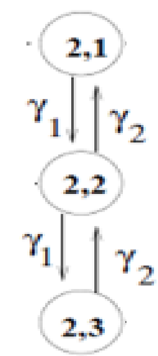
$$A = A_1 + A_2 + A_3 = \begin{pmatrix} -\gamma_1 & \gamma_1 & 0 \\ -\gamma_2 & -\gamma_1-\gamma_2 & \gamma_1 \\ 0 & \gamma_2 & -\gamma_2 \end{pmatrix}$$

$$\pi_1 = \frac{1}{1+\frac{\gamma_1}{\gamma_2}+\left(\frac{\gamma_1}{\gamma_2}\right)^2} \qquad \pi_2 = \frac{\frac{\gamma_1}{\gamma_2}}{1+\frac{\gamma_1}{\gamma_2}+\left(\frac{\gamma_1}{\gamma_2}\right)^2} \qquad \pi_3 = \frac{\left(\frac{\gamma_1}{\gamma_2}\right)^2}{1+\frac{\gamma_1}{\gamma_2}+\left(\frac{\gamma_1}{\gamma_2}\right)^2}$$

π_1

π_2

π_3



level i = 2

Mean drift to the right: $\pi A_2 e = (\pi_1 \quad \pi_2 \quad \pi_3)\, A_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda_1\pi_1 + \lambda_2\pi_3$

Mean drift to the left: $\pi A_0 e = (\pi_1 \quad \pi_2 \quad \pi_3)\, A_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mu\pi_2$

Equilibrium distribution of QBDs

Let $p_n = (p(n,0),..,p(n,m-1))$ and $p = (p_0, p_1, \ldots)$

Then **equilibrium equation** "$pQ = 0$" reads

- $p_0B_{00} + p_1B_{10} = 0,$
- $p_0B_{01} + p_1A_1 + p_2A_0 = 0$
- $p_{n-1}A_2 + p_nA_1 + p_{n+1}A_0 = 0, n \geq 2$

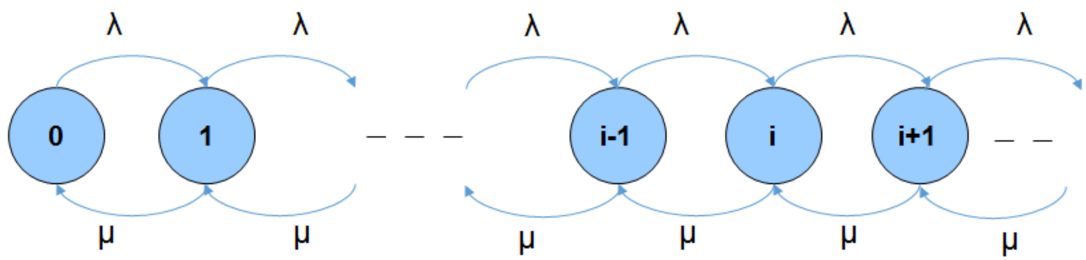
$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Theorem: if the QBD is positive recurrent, there exists a constant matrix R, such that

$$p_n = p_{n-1}R, n \geq 2 \quad \to p_n = p_1R^{n-1}, n \geq 2$$

To do: find p_0, p_1 , and R

Special case of QBD: M/M/1 queue



In that case, the "block matrices" simplify to:

$$B_{00} = (0)\,, \quad B_{01} = (\lambda)\,, \quad B_{10} = (0)$$

$$A_0 = (\mu)\,, \quad A_1 = (-\lambda - \mu)\,, \quad A_2 = (\lambda)$$

Then the equilibrium probability of state i

$$-\lambda p_0 + \mu p_1 = 0,$$

$$\lambda p_{i-1} - (\lambda + \mu) p_i + \mu p_{i+1} = 0, \qquad i = 1,2, \ldots$$

Solving equilibrium equations for $\rho = \lambda/\mu < 1$ gives:

$$p_i = p_{i-1}\rho,\ p_0 = 1 - \rho \Rightarrow p_i = (1-\rho)\rho^i, i \geq 0,$$

R-matrix

for any other non-negative solution S, we have R ≤ S

Lemma: The matrix R is the minimal nonnegative solution to the matrix equation

$$A_2 + RA_1 + R^2 A_0 = 0$$

Proof: Substituting $p_n = p_1 R^{n-1}, n \geq 2$ into the balance equations

$$p_{n-1}A_2 + p_n A_1 + p_{n+1} A_0 = 0, n \geq 2$$

implies that $p_1 R^{n-2}(A_2 + RA_1 + R^2 A_0) = 0$

- *R* is called the **rate matrix** of the Markov process Q
- *R* has spectral radius < 1, and thus, *I-R* is invertible

R-matrix for special case M/M/1

Lemma: The matrix R is the minimal nonnegative solution to the matrix equation

$$A_2 + RA_1 + R^2 A_0 = 0$$

$$B_{00} = (0)\,, \quad B_{01} = (\lambda)\,, \quad B_{10} = (0)\,, \quad e = (1)$$

$$A_0 = (\mu)\,, \quad A_1 = (-\lambda - \mu)\,, \quad A_2 = (\lambda)\,, \quad R = (\rho)$$

In M/M/1-case, the matrix R = (r) and the above equation is:

$$\lambda + r(-\lambda - \mu) + r^2 \mu = 0 \rightarrow r = 1 \ or \ r = \rho$$

smallest non-negative solution

Iterative calculation of R-matrix

Lemma: The matrix R satisfies the following equation

$$A_2 + RA_1 + R^2 A_0 = 0$$

Iterative solution to compute R

Lemma implies: $A_2 A_1^{-1} + R + R^2 A_0 A_1^{-1} = 0$

Hence: $R = -A_2 A_1^{-1} - R^2 A_0 A_1^{-1} = -V - R^2 W$

Iteration: $R_{(0)} = 0; \quad R_{(k+1)} = -V - R_{(k)}^2 W, \quad k = 1, 2, \ldots$

The iteration can be shown to converge to R (fixed point equation) , since spectral radius < 1

Calculation of p_0 and p_1

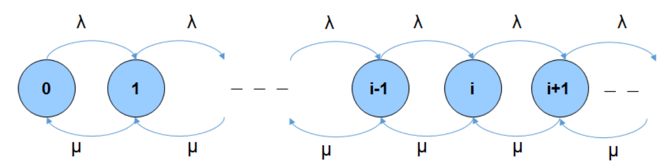
Lemma: The stationary probability vectors p_0 and p_1 are the unique solution of

- $p_0 B_{00} + p_1 B_{10} = 0$
- $p_0 B_{01} + p_1 (A_1 + R A_0) = 0$
- $p_0 e + p_1 (I - R)^{-1} e = 1$ (normalization condition)

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_2 & 0 & 0 & 0 & \cdots \\ 0 & A_0 & A_1 & A_2 & 0 & 0 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

In M/M/1-case, $B_{00} = (0)\,, \quad B_{01} = (\lambda)\,, \quad B_{10} = (0)\,, \quad e = (1)$
$A_0 = (\mu)\,, \quad A_1 = (-\lambda - \mu)\,, \quad A_2 = (\lambda)\,, \quad R = (\rho)$

Balance equations



$$-\lambda p_0 + \mu p_1 = 0,$$

$$\lambda p_{i-1} - (\lambda + \mu) p_i + \mu p_{i+1} = 0, \qquad i = 1,2, \ldots$$

Normalization

$$1 = p_0 + \frac{p_1}{1-\rho} = p_0 + p_1 + p_1\rho + p_1\rho^2 + \cdots = p_0 + p_1 + p_2 + \cdots$$

Matrix geometric method

Step 1: Verify that the matrix satisfies requirements of QBD structure

Step 2: Verify that stability condition is satisfied

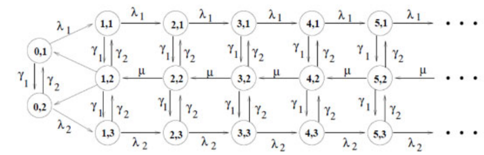
Step 3: Use recursion to compute the R-matrix

Step 4: Solve the set of equations to calculate p_0 and p_1

Step 5: Use recursion $p_n = p_{n-1}R$ to find all other p_n vectors

Example of Matrix Geometric method

Take the following parameter values for the example QBD process on page 13:



$$\lambda_1 = 1,\ \lambda_2 = .5,\ \mu = 4,\ \gamma_1 = 5,\ \gamma_2 = 3.$$

The infinitesimal generator is then given by

$$Q = \left(\begin{array}{cc|ccc|ccc|ccc|c}
-6 & 5.0 & 1 & & & & & & & & & \\
3 & -3.5 & & & .5 & & & & & & & \\
\hline
 & & -6 & 5 & & 1 & & & & & & \\
2 & 2 & 3 & -12 & 5.0 & & & & & & & \\
 & & & 3 & -3.5 & & & .5 & & & & \\
\hline
 & & & & & -6 & 5 & & 1 & & & \\
 & & & 4 & & 3 & -12 & 5.0 & & & & \\
 & & \mathbf{A_2} & & & \mathbf{A_1} & 3 & -3.5 & \mathbf{A_0} & & .5 & \\
\hline
 & & & & & & \ddots & & \ddots & & & \ddots
\end{array}\right)$$

Step 1. The matrix obviously has the correct QBD structure.

Example of MGM

Step 2: Check stability

2. We check that the system is stable by verifying Equation (8). The infinitesimal generator matrix

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} -5 & 5 & 0 \\ 3 & -8 & 5 \\ 0 & 3 & -3 \end{pmatrix}$$

has stationary probability vector

$$\pi_A = (.1837,\ .3061,\ .5102)$$

and

$$.4388 = \pi_A A_2 e \ < \ \pi_A A_0 e = 1.2245$$

Example of MGM

Recall that

$R = -A_2 A_1^{-1} - R^2 A_0 A_1^{-1} = -V - R^2 W$

Step 3: Recursion for R-matrix

3. We now initiate the iterative procedure to compute the rate matrix R. The inverse of A_1 is

$$A_1^{-1} = \begin{pmatrix} -.2466 & -.1598 & -.2283 \\ -.0959 & -.1918 & -.2740 \\ -.0822 & -.1644 & -.5205 \end{pmatrix}$$

which allows us to compute

$$V = A_2 A_1^{-1} = \begin{pmatrix} -.2466 & -.1598 & -.2283 \\ 0 & 0 & 0 \\ -.0411 & -.0822 & -.2603 \end{pmatrix}$$

$$W = A_0 A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -.3836 & -.7671 & -1.0959 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example of MGM

Recursion

$$R_{(0)} = 0; \quad R_{(k+1)} = -V - R_{(k)}^2 W, \quad k = 1, 2, \ldots$$

Step 3: Recursion for R-matrix (continued)

$$R_{(k+1)} = \begin{pmatrix} .2466 & .1598 & .2283 \\ 0 & 0 & 0 \\ .0411 & .0822 & .2603 \end{pmatrix} + R_{(k)}^2 \begin{pmatrix} 0 & 0 & 0 \\ .3836 & .7671 & 1.0959 \\ 0 & 0 & 0 \end{pmatrix}$$

and iterating successively, beginning with $R_{(0)} = 0$, we find

$$R_{(1)} = \begin{pmatrix} .2466 & .1598 & .2283 \\ 0 & 0 & 0 \\ .0411 & .0822 & .2603 \end{pmatrix}, \quad R_{(2)} = \begin{pmatrix} .2689 & .2044 & .2921 \\ 0 & 0 & 0 \\ .0518 & .1036 & .2909 \end{pmatrix},$$

$$R_{(3)} = \begin{pmatrix} .2793 & .2252 & .2921 \\ 0 & 0 & 0 \\ .0567 & .1134 & .3049 \end{pmatrix}, \quad \cdots$$

After 48 iterations, successive differences are less than 10^{-12}, at which point

$$R_{(48)} = \begin{pmatrix} .2917 & .2500 & .3571 \\ 0 & 0 & 0 \\ .0625 & .1250 & .3214 \end{pmatrix}.$$

Example of MGM

Equations for p_0 and p_1

- $p_0 B_{00} + p_1 B_{10} = 0$
- $p_0 B_{01} + p_1 (A_1 + RA_0) = 0$
- $p_0 e + p_1 (I - R)^{-1} e = 1$ (normalization condition)

Step 4: calculation of p_0 and p_1

$$(p_0, p_1)\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + RA_0 \end{pmatrix} = (p_0, p_1)\left(\begin{array}{cc|ccc} -6 & 5.0 & 1 & 0 & 0 \\ 3 & -3.5 & 0 & 0 & .5 \\ \hline 0 & 0 & -6 & 6.0 & 0 \\ 2 & 2 & 3 & -12.0 & 5.0 \\ 0 & 0 & 0 & 3.5 & -3.5 \end{array}\right) = (0, 0)$$

Solution:

$$(\pi_0, \pi_1) = (1.0,\ 1.6923,\ |\ .3974,\ .4615,\ .9011)$$

Next step: normalization

Example of MGM

Equations for p_0 and p_1

- $p_0 B_{00} + p_1 B_{10} = 0$
- $p_0 B_{01} + p_1 (A_1 + R A_0) = 0$
- $p_0 e + p_1 (I - R)^{-1} e = 1$ (normalization condition)

Step 4: normalization of p_0 and p_1

Normalization constant equals

$$\begin{aligned}
\alpha &= \pi_0 e + \pi_1 (I - R)^{-1} e \\
&= (1.0,\ 1.6923)e + (.3974,\ .4615,\ .9011) \begin{pmatrix} 1.4805 & .4675 & .7792 \\ 0 & 1 & 0 \\ .1364 & .2273 & .15455 \end{pmatrix} e \\
&= 2.6923 + 3.2657 = 5.9580
\end{aligned}$$

which allows us to compute

$$\pi_0 / \alpha = (.1678,\ .2840)$$

and

$$\pi_1 / \alpha = (.0667,\ .0775,\ .1512)$$

Example of MGM

Step 5: subcomponents of stationary distribution

$$\pi_2 = \pi_1 R = (.0667,\ .0775,\ .1512) \begin{pmatrix} .2917 & .2500 & .3571 \\ 0 & 0 & 0 \\ .0625 & .1250 & .3214 \end{pmatrix}$$

$$= (.0289,\ .0356,\ .0724)$$

and

$$\pi_3 = \pi_2 R = (.0289,\ .0356,\ .0724) \begin{pmatrix} .2917 & .2500 & .3571 \\ 0 & 0 & 0 \\ .0625 & .1250 & .3214 \end{pmatrix}$$

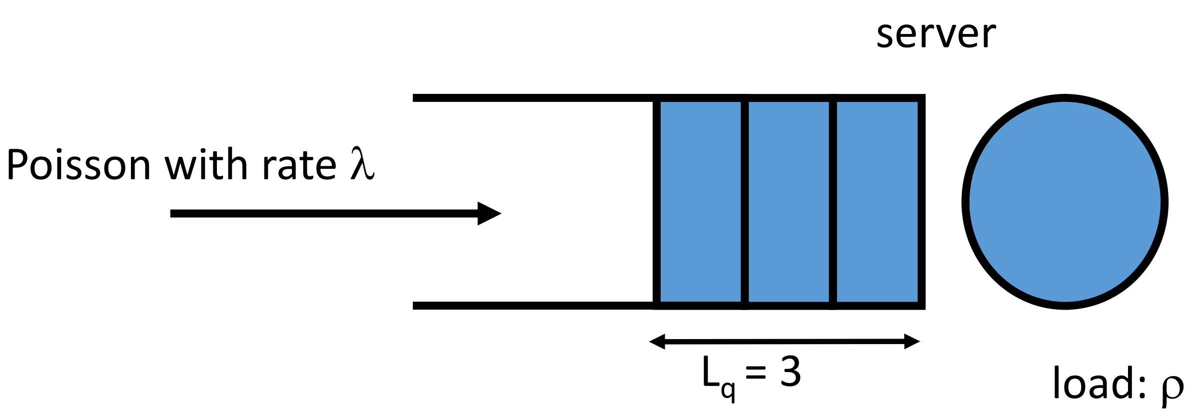
$$= (.0130,\ .0356,\ .0336)$$

and so on.

Applications of three M/M/1-type models

1. Machine with set-up times
2. Unreliable machine
3. $M/E_r/1$ model

Machine with set-up times (1)



server turned OFF when system empty

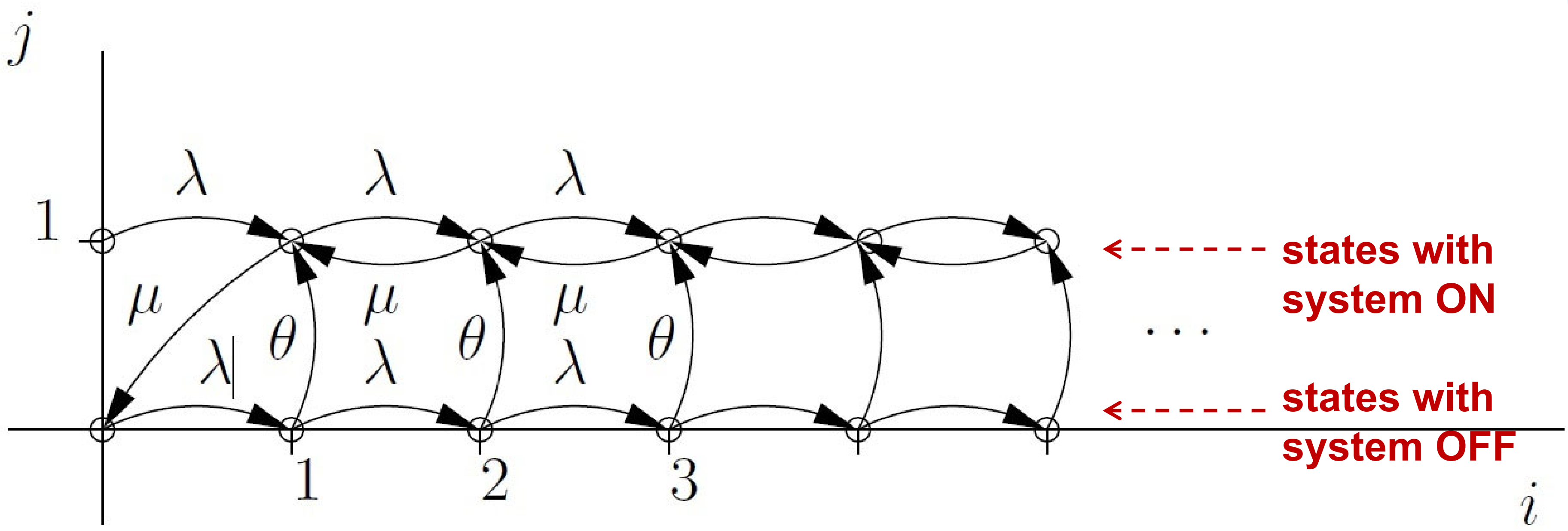
service time exponential with mean $\beta = 1/\mu$

setup-time exponential with mean $1/\theta$

load: $\rho = \lambda\beta < 1$

- In addition to assumptions on the M/M/1 system, further assume that the system is **turned off** when it is **empty**
- System is **turned on** again when a **new customer** arrives
- The **set-up time** is exponentially distributed with mean $1/\theta$

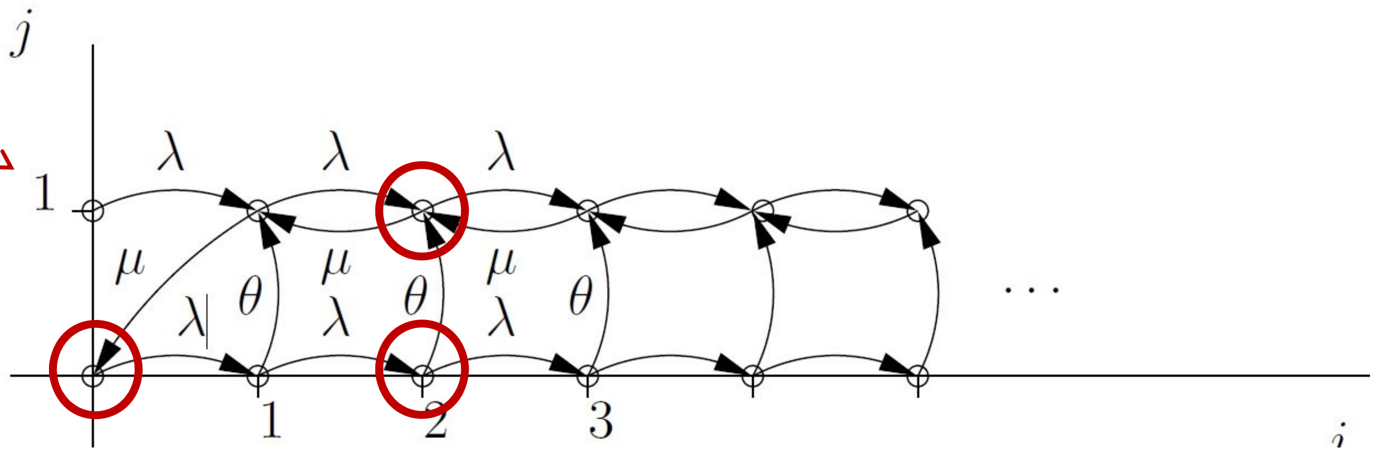
Machine with set-up time (2)



- Number of customers in system is **not a Markov process**: evolution depends on whether ON or OFF
- Two-dimensional process of state (i, j) where i is number of customers and j is system state ($j = 0$ if system is off, $j = 0$ if system is on) is Markov process

Machine with set-up time (3)

(0,1) is a transient state



- $p(i,j)$ is equilibrium probability of state $(i,j), i \geq 0, j = 0,1$

Balance equations:

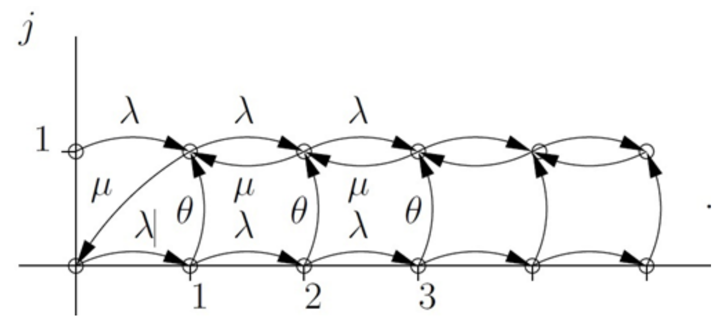
1. $p(0,0)\lambda = p(1,1)\mu$
2. $p(i,0)(\lambda + \theta) = p(i-1,0)\lambda \quad (i \geq 1)$
3. $p(i,1)(\lambda + \mu) = p(i,0)\theta + p(i+1,1)\mu + p(i-1,1)\lambda \quad (i \geq 1)$

Machine with set-up time (4)

Let $p_i = (p(i,0), p(i,1))$, then **balance equations** read

$$p_0 B_1 + p_1 B_2 = 0,$$

$$p_{i-1} A_0 + p_i A_1 + p_{i+1} A_2 = 0, i \geq 1$$



where

$$A_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}, A_1 = \begin{pmatrix} -(\lambda+\theta) & \theta \\ 0 & -(\lambda+\mu) \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}$$

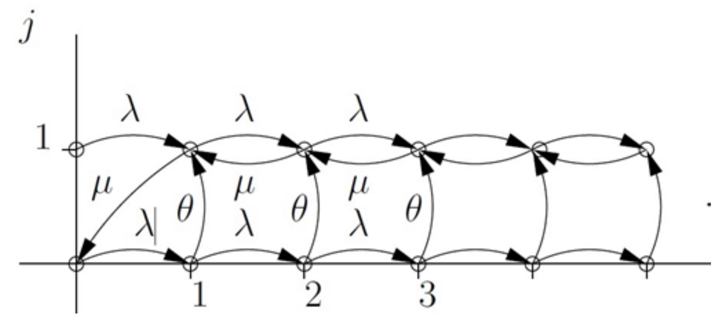
We **use the Matrix-Geometric Method (MGM)** to the find the equilibrium probability distribution

Machine with set-up time (5)

Let $p_i = (p(i,0), p(i,1))$, then **balance equations** read

$$p_0 B_1 + p_1 B_2 = 0,$$

$$p_{i-1} A_0 + p_i A_1 + p_{i+1} A_2 = 0, i \geq 1$$



Generator matrix (block structure)

$$Q = \begin{pmatrix} B_1 & A_0 & 0 & \cdots \\ B_2 & A_1 & A_0 & \ddots \\ 0 & A_2 & A_1 & \ddots \\ \vdots & \ddots & A_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$A_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}, A_1 = \begin{pmatrix} -(\lambda+\theta) & \theta \\ 0 & -(\lambda+\mu) \end{pmatrix}$$

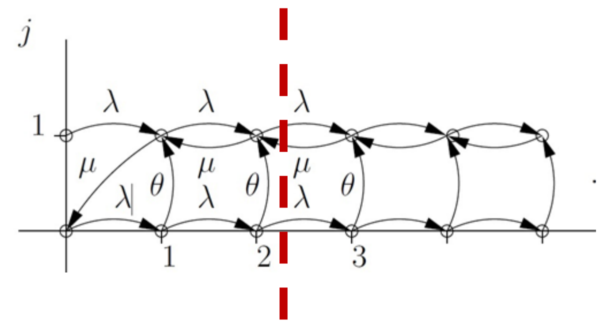
$$B_1 = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}$$

We **use the Matrix-Geometric Method (MGM)** to the find the equilibrium probability distribution

Matrix Geometric method (1)

- **Balance principle:** Global balance equations are given by equating flow from level i to $i+1$ with flow from $i+1$ to i which gives,

$$\big(p(i,0)+p(i,1)\big)\lambda = p(i+1,1)\mu, i \geq 1$$



- In matrix notation, this gives

$$p_{i+1}A_2 = p_i A_3, \text{ where } A_3 = \begin{pmatrix} \lambda & 0 \\ \lambda & 0 \end{pmatrix}$$

Recall that (balance equation)

$$p_{i-1}A_0 + p_i A_1 + p_{i+1}A_2 = 0, i \geq 1$$

- **Elimination** of p_{i+1} gives, for $i \geq 1$,

$$p_{i-1}A_0 + p_i(A_1 + A_3) = 0 \Rightarrow p_i = -p_{i-1}A_0(A_1+A_3)^{-1}$$

$$\Rightarrow R = -A_0(A_1+A_3)^{-1} = \begin{pmatrix} \lambda/(\lambda+\theta) & \lambda/\mu \\ 0 & \lambda/\mu \end{pmatrix}$$

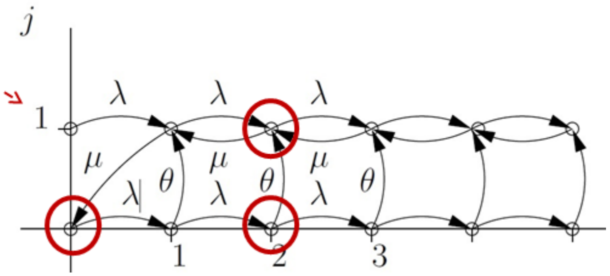
Explicit expression for R

Matrix Geometric method (2)

- **Stability condition**: absolute values of eigenvalues of R should be strictly smaller than 1

$$\lambda < \mu \text{ and } \theta > 0$$

(0,1) is a transient state



- **Normalization** condition gives

$$p_0(I + R + R^2 + \cdots)e = 1 \Rightarrow p_0(I - R)^{-1}e = 1$$

- Note that $(0,1)$ is a transient state, thus $p(0,1) = 0$.
  Normalization gives that $p(0,0) = \frac{\theta}{\theta+\lambda}\left(1 - \frac{\lambda}{\mu}\right)$

- **Mean number of customers**

$$E[L] = \sum_{i\geq 1} i p_i e = p_0 R(I - R)^{-2} e$$

Observations:

if θ → infinity, then regular M/M/1, and p(0,0)= 1 – λ/μ

if λ/μ → 1, then p(0,0) → 0

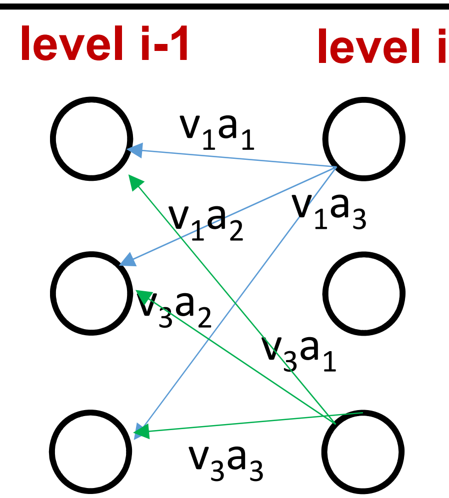
Explicit solutions in special cases

Property: In case $A_2 = v.\alpha$ is a product of two vectors where v is column vector and α is row vector with $\sum_{j=0}^{m} \alpha_j = 1$, the rate matrix reads, with e is a column vector of ones,

$$R = -A_0(A_1 + A_0 e\alpha)^{-1},$$

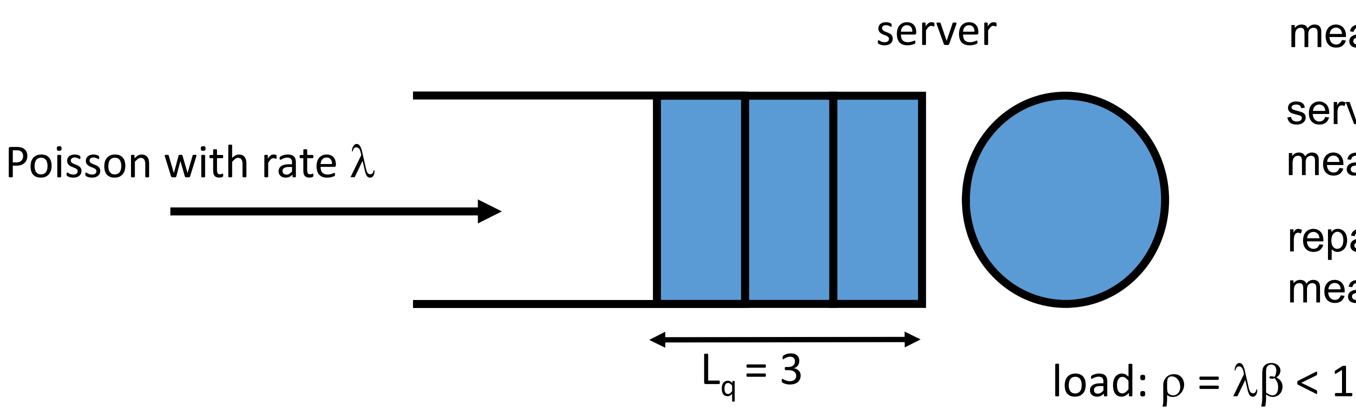
Interpretation of the assumption

When the process Q jumps from level i to level i-1, the probability of jumping to state (i-1, j) is <u>independent</u> of the starting state at level i



(see lecture notes for more details and special cases)

Unreliable machine (1)



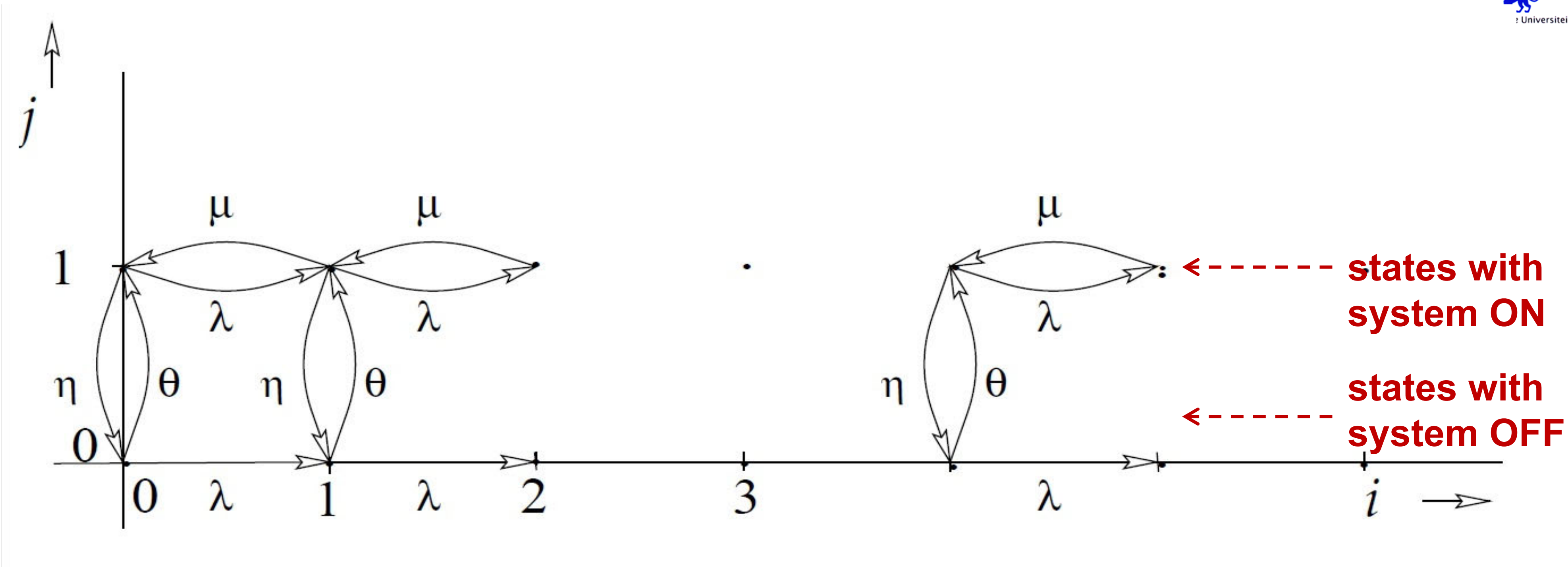
server uptime exponential with mean $1/\eta$

service time exponential with mean $1/\mu$

repair time exponential with mean $1/\theta$

- Customers arrive according to Poisson process with rate λ
- Service times is exponentially distributed of mean $1/\mu$
- Uptime of the machine is exponentially distributed with mean $1/\eta$
- Repair time is exponentially distributed with mean $1/\theta$
- **Stability condition:** load is smaller than capacity of the machine: $\lambda/\mu < P(\text{machine is up}) = \theta/(\theta+\eta)$

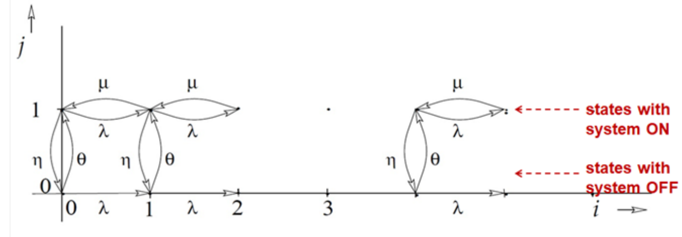
Unreliable machine (2)



- The **two-dimensional process** of state (i, j), where i number of customers, j the state of machine ($j = 1$ machine up, $j = 0$ machine down) is a Markov chain

- Note that $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \end{pmatrix} (0 \quad 1) = v\alpha$, with $v = \begin{pmatrix} 0 \\ \mu \end{pmatrix}$ and $\alpha = (0 \quad 1)$.

Unreliable machine (3)



Stability:

$$\frac{\lambda}{\mu} < \rho_U \quad \text{with} \quad \rho_U = \frac{1/\eta}{1/\eta + 1/\theta}$$ (=fraction of time that system is up)

Balance equations:

$$\begin{aligned} p(i,0)(\lambda+\theta) &= p(i-1,0)\lambda + p(i,1)\eta, \qquad i = 1,2,\dots \\ p(i,1)(\lambda+\eta+\mu) &= p(i,0)\theta + p(i+1,1)\mu + p(i-1,1)\lambda, \quad i = 1,2,\dots \end{aligned}$$

Matrix notation:

$$\begin{aligned} p_0 B_1 + p_1 A_2 &= 0, \\ p_{i-1}A_0 + p_i A_1 + p_{i+1}A_2 &= 0, \qquad i = 1,2,\dots, \end{aligned}$$

$$A_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_1 = \begin{pmatrix} -(\lambda+\theta) & \theta \\ \eta & -(\lambda+\mu+\eta) \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}$$

$$B_1 = \begin{pmatrix} -(\lambda+\theta) & \theta \\ \eta & -(\lambda+\eta) \end{pmatrix}.$$

Level probabilities:

$$p_i = (p(i,0), p(i,1))$$

Unreliable machine (3)

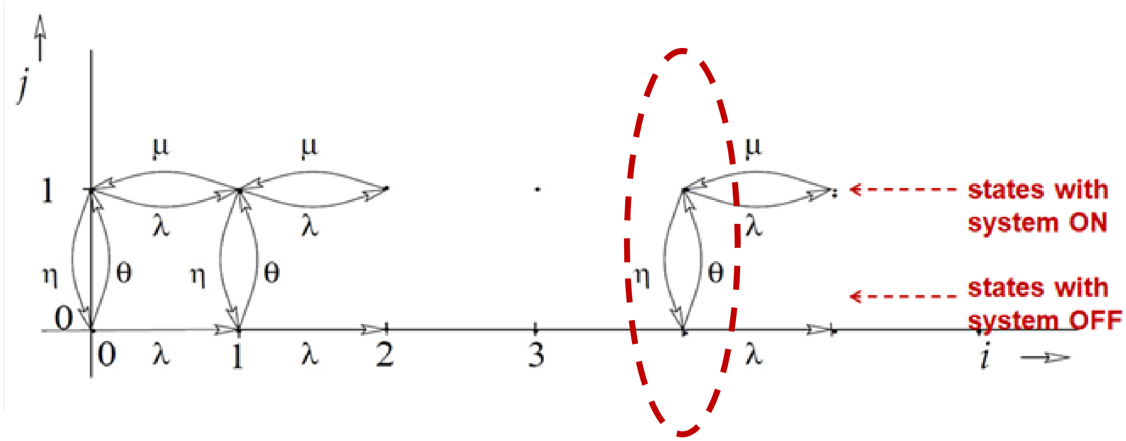
Stability:

$$\frac{\lambda}{\mu} < \rho_U \text{ with } \rho_U = \frac{1/\eta}{1/\eta + 1/\theta}$$ (=fraction of time that system is up)

$\pi_0 = \frac{\eta}{\eta+\theta}$ $\pi_1 = \frac{\theta}{\eta+\theta}$

machine down **machine up**

(solution to MC within a level)

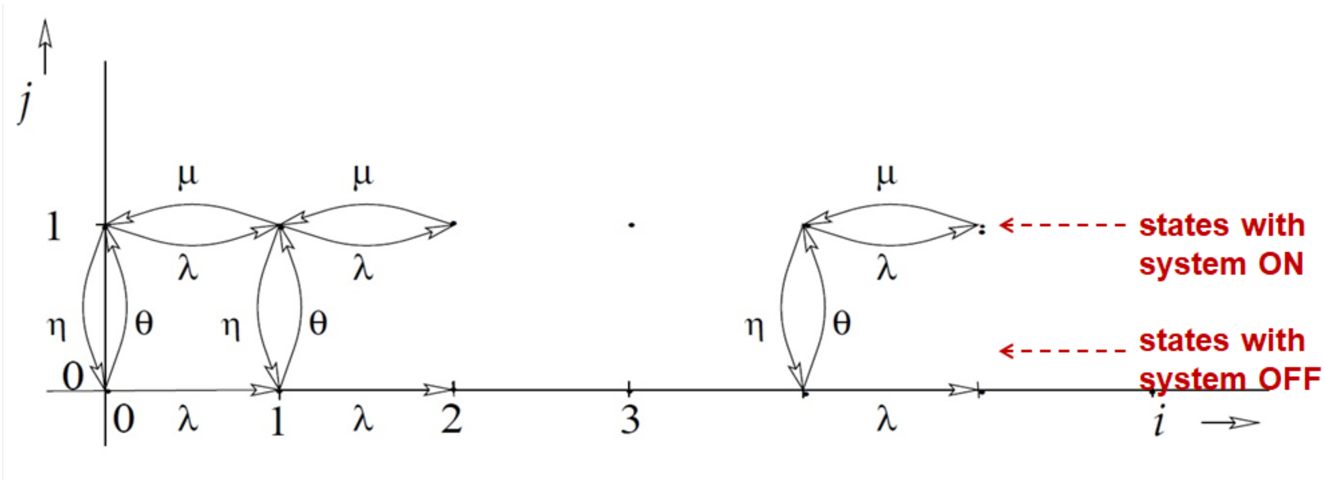


Mean drift to the left: $\pi A_2 e = (\pi_0 \quad \pi_1)\begin{pmatrix}0 & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \mu\pi_1$

Mean drift to the right: $\pi A_0 e = (\pi_0 \quad \pi_1)\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \lambda$

Neuts' drift condition: $\lambda < \mu\pi_1 = \frac{\mu\theta}{\eta+\theta}$

Unreliable machine (4)



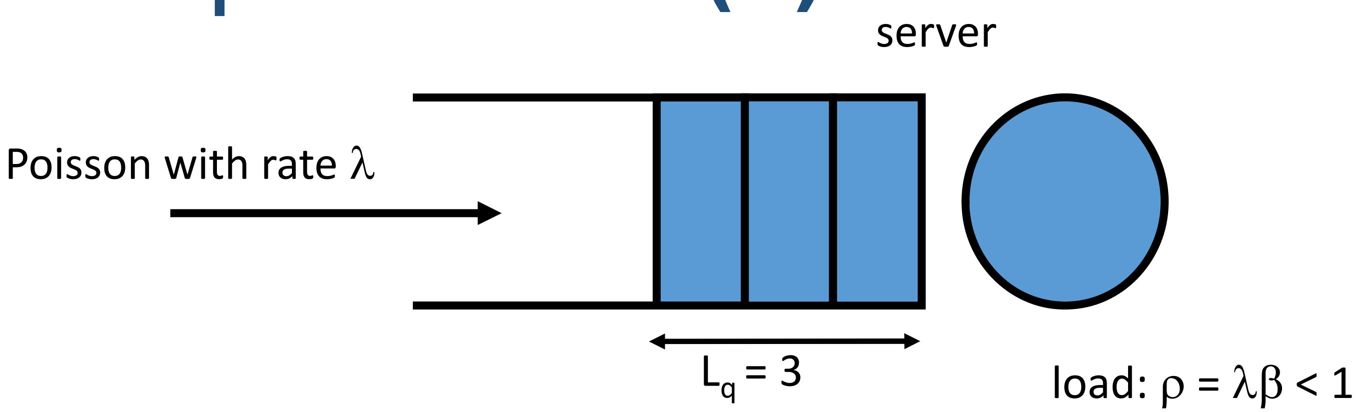
- Since $A_2 = v\alpha$, the matrix-geometric method gives

$$p_i = p_0 R^i, i \geq 1, \text{ with } R = -A_0(A_1 + A_0 e\alpha)^{-1} = \frac{\lambda}{\mu}\begin{pmatrix} \frac{\eta+\mu}{\lambda+\theta} & 1 \\ \frac{\eta}{\lambda+\theta} & 1 \end{pmatrix}$$

- Note in this case we have that $p_0(I-R)^{-1} = (1-p_u \quad p_u)$, where p_u is probability that the machine is up $\theta/(\eta+\theta)$.

- We find $p_0 = (1-p_u \quad p_u)(I-R) = \left(p_u - \frac{\lambda}{\mu}\right)\left(\frac{\eta}{\lambda+\theta} \quad 1\right)$

M/E$_r$/1 model (1)



Poisson arrivals rate λ

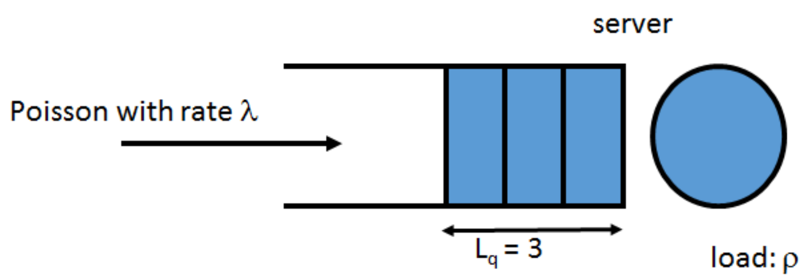
service times Erlang-r distributed with mean with mean β = r/ μ

- Poisson arrivals with rate λ
- Service times is Erlang distributed of r phases each of mean 1/μ, i.e., is sum r exponentially distributed random variable, each of rate μ
- Stability if offered load is smaller than 1:

$$\rho = \lambda r/\mu < 1$$

- **Two dimensional process** of state (i,j) where i is number of customers in the system (**excluding the customer in service**) and **j remaining phases of customer in service** is Markov process

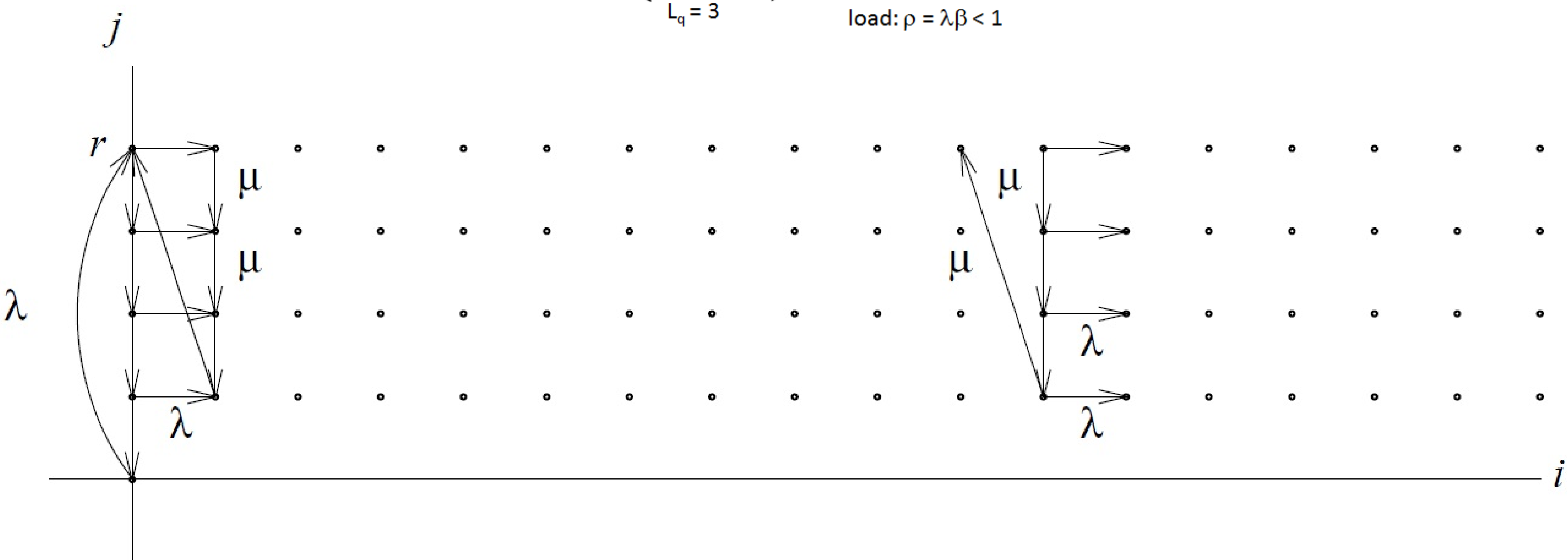
M/E$_r$/1 model (2)



Poisson arrivals rate λ

service times Erlang-r distributed with mean with mean $\beta = r/\mu$

load: $\rho = \lambda\beta < 1$

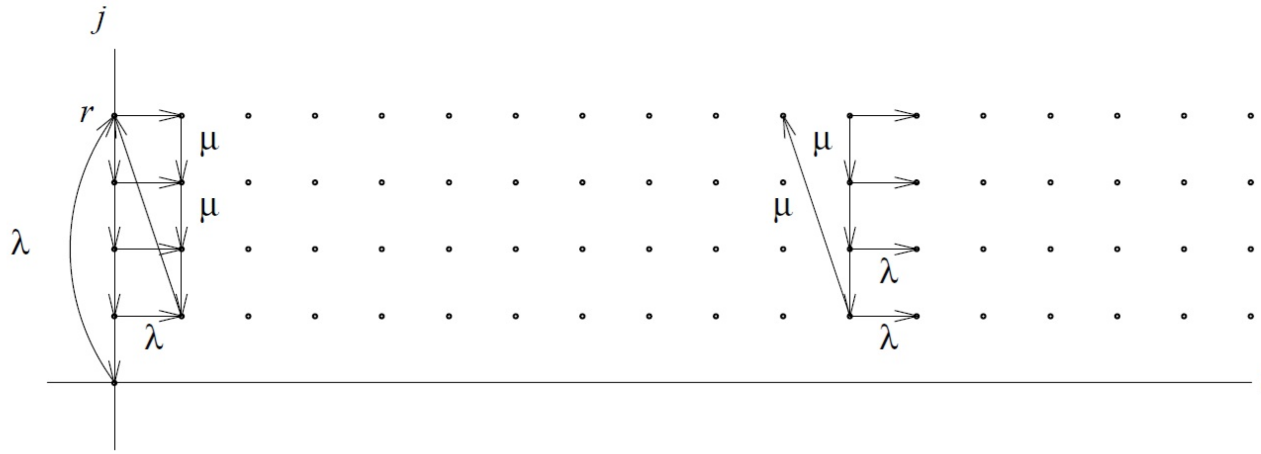


Balance equations:

$$
\begin{aligned}
p(i,j)(\lambda+\mu) &= p(i-1,j)\lambda + p(i,j+1)\mu, \qquad j = 1,\dots,r-1, \quad (i \geq 1) \\
p(i,r)(\lambda+\mu) &= p(i-1,r)\lambda + p(i+1,1)\mu,
\end{aligned}
$$

M/E$_r$/1 model (2)

State diagram:



Balance equations:

$$
\begin{aligned}
p(i,j)(\lambda+\mu) &= p(i-1,j)\lambda + p(i,j+1)\mu, \qquad j = 1,\ldots,r-1, \\
p(i,r)(\lambda+\mu) &= p(i-1,r)\lambda + p(i+1,1)\mu,
\end{aligned}
\qquad (i \geq 1)
$$

Matrix notation: $p_{i-1}A_0 + p_i A_1 + p_{i+1} A_2 = 0, \qquad i \geq 1$

where $p_i = (p(i,1),\ldots,p(i,r))$

$$
A_0 = \lambda I, A_2 = \begin{pmatrix} 0 & \cdots & 0 & \mu \\ \vdots & & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, A_1 = \mu \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 1 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & -1 \end{pmatrix} - \lambda I
$$

Matrix Geometric method

- **Balance principle:** Global balance equations are given by equating flow from level i to $i+1$ with flow from $i+1$ to i which gives,

$$(p(i,1)+\cdots+p(i,r))\lambda = p(i+1,1)\mu, i \geq 1$$

- In matrix notation this gives

$$p_i A_3 = p_{i+1} A_2, \text{ where } A_3 = \begin{pmatrix} 0 & \cdots & 0 \; \lambda \\ \vdots & \vdots & \vdots \; \lambda \\ 0 & \cdots & 0 \; \lambda \end{pmatrix}$$

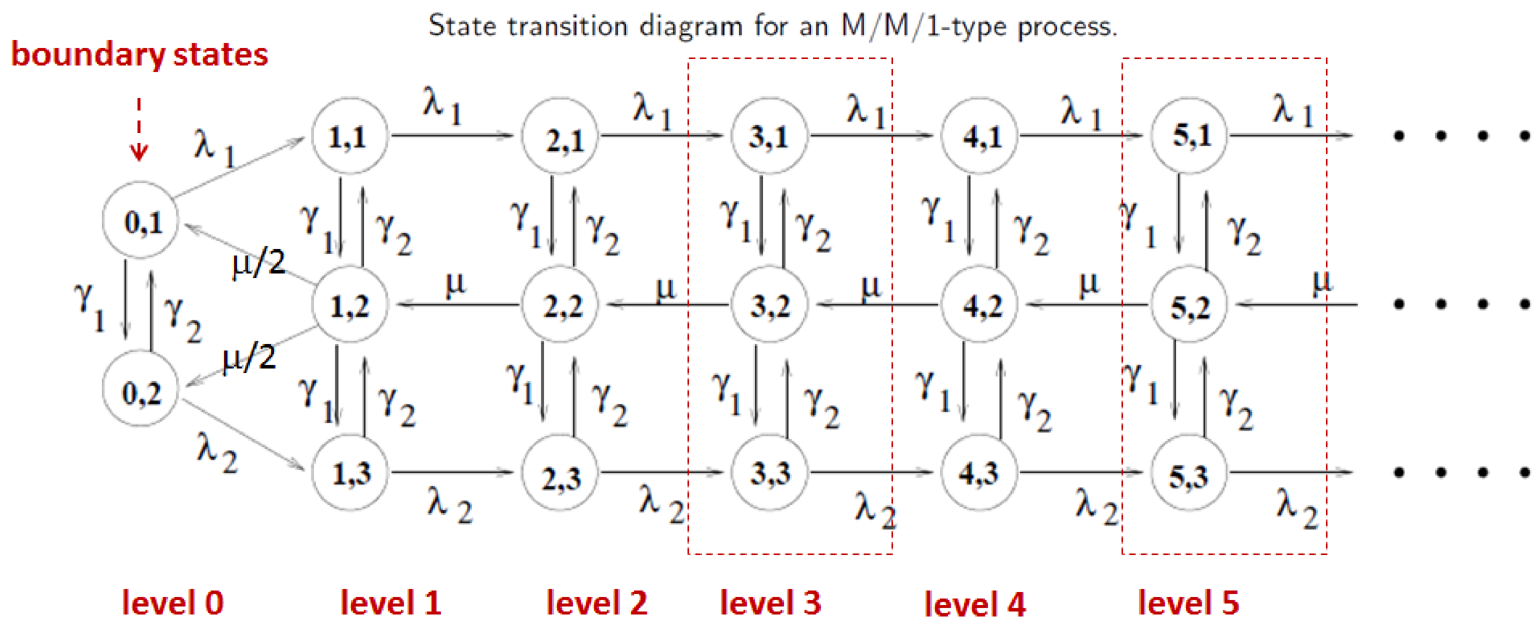
- Elimination of p_{i+1} gives, for $i \geq 1$,

$$p_{i-1}A_0 + p_i(A_1 + A_3) = 0 \Rightarrow \; p_i = -p_{i-1}A_0(A_1 + A_3)^{-1}$$

$$\Rightarrow \; R = -A_0(A_1 + A_3)^{-1}$$

Wrap up

- **Continuous-time Markov chains on a strip**
- **M/M/1-type structure, QBD-processes**



- **Equilibrium solution of the form** $p_i = p_1 R^{i-1} \ (i = 1, 2, \ldots)$
- **Matrix geometric methods**
  - Powerful numerical method
  - Closed-form expressions in special cases

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