AlQT : Lecture 1 notes

Introduction

One of the most important queueing modes is the $M/M/1$ queue.

Balance equations $\quad (\lambda+\mu)p_n = \lambda p_{n-1} + \mu p_{n+1}, \quad n \geq 1.$

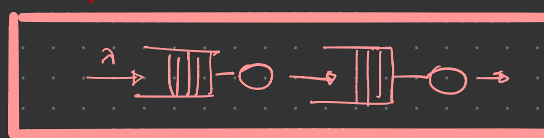
$$\lambda p_0 = \mu p_1$$

Cut equation $\quad \lambda p_n = \mu p_{n+1}, \quad n \geq 0.$

Solution : $p_n = (1-\rho)\rho^n, \quad n \geq 0, \quad \rho = \frac{\lambda}{\mu}.$

Question : Which other queueing systems have a stationary distribution with a similar form?

Answer : E.g. Jackson networks



$$c\,\rho_1^{n_1}\rho_2^{n_2}$$

$M/M/1$ type of queues

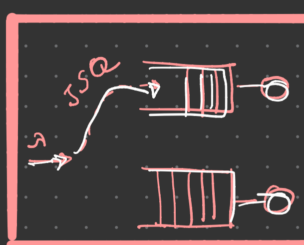
$M/M/1$ in a random environment

$\mathbb{R}^n$

$$\{(N(t), E(t)), t \geq 0\}$$

simple random walks in the quadrant with no transitions to the N, NE, E.

$$\{(Q_1(t), Q_2(t)), t \geq 0\}$$



In Lecture 1, we will delve into the class of simple random walks in the quadrant with no transitions to the N, NE, E.

Compensation approach

The model

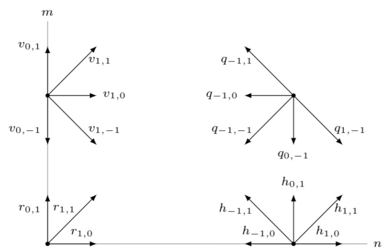


Figure 1: Transition rate diagram of the homogeneous simple random walk on the state space (n, m) with no transitions in the interior to the North, North-East, and East. Only the transitions at a few selected states are depicted as an indication.

The conditions

Conditions 1.1. *Step size: Only transitions to neighboring states are allowed;*

Forbidden steps: No transitions from interior states to the North, North-East, and East are allowed;

Homogeneity: In the interior, all transitions in the same direction occur according to the same rate.

Furthermore, in order to avoid the random walk exhibiting a trivial behavior, we consider the following assumptions.

The stability condition

$$\boldsymbol{M} = (M_x, M_y) = \Big(\sum_{n',m' \geq 0} (n'-n) p_{(n,m),(n',m')}, \sum_{n',m' \geq 0} (m'-m) p_{(n,m),(n',m')} \Big), \ n, m > 0;$$

$$\boldsymbol{M}' = (M'_x, M'_y) = \Big(\sum_{n',m' \geq 0} (n'-n) p_{(n,0),(n',m')}, \sum_{n',m' \geq 0} (m'-m) p_{(n,0),(n',m')} \Big), \ n > 0;$$

$$\boldsymbol{M}'' = (M''_x, M''_y) = \Big(\sum_{n',m' \geq 0} (n'-n) p_{(0,m),(n',m')}, \sum_{n',m' \geq 0} (m'-m) p_{(0,m),(n',m')} \Big), \ m > 0.$$

Then, when, $\boldsymbol{M} \neq 0$, the homogeneous nearest neighbor random walk is ergodic if and only if, one of the following three conditions holds:

i) $M_x < 0$, $M_y < 0$, $M_x M'_y - M_y M'_x < 0$, *and* $M_y M''_x - M_x M''_y < 0$;

ii) $M_x < 0$, $M_y \geq 0$, *and* $M_y M''_x - M_x M''_y < 0$;

iii) $M_x \geq 0$, $M_y < 0$, *and* $M_x M'_y - M_y M'_x < 0$.

The main result

Theorem 1.1. [1, Theorem 2.33] *Under the Conditions for meromorphicity 1.1 and Assumptions 1.1, and given the stability condition, there exists an $N \in \mathbb{Z}_+$, such that for $n + m > N$, the equilibrium distribution $\pi_{n,m}$ can be written*

$$\pi_{n,m} = \sum_{(\alpha_0,\beta_0)} c(\alpha_0, \beta_0) x_{n,m}(\alpha_0, \beta_0), \tag{1}$$

where (α_0, β_0) runs through the set of at most four feasible pairs and $c(\alpha_0, \beta_0)$ is an appropriately chosen coefficient and

$$x_{n,m}(\alpha_0, \beta_0) = c_0 \alpha_0^n \beta_0^m + \sum_{k=1}^{\infty} c_k \alpha_k^n (\beta_{k-1}^m + f_k \beta_k^m), \ n, m > 0, \tag{2}$$

$$x_{n,0}(\alpha_0, \beta_0) = \sum_{k=0}^{\infty} e_k \alpha_k^n, \ n > 0, \tag{3}$$

$$x_{0,m}(\alpha_0, \beta_0) = \sum_{k=0}^{\infty} d_k \beta_k^m, \ m > 0. \tag{4}$$

The equilibrium distribution of the states close to the origin, e.g. $\pi_{0,0}$, can be obtained as a function of (2)-(4) by solving the corresponding system of balance equations.

The process

Step 1: $\pi_{n,m} = \alpha^n \beta^m$, $m, n > 0$, is a solution to the balance equations in the interior if and only if α and β satisfy the following kernel equation

$$\alpha\beta(q_{-1,1} + q_{1,-1} + q_{0,-1} + q_{-1,-1} + q_{-1,0}) = \alpha^2 q_{-1,1} + \beta^2 q_{1,-1} + \alpha\beta^2 q_{0,-1} + \alpha^2\beta^2 q_{-1,-1} + \alpha^2 \beta q_{-1,0}. \tag{5}$$

Step 2: Consider a product-form $c_0\alpha_0^n\beta_0^m$ that satisfies the kernel equation (5) and also satisfies the balance equations of the horizontal boundary. Without loss of generality, we can assume that $c_0 = 1$. If the product-form $c_0\alpha_0^n\beta_0^m$ also satisfies the balance equations of the vertical boundary then this constitutes the solution of the balance equations up to a multiplicative constant that can be obtained using the normalizing equation. Otherwise, consider a linear combination of two product-forms, say $c_0\alpha_0^n\beta_0^m + c_1\alpha^n\beta^m$, $m, n > 0$, such that this combination satisfies now the balance equations of the vertical boundary. For this to happen it must be that $\beta = \beta_0$ and then $\alpha = \alpha_1$ is obtained as the solution of the kernel equation (5) for $\beta = \beta_0$.

Step 3: Finally, as long as our expression of linear combinations of product-forms violates one of the two balance equations on the boundary, we continue by adding new product-form terms satisfying the kernel equation (5). This will eventually lead to Equations (2)-(4). Of course, one still needs to show that the series expression of Equations (2)-(4) converge for all $n, m \geq 0$.

This procedure leads to the statement of Theorem 1.1.

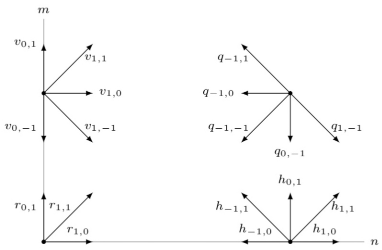
The numerical procedure

Algorithm 1 Compensation approach algorithmic implementation

1: Inputs ρ, ε and precision ε'.
2: Set $\alpha_0 = \rho^2$, $d_0 = 1$ and $N_{\text{ca}} = 1$.
3: Compute β_0 from Equation (18).
4: Set $T_{\text{ca}} = \max\{\lceil \log(\varepsilon)/\log(\beta_0) \rceil, 3\}$.
5: Compute recursively α_i, β_i, for $i = 1, \ldots, N_{\text{ca}}$, from Equation (18).
6: Compute the coefficients c_i and d_i, $i = 0, 1, \ldots, N_{\text{ca}}$, recursively from the balance equations, starting with $d_0 = 1$, cf. Step 2.
7: For all $\lfloor T_{\text{ca}}/2 \rfloor < m, n \leq T_{\text{ca}}$, compute $\pi_{m,n}^{(N_{\text{ca}})}$ from Equation (28).
8: For all $0 \leq m, n \leq \lfloor T_{\text{ca}}/2 \rfloor$, solve the linear system of the balance equations (6)-(11) and compute $\pi_{m,n}^{N_{\text{ca}}}$.
9: Normalize $\pi_{m,n}^{(N_{\text{ca}})}$.
10: Stop if Equation (29) is satisfied, else update $N_{\text{ca}} := N_{\text{ca}} + 1$ and go to Step 5.

Compensation approach example : the 2x2 switch

The general model



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The 2x2 switch model

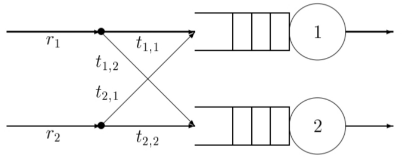

Figure 1: The 2×2 switch.

DTMC

Arrivals occur at the beg. of a slot

$t_{i,1} + t_{i,2} = 1 \quad i = 1,2$

with rate transition diagram

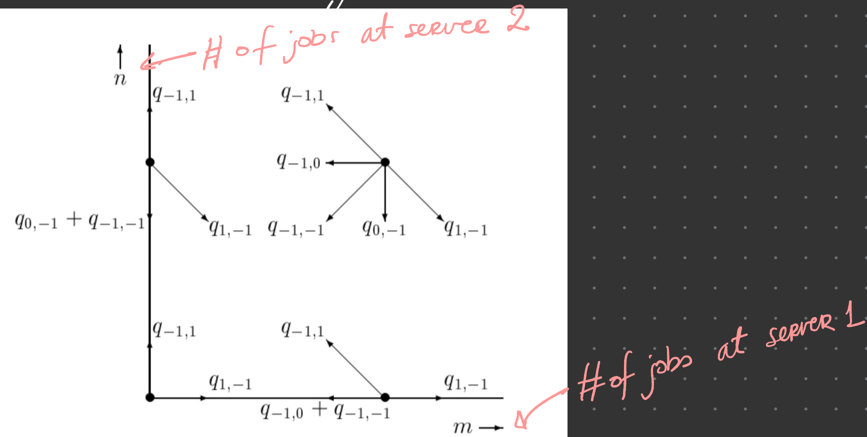

Figure 2: The one-step transition probabilities.

$$
\begin{aligned}
q_{1,-1} &= r_1 r_2 t_{1,1} t_{2,1}, & q_{0,0} &= r_1 r_2 (t_{1,1} t_{2,2} + t_{1,2} t_{2,1}), \\
q_{0,-1} &= r_1(1-r_2) t_{1,1} + r_2(1-r_1) t_{2,1}, & q_{-1,1} &= r_1 r_2 t_{1,2} t_{2,2}, \\
q_{-1,0} &= r_1(1-r_2) t_{1,2} + r_2(1-r_1) t_{2,2}, & q_{-1,-1} &= (1-r_1)(1-r_2).
\end{aligned}
$$

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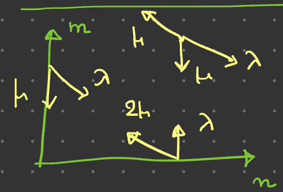
Then, when, $\boldsymbol{M} \neq 0$, *the homogeneous nearest neighbor random walk is ergodic if and only if, one of the following three conditions holds:*

i) $M_x < 0$, $M_y < 0$, $M_x M'_y - M_y M'_x < 0$, *and* $M_y M''_x - M_x M''_y < 0$;

ii) $M_x < 0$, $M_y \geq 0$, *and* $M_y M''_x - M_x M''_y < 0$;

iii) $M_x \geq 0$, $M_y < 0$, *and* $M_x M'_y - M_y M'_x < 0$.

JSQ quick application to show $\lambda < 2\mu$



$M_x = \lambda - \mu \qquad M_y = \mu - (\lambda+\mu) = -\lambda < 0$

$M''_x = \lambda \qquad M''_y = -(\lambda+\mu)$

$M'_x = -2\mu \qquad M'_y = \lambda + 2\mu$

$\Rightarrow M_y M''_x - M_x M''_y = (-\lambda)\lambda - (\lambda-\mu)(-(\lambda+\mu)) = -\lambda^2 - \mu^2 + \lambda^2 = -\mu^2 < 0$

$\Rightarrow M_x M'_y - M_y M'_x = (\lambda-\mu)(\lambda+2\mu) - (-\lambda)(-2\mu) = \lambda^2 - 2\mu^2 - \lambda\mu < 0$ for $0 < \lambda < 2\mu$

So if $\lambda < \mu$, then from condition (i) the system is stable

if $\lambda \in [\mu, 2\mu)$, then from condition (iii) the system is stable.

All in all, the stability condition is $0 < \lambda < 2\mu$

The 2x2 switch model

$r_1 t_{1,i} + r_2 t_{2,i} < 1, \quad i = 1,2$ (proof?)

The general process

Step 1: $\pi_{n,m} = \alpha^n \beta^m$, $m, n > 0$, is a solution to the balance equations in the interior if and only if α and β satisfy the following kernel equation

$$\alpha\beta(q_{-1,1} + q_{1,-1} + q_{0,-1} + q_{-1,-1} + q_{-1,0}) = \\ \alpha^2 q_{-1,1} + \beta^2 q_{1,-1} + \alpha\beta^2 q_{0,-1} + \alpha^2\beta^2 q_{-1,-1} + \alpha^2\beta q_{-1,0}. \quad (5)$$

Step 2: Consider a product-form $c_0\alpha_0^n\beta_0^m$ that satisfies the kernel equation (5) and also satisfies the balance equations of the horizontal boundary. Without loss of generality, we can assume that $c_0 = 1$. If the product-form $c_0\alpha_0^n\beta_0^m$ also satisfies the balance equations of the vertical boundary then this constitutes the solution of the balance equations up to a multiplicative constant that can be obtained using the normalizing equation. Otherwise, consider a linear combination of two product-forms, say $c_0\alpha_0^n\beta_0^m + c_1\alpha^n\beta^m$, $m, n > 0$, such that this combination satisfies now the balance equations of the vertical boundary. For this to happen it must be that $\beta = \beta_0$ and then $\alpha = \alpha_1$ is obtained as the solution of the kernel equation (5) for $\beta = \beta_0$.

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This procedure leads to the statement of Theorem 1.1.

The 2x2 switch model

$$\begin{aligned} q p_{m,n} &= q_{1,-1}p_{m-1,n+1} + q_{-1,1}p_{m+1,n-1} + q_{0,-1}p_{m,n+1} \\ &+ q_{-1,0}p_{m+1,n} + q_{-1,-1}p_{m+1,n+1}, \quad m > 0, n > 0, \end{aligned} \quad (2.2)$$

$$\begin{aligned} (q - q_{0,-1})p_{m,0} &= q_{1,-1}p_{m-1,1} + q_{1,-1}p_{m-1,0} + q_{0,-1}p_{m,1} \\ &+ (q_{-1,0} + q_{-1,-1})p_{m+1,0} + q_{-1,-1}p_{m+1,1}, \quad m > 0, n = 0, \end{aligned} \quad (2.3)$$

$$\begin{aligned} (q - q_{-1,0})p_{0,n} &= q_{-1,1}p_{1,n-1} + q_{-1,1}p_{0,n-1} + q_{-1,0}p_{1,n} \\ &+ (q_{0,-1} + q_{-1,-1})p_{0,n+1} + q_{-1,-1}p_{1,n+1}, \quad m = 0, n > 0, \end{aligned} \quad (2.4)$$

$$\begin{aligned} (q_{1,-1} + q_{-1,1})p_{0,0} &= (q_{-1,0} + q_{-1,-1})p_{1,0} + (q_{0,-1} + q_{-1,-1})p_{0,1} \\ &+ q_{-1,-1}p_{1,1}, \quad m = 0, n = 0, \end{aligned} \quad (2.5)$$

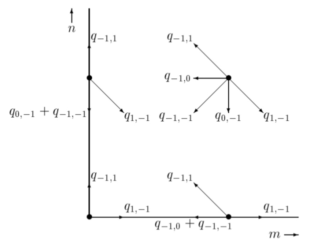
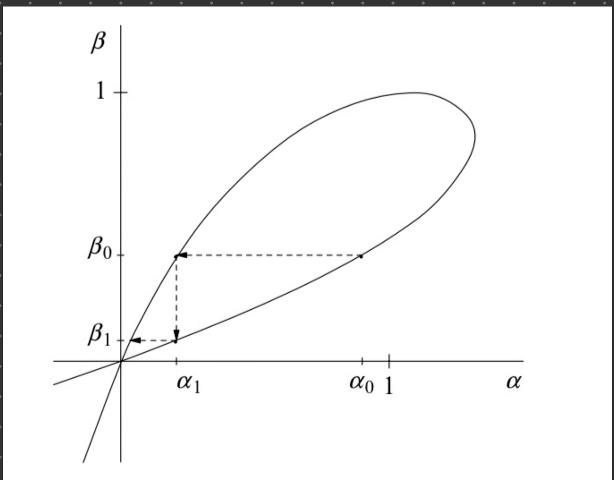


Figure 2: The one-step transition probabilities.

Setting $p_{m,n} = \alpha^m \beta^n$ yields

$$q\alpha\beta = q_{1,-1}\beta^2 + q_{-1,1}\alpha^2 + q_{0,-1}\alpha\beta^2 + q_{-1,0}\alpha^2\beta + q_{-1,-1}\alpha^2\beta^2. \quad (2.7)$$



The starting term for the horizontal :

$$(\alpha_0, \beta_0) = \left(\frac{q_{1,-1}}{q_{-1,1} + q_{-1,0} + q_{-1,-1}}, \frac{q_{-1,1}\alpha_0^2}{q_{1,-1} + q_{0,-1}\alpha_0 + q_{-1,-1}\alpha_0^2}\right). \tag{2.8}$$

the term $(\tilde{\alpha}_0, \tilde{\beta}_0)$ for the vertical boundary is symmetrical.

There is a starting term for the vertical : $(\tilde{\alpha}_0, \tilde{\beta}_0)$.

The initial solution

$$x_{m,n} = \underbrace{\overbrace{c_0\alpha_0^m\beta_0^n}^{H} + \overbrace{d_1\alpha_1^m\beta_0^n}^{H}}_{V} + \underbrace{c_1\alpha_1^m\beta_1^n + \overbrace{d_2\alpha_2^m\beta_1^n}^{H}}_{V}\overbrace{\vphantom{\beta}}^{} + c_2\alpha_2^m\beta_2^n + \cdots \tag{2.11}$$

reduces to

$$x_{m,n} = \sum_{i=0}^{\infty}(1-\beta_i)\beta_i^n[(1-\alpha_i)\alpha_i^m - (1-\alpha_{i+1})\alpha_{i+1}^m]. \tag{2.12}$$

Accounting for the second solution $\tilde{x}_{m,n}$ yields

$$p_{m,n} = x_{m,n} + \tilde{x}_{m,n}, \qquad m \geq 0, n \geq 0, m+n > 0. \tag{2.13}$$

Clearly, from this result we can derive similar expressions for performance characteristics such

The numerical procedure

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9: Normalize $\pi_{m,n}^{(N_{\text{ca}})}$.
10: Stop if Equation (29) is satisfied, else update $N_{\text{ca}} := N_{\text{ca}} + 1$ and go to Step 5.

relative normalization

OR

relative ERROR

$$\left| \frac{\sum_{m,n} \sum_{i=0}^{N_{ca}} c_i \alpha_i^m \beta_i^n - \sum_{m,n} \sum_{i=0}^{N_{ca}-1} c_i \alpha_i^m \beta_i^n}{\sum_{m,n} \sum_{i=0}^{N_{ca}} c_i \alpha_i^m \beta_i^n} \right| < \varepsilon'$$

In a nutshell

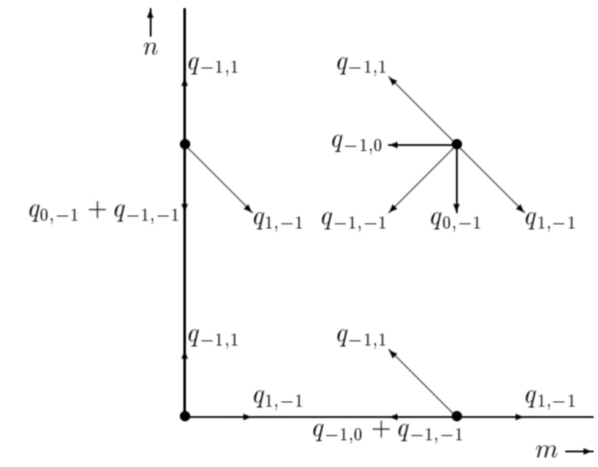

Figure 2: The one-step transition probabilities.