Deformation of tame admissible covers of curves

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Abstract

Let X be a semistable curve over a complete local ring and let $\bar{\rho} : \bar{Y} \to \bar{X}$ be a tame admissible cover of the special fiber. To lift $\bar{\rho}$ to a tame admissible cover $\rho : Y \to X$, it suffices to lift $\bar{\rho}$ locally in small neighborhoods of the singular points. The present paper gives a proof of this result using formal patching. As an application in the case of smooth curves, a proof of Grothendieck's Theorem on the tame fundamental group of smooth projective curves in positive characteristic is included.

Introduction

Let X be a semistable curve over a complete local ring and let $\bar{\rho} : \bar{Y} \to \bar{X}$ be a tame admissible cover of the special fiber. Choose a horizontal divisor $D \subset X$ lifting the branch locus $\bar{D} \subset \bar{X}$ of $\bar{\rho}$. To deform $\bar{\rho}$ to a tame admissible cover $\rho : Y \to X$, ramified along D, it suffices to lift $\bar{\rho}$ locally in small neighborhoods of the singular points. To do so, one has to choose roots of local parameters of the singular points of X. Hence, deformations of $\bar{\rho}$ exist in general only after a tamely ramified extension of the base ring and are in general not unique.

In the case of smooth curves, however, deformation of tame admissible covers is always possible and is unique. This result was first proved by Grothendieck and used in his theory of specialization of fundamental groups, [8]. In [9] deformation of mock covers is studied and applied to tame fundamental groups. These results can be reformulated in terms of deformation of tame admissible covers which are unramified over the singular points. In [23], deformation of admissible covers over complete discrete valuation rings are described. This is used to construct a specialization morphism of fundamental groups of curves with semistable reduction. The fundamental group of the special fiber, classifying admissible covers, is described by a graph of groups. There are also many results on deformation of covers of curves which are not admissible. They are all proved using some version of either formal or rigid patching. Let us only mention the result of Harbater [10] that every finite group is a Galois group over $\mathbb{Q}_p(x)$ and the proof of Abhyankar's conjecture in [22] and [11].

The deformation theorem proved in this paper is used in [13], [19] and [25] to compactify Hurwitz spaces. For this application it is important to have a precise uniqueness statement and to work over quite general complete local rings. Even though rigid patching has often shown to be the more flexible approach, in particular for covers with wild ramification, in the present situation formal patching seems to be more appropriate. The potential and the mechanisms of formal patching are certainly well known to algebraic geometers. But there seems to be no reference for this particular result which is reasonably self contained and accessible for a wider audience.

Therefore, the present paper has two goals. First, to give a rigorous proof of the general deformation theorem of admissible covers. Second, to make formal patching and its application to fundamental groups more accessible to non-specialists.

The main result

Let R be a complete noetherian local ring with separably closed residue field k and let X/R be a projective *nodal curve*. By this we mean that the special fiber $\bar{X} := X \times_R k$ has at worst ordinary double points $x_1, \ldots, x_n \in \bar{X}$ as singularities and their complete local rings on X are of the form

$$\hat{\mathcal{O}}_{X,x_i} \cong R[[u_i, v_i \mid u_i v_i = t_i]],$$

where the isomorphism is induced by elements $u_i, v_i \in \hat{\mathcal{O}}_{X,x_i}$ with $t_i := u_i v_i \in R$. The elements u_i, v_i can even be chosen from the henselian local ring $\mathcal{O}_{X,x_i} \subset \hat{\mathcal{O}}_{X,x_i}$. In other words, X is locally around x_i (in the étale topology) isomorphic to the standard nodal curve $\operatorname{Spec} R[u_i, v_i | u_i v_i = t_i]$.

Let $D \subset X$ be a *mark* on X, i.e. a horizontal divisor which is étale over $\operatorname{Spec} R$ and does not meet the singular points. A *tame admissible cover* $\rho : Y \to (X, D)$ is a finite morphism between nodal curves, which is tamely ramified along D, étale over $X^{\mathrm{sm}} - D$ and verifies the following condition over the singular points. Let $y_j \in Y$ be a point lying over one of the singular points x_i. Then y_j is a singular point of Y/R, i.e. $\hat{\mathcal{O}}_{Y,y_j} = R[[s_j, s_j | r_j s_j = \tau_j]]$ with $\tau_j := r_j s_j \in R$. Moreover, we can choose $r_j, s_j \in \hat{\mathcal{O}}_{Y,y_j}$ such that $r_j^{n_j} = u_j$ and $s_j^{n_j} = v_j$ for an integer n_j prime to the characteristic of k and $u_i, v_i \in \hat{\mathcal{O}}_{X,x_i}$ as above.

Suppose we are given a tame admissible cover $\bar{\rho} : \bar{Y} \to (\bar{X}, \bar{D})$ of the special fiber $\bar{X} := X \times_R k$ of X. For every point $y_j \in \bar{Y}$ lying over a singular point $x_i \in \bar{X}$ we can choose $\bar{r}_j, \bar{s}_j \in \hat{\mathcal{O}}_{\bar{Y},y_j}$ with $\bar{r}_j \bar{s}_j = 0$, $\bar{r}_j^{n_j} = \bar{u}_i$ and $\bar{s}_j^{n_j} = \bar{v}_j$, where u_i, v_i are as before and $\bar{u}_i, \bar{v}_i$ denote their restrictions to the special fiber. A *deformation* of $\bar{\rho}$ to R is a tame admissible cover $\rho : Y \to X$ with $\rho \times_R k = \bar{\rho}$. For every deformation ρ of $\bar{\rho}$ there are unique lifts $r_j, s_j \in \mathcal{O}_{Y,y_j}$ of $\bar{r}_j, \bar{s}_j$ with $r_j^{n_j} = u_i$, $s_j^{n_j} = v_j$ and $\tau_j := r_j s_j \in R$. Thus the deformation ρ determines a tuple $(\tau_j)_j$ of elements of R with $\tau_j^{n_j} = t_i$. Let us call the tuple (τ_j) the *deformation datum* for $\bar{\rho}$ corresponding to the deformation ρ. Our main result can be stated as follows.

Theorem: *The assignment*

$$\rho \quad \longmapsto \quad (\tau_j := r_j s_j)_j$$

induces a bijection between isomorphism classes of deformations of $\bar{\rho}$ to R and the set of tuples $(\tau_j)_j$ of elements of R with $\tau_j^{n_j} = t_i$. Moreover, the isomorphism between two deformations with the same deformation datum $(\tau_j)_j$ is unique.

This theorem appears as a claim in [13] (page 61 f) for simple covers and in [19], §3.23 in the same generality as above. In the case that R is a complete discrete valuation ring and $D = \emptyset$ it is proved in [23]. The surjectivity of the map in the theorem is proved in [12] for the case $R = k[[\tau_1, \ldots, \tau_n]]$. Other special cases of the theorem appear in various papers dealing with Galois action on fundamental groups, e.g. in [15].

Outline

Section 1 serves an an introduction to formal patching. We explain the general idea of the proof and present the necessary tools, namely étale localization, descent and Grothendieck's Existence Theorem. In Section 2 we show that every tame admissible cover is étale locally isomorphic to a cover of a certain standard shape. To show this, we introduce local coordinate systems of a nodal curve at an ordinary double point. Section 3 contains the proof of the main result. In Section 4 we give a proof of Grothendieck's theorem on the tame fundamental group of curves in positive characteristic. The appendix contains some results about étale ring extensions and henselization which are used in this paper.

Throughout, we assume that the reader is familiar with the definition of a scheme. However, we have tried to keep the references as accessible as possible. Most proofs make only references to Hartshorne's book or a standard textbook on commutative algebra.

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1 Formal patching

This section is an introduction to formal patching, as it is understood in this paper. The reader familiar with this circle of ideas can skip it without problems. The general reference for this section is [8].

We start in 1.1 with an outline of the proof of our main result from a more general point of view. More precisely, we explain those ideas of the proof which are independent of the special case of admissible covers. This outline motivates the following subsections, where the tools we need to do formal patching are presented. These are étale localization, étale descent and Grothendieck's Existence Theorem.

1.1 Outline of the proof

1.1.1 Let R be a complete local ring with residue field k. Let X be a scheme over R. We write $\bar{X} := X \times_R k$ for the special fiber. Let P be a property of morphisms of schemes which is local, in an appropriate sense. To fix ideas, we assume that a morphism with property P is finite.

Problem 1.1.1 Suppose we are given a morphism $\bar{\rho} : \bar{Y} \to \bar{X}$ with property P. Does there exist a morphism $\rho : Y \to X$ with property P such that $\bar{Y} = Y \times_R k$?

The present paper deals with this problem in the case that a morphism with property P is an admissible cover of curves. In the rest of this section we show that, under certain general conditions on the scheme X and on the property P, Problem 1.1.1 can be solved using standard techniques of algebraic geometry.

1.1.2 The main idea is to solve Problem 1.1.1 first locally and then to glue the local solutions together to a global solution. To do this, we choose an open covering $(U_i)_{i \in I}$ of X. Then $(\bar{U}_i := U_i \cap \bar{X})_i$ is an open covering of $\bar{X}$ and $(\bar{V}_i := \bar{\rho}^{-1}(\bar{U}_i))_i$ is an open covering of $\bar{Y}$. Moreover, the maps $\bar{\rho}_i : \bar{V}_i \to \bar{U}_i$ induced by $\bar{\rho}$ have property P (since P is a local property). For $i, j \in I$ we let $U_{i,j} := U_i \cap U_j$, $\bar{U}_{i,j} := \bar{U}_i \cap \bar{U}_j$ and $\bar{V}_{i,j} := \bar{V}_i \cap \bar{V}_j$.

Condition 1.1.2 If the covering $(U_i)_i$ is chosen sufficiently fine, then the following holds.

(i) The morphisms $\bar{\rho}_i : \bar{V}_i \to \bar{U}_i$ can be lifted to morphisms $\rho_i : V_i \to U_i$ with property P.

(ii) For $i \neq j$, let U be an open subset of $U_{i,j}$, and let $\bar{U} := U \cap \bar{X}$ and $\bar{V} := \bar{\rho}^{-1}(\bar{U})$. Given two morphism $\rho_1 : V_1 \to U$ and $\rho_2 : V_2 \to U$ lifting $\bar{\rho}|_{\bar{V}} : \bar{V} \to \bar{U}$ and having property P, there exists a unique isomorphism $\alpha : V_1 \xrightarrow{\sim} V_2$ with $\rho_2 \circ \alpha = \rho_1$ and $\alpha|_{\bar{V}} = \mathrm{Id}_{\bar{V}}$.

Provided we use the right notion of an open covering, Condition 1.1.2 is sufficient to solve Problem 1.1.1. Given the local lifts $\rho_i : V_i \to U_i$ of (i), Condition (ii) makes sure there are unique isomorphisms

$$\alpha_{i,j} : \rho_j^{-1}(U_{i,j}) \xrightarrow{\sim} \rho_i^{-1}(U_{i,j}) \qquad (1)$$

with $\rho_i \circ \alpha_{i,j} = \rho_j$ and $\alpha_{i,i} = \mathrm{Id}$. Their uniqueness forces the $\alpha_{i,j}$ to verify the cocycle relation

$$\alpha_{i,j}|_{\rho_j^{-1}(U_{i,j,k})} \circ \alpha_{j,k}|_{\rho_k^{-1}(U_{i,j,k})} = \alpha_{i,k}|_{\rho_k^{-1}(U_{i,j,k})}, \tag{2}$$

where $U_{i,j,k} := U_i \cap U_j \cap U_k$ for $i, j, k \in I$. In this situation we can glue the schemes V_i along the isomorphisms $\alpha_{i,j}$ and obtain a scheme Y together with a morphism $\rho : Y \to X$ such that $\rho_i = \rho|_{V_i}$. It follows that ρ is a lift of $\bar{\rho}$ with property P, solving Problem 1.1.1.

In the case that the U_i are Zariski open subsets of X, this gluing process is given as an exercise in [14] II, Exercise 2.12. But it is in general very difficult to choose a sufficiently fine Zariski covering such that Condition 1.1.2 holds. An elegant solution for this problem is to replace the Zariski topology by the finer *étale topology.* This means that we replace open subsets $U_i \subset X$ by étale morphisms $U_i \to X$ and the intersections $U_i \cap U_j$ by the fiber products $U_i \times_X U_j$ (for the moment we will keep the old notation). In the context of the étale topology, the gluing process described above can be accomplished with the theory of *descent.* This theory is a generalization of both Zariski gluing and Galois descent. We will see in Section 1.2 that the descent theorem we need here reduces immediately to an algebraic lemma dealing with faithfully flat descent of modules.

1.1.3 Whether we can find a covering $(U_i)_i$ of X verifying Condition 1.1.2 depends of course on the property P. In Section 2.3 we will show that if P means being an admissible cover of curves, then we can find an étale covering of X of a certain standard shape. For such a covering Condition 1.1.2 (i) is then easily verified. This is the only part of our formal patching process dealing with the special situation of admissible covers. In our version of formal patching, Condition 1.1.2 (ii) depends on the following étaleness condition.

Condition 1.1.3 There is a dense open subset $U_0 \subset X$ such that $\bar{\rho} : \bar{Y} \to \bar{X}$ is étale over $U_0 \cap \bar{X}$ and all lifts $\rho : Y \to X$ of $\bar{\rho}$ with property P are étale over U_0. Moreover, for a sufficiently fine open covering $(U_i)_i$ of X and for $i \neq j$ we may assume that $U_{i,j} \subset U_0$.

In the case of admissible covers of curves, one can take for U_0 the complement of the branch locus inside the smooth locus of the curve. Then Condition 1.1.3 holds if k is algebraically closed.

Assume that Condition 1.1.3 holds. Then the morphisms $\bar{\rho}_{i,j} : \bar{V}_{i,j} \to \bar{U}_{i,j}$ are étale and any lift of $\bar{\rho}_{i,j}$ to $\rho_{i,j} : V_{i,j} \to U_{i,j}$ has to be étale. The following lemma shows that if R is artinian, then Condition 1.1.3 implies Condition 1.1.2 (ii).

Lemma 1.1.4 *Let A be a ring with a nilpotent ideal I; let $\bar{A} := A/I$. Then every finite étale $\bar{A}$-algebra $\bar{B}$ lifts uniquely to a finite étale A-algebra B.*

If R is artinian, then $\mathfrak{m}^n = 0$ for $n \gg 0$. We may assume that the open subset $U \subset U_{i,j}$ of Condition 1.1.2 (ii) is affine, $U = \operatorname{Spec} A$. Then $I := \mathfrak{m}A$ is a nilpotent ideal of A and $\bar{V} = \bar{\rho}^{-1}(\bar{U}) = \operatorname{Spec} \bar{B}$ for a finite $\bar{A}$-algebra. Assume moreover that Condition 1.1.3 holds. Then we may assume that $\bar{B}$ is étale over $\bar{A}$, and Condition 1.1.2 (ii) follows from Lemma 1.1.4.

1.1.4 We have seen that we can solve Problem 1.1.1 if R is artinian and the Conditions 1.1.2 (i) and 1.1.3 hold. We would like to extend this result to the case that R is a complete noetherian local ring. The problem is that we can not apply Lemma 1.1.4 in the same way as before. Note that even if A is a finitely generated R-algebra, A is in general not complete with respect to the ideal $I := \mathfrak{m}A$. It is easy to see that the analogous version of Lemma 1.1.4 is actually false in this situation. To get around this difficulty we need a further condition.

Condition 1.1.5 The ring R is noetherian and complete and X is a projective scheme over R.

For $n \geq 0$ let $R_n := R/\mathfrak{m}^{n+1}$ and $X_n := X \times_R R_n$. Assume that the Conditions 1.1.2 (i) and 1.1.3 hold if we replace the R-scheme X by the R_n-schemes X_n. Then we can use Lemma 1.1.4 and étale descent to construct a sequence of finite morphisms $\rho_n : Y_n \to X_n$ with property P such that $Y_n = Y_{n+1} \times_{R_{n+1}} R_n$ and $\bar{Y} = Y_0$. In this situation and under Condition 1.1.5 we can apply Grothendieck's Existence Theorem (see Section 1.4). This theorem shows that there is a finite morphism $\rho : Y \to X$ such that $Y_n = Y \times_R R_n$ for all $n \geq 0$. It remains to show that ρ has property P. Since this depends strongly on P, we formulate it as the last condition.

Condition 1.1.6 If the morphisms $\rho_n : Y_n \to X_n$ all have property P, then $\rho : Y \to X$ has property P.

1.2 Etale localization

We are going to introduce some terminology related to the étale topology of a scheme. All the definitions are restricted to *affine* étale morphisms. This is all we will need and it reduces the technical background. Throughout, X denotes a separated scheme.

1.2.1 A morphism of schemes $\varphi : U \to X$ is called **affine étale**, if $U = \operatorname{Spec} A$ is affine, its image $\varphi(U)$ is contained in some affine open subset $\operatorname{Spec} B \subset X$ and the ring extension $B \to A$ induced by φ is étale (see the Appendix for a definition of 'étale').

Lemma 1.2.1 *Let* $\varphi : U = \operatorname{Spec} A \to X$ *be an affine étale map. Then for any affine open subset* $\operatorname{Spec} B \subset X$ *containing* $\varphi(U)$, $B \to A$ *is étale.*

Moreover, if $\varphi' : U' = \operatorname{Spec} A' \to X$ *is another affine étale map, the same is true for the projection of the fibered product* $U \times_X U'$ *to* X *and for any morphism* $U' \to U$ *of* X*-schemes.*

Proof: The first claim follows from the fact that 'étale' is an open condition (see the Appendix). For the second claim, take $V := \operatorname{Spec} B \cap \operatorname{Spec} B'$, where $\operatorname{Spec} B \subset X$ (resp. $\operatorname{Spec} B' \subset X$) contains $\varphi(U)$ (resp. $\varphi(U')$). By [14] II, Ex. 4.3, $V = \operatorname{Spec} B''$ is again affine (here we use that X is separated). By the construction of fibred products in [14] II, Thm. 3.3, we get $U \times_X U' = (U \cap \phi^{-1} V) \times_V (U' \cap (\phi')^{-1} V) = \operatorname{Spec} A''$ with $A'' = (A \otimes_B B'') \otimes_{B''} (B'' \otimes_{B'} A)$. The B''-algebra A'' is étale by Lemma 5.1.2 (i), proving the second claim.

If there is any X-morphism $f : U' \to U$, then we can take one $\operatorname{Spec} B \subset X$ containing both $\varphi(U)$ and $\varphi'(U')$, and then f is given by a B-algebra morphism $A' \to A$, which must be étale by Lemma 5.1.2 (ii). This proves the last claim. ∎

An **(affine étale) covering**[1] $\mathcal{U}$ of X is a family $(\varphi_i : U_i \to X)_{i \in I}$ of affine étale morphisms whose images cover X.

Given a covering $\mathcal{U} = (U_i \to X)_{i \in I}$, we will frequently use the following notation.

$$U_{i,j} := U_i \times_X U_j \,, \qquad U_{i,j,k} := U_i \times_X U_j \times_X U_k \,, \qquad i,j,k \in I.$$

Note that, if the U_i are Zariski open subsets of X, then we actually have $U_{i,j} = U_i \cap U_j$, $U_{i,j,k} = U_i \cap U_j \cap U_k$. We have lots of natural maps:

$$U \longleftarrow \underset{i}{} U_i \leftleftarrows \underset{i,j}{} U_{i,j} \;\overset{\longleftarrow}{\leftleftarrows}\; \underset{i,j,k}{} U_{i,j,k}. \tag{3}$$

The two arrows in the middle we call $p_{i,j}^{(1)}$ (projection to the first factor, U_i) and $p_{i,j}^{(2)}$ (projection to U_j). On the right hand side, we have three maps $q_{i,j,k}^{(l)}$, $l = 1, 2, 3$, for the projection leaving out the l-th factor.

By Lemma 1.2.1, $U_{i,j} = \operatorname{Spec} A_{i,j}$, $U_{i,j,k} = \operatorname{Spec} A_{i,j,k}$. Therefore (3) corresponds to a complex of ring morphisms:

$$A' := \bigoplus_i A_i \rightrightarrows A'' := \bigoplus_{i,j} A_{i,j} \;\overset{\longrightarrow}{\rightrightarrows}\; A''' := \bigoplus_{i,j,k} A_{i,j,k}. \tag{4}$$

Lemma 1.2.2 *Assume* $X = \operatorname{Spec} A$ *affine. Then:*

(i) *Canonically,* $A'' = A' \otimes_A A'$ *and* $A'' = A' \otimes_A A' \otimes_A A'$

(ii) *The natural morphism* $A \to A'$ *is faithfully flat*

Proof: By the construction of fibered products in the affine case, $A_{i,j} = A_i \otimes_A A_j$ and $A_{i,j,k} = A_i \otimes_A A_j \otimes_A A_k$. This proves (i). $A \to A'$ is étale, therefore flat. It is faithfully flat because $\operatorname{Spec} A' = {}_i U_i \to X = \operatorname{Spec} A$ is surjective ([17] 4.C (iii)). ∎

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[1] note that we give 'covering' (recouvrement, Überdeckung) a different meaning than 'cover' (revêtement, Überlagerung)

1.2.2 Let k be a field. A **geometric point** of X is a scheme morphism $x : \operatorname{Spec} k \to X$ such that k is an algebraically closed field. Let $x : \operatorname{Spec} k \to X$ be a geometric point. An **(affine étale) neighborhood** of x is a pair (U, x'), where $U \to X$ is an affine étale morphism $U \to X$ and $x' : \operatorname{Spec} k \to U$ a lift of x to U. Frequently, we will write U instead of (U, x') and call it simply a neighborhood of x. Let $\mathsf{Et}(X, x)$ denote the category of neighborhoods of x. A morphism $(U_1, x_1) \to (U_2, x_2)$ is a commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec} k & = & \operatorname{Spec} k & = & \operatorname{Spec} k \\ \downarrow {\scriptstyle x_1} & & \downarrow {\scriptstyle x_2} & & \downarrow {\scriptstyle x} \\ U_1 & \longrightarrow & U_2 & \longrightarrow & X \end{array}$$

If a morphism from U_1 to U_2 exists, we will say that U_1 is **smaller** than U_2. Let $\mathsf{Et}'(X, x)$ be the full subcategory of all connected neighborhoods.

It follows from Lemma 5.2.1 that $\mathsf{Et}'(X, x)$ is a filtered inverse system. Therefore we can define the **(strict) henselian local ring** of X at x as

$$\mathcal{O}_{X,x} := \varinjlim A\,, \qquad U = \operatorname{Spec} A \in \mathsf{Et}'(X, x) \tag{5}$$

Of course, for every neighborhood $U = \operatorname{Spec} A$ of x, $\mathcal{O}_{X,x}$ is the henselization of $A \to k$, as defined in the Appendix. We define the **(strict) complete local ring** $\mathcal{O}_{X,x}$ of X at x to be the completion of $\mathcal{O}_{X,x}$.

1.2.3 Local decomposition of finite morphisms Recall that a morphism $f : Y \to X$ of schemes is called **finite**, if for every open affine subset $U = \operatorname{Spec} A \subset X$ the preimage $f^{-1}(U) = \operatorname{Spec} B$ is affine and B is a finite A-algebra, i.e. finitely generated as A-module. Then we can write $Y = \operatorname{Spec} \mathcal{B}$, where $\mathcal{B}$ is a finite $\mathcal{O}_X$-algebra on X, i.e. a coherent sheaf with an additional algebra structure (see [14], II.3 and II.Ex. 5.17).

Let $f : Y = \operatorname{Spec} \mathcal{B} \to X$ be a finite morphism and $x : \operatorname{Spec} k \to X$ a geometric point. Since k is algebraically closed, $Y \times_X \operatorname{Spec} k \cong \operatorname{Spec} k^n$ for some $n \geq 0$. The n idempotents of k^n correspond one to one to the lifts $y_1, \ldots, y_n : \operatorname{Spec} k \to Y$ of x to Y.

If $U = \operatorname{Spec} A \to X$ is an affine étale neighborhood of x, then $V := Y \times_X U = \operatorname{Spec} B$, where B is a finite A-algebra. By Lemma 5.1.2 (i), $V = \operatorname{Spec} B \to Y$ is an affine étale map.

Lemma 1.2.3 *For every sufficiently small connected étale neighborhood $U = \operatorname{Spec} A$ of x, $V = \operatorname{Spec} B$ is a disjoint union of the form*

$$V = \bigsqcup_{i=1}^{n} V_i, \quad V_i = \operatorname{Spec} B_i,$$

such that $V_i = \operatorname{Spec} B_i \to Y$ is a connected neighborhood of y_i. Moreover,

$$\mathcal{O}_{Y,y_i} = \varinjlim B_i \otimes_A A',$$

where $A' \to k$ runs over $\mathsf{Et}'(A \to k)$.

Proof: Let $\tilde{A}$ be the henselization of A with respect to $A \to k$ and $\tilde{B} := B \otimes_A \tilde{A}$. Since $\tilde{A} \to \tilde{B}$ is finite and $\tilde{A}$ henselian, we obtain a decomposition $\tilde{B} = \oplus_{i=1}^n \tilde{B}_i$ into local factors. For a sufficiently small $A' \to k \in \mathsf{Et}'(A \to k)$, we have $B \otimes_A A' = \oplus_i B_i$ such that $\tilde{B}_i = B_i \otimes_{A'} \tilde{A}$. This proves the first claim. Checking the universal property of henselization (Remark 5.2.2) we see that $\tilde{B}_i$ is the henselization of B with respect to the ring morphisms $B \to k$ corresponding to y_i. Now the second claim follows from the definition of henselization. ∎

We will say that a neighborhood U with the properties formulated in Lemma 1.2.3 **decomposes** the finite map f. We may also state this as follows: by choosing an arbitrarily small affine étale neighborhood U of x, its inverse image on Y is the disjoint union of arbitrarily small affine étale neighborhoods V_i of y_i.

1.3 Descent

A quasi-coherent sheaf $\mathcal{M}$ on a scheme X is an $\mathcal{O}_X$-module which is, over every affine open subset $U = \operatorname{Spec} A$, represented by an A-module M. More precisely, given an affine open covering $(U_i = \operatorname{Spec} A_i \subset X)_i$ of X, $\mathcal{M}$ corresponds to a family of A_i-modules M_i together with isomorphisms $M_i \otimes_{A_i} A_{i,j} \xrightarrow{\sim} M_j \otimes_{A_j} A_{i,j}$ verifying a natural cocycle condition (where $\operatorname{Spec} A_{i,j} = U_i \times_X U_j$). The theory of descent shows that this is still true if $(U_i = \operatorname{Spec} A_i \to X)_i$ is an étale covering, as defined in the last section. Moreover, this also works for finite $\mathcal{O}_X$-algebras $\mathcal{B}$. Since the latter correspond to finite morphisms $p : Y \to X$, descent is the right tool to glue finite morphisms, as described in 1.1.2.

1.3.1 Let X be a scheme and $\mathcal{M}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules over X (see [14], II.5). Given an affine étale map $\varphi : U = \operatorname{Spec} A \to X$, the inverse image $\varphi^*\mathcal{M}$ is a quasi-coherent sheaf on $U = \operatorname{Spec} A$, so it can be written as

$$\varphi^*\mathcal{M} = M \qquad (6)$$

where M is an A-module ([14] II Proposition 5.4). If $\varphi' : U' = \operatorname{Spec} A' \to X$ is another affine étale map and $f : U' \to U$ is an X-morphism, we can write $(\varphi')^*\mathcal{M} = M'$ and the relation $\varphi' = \varphi \circ f$ induces a natural A'-linear isomorphism

$$M' \cong M \otimes_A A' \qquad (7)$$

Now let $\mathcal{U} = (\varphi_i : U_i = \operatorname{Spec} A_i \to X)_i$ be an affine étale covering and $\mathcal{M}$ a quasi-coherent sheaf on X. Then $\varphi_i^*\mathcal{M} = M_i$, where M_i is an A_i-module.

Using the notation introduced in Section 1.2, the relations $\varphi_i \circ p_{i,j}^{(1)} = \varphi_j \circ p_{i,j}^{(2)}$ induce a family $\alpha_{\mathcal{M}} = (\alpha_{i,j})_{i,j}$ of $A_{i,j}$-linear isomorphisms

$$\alpha_{i,j} : M_i \otimes_{A_i} A_{i,j} \xrightarrow{\sim} M_j \otimes_{A_j} A_{i,j}. \tag{8}$$

For each triple i, j, k, the family $(\alpha_{i,j})$ gives rise to $A_{i,j,k}$-linear isomorphisms

$$\begin{aligned} \alpha_{i,j,k}^{(1)} &: M_j \otimes_{A_j} A_{i,j,k} \xrightarrow{\sim} M_k \otimes_{A_k} A_{i,j,k} \\ \alpha_{i,j,k}^{(2)} &: M_i \otimes_{A_i} A_{i,j,k} \xrightarrow{\sim} M_k \otimes_{A_k} A_{i,j,k} \\ \alpha_{i,j,k}^{(3)} &: M_i \otimes_{A_i} A_{i,j,k} \xrightarrow{\sim} M_j \otimes_{A_j} A_{i,j,k}. \end{aligned} \tag{9}$$

For instance, $\alpha_{i,j,k}^{(3)}$ is defined by $\alpha_{i,j} \otimes_{e_{i,j}^{(3)}} A_{i,j,k}$, where $e_{i,j,k}^{(3)} : A_{i,j} \to A_{i,j,k}$ is the natural morphism and where we have identified $(M_i \otimes_{A_i} A_{i,j}) \otimes_{A_{i,j}} A_{i,j,k}$ with $M_i \otimes_{A_i} A_{i,j,k}$ and $(M_j \otimes_{A_j} A_{i,j}) \otimes_{A_{i,j}} A_{i,j,k}$ with $M_j \otimes_{A_j} A_{i,j,k}$. The other two cases are similar. A tedious but formal verification shows that the following cocycle relation holds for every triple i, j, k:

$$\alpha_{i,j,k}^{(1)} \circ \alpha_{i,j,k}^{(3)} = \alpha_{i,j,k}^{(2)}. \tag{10}$$

Conversely, let $(M_i)_i$ be a family of A_i-modules and $(\alpha_{i,j})_{i,j}$ a family of $A_{i,j}$-linear isomorphisms as in (8). The datum $(M_i, \alpha_{i,j})$ is called a **descent datum** on the covering $\mathcal{U}$ if the family of isomorphisms $\alpha_{i,j,k}^{(\mu)}$ derived from the $\alpha_{i,j}$ as in (9) verifies the cocycle relation (10). A morphism of descent data from $(M_i, \alpha_{i,j})$ to $(N_i, \beta_{i,j})$ is given by a family of A_i-linear maps $f_i : M_i \to N_i$ compatible with $\alpha_{i,j}$ and $\beta_{i,j}$ in the obvious way. Another formal verification shows that

$$\mathcal{M} \longmapsto (M_i, \alpha_{i,j}) \tag{11}$$

defines a functor from the category of quasi-coherent sheaves on X to the category of descent data on $\mathcal{U}$.

Theorem 1.3.1 (Descent for quasi-coherent sheaves) *The functor defined by (11) is an equivalence of categories. In particular, for every descent datum $(M_i, \alpha_{i,j})$ on $\mathcal{U}$ there exists a quasi-coherent sheaf $\mathcal{M}$ on X with $\varphi_i^* \mathcal{M} = M_i$.*

Proof: For a proof of this theorem and much more general results, see e.g. [20] Chapter VII or [1] Chapter 6.1. We will explain how to reduce the theorem to a problem on faithfully flat descent of modules.

First assume that $X = \operatorname{Spec} A$ is affine. Let $(M_i, \alpha_{i,j})$ be a descent datum on $\mathcal{U}$. Then $M' := \oplus_i M_i$ has a natural structure of an $A' = \oplus_i A_i$ module. Using Lemma 1.2.2 we obtain canonical A'-linear isomorphisms

$$\begin{aligned} M' \otimes_A A' &\cong \bigoplus_{i,j} M_i \otimes_A A_j \cong \bigoplus_{i,j} M_i \otimes_{A_i} A_{i,j} \\ A' \otimes_A M' &\cong \bigoplus_{i,j} M_j \otimes_A A_i \cong \bigoplus_{i,j} M_j \otimes_{A_j} A_{i,j}. \end{aligned} \tag{12}$$

Therefore a descent datum $(M_i, \alpha_{i,j})$ on $\mathcal{U}$ gives rise to a a datum (M', α). where M' is an A'-module and $\alpha : M' \otimes_A A' \xrightarrow{\sim} A' \otimes_A M'$ is an $A'' = A' \otimes_A A'$-linear isomorphism. The cocycle relation (10) corresponds to the relation

$$\alpha^{(3)} \circ \alpha^{(1)} = \alpha^{(2)}, \tag{13}$$

with A'''-linear isomorphisms

$$\begin{aligned} \alpha^{(1)} &: A' \otimes_A A' \otimes_A M' \xrightarrow{\sim} A' \otimes_A M' \otimes_A A' \\ \alpha^{(2)} &: A' \otimes_A A' \otimes_A M' \xrightarrow{\sim} M' \otimes_A A' \otimes_A A' \\ \alpha^{(3)} &: A' \otimes_A M' \otimes_A A' \xrightarrow{\sim} M' \otimes_A A' \otimes_A A' \end{aligned} \tag{14}$$

obtained by tensoring α on the left, in the middle and on the right with $\mathrm{Id}_{A'}$.

Remember that we are assuming $X = \operatorname{Spec} A$ and we can therefore identify a quasi-coherent sheaf $\mathcal{M}$ on X with the A-module $M := \Gamma(X, \mathcal{M})$. In this case we have $M' = \oplus_i M_i = M \otimes_A A'$. and α is defined by $a_1 \otimes (m \otimes a_2) \longmapsto (a_1 \otimes m) \otimes a_2$. We can therefore reformulate the statement of Theorem 1.3.1 as follows. Given a faithfully flat ring homomorphism $A \to A'$, the functor

$$M \longmapsto (M' := M \otimes_A A', \alpha) \tag{15}$$

is an equivalence between the category of A-modules and the category of descent data for the morphism $A \to A'$. For a proof of this statement, see e.g. [18] I Remark 2.21.

The general case of Theorem 1.3.1 now follows quite easily. We can cover X be affine Zariski open subsets $V_\mu = \operatorname{Spec} B_\mu$. It is easy to see that for every μ we can restrict our descent datum $(M_i, \alpha_{i,j})$ on $\mathcal{U}$ to a descent datum on the étale covering $\mathcal{U}|_{V_\mu} = (U_i \times_X V_\mu \to V_\mu)$ of $V_\mu = \operatorname{Spec} B_\mu$. Applying Theorem 1.3.1 in the affine case, we obtain quasi-coherent sheaves $\mathcal{M}_\mu$ on V_μ corresponding to B_μ-modules M_μ. Using the fully faithfulness of the functor (11) in the affine case we get isomorphisms $\mathcal{M}_\mu|_{V_\mu \cap V_\nu} \xrightarrow{\sim} \mathcal{M}_\nu|_{V_\mu \cap V_\nu}$, because both sheaves correspond to the restriction of the descent datum $(M_i, \alpha_{i,j})$ to the affine open subset $V_\mu \cap V_\nu$. The uniqueness of these isomorphisms forces them to satisfy the usual cocycle relation. By [14] II Exercise 1.22 we can glue the $\mathcal{M}_\mu$ to a quasi-coherent sheaf $\mathcal{M}$. By construction, $\mathcal{M}$ corresponds to the descent datum $(M_i, \alpha_{i,j})$. A similar argument shows the fully faithfulness of (11) in the general case. ■

1.3.2 Let X and $\mathcal{U}$ be as before and let $\mathcal{B}$ be a finite $\mathcal{O}_X$-algebra. Regarding $\mathcal{B}$ as a quasi-coherent sheaf on X, we obtain a descent datum $(B_i, \alpha_{i,j})$, where the B_i are finite A_i-algebras and the $\alpha_{i,j}$ are isomorphisms of $A_{i,j}$-algebras. We say that $(B_i, \alpha_{i,j})$ is a descent datum for finite $\mathcal{O}_X$-algebras on $\mathcal{U}$.

Corollary 1.3.2 *The functor $\mathcal{B} \mapsto (B_i, \alpha_{i,j})$ is an equivalence between the category of finite $\mathcal{O}_X$-algebras and the category of descent data for finite $\mathcal{O}_X$-algebras on $\mathcal{U}$.*

Proof: Let $(B_i, \alpha_{i,j})$ be a descent datum for finite $\mathcal{O}_X$-algebras on $\mathcal{U}$. The algebra structure of the B_i is given by A_i-linear multiplication maps $m_i : B_i \otimes_{A_i} B_i \to B_i$. These maps verify certain identities corresponding to the rules for multiplication in a ring. Using the fact that the $\alpha_{i,j}$ are algebra morphisms, the family $(m_i)_i$ is easily seen to be a morphism of descent data. By Theorem 1.3.1 we obtain a quasi-coherent sheaf $\mathcal{B}$ on X together with a morphism $m : \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{B} \to \mathcal{B}$ of quasi-coherent sheaves. The morphism m verifies the same identities as the m_i and defines thus a structure of $\mathcal{O}_X$-module on $\mathcal{B}$. It remains to show that $\mathcal{B}$ is actually a finite $\mathcal{O}_X$-module. In view of the construction of $\mathcal{B}$ in the proof of Theorem 1.3.1, this follows from the following fact. Let M be an A-module and $A \to A'$ a faithfully flat ring extension. Then M is finitely generated over A iff $M' := M \otimes_A A'$ is finitely generated over A'. ■

1.4 Grothendieck's Existence Theorem

Let R be a complete local ring with maximal ideal $\mathfrak{m}$ and let X be a scheme over R. For $n \geq 0$, let $R_n := R/\mathfrak{m}^{n+1}$ and $X_n := X \times_R R_n$. Note that X_n is a closed subscheme of X. In particular, $\bar{X} := X_0$ is the special fiber of X.

A **formal coherent sheaf** on X is a family $(\mathcal{M}_n)_{n\geq 0}$ of coherent sheaves $\mathcal{M}_n$ on X_n together with a system of isomorphisms

$$\varphi_{n,m} : \mathcal{M}_n \otimes_{R_n} R_m \xrightarrow{\sim} \mathcal{M}_m, \qquad n \geq m \tag{16}$$

such that $\varphi_{l,m} \circ (\varphi_{n,m} \otimes_{R_m} R_l) = \varphi_{n,l}$ for all $n \geq m \geq l$. For instance, given a coherent sheaf $\mathcal{M}$ on X, we can define the formal coherent sheaf $\hat{\mathcal{M}} := (\mathcal{M} \otimes_R R_n)_n$, called the **formalization** of $\mathcal{M}$. There is an obvious notion of morphisms between formal coherent sheaves. Formalization is a functor

$$\mathcal{M} \longmapsto \hat{\mathcal{M}} := (\mathcal{M} \otimes_R R_n)_n \tag{17}$$

from coherent to formal coherent sheaves. A formal coherent sheaf $\mathcal{M}'$ on X is called **algebraizable**, if there exists a coherent sheaf $\mathcal{M}$ on X with $\mathcal{M}' = \hat{\mathcal{M}}$.

Theorem 1.4.1 (Grothendieck's Existence Theorem) *Let R be a complete noetherian local ring and X a projective scheme over R. Then (17) is an equivalence of categories. In particular, every formal coherent sheaf on X is algebraizable.*

The proof of this theorem in [7], Chapter 5, uses the cohomology of coherent sheaves. In [14] this cohomological machinery is developed under less general hypotheses. Below we give a proof of Theorem 1.4.1 which follows closely the original lines of [7] but only relies on results proved in [14]. First we state the corollary which we will need to do formal patching.

Let X/R be as in Theorem 1.4.1 and let $\mathcal{B}$ be a finite $\mathcal{O}_X$-algebra (see Section 1.2.3). Regarding $\mathcal{B}$ as a coherent sheaf, we define its formalization $\mathcal{B} := (\mathcal{B}_n)$. Each $\mathcal{B}_n$ is a finite $\mathcal{O}_{X_n}$-algebra and the isomorphisms $\mathcal{B}_n \otimes_{R_n} R_m \xrightarrow{\sim} \mathcal{B}_m$ respect the algebra structure. Therefore we call $\mathcal{B}$ a **formal finite $\mathcal{O}_X$-algebra**. The following Corollary is implied by Theorem 1.4.1 in the same way as Corollary 1.3.2 is implied by Theorem 1.3.1.

Corollary 1.4.2 *Assumption as in Theorem 1.4.1. Every formal finite $\mathcal{O}_X$-algebra is uniquely algebraizable.*

1.4.1 Before passing to the proof of Theorem 1.4.1 we fix some notation and recall some general facts. Since $X \subset \mathbb{P}^r_R$ by assumption, we can consider coherent resp. formal coherent sheaves on X as coherent resp. formal coherent sheaves on $\mathbb{P}^r_R$. Therefore, we may assume that $X = \mathbb{P}^r_R$.

$X = \mathbb{P}^r_R$ has a standard affine open covering $(U_i = \operatorname{Spec} A_i)_i$, where $A_i = R[T_0/T_i, \ldots, T_r/T_i]$, $i = 0, \ldots, r$. Let $\mathcal{M}$ be a coherent sheaf on X. For every sequence $i_1, \ldots, i_p$ with $1 \le i_1 < \ldots < i_p \le r$ we put $U_{i_1,\ldots,i_p} := U_{i_1} \cap \ldots \cap U_{i_p}$ and $M_{i_1,\ldots,i_p} := \Gamma(U_{i_1,\ldots,i_p}, \mathcal{M})$. In a standard manner we obtain a complex

$$\bigoplus_i M_i \longrightarrow \bigoplus_{i,j} M_{i,j} \longrightarrow \bigoplus_{i,j,k} M_{i,j,k} \longrightarrow \ldots \tag{18}$$

of R-modules (see [14] II.4). Note that $M_i = \Gamma(U_i, \mathcal{M})$ (resp. $M_{i,j} := \Gamma(U_{i,j}, \mathcal{M})$ etc.) are finitely generated A_i-modules (resp. finitely generated $A_{i,j} = A_i[T_k/T_j]$-modules etc.). For $q \ge 0$ we define the cohomology group $H^q(X, \mathcal{M})$ as the q-th cohomology group of the complex (18). By [14] III, Thm. 4.5 this coincides with the definition of cohomology via derived functors. In particular, $H^0(X, \mathcal{M}) = \Gamma(X, \mathcal{M})$ are the global sections of $\mathcal{M}$.

Let $\mathcal{M}' = (\mathcal{M}_n)_n$ be a formal coherent sheaf. Note that we can consider $\mathcal{M}_n$ as a coherent sheaf on X. We will write $M_i^{(n)} := \Gamma(U_i, \mathcal{M}_n)$, $M_{i,j}^{(n)} := \Gamma(U_{i,j}, M_n)$ etc. For $n \ge m$ the natural morphism $\mathcal{M}_n \to \mathcal{M}_m$ induces morphisms

$$M_i^{(n)} \longrightarrow M_i^{(m)}, \quad M_{i,j}^{(n)} \longrightarrow M_{i,j}^{(m)}, \ldots \tag{19}$$

and hence a compatible system of morphisms $H^q(X, \mathcal{M}_n) \to H^q(X, \mathcal{M}_m)$. We define the cohomology of a formal coherent sheaf by

$$H^q(X, \mathcal{M}') := \varprojlim_n H^q(X, \mathcal{M}_n). \tag{20}$$

The main ingredient of the proof of GET is the Theorem on Formal Functions ([14] III, Theorem 11.1). It states that for any coherent sheaf $\mathcal{M}$ on X and $q \geq 0$ we have

$$H^q(X, \mathcal{M}) = \varprojlim_n H^q(X, \mathcal{M}_n) = H^q(X, \mathcal{M}). \qquad (21)$$

1.4.2 Let $\mathcal{M}' = (\mathcal{M}_n)_n$ be a formal coherent sheaf on $X = \mathbb{P}^r_R$. We will say that $\mathcal{M}'$ is generated by a finite number of global sections if there are elements $m_1, \dots, m_l \in H^0(X, \mathcal{M}')$ whose images generate the A_i-module $M_i^{(n)}$, for all $1 \leq i \leq r$, $n \geq 0$. Given a formal coherent sheaf $\mathcal{M}'$ and $k \in \mathbb{Z}$, we define its k-th twist of $\mathcal{M}'$ to be $\mathcal{M}'(k) := (\mathcal{M}'_n(k))_n$ (see [14] II.5).

Proposition 1.4.3 *Let $\mathcal{M}'$ be a formal coherent sheaf on $X = \mathbb{P}^r_R$. Then for $k \gg 0$, $\mathcal{M}'(k)$ is generated by a finite number of global sections.*

Proof: Let $\hat{R} := \oplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the graded ring associated to R. Consider the sheaf $\hat{\mathcal{M}} := \oplus_{n\geq 0} \mathfrak{m}^n \mathcal{M}_n$ as a quasi-coherent sheaf on $\hat{X} := \mathbb{P}^r_{\hat{R}}$. We claim that $\hat{\mathcal{M}}$ is a coherent sheaf. In fact, $\hat{\mathcal{M}}$ is generated as an $\mathcal{O}_{\hat{X}}$-module by its subsheaf $\mathcal{M}_0$, which is a finitely generated $\mathcal{O}_{X_0}$-module. Note that we have

$$H^q(\hat{X}, \hat{\mathcal{M}}) = \bigoplus_{n\geq 0} H^q(X, \mathfrak{m}^n \mathcal{M}_n). \qquad (22)$$

By [17] 10.D, $\hat{R}$ is noetherian. Hence we can apply [14] III, Theorem 5.2, to conclude that for $k \gg 0$ the coherent sheaf $\hat{\mathcal{M}}(k)$ is generated by a finite number of global sections and $H^q(\hat{X}, \hat{\mathcal{M}}(k)) = 0$ for all $q > 0$. In particular, $\mathcal{M}_0(k)$ is generated by a finite number of global sections and $H^1(X, \mathfrak{m}^n \mathcal{M}_n(k)) = 0$ for all $n \geq 0$ (use (22)).

The short exact sequence $0 \to \mathfrak{m}^{n+1}\mathcal{M}_{n+1}(k) \to \mathcal{M}_{n+1}(k) \to \mathcal{M}_n(k) \to 0$ yields an exact sequence

$$H^0(X, \mathcal{M}_{n+1}(k)) \longrightarrow H^0(X, \mathcal{M}_n(k)) \longrightarrow H^1(X, \mathfrak{m}^{n+1}\mathcal{M}_{n+1}(k)) = 0, \qquad (23)$$

showing that the first morphism is surjective. Choose a set $\bar{m}_1, \dots, \bar{m}_l \in H^0(X, \mathcal{M}_0(k))$ of global generators of $\mathcal{M}_0(k)$. Using inductively that the first map in (23) is surjective, we can lift them to global sections $m_1, \dots, m_l \in H^0(X, \mathcal{M}'(k))$. By Nakayama's Lemma, their images in $H^0(X, \mathcal{M}_n(k))$ generate $\mathcal{M}_n$ for all $n \geq 0$. This is exactly what we wanted to prove. ■

1.4.3 Now we are going to prove GET. Let $\mathcal{M}$, $\mathcal{N}$ be coherent sheaves on X. Then there is a coherent sheaf $\mathcal{H}om(\mathcal{M}, \mathcal{N})$ such that $\mathrm{Hom}(\mathcal{M}, \mathcal{N}) =$

$H^0(\mathcal{H}om(\mathcal{M},\mathcal{N}))$ ([14] II.5). Formalizing this construction we obtain a formal sheaf $\mathcal{H}om(\mathcal{M},\mathcal{N})$ such that $\mathrm{Hom}(\mathcal{M},\mathcal{N}) = H^0(\mathcal{H}om(\mathcal{M},\mathcal{N}))$. The Theorem on Formal Functions (21) implies

$$\mathrm{Hom}(\mathcal{M},\mathcal{N}) = \mathrm{Hom}(\mathcal{M},\mathcal{N}). \tag{24}$$

This proves the fully faithfulness of the formalization functor $\mathcal{M} \mapsto \mathcal{M}$.

It remains to show that for any formal coherent sheaf $\mathcal{M}'$ on X we can find a coherent sheaf $\mathcal{M}$ with $\mathcal{M}' = \mathcal{M}$. By Proposition 1.4.3 we can find k such that $\mathcal{M}'(k)$ is generated by a finite number of global sections. Therefore there exists a natural number l and a surjective morphism $\mathcal{O}_X^l \to \mathcal{M}'(k)$ of formal coherent sheaves. Twisting with $-k$ we obtain a surjective morphism $f' : \mathcal{O}_X(-k)^l \to \mathcal{M}'$. Applying this procedure once more to the formal coherent sheaf $\mathrm{Ker}(f')$, we obtain an exact sequence

$$\mathcal{O}_X(-k')^{l'} \xrightarrow{g'} \mathcal{O}_X(-k)^l \xrightarrow{f'} \mathcal{M}' \longrightarrow 0. \tag{25}$$

The first two formal sheaves are obviously algebraizable. Hence the fully faithfulness of the formalization functor guarantees the existence of a morphism $g : \mathcal{O}_X(-k')^{l'} \to \mathcal{O}_X(-k)^l$ of coherent sheaves such that $g' = g$. Let $\mathcal{M}$ be the cokernel of g. Then $\mathcal{M} \otimes_R R_n = \mathcal{M}_n$ because both sides are the cokernel of g_n. Hence $\mathcal{M}' = \mathcal{M}$. ■

2 Tame admissible covers

Tame admissible covers are finite morphism between relative nodal curves with a particular local ramification behavior. Our goal is to show that locally in the étale topology all tame admissible covers have a certain standard shape. This is done in Section 2.3. Section 2.1 and 2.2 contain some preliminary results.

Section 2.1 gives a detailed study of the strict complete local ring of a curve at an ordinary double point. The deformation theory of a curve in a neighborhood of an ordinary double point depends on the way one can choose and lift the so called *formal coordinate systems*. Since this will turn out to be crucial for the deformation theory of tame admissible covers, we do it very carefully. Note however that under the assumption of a reduced base ring (which suffices for most applications), proofs would become substantially simpler.

In Section 2.2 we use the results of 2.1 to study étale neighborhoods of ordinary double points. Technically, this means to compare the strict complete local ring of such a point to the strict henselian local ring. We also give the analogous statements for smooth points, which are much easier to prove.

2.1 Formal double points

Let A and R be complete noetherian local rings. Denote their maximal ideals by $\mathfrak{m} \lhd R$ and $\mathfrak{M} \lhd A$ and the residue field $R/\mathfrak{m}$ by k. Let $R \to A$ be a faithfully flat local ring extension.

The ring A is called a **formal double point** over R if there exists a pair u, v of elements of A such that

(i) $t := uv \in \mathfrak{m}$,

(ii) u, v induce an isomorphism $A \cong R[[u,v|uv = t]]$ of R-algebras.

In this case, the pair (u, v) is called a **(formal) coordinate system** for A/R. Any element $f \in A$ can be written uniquely in the form

$$f = a_0 + \sum_{i>0} a_i' u^i + \sum_{i>0} a_i'' v^i \tag{26}$$

with $a_0, a_i', a_i'' \in R$. We will call (26) the **(u,v)-expansion** of f.

Proposition 2.1.1 *Let A, R be as in the first paragraph of this section. Assume that $\bar{A} := A/\mathfrak{m}A$ is a formal double point over k. Then:*

(i) *Every coordinate system $(\bar{u}, \bar{v})$ of $\bar{A}/k$ lifts to a coordinate system (u, v) of A/R. In particular, A/R is a formal double point.*

(ii) *Every pair $r, s \in A$ with $rs \in \mathfrak{m}$ such that*

$$\mathfrak{M} = < r, s, \mathfrak{m} >$$

is a formal coordinate system.

Proof: Let $(\bar{u}, \bar{v})$ be a coordinate system for $\bar{A}$. Note that for every pair $u, v \in A$ lifting $\bar{u}, \bar{v}$ the induced ring homomorphism $R[[u,v]] \to A$ is surjective, since it is surjective mod $\mathfrak{m}$. We have to find lifts u, v such that $uv \in R$. To do this, we will construct inductively a sequence of lifts $u_n, v_n \in A$ of $\bar{u}, \bar{v}$ and a sequence of elements $t_n \in \mathfrak{m}$ (for $n \geq 0$) such that $u_n \equiv u_{n+1}$, $v_n \equiv v_{n+1}$ mod $\mathfrak{m}^{n+1}A$, $t_n \equiv t_{n+1} \mod \mathfrak{m}^{n+1}$ and $u_n v_n \equiv t_n \mod \mathfrak{m}^{n+1}$.

To start, take any pair u_0, v_0 lifting $\bar{u}, \bar{v}$ and let $t_0 := 0$. Suppose we have already constructed u_n, v_n, t_n for some $n \geq 0$. Then we can write

$$\begin{aligned} u_n v_n &= t_n + \sum_{i,j\geq 0} a_{i,j}\, u_n^i v_n^j \\ &\equiv t_n + a_{0,0} + \sum_{i>0} a_{i,0}\, u_n^i + \sum_{j>0} a_{0,j}\, v_n^j \mod \mathfrak{m}^{n+2}A \end{aligned} \tag{27}$$

with $a_{i,j} \in \mathfrak{m}^{n+1}$. For the first equality we have used the fact that an element of $\mathfrak{m}^{n+1}A$ can be written as a power series in u_n, v_n with coefficients in $\mathfrak{m}^{n+1}$.

For the congruence we have used $u_n v_n \in \mathfrak{m}A$. Let $u_{n+1} := u_n - \sum_{j>0} a_{0,j}\, v_n^{j-1}$ and $v_{n+1} := v_n - \sum_{i>0} a_{i,0}\, u_n^{i-1}$. Using (27) we obtain

$$\begin{aligned} u_{n+1}v_{n+1} &\equiv u_n v_n - \sum_{i>0} a_{i,0}\, u_n^i - \sum_{j>0} a_{0,j}\, v_n^j && \bmod \mathfrak{m}^{n+2}A \\ &\equiv t_n + a_{0,0} && \bmod \mathfrak{m}^{n+2}A. \end{aligned} \tag{28}$$

Hence the induction step is done if we define $t_{n+1} := t_n + a_{0,0}$. Let $u := \lim u_n$, $v := \lim v_n$ and $t := \lim t_n$. Then u, v lift $\bar{u}, \bar{v}$ and $uv = t \in \mathfrak{m}$.

The following argument is taken from [6], Lemma 2.2. Consider the short exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow R[[u,v|uv=t]] \longrightarrow A \longrightarrow 0. \tag{29}$$

The homomorphism on the right is the one induced from the above choice of $u, v \in A$. Since A is R-flat, (29) remains exact after reduction modulo $\mathfrak{m}$ (see [16] XVI, Lem. 3.3). But $k[[\bar{u},\bar{v}|\bar{u}\bar{v}=0]] \xrightarrow{\sim} A/\mathfrak{m}A$ is an isomorphism by hypothesis, hence $I = \mathfrak{m}I$. Since $R[[u,v|uv=t]]$ is noetherian, the ideal I is finitely generated and Nakayama's lemma implies $I = 0$. This proves (i).

Let r, s be as in Statement (ii) of the proposition and let $(\bar{u},\bar{v})$ be some coordinate system for $\bar{A}/k$. Denote by $(\bar{r},\bar{s})$ the reduction of (r,s) to $\bar{A}$. Write

$$\begin{aligned} \bar{r} &= a_0 + \sum_{i>0} a_i'\, \bar{u}^i + \sum_{i>0} a_i''\, \bar{v}^i, \\ \bar{s} &= b_0 + \sum_{i>0} b_i'\, \bar{u}^i + \sum_{i>0} b_i''\, \bar{v}^i, \end{aligned} \tag{30}$$

with coefficients in k. Using $\bar{r}\bar{s} = \bar{u}\bar{v} = 0$ one finds step by step that the following holds: first, $a_0 b_0 = 0$, next $a_0 = b_0 = 0$ and finally that either $a_i' = b_i'' = 0$ or $a_i'' = b_i' = 0$ for all $i > 0$ (we may assume the latter). By assumption the maximal ideal of $\bar{A}$ is generated by $\bar{r}, \bar{s}$. This can only happen if $a_1', b_1'' \neq 0$. Therefore the power series in (30) are in one variable and invertible. This proves that $(\bar{r},\bar{s})$ is a coordinate system for $\bar{A}/k$. The argument used at the end of the proof of (i) shows that (r,s) is a coordinate system for A/R. ■

Proposition 2.1.2 *Let A/R be a formal double point. Then the set of ideals $\{uA, vA\}$ and the ideal tR (with $t := uv$) are the same for any choice of a coordinate system (u,v).*

Proof: First, let us assume that there exists a coordinate system (u,v) of A/R with $uv = 0$. Under this assumption, we will prove that for every coordinate system (r,s) of A/R, we have $rs = 0$ and either $rA = uA$ and $sA = vA$ or

$rA = vA$ and $sA = uA$. Write

$$\begin{aligned} r &= a_0 + \sum_{i>0} a'_i u^i + \sum_{i>0} a''_i v^i, \\ s &= b_0 + \sum_{i>0} b'_i u^i + \sum_{i>0} b''_i v^i. \end{aligned} \tag{31}$$

Since (r, s) is a coordinate system, the (u, v)-expansion of rs consists only of a constant term. Computing the (u, v)-expansion of rs using (31) and $uv = 0$, one obtains

$$\begin{aligned} a_0 b'_n + \sum_{i=1}^{n-1} a'_i b'_{n-i} + a'_n b_0 = 0, \\ a_0 b''_n + \sum_{i=1}^{n-1} a''_i b''_{n-i} + a''_n b_0 = 0, \end{aligned} \tag{32}$$

for all $n > 0$. Reducing (31) mod $\mathfrak{m}$ and applying the arguments from the proof of Proposition 2.1.1 (ii) (following (30)) we can assume $a'_1, b''_1 \not\equiv 0 \mod \mathfrak{m}$ and $a''_1, b'_1 \equiv 0 \mod \mathfrak{m}$. For $n = 1$, (32) states

$$a_0 b'_1 + a'_1 b_0 = 0\,, \qquad a_0 b''_1 + a''_1 b_0 = 0 \tag{33}$$

Rewrite this as $b_0 = -(a'_1)^{-1} a_0 b'_1$ and $a_0 = -(b''_1)^{-1} a''_1 b_0$. Plugging the second equation into the first yields $b_0(1 - (a'_1 b''_1)^{-1} a''_1 b'_1) = 0$. But the second factor is congruent to 1 mod $\mathfrak{m}$, hence $b_0 = 0$. By symmetry we obtain $a_0 = 0$. In particular, this proves $rs = 0$.

Next we prove by induction that $a''_i, b'_i = 0$ for all $i > 0$. Assume that this is true for all $i < N$ for some $N > 0$. Then (32) with $n := N + 1$ states $a'_1 b'_N = 0$ and $a''_N b''_1 = 0$, therefore $b'_N = a''_N = 0$. We have shown that r (resp. s) is a power series in u (resp. v) starting with an invertible coefficient for the first power. This proves $rA = uA$ and $sA = vA$ and hence the Proposition in this special case.

The general case follows easily. Let (u, v) be a coordinate system and $t := uv$. The ideal $I := tR$ is the minimal one such that $\bar{u}\bar{v} = 0$ for *every* coordinate system $(\bar{u}, \bar{v})$ of A/IA over R/I. The ideals uA and vA are the inverse images of uA/tA and vA/tA. ■

Let A/R be a formal double point, (u, v) a coordinate system and n a natural number. Define

$$\begin{aligned} P_{u,n} &:= \mathrm{Ann}_A(u^n A) = \{a \in A \mid au^n = 0\} \lhd A, \\ P_{v,n} &:= \mathrm{Ann}_A(v^n A) = \{a \in A \mid av^n = 0\} \lhd A, \end{aligned} \tag{34}$$

and $P_u := P_{u,1}$, $P_v := P_{v,1}$. Using the uniqueness of the (u, v)-expansion, a straightforward verification shows:

$$P_{u,n} = \{ \sum_{i=1}^{\infty} a_i v^i \mid a_i t^i = 0 \ (i = 1, \dots, n-1), \ a_i t^n = 0 \ (i \geq n) \} \subset vA,$$
$$P_{v,n} = \{ \sum_{i=1}^{\infty} a_i u^i \mid a_i t^i = 0 \ (i = 1, \dots, n-1), \ a_i t^n = 0 \ (i \geq n) \} \subset uA, \tag{35}$$

where $t := uv \in \mathfrak{m}$. From this we see immediately that

$$P_{u,n} \cap P_{v,n} = P_{u,n} \cdot P_{v,n} = (0) \tag{36}$$

and (inside A)

$$R \cap (P_{u,n} + P_{v,n}) = (0). \tag{37}$$

Proposition 2.1.3 *(compare with [19], §3.7 and §3.8) Let (u,v) and (u',v') be coordinate systems of A/R and $n \geq 1$ such that $uA = u'A$. There are* unique *units $a, b \in A^\times$ with*

$$(u')^n = au^n \,, \ (v')^n = bv^n \,, \ ab \in R^\times$$

Proof: Put $t := uv$ and $t' := u'v'$. Consider the case $n = 1$ first. By Proposition 2.1.2 there are units $a', b' \in A^\times$ with $u' = a'u$, $v' = b'v$. Let

$$a'b' = c_0 + \sum_{i=1}^{\infty} c_i' u^i + \sum_{i=1}^{\infty} c_i'' v^i \,, \quad c_0, c_i', c_i'' \in R. \tag{38}$$

be the (u,v)-expansion of $a'b'$. Using its uniqueness and $t' = u'v' = (a'b')t \in R$ we conclude $c_i't = c_i''t = 0$ for every $i \geq 1$. Therefore (35) tells us that the sum in (38) is of the form

$$a'b' = c_0 + c_2 + c_1 \,, \quad c_0 \in R, \, c_1 \in P_u, \, c_2 \in P_v \tag{39}$$

which is unique by (36) and (37).

Every pair (a,b) of units of A with $u' = au$, $v' = bv$ is of the form

$$a = a' + \lambda, \ b = b' + \mu, \quad \lambda \in P_u, \, \mu \in P_v. \tag{40}$$

By (36) we have $\lambda\mu = 0$, so using (39) and (40), we get the decomposition

$$ab = a'b' + b'\lambda + a'\mu = c_0 + (c_1 + b'\lambda) + (c_2 + a'\lambda) \tag{41}$$

in $R + P_u + P_v$. Since it is unique, $ab \in R$ is equivalent to

$$\lambda = -(b')^{-1}c_1, \quad \mu = -(a')^{-1}c_2. \tag{42}$$

This proofs at the same time the existence and the uniqueness in the case $n = 1$.

For general n, deduce the existence of $a, b \in A^\times$ with $(u')^n = au^n$, $(v')^n = bv^n$ and $ab \in R$ by using the case $n = 1$ and taking n-th powers. Any other pair $a', b' \in A^\times$ with the same properties can be written as $a' = a + \lambda$, $b' = b + \mu$, with $\lambda \in P_{u,n}$ and $\mu \in P_{v,n}$. As above, one gets the unique decomposition $a'b' = ab + b\lambda + a\mu \in R + P_{u,n} + P_{v,n}$, and one can deduce $\lambda = \mu = 0$, which proves the uniqueness of a, b. ∎

2.2 Marked nodal curves

2.2.1 Let R be a noetherian ring. A **curve** over R is a scheme X which is flat and of finite presentation over R such that all geometric fibers of X/R are reduced curves. A **nodal curve** is a curve X/R such that all geometric fibers have at most ordinary double points as singularities. We write $X^{\mathrm{sm}} \subset X$ for the smooth locus of the morphism $X \to \operatorname{Spec} R$. A **mark** on a nodal curve X/S is a closed subscheme $D \subset X^{\mathrm{sm}}$ such that the natural morphism $D \to \operatorname{Spec} R$ is finite étale. Hence $D = \operatorname{Spec} R'$ for a finite étale R-algebra R'. We call the pair $(X/R, D)$ a marked nodal curve.

Remark 2.2.1 Being a marked nodal curve is local in the étale topology. More precisely, let X be a scheme over a noetherian ring R, $D \subset X$ a closed subscheme and $(U_i \to X)_i$ an étale covering. Then $(X/R, D)$ is a marked nodal curve if and only if $(U_i/R, D \times_X U_i)$ is a marked nodal curve, for all i.

2.2.2 Let X/R be a nodal curve, k a field and $x : \operatorname{Spec} k \to X$ a geometric point whose image is a closed point of X. Let $\bar{X} \subset X$ be the fiber of the morphism $X \to \operatorname{Spec} R$ containing the image of x. The composition of x with the morphism $X \to \operatorname{Spec} R$ corresponds to a ring morphism $R \to k$. Let $\tilde{R} := \mathcal{O}_{\operatorname{Spec} R, s}$ (resp. $\hat{R} := \mathcal{O}_{\operatorname{Spec} R, s}$) be the strict henselization (resp. strict completion) of R with respect to k.

In this subsection we assume in addition that the image of x is a singular point of $\bar{X}$. Since X/R is nodal we have $\mathcal{O}_{\bar{X},x} = k[[\bar{u}, \bar{v} | \bar{u}\bar{v} = 0]]$. We will say that x is a **geometric double point** of X/R.

Proposition 2.2.2 *Let X/R be as above and $x : \operatorname{Spec} k \to X$ a geometric double point of X/R. Then the complete local ring $\mathcal{O}_{X,x}$ is a formal double point over $\hat{R}$ (see Section 2.1).*

Proof: By the definition of a curve, there is a Zariski open subset $U = \operatorname{Spec} A \subset X$ containing the image of x such that A is flat and of finite presentation over R. By Proposition 5.2.3 (i) and (ii) the complete local ring $\mathcal{O}_{X,x}$ is a local flat $\hat{R}$-algebra. Let $\mathfrak{m} := \operatorname{Ker}(R \to k)$. Since $\bar{U} := \operatorname{Spec}(A/\mathfrak{m}A)$ is an open subset of $\bar{X}$ containing the image of x, Proposition 5.2.3 (iii) shows that $\mathcal{O}_{\bar{X},x} = \mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x}$. Hence the Proposition follows from the definition of geometric double points and Proposition 2.1.1 (i). ∎

If $x : \operatorname{Spec} k \to X$ is a rational double point, Proposition 2.2.2 shows that there exists a pair u, v of elements of $\mathcal{O}_{X,x}$ such that $t := uv \in \hat{R}$ and $\mathcal{O}_{X,x} = \hat{R}[[u, v | uv = t]]$. Such a pair (u, v) will be called a **formal coordinate system** for X/R at x. Note that the ring extensions $\tilde{R} \to \hat{R}$ and $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ are faithfully flat, in particular injective (Proposition 5.2.3 (ii)). We will call a pair u, v of elements of $\mathcal{O}_{X,x}$ a **coordinate system** for

X/R at x if $t := uv \in \check{R}$ and (u, v) (regarded as a pair of elements of $\mathcal{O}_{X,x}$) is a formal coordinate system.

Proposition 2.2.3 *Let X/R be as before and $x : \mathrm{Spec}\, k \to X$ a geometric double point. There exists a coordinate system (u, v) for X/R at x. Let (u, v) be any coordinate system for X/R at x and $R' \in \mathsf{Et}(R \to k)$ such that $t := uv \in R'$. Then for every sufficiently small affine étale neighborhood $U = \mathrm{Spec}\, A \to X$ of x, the natural morphism $R'[u, v | uv = t] \to A$ is étale.*

Proof: The first assertion is a consequence of Proposition 2.2.2 and Artin's Approximation Theorem. We will not give the details of the argument, because a much more general statement is proved in [3] XV, Corollaire 1.3.2. For the second assertion, let (u, v) be any coordinate system for X/R at x. By definition, $t := uv \in \check{R}$, hence we can find $R' \in \mathsf{Et}'(R \to k)$ with $t \in R'$. Let $U = \mathrm{Spec}\, A \to X$ be a neighborhood of x. Replacing A by $A \otimes_R R'$ we may assume that A is an R'-algebra. By the definition of $\mathcal{O}_{X,x}$ we may assume that $u, v \in A$. We obtain a natural morphism $R'[u, v | uv = t] \to A$. Since $U = \mathrm{Spec}\, A \to X$ is a neighborhood of x, A is equipped with a natural morphism $A \to k$; the composition $R'[u, v | uv = t] \to k$ sends u, v, t to 0. Taking the completion of the rings $R'[u, v | uv = t]$ and A with respect to the morphism to k, we obtain an isomorphism $\hat{R}[[u, v | uv = t]] \xrightarrow{\sim} \mathcal{O}_{X,x}$. Hence, by Proposition 5.2.3 (v) the morphism $R'[u, v | uv = t] \to A$ is étale in a neighborhood of the maximal ideal $\mathrm{Ker}(A \to k)$. Replacing $U = \mathrm{Spec}\, A$ by a Zariski open neighborhood of this point completes the proof of the Proposition. ■

Given a coordinate system (u, v) of a geometric double point on X/R, a pair $(R', U = \mathrm{Spec}\, A \to X)$ as in Proposition 2.2.3 will be called a **coordinate neighborhood** for (u, v). Frequently we will omit the ring R' from our notation.

2.2.3 Let R, X and $x : \mathrm{Spec}\, k \to X$ be as in the first paragraph of 2.2.2. Now we assume that x is a geometric smooth point on X/R. By this we mean that the image of x is a smooth point of the fibre $\bar{X} \subset X$ on which it lies. In addition, let $D \subset X$ be a mark on the nodal curve X/R. For an affine étale neighborhood $U = \mathrm{Spec}\, A \to X$ of x, let $D_U := D \times_X U$. Then (U, D_U) is again a marked nodal curve. Since $U = \mathrm{Spec}\, A$ is affine, $D_U = \mathrm{Spec}\,(A/I)$ for an ideal $I \triangleleft A$ ([14] II, Corollary 5.10). By definition, $R \to A/I$ is finite étale. There are two cases to consider. First, if the image of x lies on $X - D$, then for any sufficiently small neighborhood $U = \mathrm{Spec}\, A \to X$ of x we have $I = A$ and $D_U = \emptyset$. On the other hand, if x lies on D, Proposition 5.2.3 (iii) and (v) imply

$$\check{R} = \mathcal{O}_{D,x} = \mathcal{O}_{X,x}/I\mathcal{O}_{X,x}, \qquad \hat{R} = \mathcal{O}_{D,x} = \mathcal{O}_{X,x}/I\mathcal{O}_{X,x}. \qquad (43)$$

Proposition 2.2.4 *Let $x : \operatorname{Spec} k \to X$ be a geometric smooth point on the marked nodal curve $(X/R, D)$. There exists an element $z \in \mathcal{O}_{X,x}$ such that for every sufficiently small affine étale neighborhood $U = \operatorname{Spec} A \to X$ of x the natural morphism $R[z] \to A$ is étale. Moreover, if x lies on D, we can choose z such that D_U is defined by the equation $z = 0$.*

Proof: We will assume that x lies on D. Since x is a smooth point of $\bar{X}$ we have $\mathcal{O}_{\bar{X},x} = k[[\bar{z}]]$. Similarly to what we did in the proof of Proposition 2.1.1 (i) and Proposition 2.2.2 one shows that $\mathcal{O}_{X,x} = \hat{R}[[z]]$. Here z is any lift of $\bar{z}$ to $\mathcal{O}_{X,x}$. Let $U = \operatorname{Spec} A \to X$ be a neighborhood of x and $I \triangleleft A$ such that $D \times_X U = \operatorname{Spec}(A/I)$. Let $\hat{I} := I\mathcal{O}_{X,x}$. By (43) we have $\hat{R}[[z]]/\hat{I} \cong \hat{R}$. Therefore $z - a \in \hat{I}$ for some $a \in \hat{\mathfrak{m}} \triangleleft \hat{R}$. We may replace z by $z + a$, hence $z \in \hat{I}$. Now Nakayama's Lemma implies $\hat{I} = z\mathcal{O}_{X,x}$. Using [17] 24.E (i) we conclude that $I\mathcal{O}_{X,x} = z'\mathcal{O}_{X,x}$. Actually, we may assume that $z = z' \in \mathcal{O}_{X,x}$. After shrinking $U = \operatorname{Spec} A$ we may assume that $z \in A$ and $I = zA$. Completing the natural morphism $R[z] \to A$ with respect to k we obtain an isomorphism $\hat{R}[[z]] \xrightarrow{\sim} \mathcal{O}_{X,x}$. Hence, after shrinking $U = \operatorname{Spec} A$ a little bit more, $R[z] \to A$ will be étale. ∎

If x lies on D, an element $z \in \mathcal{O}_{X,x}$ with $\mathcal{O}_{X,x} = \hat{R}[[z]]$ and $\hat{I} = z\mathcal{O}_{X,x}$ will be called a **formal coordinate** for D at x. If z is moreover an element of $\mathcal{O}_{X,x}$, it will be called a **coordinate** for D at x.

2.3 Tame admissible covers

2.3.1 Let $(X/R, D)$ be a marked nodal curve over a noetherian ring R (see Section 2.2) and $\rho : Y \to X$ a finite morphism of schemes. Moreover, let $y : \operatorname{Spec} k \to Y$ be a closed geometric point and $x := \rho \circ y$. Since ρ is finite, it induces finite local ring extensions

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}, \quad \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}. \tag{44}$$

As in Section 2.2, $\tilde{R}$ (resp. $\hat{R}$) denotes the strict henselization (resp. the strict completion) of R with respect to k. The first (resp. second) arrow in (44) is a morphism of $\tilde{R}$-algebras (resp. $\hat{R}$-algebras).

Definition 2.3.1 *(see [13] §4) Let ρ, x and y be as above. We say that $\rho : Y \to X$ is* **tame admissible** *at y, if the following holds.*

(i) *Assume that x is a geometric double point. Then $\mathcal{O}_{Y,y}$ is a formal double point over $\hat{R}$. Moreover, there exist a formal coordinate system (u, v) of $\mathcal{O}_{X,x}$, a formal coordinate system (r, s) of $\mathcal{O}_{Y,y}$ and an integer n prime to the characteristic of k such that (44) sends u to r^n and v to s^n.*

(ii) *Assume that x is a smooth point lying on D. Then there exist a formal coordinate $z \in \mathcal{O}_{X,x}$ for D, an element $w \in \mathcal{O}_{Y,y}$ and an integer n prime* *to the characteristic of k such that $\mathcal{O}_{Y,y} = \hat{R}[[w]]$ and (44) sends z to w^n.*

(iii) *If x is a smooth point not lying on D then $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$.*

The morphism ρ is called a **tame admissible cover** *of $(X/R, D)$ if it is tame admissible at every geometric point y of Y.*

Part (ii) and (iii) of Definition 2.3.1 are usually stated as follows: the morphism ρ is étale over $X^{\mathrm{sm}} - D$ and tamely ramified along D. Let

$$\mathsf{Rev}_R^D(X) := \{ \rho : Y \to X \}$$

be the category whose objects are tame admissible covers of $(X/R, D)$ and whose morphisms between objects $Y \to X$ and $Z \to X$ are morphisms of schemes $Y \to Z$ compatible with the maps to X.

2.3.2 Consider the following situation. Let $(X/R, D)$ be a marked nodal curve and $\rho : Y \to X$ a finite morphism. Let $y : \operatorname{Spec} k \to Y$ be a geometric closed point such that $x := \rho \circ y$ is a geometric double point of X and such that ρ is admissible at y. We want to study ρ in a neighborhood of y. Let $U = \operatorname{Spec} A \to X$ be a neighborhood of x which decomposes ρ (see Section 1.2.3). Then there is a unique component $V = \operatorname{Spec} B$ of $U \times_X Y$ which is a neighborhood of y. We will refer to V as the local inverse image of U at y.

Proposition 2.3.2 *Let (u, v) be a coordinate system at x. If $U = \operatorname{Spec} A \to X$ is a sufficiently small coordinate neighborhood for (u, v) then*

$$B = A[r, s \mid r^n = u, s^n = v, rs = \tau],$$

where $V = \operatorname{Spec} B$ is the local inverse image of U at y, n an integer prime to the characteristic of k and $\tau \in \tilde{R} \cap A$. In particular, Y/R is a nodal curve in a neighborhood of y and (r, s) is a coordinate system at y.

Proof: Let $\tilde{A} := \mathcal{O}_{X,x}$, $\tilde{B} := \mathcal{O}_{Y,y}$, $\hat{A} := \mathcal{O}_{X,x}$ and $\hat{B} := \mathcal{O}_{Y,y}$. By hypothesis we have $u, v \in \tilde{A}$, $t := uv \in \tilde{R}$ and $\hat{A} = \hat{R}[[u, v | uv = t]]$. By Definition 2.3.1 there exist formal coordinate systems (u', v') for $\hat{A}$ and (r', s') for $\hat{B}$ such that $(r')^n = u'$ and $(s')^n = v'$. From Proposition 2.1.2 we know that there exist unique elements $a, b \in \hat{A}^\times$ with $u = au'$, $v = bv'$ and $ab \in \hat{R}^\times$. By Hensel's Lemma we can choose $c, d \in \hat{A}^\times$ with $c^n = a$ and $d^n = b$. Then $(cd)^n = ab \in \hat{R}^\times$, therefore we have $cd \in \hat{R}^\times$, by Hensel's Lemma. Let $r := cr'$ and $s := ds'$. Then $r^n = u$, $s^n = v$ and $\tau := rs \in \hat{R}$. But $r^n, s^n \in \tilde{B}$ and $\tau^n \in \tilde{R}$, hence by Proposition 5.2.3 (vi) we have $r, s \in \tilde{B}$ and $\tau \in \tilde{R}$.

Let $\tilde{B}_1 := \tilde{A}[r, s | r^n = u, s^n = v, rs = \tau]$. Since $\tilde{A} \to \tilde{B}_1$ is a finite local extension, $\tilde{B}_1$ is henselian. There is a natural local morphism $\tilde{B}_1 \to \tilde{B}$. Taking completions at both sides we obtain an isomorphism, because the completion of $\tilde{B}_1$ is easily seen to be isomorphic to $\hat{B} = \hat{R}[[r, s \mid rs = \tau]]$. With Proposition 5.2.3 (v) we conclude $\tilde{B} = \tilde{B}_1$. Therefore, if $U = \operatorname{Spec} A$ is sufficiently small, $V = \operatorname{Spec} B$ is as stated in the Proposition. The other statements follow easily. ■

We need a slight strengthening of Proposition 2.3.2. Let $\bar{Y} \subset Y$ be the fiber of Y/R on which y lies. Then $\mathcal{O}_{\bar{Y},y} = \mathcal{O}_{Y,y}/\hat{\mathfrak{m}}\mathcal{O}_{Y,y}$, where $\hat{\mathfrak{m}}$ is the maximal ideal of $\hat{R}$ (compare with the proof of Proposition 2.2.2). Let (u, v) be a coordinate system for X/R at x. Write $\bar{u}, \bar{v}$ for the image of u, v in $\mathcal{O}_{\bar{X},x} = \mathcal{O}_{X,x}/\hat{\mathfrak{m}}\mathcal{O}_{X,x}$.

Proposition 2.3.3 *Notation as above. Let $(\bar{r}, \bar{s})$ be a coordinate system for $\bar{Y}$ at y such that $\bar{r}^n = \bar{u}$ and $\bar{s}^n = \bar{v}$. Then there is a unique coordinate system (r, s) for Y at y lifting $(\bar{r}, \bar{s})$ such that $r^n = u$ and $s^n = v$.*

Proof: By Proposition 2.3.2 we can choose a coordinate system (r', s') for Y at y with $(r')^n = u$ and $(s')^n = v$. Then the coordinate system (r, s) we are looking for is of the form $r = ar'$, $s = bs'$ with $a, b \in \mathcal{O}_{Y,y}^{\times}$ and $ab \in \hat{R}^{\times}$. Then $r^n = u$ and $s^n = v$ is equivalent to $a^n(r')^n = (r')^n$ and $b^n(s')^n = (s')^n$. By Proposition 2.1.3 this is equivalent to $a^n = 1$ and $b^n = 1$. Applying Proposition 2.1.3 to $\mathcal{O}_{\bar{Y},y}$ we find that (r, s) is a lift of $(\bar{r}, \bar{s})$ if and only if the reductions $\bar{a}, \bar{b}$ of a, b are uniquely determined n-th roots of unity. Now the Proposition follows from Hensel's Lemma. ■

2.3.3 We continue with the notation fixed in the first paragraph of 2.3.3, but this time under the assumption that x is a smooth point of X/R lying on $D \subset X$. In analogy to Proposition 2.3.2 and Proposition 2.3.3 we can show the following.

Proposition 2.3.4 *Let z be a coordinate for D at x. If $U = \operatorname{Spec} A \to X$ is a sufficiently small coordinate neighborhood for z and $V = \operatorname{Spec} B$ its local inverse image at y, then*

$$B = A[w \mid w^n = z].$$

Here n is an integer prime to the characteristic of k. As an element of $\mathcal{O}_{Y,y}$, w is uniquely determined by z and the image of w in $\mathcal{O}_{\bar{Y},y} = \mathcal{O}_{Y,y}/\hat{\mathfrak{m}}\mathcal{O}_{Y,y}$.

3 Deformation theory

This section contains the main deformation result. In 3.1 we give the necessary notation and the precise statement (Theorem 3.1.1). Moreover, we sketch how this theorem can be generalized to the case of a non algebraically closed residue field, using Galois descent.

3.1 Statement of the main result

3.1.1 Let R be a noetherian *complete* local ring with algebraically closed residue field $k = R/\mathfrak{m}$. Let $(X/R, D)$ be a marked nodal curve where X is projective over R. We write $\bar{X} := X \times_R k$ for the special fiber and $\bar{D} := D \times_R k \subset \bar{X}$.

Let $\bar{\rho} : \bar{Y} \to \bar{X} \in \mathsf{Rev}_k^{\bar{D}}(\bar{X})$ be a tame admissible cover. A **deformation** of $\bar{\rho}$ to R is a pair (ρ, λ), where $\rho : Y \to X \in \mathsf{Rev}_R^{D}(X)$ is a tame admissible cover and $\lambda : Y \times_R k \xrightarrow{\sim} \bar{Y}$ is an isomorphism in the category $\mathsf{Rev}_k^{\bar{D}}(\bar{X})$. An isomorphism from one deformation (ρ_1, λ_1) to another (ρ_2, λ_2) is an isomorphism $f : Y_1 \xrightarrow{\sim} Y_2$ with $\rho_2 \circ f = \rho_1$ and $\lambda_2 \circ (f \times \mathrm{Id}_k) = \lambda_1$. Let

$$\mathrm{Def}_{\bar{\rho}}(R) = \{\, (\rho, \lambda) \,\}/_{\cong} \tag{45}$$

be the set of isomorphism classes of deformations of $\bar{\rho}$ to R. Most of the time we will identify the special fiber of a deformation with $\bar{Y}$, i.e. we will assume $\lambda = \mathrm{Id}_{\bar{Y}}$ and write $\rho \in \mathrm{Def}_{\bar{\rho}}(R)$.

3.1.2 Before stating the main theorem we have to fix some notations. The special fiber $\bar{X}$ has a finite number of ordinary double points, which we denote by $x_1, \ldots, x_r \in \bar{X}$ (they can be identified with the corresponding geometric points $x_i : \operatorname{Spec} k \to X$). Let $I' := \{1, \ldots, r\}$. By Proposition 2.2.2 we can choose coordinate systems (u_i, v_i) for X at x_i, $i \in I'$. Recall that u_i, v_i are elements of the strict henselian local ring $\mathcal{O}_{X,x_i}$ such that $t_i := u_i v_i$ is an element of R and $\mathcal{O}_{X,x_i} = R[[u_i, v_i | u_i v_i = t_i]]$. We write $\bar{u}_i, \bar{v}_i$ for the image of u_i, v_i in $\mathcal{O}_{\bar{X},x_i} = \mathcal{O}_{X,x_i}/\mathfrak{m}\mathcal{O}_{X,x_i}$. Then $(\bar{u}_i, \bar{v}_i)$ is a coordinate system for $\bar{X}$ at x_i.

Fix $i \in I'$ for a moment and let $\bar{\rho}^{-1}(x_i) = \{y_j \in \bar{Y} \mid j \in J'_i\}$ be the fiber of $\bar{\rho}$ over x_i, indexed by a finite set J'_i. By Proposition 2.3.2 there exist coordinate systems $(\bar{r}_j, \bar{s}_j)$ for $\bar{Y}$ at y_j such that $\bar{r}_j^{n_j} = \bar{u}_i$ and $\bar{s}_j^{n_j} = \bar{v}_j$ for integers n_j prime to the characteristic of k, for all $j \in J'_i$. Let J' be the disjoint union of the sets J'_i for $i \in I'$ and let $\kappa : J' \to I'$ be the map sending $j \in J'_i$ to i.

Let $\rho \in \mathrm{Def}_{\bar{\rho}}(R)$ be a deformation of $\bar{\rho}$ to R. For $j \in J'$ and $i := \kappa(j)$, Proposition 2.3.3 shows that there is a unique lift (r_j, s_j) of $(\bar{r}_j, \bar{s}_j)$ to a coordinate system of Y at y_j such that $r_j^{n_j} = u_i$ and $s_j^{n_j} = v_i$. Note that $\tau_j := r_j s_j$ is an element of R with $\tau_j^{n_j} = t_i$. This defines a map

$$\begin{array}{ccc} \mathrm{Def}_{\bar{\rho}}(R) & \longrightarrow & T(R) \\ (\rho, \lambda) & \longmapsto & (\tau_j := s_j r_j)_{j \in J'}, \end{array} \tag{46}$$

where $T(R) := \{(\tau_j)_{j \in J'} \mid \tau_j \in R,\ \tau_j^{n_j} = t_{\kappa(j)}\}$. An element $(\tau_j)_j$ of $T(R)$ will be called a **deformation datum**. Note that the map (46) depends on the choice of the coordinate systems (u_i, v_i) and $(\bar{r}_j, \bar{s}_j)$. Using the assumptions and notations of 3.1.1 and 3.1.2, we can now state the main theorem.

Theorem 3.1.1 *The map (46) is bijective. Moreover, if two deformations of $\bar{\rho}$ are isomorphic, then this isomorphism is unique.*

The proof of this theorem will be given in Section 3.2, following the outline given in Section 1.1.

3.1.3 In the statement of Theorem 3.1.1 we have assumed that the residue field k of R is algebraically closed. This is more than we really need. A careful inspection of the proof of Theorem 3.1.1 shows that we only have to assume that the singularities of $\bar{X}$ and the points on $\bar{D} = D \times_R k$ are k-rational and that k contains enough roots of unity. For instance, this holds if k is separably closed. But one can prove much more.

Suppose R is a complete noetherian local ring with arbitrary residue field k. Let k^s be a separable closure of k and let $\hat{R}$ be the strict completion of R with respect to k^s. Let $(X/R, D)$ be a projective marked nodal curve. Write $\bar{X} := X \times_R k$ for the special fiber and $\hat{X} := X \times_R \hat{R}$. Let $\bar{\rho} : \bar{Y} \to (\bar{X}, \bar{D})$ be a tame admissible cover and $\bar{\rho}^s$ its base change to k^s. Base change from R to $\hat{R}$ induces a map

$$\mathrm{Def}_{\bar{\rho}}(R) \longrightarrow \mathrm{Def}_{\bar{\rho}^s}(\hat{R}). \tag{47}$$

Let $G := \mathrm{Gal}(k^s/k)$ be the Galois group of k. Note that the action of G on k^s extends naturally to an action on $\hat{R}$.

Remark 3.1.2 There is a natural action of G on $\mathrm{Def}_{\bar{\rho}^s}(\hat{R})$. The map (47) induces a bijection $\mathrm{Def}_{\bar{\rho}}(R) \cong \mathrm{Def}_{\bar{\rho}^s}(\hat{R})^G$. In particular, if $\bar{\rho}$ is unramified over the singular points then there is a unique deformation ρ of $\bar{\rho}$ to R.

The first two statement of this remark are a variant of Weil's Descent Criterion and can be proved using étale descent. The third statement follows from the first two and from Theorem 3.1.1. Note that Theorem 3.1.1 can be applied to $\hat{X}/\hat{R}$. Hence we have a bijection $\mathrm{Def}_{\bar{\rho}^s}(\hat{R}) \cong T(\hat{R})$, where $T(\hat{R})$ is defined as in (46) in terms of coordinate systems of $\hat{X}$ at the singular points. The induced G-action on $T(\hat{R})$ can be determined from the natural action of G on the strict complete local rings of the singular points of Y. In particular, if X/R is smooth or if $\bar{\rho}$ is unramified over the singular points, $T(\hat{R})$ has exactly one element.

As remarked in the introduction, the results of [9] about deformation of mock covers with tame ramification can be reformulated in terms of tame admissible covers which are unramified over the singular points. Therefore, by Remark 3.1.2, Theorem 3.1.1 implies the results of [9]. However, the more general results of [10] on mock covers with wild ramification do not follow from Theorem 3.1.1.

3.1.4 Suppose that X is a smooth projective curve over a complete noetherian local ring R, and let $D \subset X$ be a mark. We will call tame admissible covers of (X, D) **tamely ramified covers.**

Corollary 3.1.3 *The functor*

$$\mathrm{Rev}_R^D(X) \xrightarrow{\sim} \mathrm{Rev}_k^{\bar{D}}(\bar{X})$$

reducing tamely ramified covers of X to the special fiber $\bar{X} := X \times_R k$ is an equivalence of categories.

Proof: In the case of smooth curves, Theorem 3.1.1 and Remark 3.1.2 show that tamely ramified covers lift uniquely. To prove Corollary 3.1.3, it remains to show the following. Let $\rho : Y \to (X, D)$ and $\eta : Z \to (X, D)$ be tamely ramified covers. Denote their reductions to $\bar{X}$ by $\bar{\rho}$ and $\bar{\eta}$. Let $\bar{f} : \bar{Z} \to \bar{Y}$ be an $\bar{X}$-morphism. Then $\bar{f}$ lifts uniquely to an X-morphism $f : Z \to Y$.

Using the result of Section 2.3, it is easy to see that $C := \rho^{-1}(D) \subset Y$ is a mark and $\bar{f}$ a tame admissible cover of $(\bar{Y}, \bar{C})$. We already know that $\bar{f}$ lifts to a tame admissible cover $f : Z' \to Y$. It follows that $\rho \circ f : Z' \to X$ is a tame admissible cover. Moreover, $\rho \circ f$ is a lift of $\bar{\eta}$. But since lifting is unique we may assume that $Z' = Z$ and $\rho \circ f = \eta$. ■

In Section 4 we will use Corollary 3.1.3 to construct a specialization morphism for tame fundamental groups.

3.2 Proof of the main theorem

In this Section we give a proof of Theorem 3.1.1 following the outline given in Section 1.1. In 3.2.1 we choose an étale covering of the curve X and fix some notation. In 3.2.2 we prove the theorem in the special case of an artinian base ring. In Section 3.2.3 we prove the general case by successively lifting $\bar{\rho}$ to the artinian rings $R_n := R/\mathfrak{m}^{n+1}$ and applying Grothendieck's Existence Theorem.

3.2.1 Notations and assumptions are as in Section 3.1.1 and 3.1.2. In addition, we assume that R is artinian. This means that $\mathfrak{m}^N = 0$ for some integer $N > 0$. The scheme $\operatorname{Spec} R$ consists of a single point and the closed embedding $\bar{X} \subset X$ induces a homeomorphism of the underlying topological spaces. Recall from 3.1.2 that x_i, $i \in I'$, are the singular points of $\bar{X}$, that y_j, $j \in J'$, are the singular points of $\bar{Y}$ and that the map $\kappa : J' \to I'$ is defined by $\bar{\rho}(y_j) = x_{\kappa(j)}$. We have chosen coordinate systems (u_i, v_i) (resp. $(\bar{r}_j, \bar{s}_j)$) for X at x_i, $i \in I'$, (resp. for $\bar{Y}$ at y_j, $j \in J'$,) such that $\bar{r}_j^{n_j} = \bar{u}_{\kappa(j)}$ and $\bar{s}_j^{n_j} = \bar{v}_{\kappa(j)}$.

The closed subscheme $\bar{D} \subset \bar{X}$ consists of a finite set of smooth points $x_{r+1}, \dots, x_s \in \bar{X}$. Let $I'' := \{r+1, \dots, s\}$. For $i \in I''$ let $\bar{\rho}^{-1}(x_i) = \{y_j \in \bar{Y} \mid j \in J_i''\}$ be the fiber over x_i, indexed by J_i''. Let $J'' := \cup_{i \in I''} J_i''$, $I := I' \cup I''$.

$J := J' \cup J''$ and extend κ to a map $\kappa : J \to I$ with $\kappa(j) = i$ for $j \in J_i''$. For all $i \in I''$, choose a coordinate $z_i \in \mathcal{O}_{X,x_i}$ for D at x_i (Proposition 2.2.4). For all $j \in J_i''$, choose an element $\bar{w}_i \in \mathcal{O}_{\bar{Y},y_j}$ such that $\mathcal{O}_{\bar{Y},y_j} = \mathcal{O}_{\bar{X},x_i}[\bar{w}_j | \bar{w}_j^{n_j} = \bar{z}_i]$, as in Proposition 2.3.4.

For $i \in I$, let $U_i = \operatorname{Spec} A_i \to X$ be an affine étale neighborhood of x_i. We may assume that U_i is a coordinate neighborhood, for the coordinate system (u_i, v_i) if $i \in I'$ and for the coordinate z_i of D if $i \in I''$. Hence, we have étale ring morphisms $R[u_i, v_i \mid u_i v_i = t_i] \to A_i$ for $i \in I'$ and $R[z_i] \to A_i$ for $i \in I''$. Let $U_0 := X - \{x_i \mid i \in I\} \subset X$ and $I_0 := I \cup \{0\}$. We may assume that $U_0 = \operatorname{Spec} A_0$ is affine. Then $\mathcal{U} := (U_i)_{i \in I_0}$ is an affine étale covering of X. In the sequel we will keep the open subset U_0 fixed and continue to shrink the neighborhoods U_i, $i \in I$, as necessary. Then $\mathcal{U}$ will always remain a covering of X. Hence we may assume that for all $i \in I$ the image of U_i on X is contained in $U_0 \cup \{x_i\}$. This implies that for $i \neq j$ the image of $U_{i,j} := U_i \times_X U_j$ on X is contained in U_0.

Let $\bar{\mathcal{U}} := (\bar{U}_i := U_i \times_X \bar{X})_{i \in I_0}$ be the restriction of $\mathcal{U}$ to $\bar{X}$. For $i \in I'$ (resp. $i \in I''$), $\bar{U}_i := \operatorname{Spec} A_i \to \bar{X}$ is a coordinate neighborhood for $(\bar{u}_i, \bar{v}_i)$ (resp. for $\bar{z}_i$), where $\bar{A}_i = A_i/\mathfrak{m}A_i$. Since $\bar{p} : \bar{Y} \to \bar{X}$ is a finite map, we can write $\bar{Y} = \operatorname{Spec} \bar{\mathcal{B}}$ for a finite $\mathcal{O}_{\bar{X}}$-algebra $\bar{\mathcal{B}}$. Let $\bar{\alpha} = (\bar{C}_i, \alpha_{i,j})$ be the descent datum for $\bar{\mathcal{B}}$ on $\bar{\mathcal{U}}$ (Section 1.3). In particular, $\bar{C}_i$ is a finite $\bar{A}_i$-algebra such that $\bar{U}_i \times_{\bar{X}} \bar{Y} = \operatorname{Spec} \bar{C}_i$. Note that the neighborhoods $\bar{U}_i$ of x_i can be made arbitrarily small by choosing U_i small. We may therefore assume that we have $\bar{C}_i = \oplus_{\kappa(j)=i} \bar{B}_j$, where

$$\bar{B}_j = \bar{A}_i[\,\bar{r}_j, \bar{s}_j \mid \bar{r}_j^{n_j} = \bar{u}_i,\ \bar{s}_j^{n_j} = \bar{v}_j,\ \bar{r}_j \bar{s}_j = 0\,] \tag{48}$$

for $j \in J_i'$ and

$$\bar{B}_j = \bar{A}_i[\,\bar{w}_j \mid \bar{w}_j^{n_j} = \bar{z}_i\,] \tag{49}$$

for $j \in J_i''$ (see Lemma 1.2.3, Proposition 2.3.2 and Proposition 2.3.4).

3.2.2 Let $\rho : Y \to X$ be a deformation of $\bar{\rho}$ to R. We can write $Y = \operatorname{Spec} \mathcal{B}$ for a finite $\mathcal{O}_{\bar{X}}$-algebra $\mathcal{B}$ with $\bar{\mathcal{B}} = \mathcal{B} \otimes_R k$. Let $\alpha = (C_i, \alpha_{i,j})$ be the descent datum for $\mathcal{B}$ on $\mathcal{U}$. It follows that $\bar{\alpha} = \alpha \otimes_R k$. By the Propositions 2.3.3 and 2.3.4 there are unique lifts (r_j, s_j) of $(\bar{r}_j, \bar{s}_j)$ (resp. w_j of $\bar{w}_j$) with $r_j^{n_j} = u_{\kappa(j)}$ and $s_j^{n_j} = v_{\kappa(j)}$, $j \in J'$ (resp. $w_j^{n_j} = z_{\kappa(j)}$, $j \in J''$). Moreover, we may assume that $C_i = \oplus_{\kappa(j)=i} B_j$, where

$$B_j = A_i[\,r_j, s_j \mid r_j^{n_j} = u_i,\ s_j^{n_j} = v_j,\ r_j s_j = \tau_j\,], \qquad \tau_j := r_j s_j \in R \tag{50}$$

for $j \in J_i'$,

$$B_j = A_i[\,w_j \mid w_j^{n_j} = z_i\,] \tag{51}$$

for $j \in J_i''$ and $\bar{B}_j = B_j \otimes_R k$ for all $j \in J$. Since ρ is étale over U_0, C_0 is an étale A_0-algebra. For the same reason and our choice of the covering $\mathcal{U}$, $C_i \otimes_{A_i} A_{i,j}$ and $C_j \otimes_{A_j} A_{i,j}$ are étale $A_{i,j}$-algebras, for $i, j \in I$, $i \neq j$.

Now let $\rho' : Y' \to X$ be another deformation of $\bar{\rho}$ to R inducing the same deformation data $(\tau_j)_j$ as ρ. Let $Y' = \operatorname{Spec} \mathcal{B}'$ and $\alpha' = (C_i', \alpha_{i,j}')$ be the corresponding descent datum for $\mathcal{B}'$ on $\mathcal{U}$. Since C_0 and C_0' are étale A_0-algebras with $C_0 \otimes_R k = \bar{C}_0 = C_0' \otimes_R k$, there is a unique A_0-linear isomorphism $f_0 : C_0 \xrightarrow{\sim} C_0'$ with $f \otimes_R k = \mathrm{Id}_{\bar{C}_0}$ (Lemma 5.1.1). After shrinking the neighborhoods U_i, $i \in I$, we find A_i-linear isomorphisms $f_i : C_i \to C_i'$ with $f_i \otimes_R k = \mathrm{Id}_{\bar{C}_i}$, because (48) and (49) depend, up to unique isomorphism, only on the deformation data $(\tau_j)_j$. Moreover, the f_i are uniquely determined by their values on the coordinates r_j, s_j and w_j. It follows from the Propositions 2.3.3 and 2.3.4 that the f_i are unique. We claim that the family (f_i) is an isomorphism of descent data between α and α'. This follows from the fact that $C_i \otimes_{A_i} A_{i,j}$, $C_j \otimes_{A_j} A_{i,j}$, $C_i' \otimes_{A_i} A_{i,j}$ and $C_j' \otimes_{A_j} A_{i,j}$ are étale $A_{i,j}$-algebras and from Lemma 5.1.1. By Corollary 1.3.2 the family (f_i) descends to a unique isomorphism $f : Y' \to Y$ between the deformations ρ' and ρ. This proves the second statement of Theorem 3.1.1 and the injectivity of the map (46) in our special case.

To prove the surjectivity of the map (46), let $(\tau_j)_j \in T(R)$. For $i \in I$, define a finite A_i-algebra $C_i = \oplus_{\kappa(j)=i} B_j$ by the expression given in (50) and (51). By Lemma 5.1.1 we can lift $\bar{C}_0$ to a finite étale A_0-algebra C_0. Using our assumptions on the covering $\mathcal{U}$, it is easy to see that for all $i \neq j$, $i, j \in I_0$, $C_i \otimes_{A_i} A_{i,j}$ and $C_j \otimes_{A_j} A_{i,j}$ are étale $A_{i,j}$-algebras. Therefore we can apply Lemma 5.1.1 once more to construct $A_{i,j}$-linear isomorphisms $\alpha_{i,j} : C_i \otimes_{A_i} A_{i,j} \xrightarrow{\sim} C_j \otimes_{A_j} A_{i,j}$ with $\alpha_{i,j} \otimes_R k = \bar{\alpha}_{i,j}$. Let $\alpha := (C_i, \alpha_{i,j})$. Since $\alpha \otimes_R k = \bar{\alpha}$, the uniqueness statement of Lemma 5.1.1 forces α to be a descent datum. By Corollary 1.3.2, α determines a finite $\mathcal{O}_X$-algebra $\mathcal{B}$, hence a finite morphism $\rho : Y = \operatorname{Spec} \mathcal{B} \to X$ lifting $\bar{\rho}$. By construction, ρ is a tame admissible cover. This completes the proof of Theorem 3.1.1 in the special case.

3.2.3 We are now going to prove Theorem 3.1.1 in the general case. For $n \geq 0$, let $R_n := R/\mathfrak{m}^{n+1}$ and $X_n := X \times_R R_n$. In particular, $X_0 = \bar{X}$. Let $\rho : Y \to X$ be a deformation of $\bar{\rho}$ to R with deformation datum $(\tau_j)_j$. Think of ρ as a finite $\mathcal{O}_X$-algebra $\mathcal{B}$ with $Y = \operatorname{Spec} \mathcal{B}$. The finite $\mathcal{O}_{X_n}$-algebra $\mathcal{B}_n := \mathcal{B} \otimes_R R_n$ corresponds to the deformation $\rho \times_R R_n$ of $\bar{\rho}$ to R_n with deformation datum $(\tau_j^{(n)})_j$. Here $\tau_j^{(n)}$ denotes the image of τ_j in R_n. Using the special case of Theorem 3.1.1 and Grothendieck's Existence Theorem (Corollary 1.4.2), we see that ρ is determined by the deformation datum $(\tau_j)_j$ up to unique isomorphism.

Conversely, let $(\tau_j)_j \in T(R)$ be a deformation datum. Applying Theorem 3.1.1 in the case of an artinian base ring, we obtain a compatible system of lifts $\rho_n : Y_n \to X_n$ with deformation datum $(\tau_j^{(n)})_j$. Write $Y_n = \operatorname{Spec} \mathcal{B}_n$, then $\mathcal{B}' =$

$(\mathcal{B}_n)_n$ is a formal finite $\mathcal{O}_X$-algebra. By Grothendieck's Existence Theorem (Corollary 1.4.2) we obtain a finite $\mathcal{O}_X$-algebra $\mathcal{B}$ with $\mathcal{B}_n = \mathcal{B} \otimes_R R_n$. Let $Y := \operatorname{Spec} \mathcal{B}$. All we have to show is that $\rho : Y \to X$ is a tame admissible cover.

Let $x \in \bar{X}$ be a singular point of the special fibre and $y \in \bar{\rho}^{-1}(x)$. Choose a formal coordinate system (u, v) for X at x and write $\hat{A} := \mathcal{O}_{X,x} = R[[u, v | uv = t]]$. Since ρ is finite, $\hat{B} := \mathcal{O}_{Y,y}$ is a finite local $\hat{A}$-algebra. It follows from Proposition 3.2.3 (iii) that $\mathcal{O}_{Y_n,y} = \hat{B}/\mathfrak{m}^{n+1}\hat{B}$. By the construction of Y_n, there exist elements $r_n, s_n \in \hat{B}/\mathfrak{m}^{n+1}\hat{B}$ with $r_n^e = u_n$, $s_n^e = v_n$, $\tau_n := r_n s_n \in R_n$ and $\hat{B}/\mathfrak{m}^{n+1}\hat{B} = R_n[[r_n, s_n \mid r_n s_n = \tau_n]]$. Moreover, we may assume that (r_{n+1}, s_{n+1}) lifts (r_n, s_n). Since $\hat{B}$ is complete, we find elements $r, s \in \hat{B}$ with image r_n, s_n in $\hat{B}/\mathfrak{m}^{n+1}$ such that $r^e = u$, $s^e = v$ and $\tau := rs \in R$. By the local criterium of flatness ([17] 20.C Theorem 49), $\hat{B}$ is a flat R-algebra. Hence, by Proposition 2.1.1, we have $\hat{B} = R[[r, s | rs = \tau]]$. We have shown that ρ is tame admissible at y (Definition 2.3.1 (i)). By similar arguments one shows that the same is true if x is a smooth point of $\bar{X}$. This completes the proof of Theorem 3.1.1. ∎

4 Tame fundamental groups of smooth curves

Let X be a smooth projective curve over a complete local ring R with algebraically closed residue field, and let $D \subset X$ be a mark. Corollary 3.1.3 states that tamely ramified covers of the special fiber $(\bar{X}, \bar{D})$ lift uniquely to tamely ramified covers of (X, D). This result was first obtained by A. Grothendieck and used to prove his famous theorem stating that the prime to p part of the tame fundamental group of a smooth projective curve over an algebraically closed field of characteristic p is the same as it would be in characteristic 0.

So far, [8] is the only complete reference for this result. For the special case of étale fundamental groups there is the more accessible account of [20]. In both expositions Grothendieck's Theorem is deduced from facts about fundamental groups of rather general schemes. We will see in this section that the case of tame covers of smooth projective curves can be handled with much less machinery. We are roughly going to prove the following. Let R be a mixed characteristic discrete valuation ring with algebraically closed residue field of characteristic $p > 0$ and X a connected smooth projective curve over R with a mark $D \subset X$. Then there is a surjective specialization morphism from the tame fundamental group of the generic geometric fiber $(X_{\bar{K}}, D_{\bar{K}})$ of X to the tame fundamental group of the special fiber $(\bar{X}, \bar{D})$. Moreover, this specialization morphism induces an isomorphism on the prime to p parts of the tame fundamental groups.

4.1 The tame fundamental group as a Galois group

Let k be an algebraically closed field and X a smooth connected projective curve over k. In this case, a mark $D \subset X$ of X/k is a finite set $D = \{x_1, \dots, x_r\}$ of closed points of X. Tamely ramified covers of (X, D) correspond to finite tamely ramified extensions of the function field $k(X)$ of X. Thus we can define the tame fundamental group $\pi_1^D(X)$ as the Galois group of the maximal algebraic extension of $k(X)$, tamely ramified over $x_1, \dots, x_r$. This is classical valuation theory for function fields in one variable. Moreover, if k is of characteristic 0, $\pi_1^D(X)$ is the profinite completion of the topological fundamental group $\pi_1^{\mathrm{top}}(X_{\mathbb{C}} - \{x_1, \dots, x_r\})$. This follows from the Riemann Existence Theorem. In this subsection, we list all the facts we need about tame fundamental groups of smooth projective curves over an algebraically closed field.

4.1.1 If $a \in X$ is a closed point, the local ring $\mathcal{O}_{X,a}$ is a discrete valuation ring with quotient field $k(X)$. This induces a bijection between closed points of X and discrete valuations of $k(X)$ which are trivial on k. If $z \in k(X)$ is a local parameter for a, the complete local ring is of the form $\hat{\mathcal{O}}_{X,a} = k[[z]]$.

If $\rho : Y \to (X, D) \in \mathsf{Rev}_k^D(X)$ is a tamely ramified cover (see Definition 2.3.1) and Y is connected, the function field $k(Y)$ is a finite extension of $k(X)$, tamely ramified over the valuations corresponding to the points $x_1, \dots, x_r$. In fact, let $y \in \rho^{-1}(x_i)$ and consider the natural extension $\mathcal{O}_{X,x_i} \to \mathcal{O}_{Y,y}$ of valuation rings; passing to the complete local rings, we obtain $\hat{\mathcal{O}}_{Y,y} = k[[w]] = \hat{\mathcal{O}}_{X,x_i}[w|w^n = z]$ for a suitable choice of local parameters z and w and with n prime to the characteristic of k (see Definition 2.3.1 (ii)).

Conversely, let $L/k(X)$ be a finite field extension which is tamely ramified over $x_1, \dots, x_r$ and unramified everywhere else. Then $L = k(Y)$ for a smooth connected projective curve Y over k. Moreover, the birational map induced by the inclusion $k(X) \hookrightarrow L = k(Y)$ extends uniquely to a tamely ramified cover $\rho : Y \to (X, D)$.

This correspondence between function fields and connected curves carries over to non connected curves. Let $\rho : Y \to (X, D)$ be any tamely ramified cover and let $Y_1, \dots, Y_s$ be the connected components of Y. The function ring of Y is defined as the finite $k(X)$-algebra $k(Y) := k(Y_1) \oplus \dots \oplus k(Y_s)$. The extension $k(Y)/k(X)$ is tamely ramified over $x_1, \dots, x_r$ and unramified everywhere else. We obtain an equivalence between $\mathsf{Rev}_k^D(X)$ and the category of finite $k(X)$-algebras, tamely ramified over $x_1, \dots, x_r$ (from now on we will tacitly understand that 'tamely ramified over $x_1, \dots, x_r$' implies 'unramified everywhere else').

4.1.2 Choose an algebraic closure $k(X) \hookrightarrow \bar{\Omega}$ and let $\Omega \subset \bar{\Omega}$ be the maximal subextension tamely ramified over $x_1, \dots, x_r$. Then $\Omega/k(X)$ is a Galois extension. Choose a closed point $a \in X - \{x_1, \dots, x_r\}$ and a discrete valuation $\tilde{a}$ of Ω extending the valuation of $k(X)$ corresponding to a. Define the **tame fundamental group** of (X, D) with **base point** a as the Galois group of Ω over $k(X)$:

$$\pi_1^D(X, a) := \mathrm{Gal}(\Omega/k(X)). \tag{52}$$

It may seem strange that $\pi_1^D(X, a)$ does not depend on a. But the choice of a and $\tilde{a}$ determines the way in which $\pi_1^D(X, a)$ classifies all tamely ramified covers of (X, D). For our purposes it suffices to consider only Galois covers of (X, D).

Let G be a finite group. A **G-Galois cover** of (X, D, a) is a tamely ramified cover $\rho : Y \to (X, D) \in \mathfrak{Rev}_k^D(X)$ together with an isomorphism $G \xrightarrow{\sim} \mathrm{Aut}_X(Y)$ such that Y is connected and G (as automorphism group of the cover ρ) acts transitively and without fixed points on the fiber $\rho^{-1}(a)$. A **pointed G-Galois cover** of (X, D, a) is a G-Galois cover $\rho : Y \to (X, D)$ together with a choice of an element $b \in \rho^{-1}(a)$.

Let $\rho : Y \to (X, D)$ be a G-Galois cover. Then $k(Y)/k(X)$ is a finite field extension and G acts as a group of $k(X)$-automorphisms on $k(Y)$. We can identify $\rho^{-1}(a)$ with the set of valuations of $k(Y)$ extending a. Since G acts transitively and fixed point free on this set, $k(Y)/k(X)$ is a Galois extension with Galois group G. Moreover, if we choose an element $b \in \rho^{-1}(a)$ there is a unique embedding $\lambda_b : k(Y) \hookrightarrow \Omega$ over $k(X)$ such that $\lambda_b^{-1}(\tilde{a}) = b$ (as valuation of $k(Y)$). The restriction map on the Galois groups induced by λ_b is a surjective continuous homomorphism $\pi_1^D(X, a) \twoheadrightarrow G$.

Conversely, any surjective and continuous homomorphism $\pi_1^D(X, a) \twoheadrightarrow G$ corresponds to a subfield $L \subset \Omega$ which is Galois over $k(X)$ with Galois group G. If Y is the smooth projective model of L, i.e. $L = k(Y)$, we obtain a G-Galois cover $\rho : Y \to (X, D)$. Note that G acts without fixed points on $\rho^{-1}(a)$ because a is unramified in $L = k(Y)$. Moreover, there is a unique distinguished element $b \in \rho^{-1}(a)$ corresponding to the restriction of $\tilde{a}$ on $L = k(Y)$. We have proved the following:

Proposition 4.1.1 *The choice of $\tilde{a}$ induces a bijection between isomorphism classes of pointed G-Galois covers of (X, D, a) and surjective continuous morphisms $\pi_1^D(X, a) \twoheadrightarrow G$.*

Let $g, r \geq 0$. Let $\Gamma_{g,r}$ be the profinite group with $2g + r$ generators $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r$ and the single relation

$$[a_1, b_1] \cdot \ldots \cdot [a_g, b_g] \cdot c_1 \cdot \ldots \cdot c_r = 1. \tag{53}$$

Remark 4.1.2 If the ground field k is of characteristic 0 and the curve X of genus g, we obtain an isomorphism

$$\Gamma_{g,r} \xrightarrow{\sim} \pi_1^D(X, a).$$

To prove this fact, one may assume that k is a subfield of $\mathbb{C}$. One can show that the natural homomorphism $\pi_1^D(X_{\mathbb{C}}, a) \to \pi_1^D(X, a)$ is actually an isomorphism, because any G-Galois cover of $(X_{\mathbb{C}}, D_{\mathbb{C}})$ is already defined over k. By the Riemann Existence Theorem, $\pi_1^D(X_{\mathbb{C}}, a)$ is isomorphic to the profinite completion of the topological fundamental group $\pi_1^{\mathrm{top}}(X_{\mathbb{C}} - \{x_1, \ldots, x_r\}, a)$, which is $\Gamma_{g,r}$. See e.g. [24] for more details.

4.2 Tame covers over complete discrete valuation rings

In this section R will be a complete discrete valuation ring with algebraically closed residue field k. We denote by K the quotient field of R and by $\bar{K}$ an algebraic closure of K. We work out several details about smooth projective curves and tamely ramified covers over R which are used in the proof of Grothendieck's Theorem.

4.2.1 Purity of branch locus First we need some preliminaries about regular local rings of dimension 2. See [17], Chapter 17, or [2] VIII §3 for more details.

Let A be a noetherian local domain; denote its residue field by k, its maximal ideal by $\mathfrak{m}$ and its quotient field by K. The ring A is regular of dimension d iff its completion $\hat{A}$ is.

From now on, assume A to be regular of dimension 2. Let S_A be the set of discrete valuations v of K such that the valuation ring $\mathcal{O}_v$ contains A. If $v \in S_A$ then $\mathfrak{p} := A \cap \mathfrak{m}_v$ is a prime ideal of height 1. The localization A is a discrete valuation ring, hence $\mathcal{O}_v = A$. Regular local rings are normal, therefore equal to the intersection of their localizations at primes of height one. We see that

$$A = \bigcap_{v \in S_A} \mathcal{O}_v. \tag{54}$$

Let $v \in S_A$ and $\mathfrak{p} := A \cap \mathfrak{m}_v$ be as above and choose $a \in \mathfrak{p} - \mathfrak{m}^2$. By [2] VIII, 5.3 Proposition 2 and Corollaire 1 we conclude that $\bar{A} := A/aA$ is regular and that (a) is a prime ideal. Since $\dim A = 2$ we have in fact $\mathfrak{p} = (a)$, and $\bar{A}$ is a discrete valuation ring. Moreover, we have $v'(a) = 0$ for all $v' \in S_A - \{v\}$. We will call an element $a \in A$ with the above properties a **parameter** of A at v.

The following lemma is a special case of the Purity Theorem of Nagata and Zariski, also called 'Purity of branch locus'. Its proof is an adaption of a proof given in the Appendix of [4]. For much more general versions of Purity, see e.g. [8] X.3.

Lemma 4.2.1 *Let A be a noetherian regular local domain of dimension 2, K the quotient field of A and L/K a finite extension. Then the integral closure*

B of A in L is a finite A-module of rank $[L : K]$. Moreover, B is étale over A iff L/K is unramified over S_A (the set of valuations dominating A).

Proof: Let S_B be the set of discrete valuations w of L lying over some valuation $v \in S_A$. Denote the corresponding valuation ring by $\mathcal{O}_w$ and put

$$B := \bigcap_{w \in S_B} \mathcal{O}_w \qquad (\subset L) \tag{55}$$

From (54) we can see that $A \subset B$. As intersection of integrally closed domains of L, B is itself an integrally closed domain of L. Hence, if we can show that B is a finite A-module then it will follow that B is the integral closure of A in L.

Let $v \in S_A$ be any valuation of K dominating A and choose a parameter a for A at v. Since the localization $A_{(a)}$ is a discrete valuation ring, $B_{(a)} := \cap_{w|v_a} \mathcal{O}_w$ is the integral closure of $A_{(a)}$ in L. By standard valuation theory $B_{(a)}$ is a free $A_{(a)}$-module of rank $[L : K]$. Therefore $B_{(a)}/aB_{(a)}$ is a free $A_{(a)}/aA_{(a)}$-module of rank $[L : K]$. Let $\bar{A} := A/aA$ and $\bar{B} := B/aB$. Using the fact that $v(a) = 0$ for all $v \in S_A - \{v_a\}$ it is easy to see that the $\bar{A}$-algebra $\bar{B} := B/aB$ injects into $B_{(a)}/aB_{(a)}$. Since $\bar{A}$ is a discrete valuation ring, it follows that $\bar{B}$ is a free $\bar{A}$-module of rank $\leq [L : K]$. Nakayama's Lemma implies that B can be generated (as an A-module) by $[L : K]$ elements. But L is the quotient field of B, therefore B is free over A of rank exactly $[L : K]$.

Let $b_1, \dots, b_n$ be an A-basis of B and δ be the discriminant of B over A with respect to $b_1, \dots, b_n$. Now L/K is unramified over S_A iff $v(\delta) = 0$ for all $v \in S_A$ iff $\delta \in A^*$ iff B is étale over A (Lemma 5.1.2 (iv)). This completes the proof of the lemma. ∎

If L/K is tamely ramified in only one place of S_A, we can use Abhyankar's Lemma to prove another version of purity. Its formulation is simpler if we work with strict complete local rings.

Lemma 4.2.2 *In the situation of Lemma 4.2.1, assume that A is complete with algebraically closed residue field. Assume moreover that L/K is tamely ramified over some valuation $v_0 \in S_A$ and unramified over $S_A - \{v_0\}$. Then L/K is purely ramified at v_0 and the integral closure of A in L is of the form*

$$B = A[b \mid b^e = a].$$

Here e is the ramification index of v_0 in L and $a \in A$ is a parameter of A at v_0.

Proof: For any natural number e prime to the residue characteristic of v_0 we put $K_e := K[b|b^e = a]$; this is a finite extension of K of degree e, purely ramified in v_0 and unramified over $S_A - \{v_0\}$. Moreover, $A_e := A[b \mid b^e = a]$ is the integral closure of A in K_e; it is a complete regular local domain of dimension two (see [2] VIII, 5.4).

Let $L_e := L \cdot K_e$ (inside some algebraic closure of K). By Abhyankar's Lemma (see e.g. [20], Appendix to Chapter IX) and the hypothesis we can choose e such that L_e/K_e is unramified at the unique extension of v_0 to K_e. Hence it is unramified at all discrete valuations dominating A_e. Now Lemma 4.2.1 applies, showing that the integral closure B_e of A_e in L_e is finite étale over A_e. But A_e is strictly henselian, therefore we have $B_e = A_e$ (Proposition 5.2.3 (v)) and $L \subset K_e$. In particular, v_0 is purely ramified in L. Hence, if we choose e to be its ramification index, then $L = K_e$ and $B = A_e$. ■

4.2.2 Specialization Let X be a connected smooth projective curve over R. It is a regular scheme of dimension 2. We will use the notation $\bar{X} := X \times_R k$ for the special fiber, $X_K := X \times_R K$ for the generic fiber and $X_{\bar{K}} := X \times_R \bar{K}$ for the geometric generic fiber. Note that X is an integral scheme, because it is both regular and connected (use [14] II Proposition 3.1 and the fact that a regular connected scheme is irreducible). We will denote its function field by $K(X)$.

Let x be a closed point of the generic fiber X_K (which we consider as an open subset of X). The point x corresponds to a discrete valuation of $K(X)$ which is trivial on K. The local ring $\mathcal{O}_{X,x}$ is the corresponding valuation ring. Let K' be the residue field of x and R' the integral closure of R in K'. Then R' is a complete discrete valuation ring, purely ramified over R. Since X/R is projective, the morphism $\operatorname{Spec} K' \to X$ corresponding to x extends uniquely to a morphism $\operatorname{Spec} R' \to X$ ([14] II, Th. 4.7 and Th. 4.9). Let $\bar{x}$ be the image of the special point of $\operatorname{Spec} R'$ on X. The point $\bar{x}$ is the unique closed point of the special fiber $\bar{X}$ contained in the Zariski closure of x on X. We will call $\bar{x}$ the **specialization** of x and write $x \rightsquigarrow \bar{x}$.

Let $\bar{x} \in \bar{X} \subset X$ be a closed point of the special fiber. The local ring $\mathcal{O}_{X,\bar{x}}$ is a regular local ring of dimension 2 with quotient field $K(X)$. Let $S_{\bar{x}}$ be the set of discrete valuations of $K(X)$ dominating $\mathcal{O}_{X,\bar{x}}$. A valuation $v \in S_{\bar{x}}$ corresponds to a point on X of codimension 1 whose Zariski closure contains $\bar{x}$. Hence we can identify $S_{\bar{x}}$ with the set $\{x \in X_K \mid x \rightsquigarrow \bar{x}\} \cup \{\eta\}$, where η is the generic point of the unique irreducible component of $\bar{X}$ containing $\bar{x}$. Therefore (54) becomes

$$\mathcal{O}_{X,\bar{x}} = \Big(\bigcap_{x \rightsquigarrow \bar{x}} \mathcal{O}_{X,x}\Big) \cap \mathcal{O}_{X,\eta}. \qquad (56)$$

Let $D \subset X$ be a mark on X (see Section 2.2). In particular, $D = \operatorname{Spec} R'$ for a finite étale R-algebra R'. But since R is strictly henselian we have $R' = R \oplus \ldots \oplus R$. Therefore D is the Zariski closure of a finite set $\{x_1, \ldots, x_r\} \subset X_K$ of K-rational points whose specializations $\bar{x}_1, \ldots, \bar{x}_r \in \bar{X}$ are pairwise distinct. We will identify D with the set $\{x_1, \ldots, x_r\}$ and write $X(K)$ for the set of K-rational points of $X(K)$.

Lemma 4.2.3 *Let X be a smooth projective curve over R and let $D = \{x_1, \dots, x_r\}$ be a mark on X.*

(i) *For every closed point $\bar{x} \in \bar{X}$ there exists a point $x \in X(K)$ with $x \rightsquigarrow \bar{x}$.*

(ii) *Let $\rho : Y \to (X, D)$ be a tamely ramified cover and $a \in X(K)$ such that its specialization $\bar{a}$ is not an element of the reduction $\{\bar{x}_1, \dots, \bar{x}_r\}$ of D. Then for any point $\bar{b} \in \rho^{-1}(\bar{a})$ there is a unique point $b \in Y(K)$ with $b \rightsquigarrow \bar{b}$ and $\rho(b) = a$.*

Proof: Let $\bar{a} \in \bar{X}$. By Proposition 2.2.3 we have $\mathcal{O}_{X,\bar{a}} \xrightarrow{\sim} R[[z]]$ for some local coordinate z. Hence we can embed $K(X)$ (which is the quotient field of $\mathcal{O}_{X,\bar{a}}$) into $K((z))$. This defines a discrete valuation v_x with residue field K on $K(X)$. It is clear that the point $x \in X(K)$ corresponding to v_x specializes to $\bar{x}$. This proves the first claim. In the situation of (ii), ρ is étale at $\bar{b}$. Therefore $\mathcal{O}_{X,\bar{a}} \xrightarrow{\sim} \mathcal{O}_{Y,\bar{b}}$, proving the second claim. ∎

4.2.3 Reduction Now we assume in addition that the geometric fibers $X_{\bar{K}}$ and $\bar{X}$ are connected. For any finite extension K'/K, let R' be the integral closure of R in K'. Then $X' := X \times_R R'$ is a smooth projective curve over R'. The scheme X' is still connected, has special fiber $\bar{X}$ and function field $K'(X') = K(X) \otimes_K K'$.

Proposition 4.2.4 *Let $\rho_{\bar{K}} : Y_{\bar{K}} \to (X_{\bar{K}}, D_{\bar{K}})$ be a tamely ramified G-Galois cover. Assume that the order of G is prime to the characteristic of k. Then there exists a tamely ramified cover $\rho : Y \to (X, D)$ with $\rho_{\bar{K}} = \rho \times_R \bar{K}$. Its reduction $\bar{\rho} : \bar{Y} \to (\bar{X}, \bar{D})$ is a G-Galois cover.*

Proof: By assumption $Y_{\bar{K}}$ is connected. The G-Galois cover $\rho_{\bar{K}}$ is already defined over some finite extension K' of K, i.e. there is a tamely ramified G-Galois cover $\rho_{K'} : Y_{K'} \to (X_{K'}, D_{K'})$ with $\rho_{\bar{K}} = \rho_{K'} \times_{K'} \bar{K}$. Note that the corresponding extension of function fields $K'(Y_{K'})/K'(X_{K'})$ is a G-Galois extension. Our assumption on G allows us to apply Abhyankar's Lemma. More precisely, after a tamely ramified extension of K' we may assume that the discrete valuation of $K'(X')$ corresponding to the generic point of $\bar{X}$ is unramified in $K'(Y_{K'})$.

Let Y' be the normalization of X' in $K'(Y_{K'})$ and $\rho' : Y' \to X'$ the natural map ([14] II Ex. 3.8). By construction Y' is a normal integral scheme with function field $K'(Y') = K'(Y_{K'})$. We claim that ρ' is a tamely ramified cover of (X', D').

Let $\bar{x} \in \bar{X}$ be a closed point and let B be the integral closure of $\mathcal{O}_{X',\bar{x}}$ in $K'(Y_{K'})$. By Lemma 4.2.1 B is finite over $\mathcal{O}_{X',\bar{x}}$. Therefore the fiber $(\rho')^{-1}(\bar{x}) = \{\bar{y}_1, \dots, \bar{y}_s\}$ is a finite set and B is its semi-local ring (i.e. its localizations are exactly $\mathcal{O}_{Y',\bar{y}_j}$). In particular, ρ' is a finite morphism.

Let $S_{\bar{x}} := \{x \in X_{K'} \mid x \rightsquigarrow \bar{x}\} \cup \{\eta\}$ be the set of discrete valuations of $K'(X_{K'})$ specializing to $\bar{x}$. If $\bar{x} \notin \{\bar{x}_1, \ldots, \bar{x}_r\}$ then $K'(Y_{K'})/K'(X_{K'})$ is unramified over $S_{\bar{x}}$. If $\bar{x} = \bar{x}_i$ then it is tamely ramified over x_i and unramified over $S_{\bar{x}} - \{x_i\}$. Using Lemma 4.2.1 and Lemma 4.2.2, we conclude that ρ' is a tamely ramified cover of (X', D').

Let $\bar{\rho} := \rho' \times_{R'} k$ be the reduction to the special fiber; it is a tamely ramified cover of $(\bar{X}, \bar{D})$. By Corollary 3.1.3 it lifts uniquely to a tamely ramified cover $\rho : Y \to (X, D)$. The uniqueness of lifting implies $\rho' = \rho \times_R R'$. Therefore $\rho_{\bar{K}} = \rho \times_R \bar{K}$.

Note that there is a natural G-action on Y' inducing a G-action on Y and hence on $\bar{Y}$. To prove that $\bar{\rho} : \bar{Y} \to \bar{X}$ is in fact a G-Galois cover, it remains to show that $\bar{Y}$ is connected. Assume that $\bar{Y}$ is the disjoint union of two open subset $\bar{Y}_1, \bar{Y}_2$. Then the natural maps $\bar{Y}_i \to (\bar{X}, \bar{D})$ are still tamely ramified covers, for $i = 1, 2$. Lift them to tamely ramified covers $Y_i \to (X, D)$, $i = 1, 2$. By the uniqueness of lifting, we conclude that Y is the disjoint union of Y_1 and Y_2. But Y is an integral normal scheme and hence connected ([14] II Proposition 3.1). Therefore one of the Y_i must be empty, proving that $\bar{Y}$ is connected. ∎

4.3 The specialization morphism

Let G be a profinite group. We define the **prime to p part** of G as the inverse limit over those finite quotients of G which are of order prime to p and denote it by $G^{p'}$. It is a profinite quotient of G. If $\phi : G \to H$ is a continuous morphisms of profinite groups, ϕ factors to a continuous morphism $\phi^{p'} : G^{p'} \to H^{p'}$. Note that surjectivity of ϕ implies surjectivity of $\phi^{p'}$. Remember that $\Gamma_{g,r}$ denotes the tame fundamental group of a smooth projective curve of genus g with r marked points in characteristic zero (Remark 4.1.2).

Theorem 4.3.1 (Grothendieck) *Let $\bar{X}$ be a smooth projective curve of genus g over an algebraically closed field k of characteristic $p > 0$ and $a_0, \bar{x}_1, \ldots, \bar{x}_r$ be pairwise distinct closed points. Let $\bar{D} := \{\bar{x}_1, \ldots, \bar{x}_r\}$. Then there is a surjective homomorphism of profinite groups*

$$\Gamma_{g,r} \longrightarrow \pi_1^{\bar{D}}(\bar{X}, a_0)$$

inducing an isomorphism on the prime to p parts, i.e. $\Gamma_{g,r}^{p'} \xrightarrow{\sim} \pi_1^{\bar{D}}(\bar{X}, a_0)^{p'}$

The proof of this theorem will be given in the rest of this section. In 4.3.1 we construct a lift of $\bar{X}$ to characteristic 0. In 4.3.2 we construct the specialization morphism between tame fundamental groups. In 4.3.3 it is shown that this specialization morphism is injective on the prime to p parts.

4.3.1 Let $\bar{X}$ be a smooth projective curve over an algebraically closed field of characteristic $p > 0$ and let $\bar{D} = \{\bar{x}_1, \ldots, \bar{x}_r\}$ be a mark on $\bar{X}$. Let

$R := W(k)$ be the ring of Witt vectors over k; it is a complete discrete valuation ring of characteristic 0. We adopt the notations K, $\bar{K}$ etc. from Section 4.2. By Proposition 4.3.2 below we can lift $\bar{X}$ to a smooth projective curve X over R; its geometric generic fiber $X_{\bar{K}}$ is connected. By Lemma 4.2.3 we can lift $\bar{x}_1, \ldots, \bar{x}_r \in \bar{X}$ to K-rational points $x_1, \ldots, x_r \in X(K)$. This defines a mark $D := \{x_1, \ldots, x_r\}$ on X.

Proposition 4.3.2 *Let $\bar{X}$ be a smooth projective curve over k.*

(i) *There exists a smooth projective curve X over R with $\bar{X} = X \times_R k$.*

(ii) *If $\bar{X}$ is connected, then for any X as in (i) the generic geometric fiber $X_{\bar{K}} := X \times_R \bar{K}$ is connected.*

Proof: We will give a proof of (i) only in characteristic different from 2. For the general case, see e.g. [8] III. If $\mathrm{char}(k) \neq 2$ then there exists a *simple morphism* $\bar{f} : \bar{X} \to \mathbb{P}^1_k$ (see [5] Proposition 8.1). In particular, $\bar{f}$ is tamely ramified, having only ramification of order 2. Therefore we can lift $\bar{f}$ to a tamely ramified cover $f : X \to \mathbb{P}^1_R$ (Corollary 3.1.3). This proves (i) in this special case.

Assume that $\bar{X}$ is connected and that X is a lift of $\bar{X}$ as in (i). Let $\bar{R}$ be the integral closure of R in $\bar{K}$. Then $X_{\bar{R}} := X \times_R \bar{R}$ is a smooth projective curve over the valuation ring $\bar{R}$ with special fiber $\bar{X}$. Since the natural morphism $X_{\bar{R}} \to \mathrm{Spec}\,\bar{R}$ is closed ([14] II Theorem 4.9), every closed subset of $X_{\bar{R}}$ must intersect the special fiber $\bar{X}$. Therefore $X_{\bar{R}}$ is connected. A connected regular scheme is integral. Therefore $X_{\bar{K}} \subset X_{\bar{R}}$ is integral and in particular connected. ■

4.3.2 Let (X, D) be as constructed in 4.3.1. Choose a closed point $a_0 \in \bar{X} - \{\bar{x}_1, \ldots, \bar{x}_r\}$. By Lemma 4.2.3 (i) we can choose a point $a_1 \in X(K) - \{x_1, \ldots, x_r\}$ which specializes to a_0. We will regard a_0 (resp. a_1) as base points on $\bar{X}$ (resp. $X_{\bar{K}}$). As in 4.1.2, choose an algebraic closure $k(\bar{X}) \hookrightarrow \Omega_0$ and let Ω_0 be the maximal subextension of Ω_0 which is tamely ramified over $\bar{x}_1, \ldots, \bar{x}_r$. Extend the valuation on $k(\bar{X})$ corresponding to a_0 to a valuation $\tilde{a}_0$ on Ω_0. Similarly for $X_{\bar{K}}$: let Ω_1 be an algebraic closure of $\bar{K}(X)$, Ω_1 the maximal subextension tamely ramified over $x_1, \ldots, x_r$ and $\tilde{a}_1$ a valuation of Ω_1 lying over a_1. As in 4.1.2, we define the fundamental groups

$$\pi_1^{D_{\bar{K}}}(X_{\bar{K}}, a_1) := \mathrm{Gal}(\Omega_1/\bar{K}(X)), \quad \pi_1^{\bar{D}}(\bar{X}, a_0) := \mathrm{Gal}(\Omega_0/k(\bar{X})) \qquad (57)$$

We can interpret Lemma 4.2.3 as follows. The process of specialization $a_1 \rightsquigarrow a_0$ is a 'path' in $X - D$ connecting a_0 with a_1, and tamely ramified covers of (X, D) have a unique lifting property with respect to such paths. In close analogy to topological covering space theory, this lifting property defines a morphism $\pi_1^{D_{\bar{K}}}(X_{\bar{K}}, a_1) \longrightarrow \pi_1^{\bar{D}}(\bar{X}, a_0)$ of fundamental groups.

Proposition 4.3.3 *There exists a natural surjective morphism of profinite groups*

$$\phi : \pi_1^{D_{\bar{K}}}(X_{\bar{K}}, a_1) \longrightarrow \pi_1^{\bar{D}}(\bar{X}, a_0).$$

Proof: Let $G = \pi_1^{\bar{D}}(\bar{X}, a_0)/H$ be a finite quotient. By Proposition 4.1.1 it corresponds to a pointed G-Galois cover $\bar{\rho} : \bar{Y} \to (\bar{X}, \bar{D})$. Let $b_0 \in \bar{\rho}^{-1}(a_0)$ be the distinguished point.

Corollary 3.1.3 states that we can lift $\bar{\rho}$ to a tamely ramified cover $\rho : Y \to (X, D)$ with automorphism group G. In particular, G acts on $Y_{\bar{K}}$. Lemma 4.2.3 (ii) shows that specialization defines a G-equivariant bijection $\rho_{\bar{K}}^{-1}(a_1) \xrightarrow{\sim} \bar{\rho}^{-1}(a_0)$. Hence G acts transitively and fixed point free on $\rho_{\bar{K}}^{-1}(a_1)$. By Proposition 4.3.2 (ii) $Y_{\bar{K}}$ is connected. Hence $\rho_{\bar{K}} : Y_{\bar{K}} \to X_{\bar{K}}$ is a G-Galois cover. Let $b_1 \in \rho_{\bar{K}}^{-1}(a_1)$ be the unique lift of b_0 over a_1 (Lemma 4.2.3 (ii)). Consider $\rho_{\bar{K}}$ as a pointed G-Galois cover with distinguished point b_1. By Proposition 4.1.1, $\rho_{\bar{K}}$ corresponds to a surjective continuous homomorphism $\phi_H : \pi_1^{D_{\bar{K}}}(X_{\bar{K}}, a_1) \to G$.

We claim that the maps ϕ_H are compatible (where H runs over all normal subgroups of finite index) and lift to the desired surjective morphism ϕ. To prove this we have to show the following: if $G' = \pi_1^{\bar{D}}(\bar{X}, a_0)/H'$ is a smaller quotient than G, i.e. $H \subset H'$, and $\psi : G \to G'$ is the natural map, then $\psi \circ \phi_H = \phi_{H'}$.

Let $\bar{\rho}' : \bar{Y}' \to \bar{X}$ be the pointed G'-Galois cover corresponding to the natural map $\pi_1^{\bar{D}}(\bar{X}, a_0) \to G'$ (with distinguished point b_0'). Let $\rho' : Y' \to X$ be its lift to R and b_1' be the lift of b_0' in $(\rho')^{-1}(a_1)$. Let $\lambda_{b_1'} : \bar{K}(Y') \hookrightarrow \Omega_1$ be the unique embedding such that $\lambda_{b_1'}^{-1}(\tilde{a}_1)$ is the valuation on $\bar{K}(Y')$ corresponding to the point b_1'. The map $\phi_{H'}$ is the restriction map of Galois groups corresponding to $\lambda_{b_1'}$.

By the definition of G and G' we have a natural inclusion $k(\bar{Y}') \subset k(\bar{Y}) \subset \Omega_0$ of Galois extensions, such that the restriction map from $G = \mathrm{Gal}(k(\bar{Y})/k(\bar{X}))$ to $G' = \mathrm{Gal}(k(\bar{Y}')/k(\bar{X}))$ is the natural map. It induces an $\bar{X}$-morphism $\bar{f} : \bar{Y} \to \bar{Y}'$ such that $\bar{f}(b_0) = b_0'$. By Corollary 3.1.3 $\bar{f}$ lifts uniquely to an X-morphism $f : Y \to Y'$. The uniqueness statement of Lemma 4.2.3 (ii) implies that $f(b_1) = b_1'$. Let $\mu : \bar{K}(Y') \hookrightarrow \bar{K}(Y)$ be the inclusion of function fields induced by f. We have

$$\lambda_{b_1'} = \lambda_{b_1} \circ \mu, \tag{58}$$

because both embeddings restrict $\tilde{a}_1$ to the valuation corresponding to b_1'. If we translate (58) into the corresponding equation for maps between Galois groups, we obtain $\psi \circ \phi_H = \phi_{H'}$. This proves the proposition. ■

4.3.3 We are now going to finish the proof of Theorem 4.3.1. Remember that $\bar{K}$ is an algebraically closed field of characteristic 0. By Remark 4.1.2

we can identify $\pi_1^{D_{\bar{K}}}(X_{\bar{K}}, a_1)$ with $\Gamma_{g,r}$. Thus, the proof of Theorem 4.3.1 is complete if we are able to show that the morphism ϕ given in Proposition 4.3.3 is injective on the prime to p parts.

Reconsidering the construction of the specialization morphism, we see that we have to prove the following. Let G be a finite quotient of $\pi_1^{D_{\bar{K}}}(X_{\bar{K}}, a_1)$ of order prime to p. Then the natural map $\psi : \pi_1^{D_{\bar{K}}}(X_{\bar{K}}, a_1) \to G$ factors over ϕ.

ψ corresponds to a punctured G-Galois cover $\rho_{\bar{K}} : Y_{\bar{K}} \to X_{\bar{K}}$ (with distinguished point $b_1 \in \rho_{\bar{K}}^{-1}(a_1)$. By Proposition 4.2.4 we can find a tamely ramified cover $\rho : Y \to X$ with $\rho_{\bar{K}} = \rho \times_R \bar{K}$. Its reduction $\bar{\rho} := \rho \times_R k$ is a G-Galois cover, which we consider as punctured by the specialization b_0 of b_1. Therefore $\bar{\rho}$ corresponds to a surjective map $\chi : \pi_1^{\bar{D}}(\bar{X}, a_0) \to G$. By the construction of ϕ we have $\psi = \chi \circ \phi$. ■

5 Appendix

For the convenience of the reader, this appendix contains some results about étale ring extensions and henselian rings which are used in this paper. Our main reference is [21]. See also [18] Chapter I, §3.

5.1 Let A, B be rings, $\varphi : A \to B$ a morphism and $\mathfrak{q} \in \operatorname{Spec} B$. $A \to B$ is called **unramified** at $\mathfrak{q}$, if $\mathfrak{p}B = \mathfrak{q}$ and the residue field extension $A/\mathfrak{p}A \to B/\mathfrak{q}B$ is separable (where $\mathfrak{p} := \varphi^{-1}(\mathfrak{q}) \in \operatorname{Spec} A$). We say that B/A is unramified, if it is unramified for all $\mathfrak{q} \in \operatorname{Spec} B$. Assume for a moment that B is finitely generated as an A-algebra. Then $A \to B$ is unramified at $\mathfrak{q}$ iff $A \to B$ is **'net'** at $\mathfrak{q}$ (in Raynaud's terminology, see [21] I, Def. 4) iff $(\Omega_{B/A}) = 0$ (see [21] III, §4, Prop. 9 u. Ex. 1). Since $\Omega_{B/A}$ is a finitely generated B-module, being unramified is an open property on Spec B (supp $\Omega_{B/A} \subset \operatorname{Spec} B$ is closed).

$A \to B$ is called **étale**, if it is of finite presentation, flat and unramified. $A \to B$ is called **formally étale** if for every A-algebra C and ideal $I \lhd C$ with $I^2 = 0$ the canonical map

$$\operatorname{Hom}_A(B, C) \longrightarrow \operatorname{Hom}_A(B, C/I) \qquad (59)$$

is a bijection. By [21], Chapitre I, Definition 2 and Chapitre V, Corollaire 1, $A \to B$ is étale if and only if it is of finite presentation and formally étale.

Lemma 5.1.1 *Let A be a ring with a nilpotent ideal I; let $\bar{A} := A/I$. Then every finite étale $\bar{A}$-algebra $\bar{B}$ lifts uniquely to a finite étale A-algebra B.*

Proof: This follows from [21] V, Thm. 4, and from Nakayama's Lemma. ■

Lemma 5.1.2 *Let A be a ring and B, C be A-algebras.*

(i) *If* $A \to B$ *is étale, then so is* $C \to B \otimes_A C$.

(ii) *Let* $B \to C$ *be an A-algebra morphism. If* $A \to B$ *and* $A \to C$ *are étale, then* $B \to C$ *is étale.*

(iii) *If* $A \to B$ *and* $B \to C$ *are étale, then* $A \to C$ *is étale.*

(iv) *Assume that B is a finite free A-algebra and let* $(\delta_{B/A}) \lhd A$ *be the discriminant ideal. Then* $A \to B$ *is étale if and only if* $\delta_{B/A} \in A^\times$.

Proof: As remarked above, a ring extension is étale if and only if it is formally étale and of finite presentation. This implies the Claims (i), (ii) and (iii), because the analogous statements hold both for formally étale and for finitely presented morphisms. To prove (iv), note that $A \to B$ is étale if and only if it is unramified. By [21] III Proposition 10 and 11, $A \to B$ is unramified if and only if $B \otimes_A (A/\mathfrak{p}A)$ is a separable $A/\mathfrak{p}A$-algebra, for all $\mathfrak{p} \in \operatorname{Spec} A$. It is well known that the latter holds if and only if $\delta_{B/A} \notin \mathfrak{p}$ (see e.g. [18] III Proposition 3.1). ∎

5.2 Let A be a ring and $A \to k$ a morphism to some field k. Denote by $\mathsf{Et}(A \to k)$ the category of étale A-algebras B equipped with an A-algebra morphism $B \to k$. Morphisms between objects $B \to k$ and $C \to k$ are A-algebra morphisms $B \to C$ compatible with the maps to k. If such a morphism exists, then we say that $C \to k$ is **smaller** than $B \to k$. A ring is called **indecomposable** if it is not the direct sum of two rings (equivalently, $\operatorname{Spec} A$ is connected, [14] II Ex. 2.19). Let $\mathsf{Et}'(A \to k)$ be the full subcategory of indecomposable objects of $\mathsf{Et}(A \to k)$.

Lemma 5.2.1 *Let* $A \to k$ *be as above and* $B \to k$, $C \to k \in \mathsf{Et}(A \to k)$. *If C is indecomposable, then there is at most one morphism from* $B \to k$ *to* $C \to k$. *In any case, there exists an indecomposable* $D \to k \in \mathsf{Et}'(A \to k)$, *smaller than* $B \to k$ *and* $C \to k$.

Proof: See [21] VIII §2, Prop. 2. ∎

The Lemmas 5.1.2 and 5.2.1 allow us to define the direct limit

$$\tilde{A} := \varinjlim A', \qquad A' \to k \in \mathsf{Et}'(A \to k). \tag{60}$$

The ring $\tilde{A}$ is called **the henselization** of A with respect to $A \to k$. There is a natural A-algebra morphism $\tilde{A} \to k$. If k is separably closed, then $\tilde{A}$ is called a **strict henselization** of A at $\operatorname{Ker}(A \to k)$.

A local ring A is called **henselian**, if every finite A-algebra is the direct sum of local factors (see [21], I, Def. 1). Or, equivalently, if A verifies Hensel's Lemma ([21] I, Prop. 5). The ring A is called **strictly henselian** if it is henselian and its residue field is separably closed. From [21] Chap. VIII, we get

Remark 5.2.2 $\tilde{A}$ is a local henselian ring and its residue field $\tilde{A}/\tilde{\mathfrak{m}}$ is the separable closure of $A/\mathfrak{p}$ inside k. $A \to \tilde{A}$ is flat and unramified. If A is noetherian, $\tilde{A}$ will be noetherian, too. $\tilde{A} \to k$ satisfies the following universal property: for every henselian local A-algebra B, equipped with an A-algebra morphism $B \to k$, and whose residue field contains the separable closure of $A/\mathfrak{p}$ inside k, there is a unique A-algebra morphism $\tilde{A} \to B$ compatible with the maps to k.

We define the **completion** $\hat{A}$ of A with respect to $A \to k$ as the completion of $\tilde{A}$ at its maximal ideal. If k is separably closed, $\hat{A}$ is called the **strict completion** of A at $\mathrm{Ker}(A \to k)$.

Proposition 5.2.3 *Let A be a noetherian ring, $A \to B$ be a finitely presented morphism and $B \to k$ a morphism to a field k. Then:*

(i) *There are natural local morphisms $\tilde{A} \to \tilde{B}$ (resp. $\hat{A} \to \hat{B}$) between the henselizations (resp. completions) with respect to k.*

(ii) *If $A \to B$ is flat, $\tilde{A} \to \tilde{B}$ and $\hat{A} \to \hat{B}$ are faithfully flat.*

(iii) *If $B = A/I$ for some ideal $I \triangleleft A$, then $\tilde{B} = \tilde{A}/I\tilde{A}$ and $\hat{B} = \hat{A}/I\hat{A}$.*

(iv) *$A \to B$ is unramified at $\mathfrak{q} := \ker(B \to k)$* iff *$\tilde{A} \to \tilde{B}$ is surjective* iff *$\hat{A} \to \hat{B}$ is surjective.*

(v) *$A \to B$ is étale at $\mathfrak{q}$* iff *$\tilde{A} \xrightarrow{\sim} \tilde{B}$ is an isomorphism* iff *$\hat{A} \xrightarrow{\sim} \hat{B}$ is an isomorphism.*

(vi) *$\tilde{A}$ is integrally closed in $\hat{A}$.*

Proof:

(i): $\tilde{A} \to \tilde{B}$ exists by the universal property of henselization (see Remark 5.2.2). The existence of $\hat{A} \to \hat{B}$ is a direct consequence.

(ii): If $A \to B$ is flat, its base change $\tilde{A} \to B \otimes_A \tilde{A}$ is flat, too. But $B \otimes_A \tilde{A}$ is the limit of the $B \otimes_A A' \to k \in \mathsf{Et}(B \to k)$, where $A' \to k$ runs over $\mathsf{Et}'(A \to k)$. Conclude with [21] VIII, §2, Prop. 5 that $\tilde{B}$ is also the henselization of $B \otimes_A \tilde{A}$ w.r.t. its natural morphism to k, i.e. $B \otimes_A \tilde{A} \to \tilde{B}$ is flat. Now transitivity of flatness and [17] 4.A shows that $\tilde{A} \to \tilde{B}$ is faithfully flat. As A is noetherian, all other rings we use are noetherian, too, and $\tilde{A} \hookrightarrow \hat{A}$, $\tilde{B} \hookrightarrow \hat{B}$ are faithfully flat ([17] 23.L Cor. 1). Look at the factorization $\hat{A} \to \hat{A} \otimes_{\tilde{A}} \tilde{B} \to \hat{B}$. The first arrow is faithfully flat by the base change rule, the second one is completion at the maximal ideal of $\hat{A} \otimes_{\tilde{A}} \tilde{B}$, hence faithfully flat, too. Again, transitivity shows that $\hat{A} \to \hat{B}$ is faithfully flat.

(iii): We have a natural morphism $\tilde{A}/I\tilde{A} \to \tilde{B}$. Since $\tilde{A}$ is henselian and $\tilde{A} \to \tilde{A}/I\tilde{A}$ a finite local morphism, $\tilde{A}/I\tilde{A}$ must be henselian. By the universal property of henselization, there is a morphism $\tilde{B} \to \tilde{A}/I\tilde{A}$, which is easily seen to be the inverse of the morphism of the first sentence.

(iv): If $A \to B$ is unramified at $\mathfrak{q}$, then it is net in a neighborhood of $\mathfrak{q}$. By [21] V, §1 Thm. 1, we can find $A' \to k \in \mathrm{Et}'(A \to k)$ and $B' \to k \in \mathrm{Et}'(B \to k)$ with $B' = A'/I'$. Now (iii) implies $\tilde{B} = \tilde{A}/I\tilde{A}$. The surjectivity of $\tilde{A} \to \tilde{B}$ immediately implies the surjectivity of $\hat{A} \to \hat{B}$. Now assume the latter. Then $A \to \hat{A}$ and $\hat{A} \to \hat{B}$ are unramified, hence there composition is unramified, too. Thus, $A/\mathfrak{p} \to B/\mathfrak{q} \hookrightarrow \hat{B}/\hat{\mathfrak{m}}_B$ is separable and $\mathfrak{p}\hat{B} = \hat{\mathfrak{m}}_B$ (where $\hat{\mathfrak{m}}_A$, $\hat{\mathfrak{m}}_B$ are the maximal ideals of $\hat{A}$, $\hat{B}$ and $\mathfrak{p} := \ker(A \to k)$). Now the faithful flatness of $B \to \hat{B}$ and $\hat{\mathfrak{m}}_B = \mathfrak{q}\hat{B}$ implies $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}$, hence $A \to B$ is unramified at $\mathfrak{q}$.

(v): If $A \to B$ is étale at (and then also around) $\mathfrak{q}$, the definition of the henselization immediately shows that $\tilde{A} \cong \tilde{B}$, which trivially implies $\hat{A} \cong \hat{B}$. Assuming the latter, we conclude as in the proof of (iv) that $A \to B$ is unramified at $\mathfrak{q}$. Now the faithful flatness of $A \to \hat{A}$ and $B \to \hat{B}$ and [17] 4.B show that $A \to B$ is flat, therefore $A \to B$ is étale in a neighborhood of $\mathfrak{q}$.

(vi): Let $a \in \hat{A}$ be an element which is integral over $\tilde{A}$. Then $A' := \tilde{A}[a] \subset \hat{A}$ is a finite local extension of $\tilde{A}$. But the completion of $\tilde{A}$ and A' is $\hat{A}$ for both rings. By (v), $\tilde{A} \to A'$ must be étale. Therefore $A' = \tilde{A}$ and $a \in \tilde{A}$.

■

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