# Ordered set partitions and the 0-Hecke algebra

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## Iwahori-Hecke Algebra

- The symmetric group $\mathfrak{S}_n$ is generated by $s_1, \dots, s_{n-1}$, where $s_i := (i, i+1)$.
- The *(Iwahori-)Hecke algebra* $H_n(q)$ of $\mathfrak{S}_n$ over a field $\mathbb{F}$ is an $\mathbb{F}(q)$-algebra generated by $T_1, \dots, T_{n-1}$ with *quadratic relations* $(T_i + 1)(T_i - q) = 0$ for $i = 1, 2, \dots, n-1$ and *braid relations*
$$\begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \le i \le n-2, \\ T_i T_j = T_j T_i, & |i-j| > 1. \end{cases}$$
- The algebra $H_n(q)$ has an $\mathbb{F}(q)$-basis $\{T_w : w \in \mathfrak{S}_n\}$, where $T_w := T_{i_1} \cdots T_{i_k}$ is well defined for any expression $w = s_{i_1} \cdots s_{i_k}$ with $k$ minimum.
- Set $q = 1$: $H_n(q) \to \mathbb{F}\mathfrak{S}_n$, $T_i \to s_i$, $T_w \to w$, $s_i^2 = 1$, same braid relations.
- When $\mathbb{F} = \mathbb{C}$, Tits showed that $H_n(q) \cong \mathbb{C}\mathfrak{S}_n$ unless $q \in \{0, \text{roots of unity}\}$.
- Set $q = 0$: $H_n(q) \to H_n(0)$, $T_i \to \overline{\pi}_i$, $T_w \to \overline{\pi}_w$, $\overline{\pi}_i^2 = -\overline{\pi}_i$, same braid relations.
- $H_n(0)$ is also generated by $\pi_i := \overline{\pi}_i + 1$ with $\pi_i^2 = \pi_i$ and the same braid relations.

## Representation theory of $\mathfrak{S}_n$

- Every $\mathfrak{S}_n$-module (over $\mathbb{Q}$) is a direct sum of simple/irreducible modules.
- The simple $\mathfrak{S}_n$-modules $S^\lambda$ are indexed by *partitions* $\lambda \vdash n$, where $\lambda = (\lambda_1, \dots, \lambda_k)$ is a decreasing sequence of positive integers whose sum is $n$.
- Representations of $\mathfrak{S}_n$ correspond to symmetric functions via the *Frobenius characteristic map* Frob : $S^\lambda \mapsto s_\lambda$, where $s_\lambda$ is a *Schur function*.
- Symmetric functions form a graded Hopf algebra Sym with a self-dual basis $\{s_\lambda\}$, and Frob is a Hopf algebra isomorphism.

## Representation theory of $H_n(0)$

- A *composition* of $n$, denoted by $\alpha \models n$, is a sequence $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of positive integers whose sum is $n$. It can be identified with its *descent set* $\mathrm{Des}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{\ell-1}\} \subseteq \{1, \dots, n-1\}$.
- Norton showed $H_n(0) = \bigoplus_{\alpha \models n} \mathbf{P}_\alpha$, so every indecomposable projective (simple, resp.) $H_n(0)$-module is isomorphic to some $\mathbf{P}_\alpha$ ($\mathbf{C}_\alpha := \mathbf{P}_\alpha / \mathrm{rad}\, \mathbf{P}_\alpha$, resp.).
- The graded Hopf algebras QSym (*quasisymmetric functions*) and **NSym** (*noncommutative symmetric functions*) have dual bases $\{F_\alpha\}$ and $\{\mathbf{s}_\alpha\}$ indexed by compositions $\alpha$. We have $\mathbf{NSym} \twoheadrightarrow \mathrm{Sym} \hookrightarrow \mathrm{QSym}$.
- Krob and Thibon introduced two Hopf algebra isomorphisms:
  - Ch : $H_n(0)$-modules $\longleftrightarrow$ QSym, $\mathbf{C}_\alpha \mapsto F_\alpha$ (up to composition factors),
  - **ch**: projective $H_n(0)$-modules $\longleftrightarrow$ **NSym**, $\mathbf{P}_\alpha \mapsto \mathbf{s}_\alpha$.

## Coinvariant algebra as an $\mathfrak{S}_n$-module

- The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \dots, x_n]$ by variable permutation.
- The corresponding *invariant algebra* is generated by the *elementary symmetric functions* $e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n)$.
- The *coinvariant algebra* $R_n := \mathbb{Q}[\mathbf{x}_n]/I_n$ is the quotient of $\mathbb{Q}[\mathbf{x}_n]$ by the ideal $I_n := \langle e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n) \rangle$.
- The coinvariant algebra $R_n$ has $\mathbb{Q}$-dimension $n!$ and has various $\mathbb{Q}$-bases constructed by Artin, Garsia–Stanton, and others.
- Chevalley showed that $R_n$ is isomorphic to the regular $\mathfrak{S}_n$-representation.
- Lusztig and Stanley described the *graded* $\mathfrak{S}_n$-module structure of $R_n$ using the major index statistic on standard Young tableaux.

## Coinvariant Algebra as an $H_n(0)$-module

- The algebra $H_n(0)$ acts on $\mathbb{F}[\mathbf{x}_n]$ by the *isobaric Demazure operators*:
$$\pi_i(f) := \frac{x_i f - x_{i+1}(s_i(f))}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[\mathbf{x}_n], \quad 1 \le i \le n-1.$$
- This action has the same invariant/coinvariant algebra as $\mathfrak{S}_n$.
- The first author showed that $R_n := \mathbb{F}[\mathbf{x}_n]/I_n$ is isomorphic to the regular $H_n(0)$-representation with bigraded characteristics
$$\mathrm{Ch}_{q,t}(R_n) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} F_{\mathrm{Des}(w^{-1})}.$$

($q$-grading: *length filtration* of $H_n(0)$; $t$-grading: polynomial degree)

## Ordered set partitions

- Let $\mathcal{OP}_{n,k}$ be the set of *ordered set partitions* of $[n] := \{1, 2, \dots, n\}$ with $k$ blocks. Let $\mathbb{F}[\mathcal{OP}_{n,k}]$ be the vector space with $\mathcal{OP}_{n,k}$ as a basis.
- The symmetric group $\mathfrak{S}_n$ actions on $\mathbb{F}[\mathcal{OP}_{n,k}]$ by permuting $1, 2, \dots, n$.
- The Hecke algebra $H_n(0)$ acts on $\mathbb{F}[\mathcal{OP}_{n,k}]$ by the rule
$$\overline{\pi}_i(\sigma) := \begin{cases} -\sigma, & \text{if } i+1 \text{ appears in a block to the left of } i \text{ in } \sigma, \\ s_i(\sigma) & \text{if } i+1 \text{ appears in a block to the right of } i \text{ in } \sigma. \\ 0, & \text{if } i+1 \text{ appears in the same block as } i \text{ in } \sigma, \end{cases}$$
- Example: $\sigma = (245|6|13)$, $\overline{\pi}_1(\sigma) = -\sigma$, $\overline{\pi}_2(\sigma) = (345|6|12)$, $\overline{\pi}_4(\sigma) = 0$.
- An element of $\mathcal{OP}_{n,k}$ can be written as $\sigma = (w, \alpha)$, where $w \in \mathfrak{S}_n$ and $\alpha \models n$ satisfy $\mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha)$. For example, $(245|6|13) = (245613, (3,1,2))$.
- For $\sigma = (B_1 | \cdots | B_k) = (w, \alpha) \in \mathcal{OP}_{n,k}$, define
$$\mathrm{maj}(\sigma) = \mathrm{maj}(w, \alpha) := \mathrm{maj}(w) + \sum_{i : \max(B_i) < \min(B_{i+1})} (\alpha_1 + \cdots + \alpha_i - i).$$
- Example: $\mathrm{maj}(24|57|136|8) = 4 + (2-1) + (2+2+3-3) = 9$.
- We have $|\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k)$, where $\mathrm{Stir}(n,k)$ is the *(signless) Stirling number of the second kind* counting $k$-block set partitions of $[n]$. Moreover, if $\mathrm{Stir}_q(0,k) := \delta_{0,k}$, $\mathrm{Stir}_q(n,k) := \mathrm{Stir}_q(n-1,k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1,k)$ for $n \ge 1$, and $\mathrm{rev}_q$ reverses the sequence of $q$-coefficients, then
$$\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{maj}(\sigma)} = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k)).$$

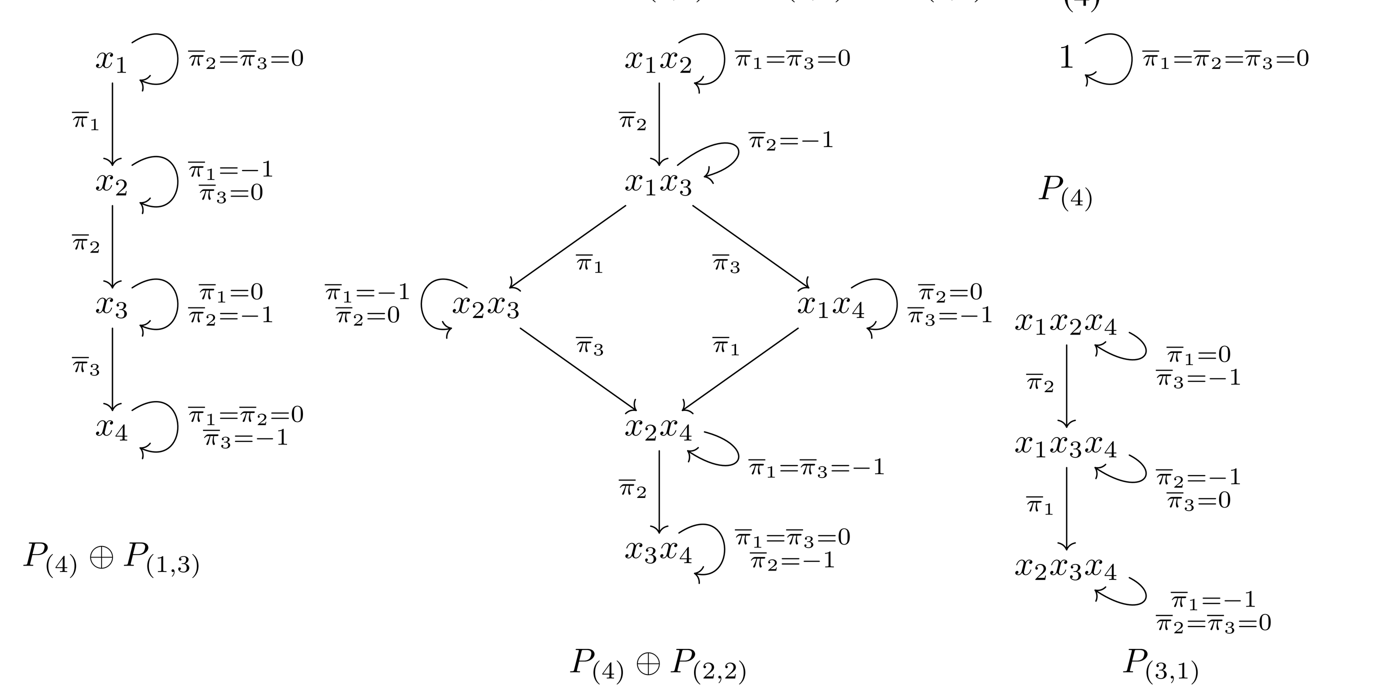
## Generalized Coinvariant Algebra

- Let $k$ and $n$ be two integers with $1 \le k \le n$ throughout the end of the poster.
- Haglund, Rhoades, and Shimozono introduced an $\mathfrak{S}_n$-stable homogeneous ideal of $\mathbb{Q}[\mathbf{x}_n]$ generated by $x_1^k, x_2^k, \dots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n)$.
- The quotient $R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k}$ reduces to the coinvariant algebra $R_n$ if $k = n$.
- The Artin basis and Garsia-Stanton basis for $R_n$ can be generalized to $R_{n,k}$.
- As an ungraded $\mathfrak{S}_n$-module, $R_{n,k}$ is isomorphic to $\mathbb{Q}[\mathcal{OP}_{n,k}]$.
- Haglund, Rhoades, and Shimozono gave an explicit descriptions of the *graded* $\mathfrak{S}_n$-module structure of $R_{n,k}$, generalizing the work of Lusztig–Stanley.
- In general, the ideal $I_{n,k}$ is not $H_n(0)$-stable. To remedy this, replace $x_i^k$ with the *complete homogeneous symmetric function* $h_k(\mathbf{x}_i)$ for all $i$ and get an ideal $J_{n,k}$.
- The span of $x_1^k, \dots, x_n^k$ is isomorphic to the defining representation of $\mathfrak{S}_n$ and the span of $h_k(\mathbf{x}_1), \dots, h_k(\mathbf{x}_n)$ is isomorphic to the defining representation of $H_n(0)$.

$$\begin{array}{ccccccc} 1 & \overset{s_1}{\longleftrightarrow} & 2 & \overset{s_2}{\longleftrightarrow} & \cdots & \overset{s_{n-1}}{\longleftrightarrow} & n \\ x_1^k & \overset{s_1}{\longleftrightarrow} & x_2^k & \overset{s_2}{\longleftrightarrow} & \cdots & \overset{s_{n-1}}{\longleftrightarrow} & x_n^k \end{array} \qquad \begin{array}{ccccccc} 1 & \overset{\pi_1}{\longrightarrow} & 2 & \overset{\pi_2}{\longrightarrow} & \cdots & \overset{\pi_{n-1}}{\longrightarrow} & n \\ h_k(\mathbf{x}_1) & \overset{\pi_1}{\longrightarrow} & h_k(\mathbf{x}_2) & \overset{\pi_2}{\longrightarrow} & \cdots & \overset{\pi_{n-1}}{\longrightarrow} & h_k(\mathbf{x}_n) \end{array}$$

## Our Main Results

- The quotient $S_{n,k} := \mathbb{F}[\mathbf{x}_n]/J_{n,k}$ is a graded $H_n(0)$-module having the same $\mathbb{F}$-dimension and similar $\mathbb{F}$-bases as $R_{n,k}$. Moreover, its Hilbert series is
$$\mathrm{Hilb}(S_{n,k}; q) = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k)) = \sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{maj}(\sigma)}$$
- As an ungraded $H_n(0)$-module, $S_{n,k}$ is projective and isomorphic to $\mathbb{F}[\mathcal{OP}_{n,k}]$ with the following direct sum decomposition, where $N_{\alpha,\mathbf{i}}$ is an $H_n(0)$-submodule of $S_{n,k}$ and $(\alpha, \mathbf{i}) \in A_{n,k}$ means $\alpha = (\alpha_1, \dots, \alpha_\ell) \models n$, $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$, $\alpha_1 > n-k$, and $k - \ell \ge i_1 \ge \cdots \ge i_{n-k} \ge 0 = i_{n-k+1} = \cdots = i_n$.
$$S_{n,k} = \bigoplus_{(\alpha,\mathbf{i}) \in A_{n,k}} N_{\alpha,\mathbf{i}} \cong \bigoplus_{\beta \models n} P_\beta^{\oplus \binom{n-\ell(\beta)}{k-\ell(\beta)}}$$
- The figure below illustrates $S_{4,2} \cong P_{(2,2)} \oplus P_{(1,3)} \oplus P_{(3,1)} \oplus P_{(4)}^{\oplus 3} \cong \mathbb{F}[\mathcal{OP}_{4,2}]$.




- It follows that
$$\mathbf{ch}_t(S_{n,k}) = \sum_{\alpha \models n} t^{\mathrm{maj}(\alpha)} \begin{bmatrix} n - \ell(\alpha) \\ k - \ell(\alpha) \end{bmatrix}_t \mathbf{s}_\alpha \quad \text{and}$$
$$\mathrm{Ch}_{q,t}(S_{n,k}) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} \begin{bmatrix} n - \mathrm{des}(w) - 1 \\ k - \mathrm{des}(w) - 1 \end{bmatrix}_t F_{\mathrm{Des}(w^{-1})}$$
$$= \sum_{(w,\alpha) \in \mathcal{OP}_{n,k}} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w,\alpha)} F_{\mathrm{Des}(w^{-1})}.$$

Setting $q = 1$ and comparing with work of Haglund–Rhoades–Shimozono give
$$\mathrm{Ch}_t(S_{n,k}) = \sum_{Q \in \mathrm{SYT}(n)} t^{\mathrm{maj}(Q)} \begin{bmatrix} n - \mathrm{des}(Q) - 1 \\ k - \mathrm{des}(Q) - 1 \end{bmatrix}_t s_{\mathrm{shape}(Q)} = \mathrm{grFrob}_t(R_{n,k}).$$

## Connections with Macdonald polynomials

- Let $\widetilde{H}_\mu$ be the *modified Macdonald symmetric function* indexed by a partition $\mu$, and let $B_\mu := \sum q^i t^j$, where $(i,j)$ ranges over the coordinates of the cells of $\mu$.
- For $F \in \mathrm{Sym}$, the *delta operator* $\Delta'_F : \mathrm{Sym} \to \mathrm{Sym}$ is defined (using plethystic notation) by $\Delta'_F : \widetilde{H}_\mu \mapsto F[B_\mu(q,t) - 1] \cdot \widetilde{H}_\mu$.
- Delta Conjecture: $\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t)$ where $\mathrm{Rise}(\mathbf{x}; q, t)$ and $\mathrm{Val}(\mathbf{x}; q, t)$ are certain formal power series in $\mathbf{x} = (x_1, x_2, \dots)$.
- The Delta Conjecture becomes the Shuffle Theorem when $k = n$.
- Garsia–Haglund–Remmel–Yoo prove the Delta Conjecture when $q = 0$ or $t = 0$. Combining this with results of the second author and Wilson gives
$$\Delta'_{e_{k-1}} e_n|_{q=0} = \Delta'_{e_{k-1}} e_n|_{t=0,\, q=t} = \mathrm{Rise}_{n,k}(\mathbf{x}; 0, t) = \mathrm{Rise}_{n,k}(\mathbf{x}; t, 0)$$
$$= \mathrm{Val}_{n,k}(\mathbf{x}; 0, t) = \mathrm{Val}_{n,k}(\mathbf{x}; t, 0).$$
- The above symmetric function equals $(\omega \circ \mathrm{rev}_t)(\mathrm{Ch}_t(S_{n,k}))$ by our results and earlier work of Haglund–Remmel–Wilson, where $\omega$ interchanges $e_n$ and $h_n$.