An Invitation to Integration in Finite Terms

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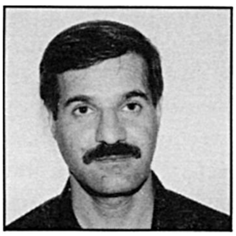

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Students in elementary calculus classes learn that $\ln x$ has an antiderivative that is an elementary function: $\int \ln x\,dx = x \ln x - x + C$. They are naturally curious about how to determine whether a given function can be integrated. For example, how can they know whether the error function $\int e^{-x^2}dx$ is expressible as a finite combination of familiar functions?

The Risch algorithm provides the ultimate answer, for certain functions, to the question of integration in finite terms. However, this algorithm and its proof require advanced techniques that are inappropriate for an elementary calculus class. Still, for determining if a function is integrable in finite terms there are methods that are accessible to those studying calculus. As a bonus, the study of these techniques provides interesting insights into the history of mathematics.

In this paper we show how one can prove many integrals to be nonelementary, and we provide some interesting examples and methods of integration in finite terms. Most of our examples are suitable for the elementary calculus class.

History

The two principal inventors of calculus, Newton and Leibniz, had very distinct approaches to integration [**4**, pp. 266–267]. Newton allowed for infinite series solutions, although he did not consider integration solely in this way. In contrast, Leibniz favored solutions in finite terms.

Newton rejected integration in terms of transcendental quantities like logarithms. His 1704 treatise, *De Quadratura Curvarum*, contains a list of infinite series solutions for integrals of functions like

$$\frac{1}{a+bx} \quad \text{and} \quad \sqrt{a+bx+cx^2}.$$

Newton's avoidance of nonalgebraic forms was very much in the tradition of Descartes, who favored functions with algebraic rather than transcendental representations. Leibniz pursued a different direction, admitting integration using transcendental functions.

Well into the eighteenth century, mathematicians expressed different preferences (finite vs. infinite series) for representations of indefinite integrals. In the nineteenth century, Laplace, Abel, Liouville, Chebyshev, Maximovich, and Roengsberger, among others, worked on the problem of discovering methods to calculate integrals in finite terms. By 1841, Joseph Liouville had developed a theory of integration that settled the question of integration in finite terms for many important cases [**9**], [**11**].

Much of the important work in the twentieth century on integration in finite terms has been based on Liouville's theory. In 1948, Joseph Ritt published *Integration in Finite Terms: Liouville's Theory of Elementary Methods* [**18**], a book that has come to be regarded as the classical account of integration in finite terms [7, p. 199]. In 1946, A. Ostrowski used the idea of field extension to generalize Liouville's theory to a wider class of functions. Ostrowski's work provided the germ of the algebraic approach that finally succeeded in solving the problem of integration in finite terms. M. Rosenlicht, in 1968, published the first purely algebraic version of Liouville's theory [**19**], [**20**]. Finally in 1970, R. H. Risch, building on the work of Liouville, Ritt, Rosenlicht, and Ostrowski, showed that the general problem of integration in finite terms can be reduced to a decidable question in the theory of algebraic functions [**16**], [**17**]. Today, research continues on the topic of integration in finite terms and related questions (see [**2**, pp. 186–210]), which is an area of interest not only to mathematicians, but, because of the algorithmic nature of solutions, to computer scientists as well.

Laplace's Theorem

Our first theorem, which says that a rational function has an antiderivative that is the finite sum of a rational function and the logarithms of rational functions, may be familiar to the reader.

Laplace's theorem (1812). *The integral of a rational function is always an elementary function. In fact, it is either rational or the sum of a rational function and a finite number of constant multiples of logarithms of rational functions.*

Laplace proved this theorem by decomposing the integrand into partial fractions with possibly complex number coefficients [**5**]. The following example illustrates Laplace's theorem.

Example 1.

$$\begin{aligned}\int \frac{(x^2+1)^2+x}{x(x^2+1)}\,dx &= \int x\,dx + \int \frac{1}{x}\,dx + \int \frac{1}{x^2+1}\,dx \\ &= \frac{1}{2}x^2 + \ln|x| + \int\left(\frac{i}{2(x+i)} - \frac{i}{2(x-i)}\right)dx \\ &= \frac{1}{2}x^2 + \ln|x| + \frac{i}{2}\ln\left|\frac{x+i}{x-i}\right| + C.\end{aligned}$$

Liouville's Theorems

In this section we present some of the major theorems that serve as bases for determining whether a certain class of functions can be integrated in finite terms, and we give examples of functions whose nonintegrability in terms of elementary functions can be established using the given theorems. We begin with some definitions.

An *algebraic function* $y = f(x)$ is a root of a polynomial in y whose coefficients are themselves polynomials in x with constant coefficients. Algebraic functions may be presented explicitly, such as

$$f(x) = x^2 - 5x + 1 \quad \text{and} \quad f(x) = \frac{10x^3}{\sqrt{x+1}},$$

or implicitly by an equation such as

$$(x^2+1)[f(x)]^7 - xf(x) + 3 = 0 \quad \text{and} \quad [f(x)]^3 + x = 1.$$

An *elementary function* is a function of one variable that can be constructed, using that variable and constants, by a finite number of repeated operations of addition, subtraction, multiplication, division, composition, raising to powers, taking roots, forming trigonometric functions and their inverses, and taking of exponentials and logarithms. In this paper, we assume real variables and complex coefficients. Some examples of elementary functions are

$$\sin x; \quad \arcsin x; \quad x^2 - 5x + 1;$$
$$x^x = \exp(x \ln x); \quad \tan\left[\cos^2(x^{3/2}+1) + x - 1\right].$$

The basic differentiation rules (product rule, chain rule, etc.) imply that the derivative of an elementary function is also elementary. However, integration is a much harder problem than differentiation; the integral of an elementary function may or may not be elementary. Also, it is possible, for example, for the integral of $f(x) + g(x)$ to be elementary when neither the integral of $f(x)$ nor the integral of $g(x)$ is elementary. Consider $\int (x^x + x^x \ln x)\,dx = x^x + C$: The integral of x^x and the integral of $x^x \ln x$ are not elementary even though the integral of the sum of these two functions is elementary. The class of integrals that are elementary is very small compared with nonelementary integrals.

Liouville's theorem is reminiscent of Laplace's theorem. It says that *if* an algebraic function is integrable in finite terms, its antiderivative is the finite sum of an algebraic function and the logarithms of algebraic functions.

Liouville's 1834 theorem. *If $f(x)$ is an algebraic function of x and if $\int f(x)\,dx$ is elementary, then*

$$\int f(x)\,dx = U_0 + \sum_{j=1}^{n} C_j \ln(U_j)$$

where the C_j's are constants and the U_j's are algebraic functions of x.

The proof of this theorem is based on the fact that the derivative of an exponential term is exponential, and the derivative of a logarithmic term of order higher than one is logarithmic. Thus no exponential terms, and no logarithmic terms except in linear form, can be part of the integral of the algebraic function $f(x)$. Trigonometric functions present no new difficulties because, using complex variables, they can be written as exponential functions. [From Euler's identity, $e^{ix} = \cos x + i \sin x$, it follows that $\sin x = (e^{ix} - e^{-ix})/(2i)$ and $\cos x = (e^{ix} + e^{-ix})/2$.] Consequently, the integral of an algebraic function cannot contain a trigonometric term. But, as we see in Example 1 and the following Example 2, the integral of an algebraic function can be an inverse trigonometric function, which can be written as the logarithm of an algebraic function.

Example 2. $\int 1/\sqrt{1-x^2}\, dx = \arcsin x + C = -i \ln(ix + \sqrt{1-x^2}) + C$. To see why the last equality holds, replace $\sin y$ by x in the identity $\sin y = (e^{iy} - e^{-iy})/(2i)$ to get $e^{2iy} - 2ixe^{iy} - 1 = 0$. The quadratic formula then yields

$$e^{iy} = ix + \sqrt{1-x^2}\,.$$

(For complex numbers the square root is two-valued.) Taking logarithms of each side then gives the required identity.

For a more detailed proof of Liouville's 1834 theorem, see [**18**, p. 21].

In 1835, Liouville generalized this theorem to several variables, and thereby greatly extended the class of functions one can prove to have nonelementary integrals.

Strong Liouville theorem (1835).

(a) *If F is an algebraic function of $x, y_1, \dots, y_m$, where $y_1, \dots, y_m$ are functions of x whose derivatives $dy_1/dx, \dots, dy_m/dx$ are each algebraic functions of $x, y_1, \dots, y_m$, then $\int F(x, y_1, y_2, \dots, y_m)\, dx$ is elementary if and only if*

$$\int F(x, y_1, y_2, \dots, y_m)\, dx = U_0 + \sum_{j=1}^{n} C_j \ln(U_j)$$

where the C_j's are constants, and the U_j's are algebraic functions of $x, y_1, \dots, y_m$.

(b) *If $F(x, y_1, \dots, y_m)$ is a rational function and $dy_1/dx, \dots, dy_m/dx$ are rational functions of $x, y_1, \dots, y_m$, then the U_j's in part* (a) *must be rational functions of $x, y_1, \dots, y_m$.*

The proof of the strong Liouville theorem is basically the same as the proof of the 1834 theorem. For details, see [**18**].

Example 3. The strong Liouville theorem, part (a), applies to the integrand

$$F(x, y_1, y_2, \dots, y_7) = F(x, e^x, \ln x, \exp e^x, \ln(\ln x), \sin x, \cos x, \cos e^x)$$

where F is an algebraic function of its arguments, since

$$\frac{dy_1}{dx} = y_1, \quad \frac{dy_2}{dx} = \frac{1}{x}, \quad \frac{dy_3}{dx} = y_1 y_3, \quad \frac{dy_4}{dx} = \frac{1}{xy_2},$$

$$\frac{dy_5}{dx} = y_6, \quad \frac{dy_6}{dx} = -y_5, \quad \text{and} \quad \frac{dy_7}{dx} = -y_1\sqrt{1-y_7^2}$$

are algebraic functions of $x, y_1, y_2, \dots, y_7$.

Note that the strong Liouville theorem, part (b), does not apply to F because the derivative of its last argument is not a rational function of $x, y_1, y_2, \ldots, y_7$. However, if we choose

$$G(x, y_1, y_2, \ldots, y_6) = G(x, e^x, \ln x, \exp e^x, \ln(\ln x), \sin x, \cos x)$$

where G is a rational function of its arguments, then part (b) of the strong Liouville theorem applies to the integrand G. In this case the integral of G is elementary if and only if

$$\int G(x, y_1, y_2, \ldots, y_6)\, dx = U_0 + \sum_{j=1}^{n} C_j \ln(U_j)$$

where C_j's are constants and U_j's are rational functions of $x, y_1, y_2, \ldots, y_6$.

As an exercise, determine whether the following arguments meet the criteria for part (a) and for part (b) of the strong Liouville's theorem.

1) $F(x, \sin x)$ 2) $F(x, \sin x, \cos x)$ 3) $F(x, e^x, \sin x)$

4) $F(x, e^x, \ln x)$ 5) $F(x, e^x, \sin x, \cos x, \ln x)$

Answers: 1) Part (a) only; 2) both parts; 3) part (a) only; 4) both parts; 5) both parts.

The strong Liouville theorem was the basis of much of the work done on the problem of integration in finite terms during the twentieth century. Its value rests on the fact that it gives a condition that is necessary for the integrability in finite terms of any one of a rather large class of functions. Using this theorem, we can determine whether certain functions belong to the class of functions that *cannot* be integrated in finite terms.

To better appreciate the strong Liouville theorem, it is instructive to formulate and consider special cases. The first special case we consider is

$$\int f(x) e^{g(x)}\, dx$$

where $f(x)$ and $g(x)$ are rational functions. This integral is of the form required by part (b) of the strong Liouville theorem where $F(x, y_1) = xy_1$ and $y_1 = e^{g(x)}$. The integrand satisfies the assumptions of the strong Liouville theorem, part (b), since

$$\frac{dy_1}{dx} = \frac{d}{dx}(e^{g(x)}) = g'(x) e^{g(x)} = g'(x) y_1$$

is a rational function of x and y_1. The theorem therefore asserts that if $\int f(x) e^{g(x)}\, dx$ is elementary, then it must be the sum of a rational function of $(x, e^{g(x)})$ and a finite number of logarithms of such functions. Further analysis shows that in fact it must be of the form $R(x) e^{g(x)} + C$ for some rational function $R(x)$. The formal proof of this last statement uses advanced mathematics. We refer the interested reader to [**9**, p. 114] which uses the idea of analytic branches, or to [**18**, p. 47] which uses ideas from the theory of differential fields. Conversely,

$$\frac{d}{dx}\left[R(x) e^{g(x)} + C\right] = f(x) e^{g(x)}.$$

The following special case of the strong Liouville theorem results.

Strong Liouville theorem (special case, 1835). *If $f(x)$ and $g(x)$ are rational functions with $g(x)$ nonconstant, then $\int f(x)e^{g(x)}\,dx$ is elementary if and only if there exists a rational function $R(x)$ such that $f(x) = R'(x) + R(x)g'(x)$.*

We are now in a position to prove that certain integrals are nonelementary using this special case of the strong Liouville theorem.

Example 4. $\int x^{2n}e^{ax^2}\,dx$, for n an integer, is nonelementary for $a \neq 0$, because $x^{2n} = R'(x) + 2axR(x)$ has no rational solution $R(x)$ in the field of rational functions over $\mathbb{C}$ (proof below.) For $n = 0$ and $a = -1$, this is the error function mentioned at the beginning of this article.

Proof that $x^{2n} = R'(x) + 2axR(x)$ has no rational solution. Assume $x^{2n} = R'(x) + 2axR(x)$, where $R(x) = p(x)/q(x)$, for $p(x)$ and $q(x)$ relatively prime polynomials. Since

$$R'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q^2(x)},$$

we get

$$x^{2n}q^2(x) = p'(x)q(x) - p(x)q'(x) + 2axp(x)q(x). \tag{1}$$

This can be written as

$$\left[x^{2n}q(x) - p'(x) - 2axp(x)\right]q(x) = -p(x)q'(x). \tag{2}$$

If x_0 is a zero of multiplicity $k \geq 1$ of $q(x)$, then x_0 is a zero of the left side of (2) with multiplicity greater than or equal to k. But since $p(x)$ and $q(x)$ are relatively prime, x_0 is a zero of multiplicity $k-1$ of the right side of (2). This is a contradiction, so $q(x)$ has no zeros; i.e., $q(x)$ is a constant.

Without loss of generality, let $q(x) \equiv 1$. Then (1) becomes

$$x^{2n} = p'(x) + 2axp(x). \tag{3}$$

We will show (3) has no solution by comparing coefficients. Since p is a polynomial in x, then for $n \geq 1$, the degree of $p(x)$ must be $2n-1$. Let $p(x) = \sum_{j=0}^{2n-1} c_j x^j$. Substituting this in (3) we get

$$\begin{aligned} x^{2n} &= \sum_{j=0}^{2n-1} jc_j x^{j-1} + \sum_{j=0}^{2n-1} 2ac_j x^{j+1} \\ &= \sum_{j=0}^{2n-2} (j+1)c_{j+1}x^j + \sum_{j=1}^{2n} 2ac_{j-1}x^j \\ &= c_1 + \sum_{j=1}^{2n-2} \left[(j+1)c_{j+1} + 2ac_{j-1}\right]x^j + 2ac_{2n-2}x^{2n-1} + 2ac_{2n-1}x^{2n}. \end{aligned}$$

We must conclude that $c_1 = 0$, $c_{2n-2} = 0$, $2ac_{2n-1} = 1$, and $(j+1)c_{j+1} + 2ac_{j-1} = 0$ for $j = 1, 2, \ldots, 2n-2$. These last equations imply that since $c_1 = 0$ then $c_3 = 0$, $c_5 = 0, \ldots, c_{2n-1} = 0$. But this contradicts the third equation: $c_{2n-1} = 1/(2a)$. So there is no polynomial $p(x)$ satisfying (3) for $n \geq 1$. If $n \leq 0$ then clearly (3) does not have a polynomial solution. Hence there is no rational function $R(x)$ satisfying the given differential equation. ∎

For a shorter proof that uses concepts from complex analysis, see **[18]**.

Example 5. $\int x^{-n} e^{cx}\, dx$, for n a positive integer and c a nonzero constant, is nonelementary since $x^{-n} = R'(x) + cR(x)$ has no solution $R(x)$ in the field of rational functions over $\mathbb{C}$.

Proof that $x^{-n} = R'(x) + cR(x)$ has no rational solution. Assume $x^{-n} = R'(x) + cR(x)$, where $R(x) = p(x)/q(x)$, for $p(x)$ and $q(x)$ relatively prime polynomials. Then $1/x^n = (p'q - q'p)/q^2 + cp/q$ which can be written as

$$q(-q + x^n p' + cx^n p) = x^n pq'. \tag{4}$$

Assume that the degree of q is positive. Let x_0 be a zero of $q(x)$ with multiplicity r. If $x_0 \neq 0$ then x_0 is a zero of the left side of (4) of multiplicity at least r, while x_0 is a zero of multiplicity $r - 1$ of the right side of (4). This is a contradiction. Thus x_0 must be zero and $q(x) = kx^r$ for some nonzero constant k. Substituting $q = kx^r$ into (4) we get

$$x^r(-kx^r + p'x^n + cpx^n) = rpx^{n+r-1}. \tag{5}$$

Case 1: If $n < r$ then zero is a root of multiplicity at least $r + n$ for the left side of (5) while it is a root of multiplicity $n + r - 1$ for the right side of (5). This is a contradiction.
Case 2: If $n \geq r$ and $n \neq r + 1$ then zero is a root of multiplicity $2r$ for the left side of (5) while it is a root of multiplicity $n + r - 1$ for the right side of (5). So $2r = n + r - 1$, again a contradiction.
Case 3: If $n = r + 1$ then (5) reduces to $xp' = k + rp - cxp$. But the two sides of this equation are polynomials of different degrees, another contradiction.

So our assumption that q has positive degree is no longer valid. Thus q must be a constant. Without loss of generality, let $q \equiv 1$. Then (4) becomes

$$x^n p' + cx^n p = 1. \tag{6}$$

No polynomial $p(x)$ satisfies (6), since n is positive. Consequently $x^{-n} = R'(x) + cR(x)$ has no rational solution. ■

Examples 4 and 5 treat two special cases of integrals of the form

$$\int x^n e^{ax^m}\, dx \tag{7}$$

where a is a nonzero constant and m and n are integers. In fact, using the Euler or hyperbolic identities, many other integrals such as

$$\int x^n \cos(ax^m)\, dx, \quad \int x^n \cosh(ax^m)\, dx, \quad \text{and} \quad \int x^n \sin^k(ax^m)\, dx$$

can be expressed as sums of integrals in the form (7). For example,

$$\int x^n \cos(ax^m)\, dx = \mathrm{Re}\left[\int x^n e^{iax^m}\, dx\right]$$

and

$$\int x^n \cosh(ax^m)\, dx = \frac{1}{2}\int x^n e^{ax^m}\, dx + \frac{1}{2}\int x^n e^{-ax^m}\, dx.$$

The interested reader can find the necessary condition(s) on m and n that will make the integral (7) nonelementary. Once we see that an integral cannot be expressed in terms of a finite number of elementary functions, we can view it as a new transcendental (i.e., nonalgebraic) function. For example, $f(x) = \int e^{x^3} dx$ is transcendental, as there is no rational function $R(x)$ such that $1 = R'(x) + 3x^2 R(x)$. The interested reader should verify this.

Many integrals can be reduced to the special forms discussed in Examples 4 and 5, by a change of variable, by applying integration by parts, or by separating the real and imaginary parts of complex-valued functions. For instance, the following examples are shown to be nonelementary by reducing them to $\int t^{2n} e^{at^2} dt$, n an integer, $a \neq 0$ (Example 4) by the indicated change of variable.

Example 6. $\int \sqrt{\ln x}\, dx = \int 2t^2 e^{t^2} dt$, where $t^2 = \ln x$.

Example 7. $\int \frac{1}{\sqrt{\ln x}} dx = \int 2e^{t^2} dt$, where $t^2 = \ln x$.

Example 8. $\int \frac{e^{ax}}{\sqrt{x}} dx = \int 2e^{at^2} dt$, where $t^2 = x$.

The following examples are shown to be nonelementary by Example 5:

Example 9. $\int e^{e^x} dx = \int \frac{e^t}{t} dt$, where $t = e^x$.

Example 10. $\int \frac{1}{\ln x} dx = \int \frac{e^t}{t} dt$, where $t = \ln x$.

Example 11. $\int \ln(\ln x)\, dx = x \ln(\ln x) - \int \frac{1}{\ln x} dx$. (Use integration by parts and Example 10.)

Example 12. $\int \frac{\sin x}{x} dx = \operatorname{Im}\left(\int \frac{e^{ix}}{x} dx \right)$. (Use the Euler identity, and notice that if the integral of $f(x)$ is elementary, then both its real and imaginary parts are elementary.)

We next consider another special case of the strong Liouville theorem, where the integrand has the form $f(x) \ln x$, for a rational function $f(x)$. Set $y_1 = \ln x$ and $F(x, y_1) = f(x) y_1$. Notice that F is a rational function of its arguments and $dy_1/dx = 1/x$ is a rational function of x. By the strong Liouville theorem, part (b), if the integral of F is elementary, then

$$\int f(x) \ln x\, dx = U_0(x, \ln x) + \sum_{j=1}^{n} C_j \ln\big(U_j(x, \ln x)\big)$$

where all the U_j's are rational functions of their arguments. Differentiating both sides, we get

$$f(x) \ln x = \frac{d}{dx}\left[U_0(x, \ln x) + \sum_{j=1}^{n} C_j \ln\big(U_j(x, \ln x)\big) \right]. \tag{8}$$

Consider the Taylor's expansion of $U_0(x, \ln x)$ about zero for its second argument:

$$U_0(x, \ln x) = U_0(x,0) + D_2U_0(x,0)\ln x + \frac{1}{2!}D_2^2U_0(x,0)(\ln x)^2 + \cdots + \frac{1}{n!}D_2^nU_0(x,0)(\ln x)^n + \frac{1}{(n+1)!}D_2^{n+1}U_0(x,\xi)(\ln x)^{n+1} \quad (9)$$

where ξ is between 0 and $\ln x$, and $D_2^kU_0$ denotes the kth partial derivative of U_0 with respect to its second variable. On the left-hand side of (8), $\ln x$ occurs only in linear form multiplied by a rational function. It is not hard to see from (8) and (9) that then $U_0(x, \ln x)$ must be of the form

$$W(x) + V(x)\ln x + U(x)(\ln x)^2$$

where U, V, and W are rational functions of x (i.e., terms of degree higher than two in the Taylor series (9) must vanish). With a little more work, we see that the U_j's must be rational functions of x only; in fact

$$\sum_{j=1}^{n} C_j \ln\big(U_j(x, \ln x)\big) = \sum_{j=1}^{M} b_j \ln(x - a_j)$$

where the b_j's and a_j's are constants. Therefore

$$\int f(x)\ln x\,dx = U(x)(\ln x)^2 + V(x)\ln x + W(x) + \sum_{j=1}^{M} b_j \ln(x - a_j) \quad (10)$$

Differentiating the right-hand side and comparing it with the integrand, we get

$$U'(x) = 0, \quad \frac{2U(x)}{x} + V'(x) = f(x), \quad \frac{V(x)}{x} + W'(x) + \sum_{j=1}^{M} \frac{b_j}{x - a_j} = 0.$$

Integrating the first and third of these equations to find $U(x)$ and $W(x)$, and substituting the results into (10), gives

$$\int f(x)\ln x\,dx = \frac{C}{2}(\ln x)^2 + V(x)\ln x - \int \frac{V(x)}{x}\,dx$$

where C is a constant. By Laplace's theorem the integral on the right-hand side is elementary since $V(x)$ [and hence $V(x)/x$] is a rational function. This special case of the strong Liouville theorem was obtained by G. H. Hardy [**5**, p. 60], and may be stated as follows [giving the rational function $V(x)$ a new name, $g(x)$].

Liouville–Hardy theorem (1905). *If $f(x)$ is a rational function, then $\int f(x)\ln x\,dx$ is elementary if and only if there exists a rational function $g(x)$ and a constant C such that $f(x) = C/x + g'(x)$.*

We still need to show that if $f(x) = C/x + g'(x)$, the integral $\int f(x)\ln x\,dx$ is elementary. But this is easy. Integration by parts on the second term in the integral of $(C/x + g'(x))\ln x$ gives

$$\int \left(\frac{C}{x}\ln x + g'(x)\ln x\right)dx = \frac{C}{2}(\ln x)^2 + g(x)\ln x - \int \frac{1}{x}g(x)\,dx.$$

The last term, $\int(1/x)g(x)\,dx$, is elementary by Laplace's theorem since the integrand is a rational function.

The Liouville–Hardy theorem provides us with another test for integrability in finite terms. Consider this example:

Example 13. $\int \frac{\ln x}{x-a}\,dx$, where a is a nonzero constant, is nonelementary.

Proof. $1/(x-a) = C/x + g'(x)$ implies $g(x) = \ln(x-a) - C\ln x + c$, which is not a rational function of x for any value of C. ■

We can generalize Example 13 to treat

$$f(x) = \prod_{j=1}^{N} (x - a_j)^{-1}$$

for distinct nonzero a_j's. Thus, decomposing by partial fractions, it can be shown that

$$\int \left[\prod_{j=1}^{N} (x - a_j)^{-1} \right] \ln x\,dx$$

is nonelementary, since

$$\sum_{j=1}^{N} b_j \ln(x - a_j) - C\ln x + c$$

is not a rational function for any constants C and c. Taking, for example, $a_i = i$ and $a_2 = -i$, we have that

$$\int \frac{\ln x}{x^2+1}\,dx$$

is nonelementary. Once we know this, we can prove that $\int(\operatorname{arcsec} x)^2\,dx$ is nonelementary. Using integration by parts,

$$\begin{aligned}\int(\operatorname{arcsec} x)^2\,dx &= x(\operatorname{arcsec} x)^2 - 2\int \frac{\operatorname{arcsec} x}{\sqrt{x^2-1}}\,dx \\ &= x(\operatorname{arcsec} x)^2 - 2\int t\sec t\,dt \\ &= x(\operatorname{arcsec} x)^2 + 4\int \frac{\ln s}{s^2+1}\,ds,\end{aligned}$$

with the change of variables $t = \operatorname{arcsec} x$ and $s = e^{it}$.

Note that a common thread in the special cases of the strong Liouville theorem is that the given integral is elementary if and only if a certain associated differential equation has a rational solution.

Chebyshev's Theorem

For what choices of k (k a rational number) is the arc length of $y = x^k$ integrable in finite terms? To answer this question, we turn to a result of P. L. Chebyshev, whose work in the area of integration of algebraic functions was closely associated with the work of Abel and Liouville. While Liouville focused on general algebraic functions, Chebyshev dealt with specific forms of algebraic functions. In an 1853 paper, Chebyshev gave a complete solution for the logarithmic part of

$$\int \frac{f(x)}{g(x)} [h(x)]^{-1/n}\, dx$$

where f, g, and h are polynomials in x and n is a positive integer [**1**]. [See part (a) of the strong Liouville theorem.] The following binomial-type integral theorem of Chebyshev, which we can use to solve the arc length problem stated above, generalizes an earlier result of Newton.

Chebyshev's theorem (1853). *If p, q, and r are rational numbers and a, b are real numbers with $a, b, r \neq 0$, then $\int x^p(a + bx^r)^q\, dx$ is elementary if and only if at least one of $(p+1)/r$, q, or $(p+1)/r + q$ is an integer.*

Proof. Proof of sufficiency (due to Goldbach and Euler) proceeds from letting $bx^r = at$ and neglecting the constant factor. The integral reduces to $\int t^m(1+t)^q\, dt$, where $m = (p+1)/r - 1$, which is a rational number.

i) If $(p+1)/r$ (and hence m) is an integer and $q = h/k$ where h is an integer, make the substitution $1 + t = u^k$.
ii) If q is an integer and $m = j/k$ where j and h are integers, make the substitution $t = u^k$.
iii) If $(p+1)/r + q$ (and hence $m + q$) is an integer, and $q = h/k$, make the substitution $1 + t = tu^k$.

In each case $\int t^m(1+t)^q\, dt$ transforms into an integral whose integrand is a rational function of u. Hence, it is elementary by Laplace's theorem. The proof of necessity (due to Chebyshev) is more involved and uses the idea of analytic branches. For this we refer the interested reader to [**18**, pp. 37–39]. ■

Example 14. $\int \sqrt[3]{1 + x^2}\, dx$ is nonelementary by Chebyshev's theorem. Here, $p = 0$, $r = 2$, and $q = 1/3$, so $(p+1)/r$, q, and $(p+1)/r + q$ are all nonintegral.

Chebyshev's theorem provides tests for determining whether the integrals for arc length and for the surface area of solids of revolution of functions of the form x^k, k rational, are elementary. Examples 15 and 16 describe these tests.

Example 15. Consider the arc length of the graph of $f(x) = x^k$, given by $\int \sqrt{1 + k^2x^{2k-2}}\, dx$. This integral is elementary if and only if either $1/(2k-2)$ or $1/(2k-2) + 1/2$ is an integer, where $k \neq 1$. Thus, the related arc length integral for $f(x) = x^k$ is elementary if and only if $k = 1$ or $k = 1 + 1/n$, where n is an integer. It follows that, for example, $\int \sqrt{1 + x^3}\, dx$ and $\int \sqrt{1 + x^{-4}}\, dx$ are nonelementary integrals. [This last integral is the arc length integral for $f(x) = 1/x$.]

Example 16. A similar calculation can be performed for integrals representing the area of the surface obtained by revolving the graph of $f(x) = x^k$ about the x-axis. These integrals are elementary if and only if $k = 1$ or $k = 1 + 2/n$, where n is an integer.

Example 17. $\int\sqrt{\sin x}\, dx$ and $\int\sqrt{\cos x}\, dx$ are nonelementary by Chebyshev's theorem. This can be seen by a change of variable. We can, for example, use $\sin x = u$ to get $\int\sqrt{\sin x}\, dx = \int u^{1/2}(1-u^2)^{-1/2}\, du$.

In fact, if m and n are integers, one can easily prove the following [**23**]:

1. $\int(1-x^n)^{1/m}\, dx$ is elementary if and only if $m = \pm 1$, or $n = \pm 1$, or $m = n = 2$, or $m = -n$.

2. $\int(\sin x)^m(\cos x)^n\, dx$ is elementary if and only if m is odd, or n is odd, or both m and n are even. Since these are all possible cases, the integral is elementary for all integral m and n.

Example 18. $\int\sqrt{\tan x}\, dx$ is elementary by Chebyshev's theorem. (Let $u^2 = \tan x$.)

Example 19. We can even use Chebyshev's theorem to prove *Fermat's Last Theorem for polynomial functions*: *If n is an integer greater than 2 then there are no polynomial functions, $p(t)$, $q(t)$, and $r(t)$, where p/r and q/r are nonconstant rational functions, such that $[p(t)]^n + [q(t)]^n = [r(t)]^n$.*

Proof. It is sufficient to show that for $n > 2$, there are no nonconstant rational functions, $f(t)$ and $g(t)$ such that $[f(t)]^n + [g(t)]^n = 1$. If there are such rational functions $f(t)$ and $g(t)$, then $g(t)f'(t)$ is a rational function, and its integral is elementary by Laplace's theorem. Consider

$$\int g(t)f'(t)\, dt = \int\left(1-[f(t)]^n\right)^{1/n} f'(t)\, dt = \int(1-f^n)^{1/n}\, df.$$

By Chebyshev's theorem, the integral on the right side of the equation is elementary only if $1/n$ or $2/n$ is an integer. This is not possible since $n > 2$. ■

Integrals of Inverse Functions

Our final result is an interesting application of integration by parts. Let $f(x)$ and $f^{-1}(x)$ be inverses of one another on some closed interval $[a, b]$. Then, using integration by parts, we get

$$\begin{aligned}\int f(x)\, dx &= xf(x) - \int xf'(x)\, dx\\ &= xf(x) - \int f^{-1}(f(x))f'(x)\, dx\\ &= xf(x) - G(f(x))\end{aligned}$$

where $G(x) = \int f^{-1}(x)\, dx$. The origin of this observation is not known to us. It was known to Liouville in writing his 1841 paper on the Riccati equation [**10**, p. 4], and it also appears in Parker [**14**], Staib [**22**], and a recent note by Key [**8**]. We state it as follows.

Integrals of inverse functions theorem.

(*a*) *If* f *and* f^{-1} *are inverses of one another on some closed interval then* $\int f(x)\,dx = xf(x) - G(f(x))$, *where* $G(x) = \int f^{-1}(x)\,dx$.

(*b*) *If* f *and* f^{-1} *are elementary functions over some closed interval, then* $\int f(x)\,dx$ *is elementary if and only if* $\int f^{-1}(x)\,dx$ *is elementary.*

The following examples illustrate this theorem.

Example 20. $\int\sqrt{\ln x}\,dx$ is nonelementary since the integral of the inverse function of its integrand, $\int e^{x^2}\,dx$, is nonelementary by our first special case of the strong Liouville theorem. See Example 4 and compare Example 6.

Example 21. $\int 1/\ln x\,dx$ is nonelementary since $\int e^{1/x}\,dx$ is nonelementary by our first special case of the strong Liouville theorem. (See Example 5 and use the change of variable $u = 1/x$ to transform the integral into $\int -u^{-2}e^{u}\,du$.)

Example 22. $\int f(x)\,dx$ is elementary when, for example, $y = f(x)$ is a solution of any of the following equations: $y - e^{y} = x$, $y^5 - xy = x + 1$, or $ye^{y} = x$.

Example 23. By Example 5, $\int f(x)\,dx$ is nonelementary when, for example, $y = f(x)$ is a solution of $e^{y} = xy$.

Example 24. By Example 4, $\int f(x)\,dx$ is nonelementary when, for example, $y = f(x)$ is a solution of $x = y^2e^{y^2}$.

The integrals of inverse functions theorem not only establishes a test of integrability in finite terms, it also provides a practical method for integrating certain functions by means of their inverses.

Example 25. $\int\sqrt{x/(1-x)}\,dx$ is elementary by Chebyshev's theorem. To evaluate this integral we use the formula $\int f(x)\,dx = xf(x) - G(f(x))$ obtained earlier in deriving the theorem. We get

$$\begin{aligned} G(x) &= \int f^{-1}(x)\,dx = \int \frac{x^2}{1+x^2}\,dx \\ &= \int\left(1 - \frac{1}{1+x^2}\right)dx = x - \arctan x + C. \end{aligned}$$

Thus

$$\int\sqrt{x/(1-x)}\,dx = (x-1)\sqrt{x/(1-x)} + \arctan\left(\sqrt{x/(1-x)}\right) + C.$$

Conclusion

We have reviewed several theorems that can be used to determine whether a wide variety of functions can be integrated in finite terms. Many of these results are accessible to students of calculus, and we recommend that instructors introduce them to their students. The theory of integration in finite terms and the methods we have described can be used to show calculus students that mathematics is an evolving discipline, with a history and a future (see e.g. [**21**] and [**2**, pp. 186–210]), and that results determined in one century can provide direction for research and useful techniques in centuries that follow. We include a reference list that should serve as a good starting point for future reading.

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Lead Kindly Light

A teacher in Sumatra, enraged by complaints from his pupils thay they did not understand his mathematics lesson, beat two children unconscious and injured thirteen others.

The London Times (Australia), March 23, 1981
Contributed by Kay Wagner,
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