DEFINING BASIC KINDS OF PROPERTIES

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Abstract: This paper follows in Pavel Tichý's concept of distinguishing between trivial (i.e. constant) and non-trivial properties. This classification has been extended by Pavel Cmorej who distinguished two kinds of non-trivial properties, namely purely empirical and partly essential (which are partly empirical) properties (partly essential property is essential for certain individual(s), but that it is not for other(s)). The present study provides rigorous formal definitions of trivial / non-trivial, essential / non-essential, and purely empirical / partly essential / purely essential properties with respect to the possible partiality of these properties. In addition, a definition of trivially void properties completing the classification of the kinds of properties to form a quadruplet is proposed. Furthermore, the introduction of the concept of accidental properties gives rise to another (yet not equivalent) quadruplet: purely accidental / partly essential (i.e. partly accidental) / purely essential / void properties.

Pavel Tichý, a logician who developed a specific intensional logic, namely transparent intensional logic (hereafter TIL), used his logic not only for a logico-semantic analysis of natural language but he also applied it to problems in several philosophical areas. The field we are concerned with in this paper can be called 'the logic of properties'. Intensional logic may be viewed as a conceptual tool helping to develop intensional metaphysics. According to its main assumption, an individual can instantiate various properties. Properties are modelled as functions from possible worlds and have classes of individuals as their values (extensions). Since possessing of a property by an individual is what constitutes a fact, a precise conceptual handling of properties is of great importance for the core of metaphysics.

Tichý himself used some classifications of properties of his own but he did not write a systematic paper on it (his key classification emerged from his comments on the doctrine of bare individuals). Pavel Cmorej, a follower of Tichý, paid close attention to properties and defined other kinds of them. Naturally, properties can be classified according to various criteria but only some of these have been discussed by the followers of Tichý. The topic is therefore still open for further development. The aim of this paper is to summarize and extend up-to-date knowledge of this area and offer exact definitions of the kinds of properties (with respect to partiality), which is a necessary step for further investigations. The conceptual construction is based on a few simple logical concepts like identity, connectives and quantifiers. The gradual involvement in the question helps us to guarantee the adequacy of proposed definitions. The whole construction culminates in the definitions of three related categories of properties (which are further divided): trivial / non-trivial properties (i.e.

properties with constant or with non-constant range of extensions), essential / non-essential properties (i.e. properties which are, or which are not, necessarily possessed by a certain individual), and purely empirical / partly essential / purely essential / trivially void properties (i.e. properties with entirely contingent extensions, properties possessed by some individuals necessarily but by others contingently, properties instantiated by an individual necessarily, properties not instantiable by any individual). Special attention is paid to partly essential properties which have already been investigated by Cmorej.[1] The two kinds of void properties and accidental / non-accidental properties represent a new contribution extending Cmorej's classification. They do so not only in completing the originally triple division into the above mentioned quadruplet of kinds of properties because there is also another quadruplet conceivable. This quadruplet divides properties into purely accidental / partly essential (i.e. partly accidental) / purely essential / void properties (now non-trivially void properties are not covered by the first category as in the preceding quadruple, but by the last one).

TRANSPARENT INTENSIONAL LOGIC

Natural language by means of which we formulate our theories was not originally designed for such artificial service. It was spontaneously developed by ordinary people for easy communication of facts without going into subtleties we need to sort out. The danger of natural language ambiguity for philosophers' theories was stressed by Francis Bacon. Subsequently, Georg Wilhelm Leibniz promoted the idea of 'conceptual script', an instrument designed to help theoreticians to solve their abstract questions. This movement was joined by Gottlob Frege, who developed the first modern 'shorthand' writing. Nevertheless, Frege's original proposal has to be modified because predicate logic cannot sufficiently handle items within realistic empirical framework, which is investigated by intensional metaphysics.

Below we are using the notation of transparent intensional logic developed by Tichý at the very beginning of 1970th (see [Tichý 2004], [Tichý 1988]). TIL can be characterized as a higher order intensional logic with a temporal parameter. It is higher order logic for it quantifies not only over individuals but also over properties of individuals or over properties of properties, etc. Predicate logic can be viewed as a calculus whose (model-theoretic) interpretation is based on objects belonging to the category of *individuals* (marked ι), or *truth-values* (marked o). TIL also adopts atomic categories of *possible worlds* (ω) and real numbers

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[1] The author is happy to express here his warmest congratulations to Pavel Cmorej on the occasion of his 70th birthday and is pleased to dedicate this paper to him.

(serving, inter alia, for the representation of *time-moments*; collection τ).[2] The term intensional does not mean that TIL lacks good features of predicate logic such as the functional principle of compositionality, etc. In the philosophical background of TIL the idea of a fixed domain of individuals has been adopted (the idea of varying domains seems to be based on a mistaken conflation of individuals with s.c. individual offices). Clearly, possible worlds are individuated by different distributions of attributes through a definite colleciton of individuals – worlds differ in these distributions, not in having distinct individuals.[3]

TIL can be easily adopted not only for its suitable incorporation of predicate logic (without several disadvantages of other intensional logics), but also for its foundation in the typed λ-calculus. The λ-calculi comprise, over and above constants and variables, only two kinds of forming terms. The first one is *application* of a function to an argument, schematically [F $A_1...A_n$] (where F is a certain function and the sequence $A_1...A_n$ is an *n*-tuple serving as an argument of appropriate type). The second one is λ-*abstraction* consisting in a closing an open application such as [F x] (being open, its results depends on particular valuations for x) by an abstraction over particular valuations for x and by a completion into a term generating the function itself, λx [F x] (btw. this term is η-reducible to F).

A characteristic feature of TIL is formal capturing of propositions by means of explicit use of possible world variable w. A propositional matrix is then of form λw [...w...], i.e. if not η-reduced, all terms standing for functions from possible worlds are λ-abstractions abstracting over possible worlds. Therefore, the difference between belonging of an individual I_1 to class C – which is entirely a non-contingent matter – and a typical contingent fact such as that I_1 is (an) F (where "F" is a property of individuals) will be explicitly manifested by the difference between the formula [C I_1] on the one hand and the formula λw [[F w] I_1] on the other hand. A term such as λw [[F w] I_1] stands for the following 'recipe': to any value of the variable w assign the result of [[F w] I_1]; the particular results of [[F w] I_1] are obtained by executing of [F w], namely by taking the property "F" and checking its extension in particular world w, and by an ascertaining whether I_1 is, or is not, in the respective extension (i.e. a certain class) of "F". In other words, we cannot directly apply "F" to I_1; instead, we need to carry out a descent to the value of "F" in a particular world.

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[2] The temporal factor shall be ignored throughout the paper to provide a simpler consideration of properties and concise records of constructions.
[3] There is really no approval of the inhabitation of worlds by individuals: possible worlds are not collections of particular things, but collections of state-of-affairs; i.e. individuals and possible worlds are categorically distinct entities.

The handy notation sketched above can aptly distinguish the cases when we speak about the property itself – F does not occur as applied to a possible world variable – and when we need to handle the extension of it – F does occur as applied to a possible world variable. The same holds for s.c. individual offices which are modelled as functions from possible worlds to individuals.[4] When we say, for example, that the individual office "the U.S. president" is a remarkable political post, we talk about that office (sign it 'A' and the property 'RPP'). Within our conceptual script we will write it simply as λw [[RPP w] A] (here A stands in the supposition *de dicto* in Tichý's sense). On the other hand, when we use the description 'the U.S. president' to pick out whichever holder of that office, let us say George W. Bush, to attribute to him that he is, for example, a man, (which is the most typical usage of descriptions), then A occurs in application to w, i.e. λw [[M w] [A w]] (here A stands in the supposition *de re* in Tichý's sense).

Nonetheless, TIL is not only an intensional logic but it has also a hyperintensional stage. Its hyperintensional level enables us to explicate suitably notions such as concept or thought. A concept, or more evidently a thought, is often conceived as structured. The compound concept that Alan is a man consists in a certain structure combining certain basic items, namely Alan and a man. Standard intensional explication of propositions does not work here for a proposition is only a set of possible worlds, thus no proposition can really involve Alan or a man.[5] Thoughts are more fine-grained than propositions and concepts are more fine-grained than non-propositional intensions. Significantly, TIL offers a rigorous tool for their explication. The λ-terms like [+ 2 3] or [$\sqrt{}$ 25] are usually understood as standing directly for the result, the number 5. The proposal of TIL, however, is to understand these terms as not standing just and only for the result, but as standing for the structured and unique calculations, *constructions* (abstract procedures), of the number 5. As a striking example, consider the only proposition true in all possible worlds. From the viewpoint of intensional logic, all mathematical theorems and logical tautologies collapse into this one proposition. According to hyperintensional level of TIL, however, all these tautologies and theorems are distinct structured procedures differing in the way they construct this proposition. Briefly, λ-terms will be conceived as records of these constructions. In a certain sense, we will not utilize full power of the hyperintensional level of TIL in our paper, we need only to distinguish the constructions (or 'concepts') of property, etc., from the property conceived as mere intension

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[4] Tichý sometimes called intensions, having ξ-objects (where ξ is any type) as their values, ξ-offices. Properties are then offices occupiable by classes of individuals, whereas propositions are offices occupiable by truth-values.
[5] When explicating propositions, possible worlds must be primitive, unanalyzed entities, not sums of propositions; otherwise it would be a moving circle.

(which is a flat mapping). Now let us expose also our semantical triangle: expressions *express* constructions (concepts, thoughts), which construct *denotata* of those expressions (intensions or non-intensions); note, however, that *reference* of a certain description is outside this scheme because the reference of a description is the value of the denoted intension in a certain world (it must be discovered by empirical investigation).

Without providing a precise definition of the simple theory of types embedded in TIL, let us simply state that over our basis $\{\iota, o, \omega\}$ there is a range of *n*-ary *total* and also *partial functions*.[6] *Intensions* are of type $(\xi\omega)$ (where ξ is any type) and we will write this type 'ξ_ω' (but in superscript as 'ξ_ω' still). Well known intensions include: *propositions*, i.e. mappings from possible worlds to truth-values, o_ω-objects, *individual offices*, i.e. mappings from possible worlds to individuals, ι_ω-objects, *properties of individuals*, i.e. mappings from possible worlds to classes of individuals, $(o\iota)_\omega$-objects, *properties of such properties* $((o(o\iota)_\omega)_\omega$-objects), etc. Intensions are frequently partial mappings; consider, for instance, the office denoted by the description 'the king of France' and the actually valueless proposition denoted by the sentence 'The king of France is bald'. Among non-intensions there are o-objects T and F (the truth-values 'true' and 'false'). Standard unary and binary truth-functions such as $\neg$, $\wedge$ (or $\vee$, $\rightarrow$, $\leftrightarrow$, etc.) are of types (oo), (ooo) respectively. Classes are $(o\xi)$-objects. The notion of class is not a primary notion in our entirely functionally based explicative system; we will use the term 'class' instead of 'characteristic function' only for easier understanding. Quantifiers, i.e. $(o(o\xi))$-objects, represent sub-classes of certain classes. Finally, let 'sing' stands for 'singularization', where "singularization" is a partial mapping returning the only member of a singleton, undefined otherwise (it is an $(\xi(o\xi))$-object). The proper type of $\exists$ (or $\forall$), or = (the well-known $(o\xi\xi)$-object), or singularization, can be easily gathered from the surrounding context.

Similarly as algorithms, Tichý's constructions are not purely set-theoretical entities.[7] There are four basic kinds of constructions: a) *variables* (genuine procedures, which are objectual pendants of variable-letters); b) *trivializations* (one-step procedures, which are objectual pendants of constants); c) *compositions* (objectual pendants of λ-terms called applications); d) *closures* (objectual pendants of λ-terms called λ-abstractions). The only kind of constructions which can be incomplete (improper) in the sense that they are not able to

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[6] Some theoreticians conceive total functions as special cases of partial functions, but we will follow Tichý's terminology.
[7] Btw. they belong to higher type categories marked $*_j$, where $_j$ is any natural number from 1 to *n* (in TIL we have a certain version of the ramified theory of types).

construct an object are compositions due to the fact that a given function may be undefined on given argument.

Trivializations serving for direct grasping of entities (individuals, functions or other entities) should be written as ^0X (whereas X is any object or construction), but we omit the sign '^{0}' in this paper. Compositions of the form $[X\ w]$ (X is any construction) will be written as X_w. We will frequently omit some pairs of brackets. Instead of $[\exists[\lambda x\ [...]\ ]]$ we will simply write $[\exists.\lambda x\ [...]]$ evoking here the dot convention (the respective right bracket should be imagined as far on the right as it is consistent). Compositions of the form $[\#\ X_1\ X_2]$, where $\#$ is (a construction of) any binary operation, will be written as $[X_1\ \#\ X_2]$.

Depending on valuation v, variable x (or x', or y) v-constructs ι-objects (individuals), variable w (or w', w'', w''', w'''') v-constructs ω-objects (possible worlds), variable s (or s') v-constructs $(o\iota)$-objects (classes of individuals), variable f (or f', or g) v-constructs $(o\iota)_\omega$-objects (properties of individuals), variable h (or h') v-constructs $(o(o\iota)_\omega)_\omega$-objects (properties of properties of individuals), variable p v-constructs o_ω-objects (propositions). The type $(o\iota)_\omega$ will be written as φ, the type o_ω as π.

Formal definitions of our concepts may be called *objectual definitions* for they are not mere linguistic shortcuts − we define concepts by means of other, more basic, concepts. We usually offer several equivalent alternatives (we do not suggest any particular conceptual system as given by its primitive concepts; a reader can imagine various conceptual systems if equivalent definitions are proposed).[8] The sense of each such definition is to specify which object would be constructed by the construction on the left side. Both constructions related by $\equiv^\xi$ construct, depending on any valuation, the very same object (if they construct, with respect to a particular valuation, anything at all), thus they are equivalent. Constructions on both sides are open constructions – variables freely occurring in both constructions are not bound by the respective λ operators. To facilitate understanding of the paper, behind '//' we will write the missing binding string such as '$\lambda w\ [\lambda x f$', which may close each of the open constructions on both sides of the definition.[9] In definitions following certain previously stated definitions, η-reduced (even η-normalized) forms of constructions from the preceding definitions will be used. For instance, when the definition contains an open construction like $[X_w\ x\ f]$ and it

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[8] It is noteworthy that in Tichý's system of deduction (see the respective papers in [Tichý 2004]) inference among properties is enabled. We do not investigate deduction rules, thus inference among properties constructed by equivalent constructions is not discussed in detail here (this is left to the logical insight of the reader).

[9] If constructions in the immediately following definition should be closed by the same binding string, we do not repeat this indication. Of course, the reader should complete in his/her mind the proper record of a construction by the respective number of the right brackets in the right places.

should be closed by $\lambda w\ [\lambda xf$, then the closed construction $\lambda w\ [\lambda xf\ [X_w\ xf]]]$ will occur in certain further definitions in its η-reduced form, namely X (because $\lambda w\ [\lambda xf\ [X_w\ xf]]] \equiv_\eta \lambda w\ [X_w]]$ and then $\lambda w\ [X_w]] \equiv_\eta X$).

The type ξ indicated in '$\equiv^\xi$' is a type of object constructed by the construction (on each side) after its closuring by the respective binding item (i.e. a construction written in bold is η-normalized construction of a ξ-object). Note, however, that the equality "$\equiv^\xi$" does not relate just ξ-objects, but certain ζ-objects which are constructed by open constructions on both sides; thus the type of "$\equiv^\xi$" is in fact (o$\zeta\zeta$). Nevertheless, the inscription '$\equiv^\xi$' indicates which type the type ζ precisely is: when ξ is, for instance, $(o(o\iota)\varphi)_\omega$ and we read '// $\lambda w\ [\lambda sf$' behind the definition, then ζ is $(o(o\iota)\varphi)_\omega$ minus ω (due to '$\lambda w\ [$') and minus $(o\iota)\varphi$ (due to 'λsf'), thus ζ is just o, i.e. '$\equiv^{(o(o\iota)\varphi)\omega}$' stands here for the equality of type (ooo). In most cases "$\equiv^\xi$" is just the standard "$\leftrightarrow$".

In many definitions we will use (on the left part of the definition) vertical bars surrounding the variable w ('$_{|w|}$'). We indicate by it that a given property of properties is a constant intension, that it has one and the same extension in all possible worlds. We can then speak directly about a class of such-and-such properties instead of about properties having such-and-such property of properties.

The temporal versions of our definitions are easy to obtain.[10] The proposed definitions can also be easily reworked into definitions of kinds of properties instantiated by other types of objects than individuals or properties of individuals.

TRIVIAL / NON-TRIVIAL PROPERTIES

Total / partial properties

Our first definitions distinguish the 'non-totalizing' predicate 'true' applicable to propositions which do not yield any truth-value when the proposition p is undefined in a given particular world w ('P' in superscript indicates partiality; the right side can be simplified to just p_w):

$[\mathbf{True^P}_w\ p] \equiv^{(o\pi)\omega} [p_w = \mathsf{T}]$　　　　　　　　　　　　　　// $\lambda w\ [\lambda p$

and the 'totalizing' predicate 'true' defined by (the variable o ranges the type of truth-values, o):[11]

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[10] It would be sufficient just to write '$\lambda w[\lambda t$' instead of 'λw' and '$..._{wt}...$' instead of '$..._w...$' (making use of the convention that $[[X\ w]\ t]$ is abbreviated as X_{wt}). Of course, when, for example, the variable w'' is used, then we should use the variable t''; and when we quantify over w we should also quantify over t by means of the same quantifier.

a proposition *p* is *true* $=_{df}$ there exists a truth-value *o* which is identical with the value of
  proposition *p* in *w* and *o* is identical with T

$$[\textbf{True}_w\, p] \equiv^{(o\pi)\omega} [\exists.\lambda o\, [\, [o = p_w] \wedge [o = \mathsf{T}]\,]]$$

The 'totalizing' of this predicate is caused by the existential quantifier, which returns F when
p is false or undefined in a given particular *w*. "True" is a property of proposition, thus it is an
$(o\pi)_\omega$-object.

We will frequently need to speak about an extension of a certain property, thus we
define:

  the *extension of f* in *w* $=_{df}$ the only class *s* such that it is identical with the value of *f* in *w*

$$[\textbf{ExtensionOf}_w\, f] \equiv^{((o\iota)\varphi)\omega} [\text{sing}.\lambda s\, [s = f_w]] \hspace{4em} (\lambda w.\lambda f.)$$

It apparently holds that:

$$[\text{sing}.\lambda s\, [s = f_w]] \equiv^{((o\iota)\varphi)\omega} f_w$$

It needs to be taken into account that when there is no extension of *f* in *w*, then [ExtensionOf_w
f] or [sing.λ*s* [*s* = *f*_w]] or *f*_w construct nothing at all – these constructions are *v*-improper for the
respective valuation *v*. "ExtensionOf" is not a property of properties but a (partial) mapping,
which assigns (depending on worlds) classes to properties. Such mapping should not be
confused with the (world-dependent) relation between classes and properties:

  a class *s* is the *extension* of *f* in *w* $=_{df}$ a class *s* identical with the value of *f* in *w*

$$[\textbf{Extension}_w\, s\, f] \equiv^{(o(o\iota)\varphi)\omega} [s = f_w] \hspace{4em} (\lambda w.\lambda sf.)$$

('λ*sf*' indicates abstraction over class-property couples).

The concept of 'ExtensionOf' allows us to formulate the *extensionality principle* (for
properties of individuals) in a precise manner. This claim (not a definition) denotes the
proposition that is true in all possible worlds (words 'proposition', 'at the same time' are used
only for easier understanding of the sentential structure):[12]

  For any property *f* and *g* and every possible world *w'*, if the proposition that there is no
  class *s* identical with the extension of *f* (in *w'*) is equivalent with the proposition that there
  is no class *s'* identical with the extension of *g* (in *w'*) and at the same time if there exists
  the extension of *f* (in *w'*), then it is true in *w'* that the extension of *f* (in *w'*) is identical with
  the extension of *g* (in *w'*), so then *f* is identical with *g*.

  λ*w* [∀.λ*f* [∀.λ*g* [

    [∀.λ*w'*[[¬[∃.λ*s* [*s*=[ExtensionOf_{w'} *f*]]] ↔ ¬[∃.λ*s'* [*s'*=[ExtensionOf_{w'} *g*]]]]

        ∧ [[∃.λ*s* [*s*=[ExtensionOf_{w'} *f*]]]

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[11] Verbal formulations of definitions are used to help non-technical readers. These expressions are not analyzed
by the underlying formulas.

[12] In the case of equivalence (within the definition), the negations on both sides are eliminable.

$$\rightarrow [\text{True}_{w'} [\lambda w'' [[\text{ExtensionOf}_{w''} f] = [\text{ExtensionOf}_{w''} g]]]] \ ] \ ]]$$

$$\rightarrow [f = g] \quad]]]$$

We can also easily define the concept of total or partial property:[15]

   a property *f* is *total* $=_{df}$ in every world *w'* there exists a class *s* which is identical with the

   extension of *f* in *w'*

$$[\textbf{Total}_{|w|} f] \equiv^{(o\varphi)\omega} \quad [\forall.\lambda w' [\exists.\lambda s [s = [\text{ExtensionOf}_{w'} f]]]] \qquad\qquad (\lambda w.\lambda f.)$$

   a property *f* is *partial* $=_{df}$ there exists a world *w'* which is such that there does not exist

   a class *s* which is identical with the extension of *f* in *w'*

$$[\textbf{Partial}_{|w|} f] \equiv^{(o\varphi)\omega} [\exists.\lambda w' \neg[\exists.\lambda s [s = [\text{ExtensionOf}_{w'} f]]]]$$

Both these properties are complementary (*cf.* our definition of the concept Complementary

below). Since "total" is not a partial function, it holds that:

$$[\text{Partial}_{|w|} f] \equiv^{(o\varphi)\omega} \neg[\text{Total}_w f]$$

(Can) instantiate / (can) lack

   In many studies reflecting properties or individuals there frequently appear two

concepts, namely that of instantiating and that of lacking (of a certain property). It seems to be

seductive to relate these two concepts explicitly to kinds of properties as we will demonstrate

below. Since their 'modalized' versions are not in *prima facie* obvious relationship, this

definitional intermezzo eventually turned out to be longer than the present author originally

expected.

   The statement that an individual I_1 *is an F* (formally $\lambda w [F_w I_1]$) can be without a truth-

value when the property "F" is undefined in a given world.[16] To allow such an attributing of

"F" to I_1 which would be definitely true or false but not without a truth-value, it would be

convenient to use a predicate which enables us to get over the partiality in question. We will

use the 'totalizing' predicate 'true' for that purpose and we formulate the following definition

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[13] The use of True is indispensable here: mere [[ExtensionOf$_{w'}$ *f*] = [ExtensionOf$_{w'}$ *g*]] would not construct a truth-value when *f* (or *g*) is undefined in a certain world. Tus ∨ would not give us a truth-value (therefore ∀ will return F). However, we need to get T even for function *f* (or *g*) which lacks a value on some argument.

[14] Notice also that the use of variable *w'* (or *w''*) is, in fact, superfluous here. For that reason it can be renamed – evoking α-reduction here – in all its occurrences to *w*; we are using *w'* only for an easier understanding of the fact that it is bound just by λ*w'* and not by λ*w*. On the other hand, in distinct cases (some of them are shown below) an attempt to rename *w'* would not be acceptable for it would lead to a non-equivalent construction (α-reduction should be collisionless). It is also important to realize that one of the possible worlds which the bound variable *w'* may construct is the same as the one from possible worlds assigned by valuation(s) to the free variable *w* (in no formula of ours the condition *w≠w'* is used); analogously, the same holds for *s*, *s'*, etc.

[15] Among total properties there is also such property whose stable extension is a characteristic function that is undefined for all arguments (see also the discussion concerned with partial characteristic functions below).

[16] Alternatively, consider such characteristic function which is undefined for I_1 (compare with the previous footnote and the discussion of partial characteristic functions below).

of total (world-dependent) relation between individuals and properties "instantiate" (it can be considered that "to have F", i.e. "to have the property F", is analogous):

an individual *x instantiates* in *w* a property *f* =$_{df}$ it is true that an individual *x* is an *f* in *w*

$[\textbf{Instantiate}_w\, x\, f] \equiv^{(o\iota\varphi)\omega} [\text{True}_w\, [\lambda w'\, [f_{w'}\, x]]]$ ($\lambda w.\lambda x f.$) [17]

And further:

an individual *x lacks* a property *f* in *w* =$_{df}$ an individual *x* does not instantiate *f* in *w*

$[\textbf{Lack}_w\, x\, f] \equiv^{(o\iota\varphi)\omega} \neg[\text{Instantiate}_w\, x\, f]$

Of course, the quality of being a bearer of a certain property can be also reasonably explicated in the 'total' manner; thus we would define simply:

an individual *x* is a *bearer* of a property *f* in *w* =$_{df}$ an individual *x* instantiates *f* in *w*

$[\textbf{Bearer}_w\, x\, f] \equiv^{(o\iota\varphi)\omega} [\text{Instantiate}_w\, x\, f]$

It would be also fruitful to embed a concept of the relation "*can lack*" into several of our definitions. It seems that this predicate presupposes that it is possible for an individual to have the property in question. By way of illustration consider the trivial undefined property denoted, *inter alia*, by the expression 'being an individual such that 3÷0=0' (formalized as λw $[\lambda x\ [3÷0=0]]$). It is apparent that no individual may possibly have this property, thus to say that an individual can lack it (i.e. that there exists a possible world in which this individual ceases to have this property) amounts to saying an undoubtedly false claim. Let us therefore assume the concept CanLack, which may be expressed also by the phrase 'being possible for *x* to lack *f*', while being possible is explained by means of existence of at least one possible world (in which something happens):

an individual *x can lack* a property *f* =$_{df}$ there exists a possible world *w'* such that *x* is an *f*
    in *w'* and there exists a possible world *w''* such that *x* lacks *f* in *w''*

$[\textbf{CanLack}_{|w|}\, x\, f] \equiv^{(o\iota\varphi)\omega} [\ [\exists.\lambda w'\ [f_{w'}\, x]] \wedge [\exists.\lambda w''\ [\text{Lack}_{w''}\, x\, f]]\ ]$ [18]

which is equivalent to:

$\equiv^{(o\iota\varphi)\omega} [\ [\exists.\lambda w'\ [f_{w'}\, x]] \wedge [\exists.\lambda w''\ \neg[\text{True}_{w''}\, [\lambda w'''[f_{w'''}\, x]]]]\ ]$ [19]

Below we will use also CannotLack which we can define simply as (for "canlack" is total):

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[17] Although the property "instantiate" is total, the construction [Instantiate$_w$ D$_w$ *f*] may be even improper (*v*-constructing no truth-value a truth-value) when D is a construction expressed by a description which does not refer to any individual in a particular world *w* (consider 'the king of France'). We do not obtain here a definite truth-value, however, it is so not because "instantiate" is partial, but because this function does not obtain an argument as there is no individual which should be constructed by D$_w$ (thus there is no couple <individual, property> that would be an argument for "instantiate"). The same holds for other such cases.
[18] We use Lack to allow for cases when an individual is not in the extension of *f* because it is not in that class or for the reason that *f* has no extension in the particular world.
[19] Realize that we evaluate [True$_{w''}$ [$\lambda w'''$ [... $_{w'''}$...]]] in the sense of [$\exists.\lambda o$ [[$o = [\lambda w'''$ [...$_{w'''}$...]]$_{w''}$] $\wedge$ ... what is equivalent (by the I. rule of λ-conversion, substituting *w''* for *w'''*) to [$\exists.\lambda o$ [[$o = [...f_{w''}...]$] $\wedge$ ... .

$[\textbf{CannotLack}_{|w|} \; x \; f] \equiv^{(o\iota\varphi)\omega} \neg[\text{CanLack}_w \; x \; f]$

When performing equivalences accordingly to our above definitions and common logical laws one should, however, be careful about De Morgan's laws for quantifiers since partiality may cause non-equivalent result. Therefore, when we convert a formula with one quantifier to a formula with the second one, the 'condition' after the quantifier should be 'totalized' by the predicate 'True'; i.e. $\neg[\exists.\lambda w \; [\text{True}_w \; [\lambda w' \; [..._{w'}...]]]] \leftrightarrow [\forall.\lambda w \; \neg[\text{True}_w \; [\lambda w' \; [..._{w'}...]]]]$. The first equivalence to CannotLack:[20]

$$\equiv^{(o\iota\varphi)\omega} [\; \neg[\exists.\lambda w' \; [f_{w'} \; x]] \vee \neg[\exists.\lambda w'' \; [\text{Lack}_{w''} \; x \; f]] \;]$$

is correct, however, the next one must not be just $[\; [\forall.\lambda w' \; [\neg[f_{w'} \; x]]] \vee [\forall.\lambda w'' \; [\neg[\text{Lack}_{w''} \; x \; f]]] \;]$, but:

$$\equiv^{(o\iota\varphi)\omega} [\; [\forall.\lambda w' \; [\text{True}_{w'} \; [\lambda w'' \; [\neg[f_{w''} \; x]]]]]$$
$$\vee [\forall.\lambda w''' \; [\text{True}_{w'''} \; [\lambda w'''' \; [\neg[\text{Lack}_{w''''} \; x \; f]]]]] \;]$$

In the second part of the disjunction, the use of True is not necessary here because the second part of the disjunction would not be without a truth-value anyway[21] (*cf.* also footnote 12). In the present case we are safe when using True in this place also for the reason that the definition equivalent of Lack already use it, thus the partiality caused by unfitting description not singling out (in the respective world) an individual or a property is overcome anyway:

$$\equiv^{(o\iota\varphi)\omega} [\; [\forall.\lambda w' \; [\text{True}_{w'} \; [\lambda w'' \; \neg[f_{w''} \; x]]]] \vee [\forall.\lambda w''' \; [\text{True}_{w'''} \; [\lambda w'''' \; [f_{w''''} \; x]]]] \;]$$

The last equivalence can be simplified for the second part of the disjunction should assign T to trivial properties which the individual has in all worlds:

$$\equiv^{(o\iota\varphi)\omega} [\; [\forall.\lambda w' \; [\text{True}_{w'} \; [\lambda w'' \; \neg[f_{w''} \; x]]]] \vee [\forall.\lambda w''' \; [f_{w'''} \; x]] \;]$$

Notably, with the help of True the first part of the disjunction returns T not only for the property whose invariant extension is the empty' class but mainly for the property which is undefined in all worlds (thus any individual cannot lack it); without the use of True we would obtain F for this property.

We can add also:

an individual *x can instantiate* a property $f =_{df}$ there exists a possible world w' such that
an individual *x* is an *f* in *w'*

$[\textbf{CanInstantiate}_{|w|} \; x \; f] \equiv^{(o\iota\varphi)\omega} [\exists.\lambda w' \; [f_{w'} \; x]]$

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[20] The (construction of) binary truth-function xor, the exclusive or, can be used instead of $\vee$ (analogously also in other formal definitions containing $\vee$).

[21] The construction expressed by a description like 'the king of France' or 'the only favourite property of the king of France' standing instead of *x* or *f* may lead in w'''' to nothing.

In the case of CanInstantiate the 'condition' $[f_{w'}\,x]$ should return T to obtain the sought class of properties that an individual can instantiate. Thus we can reasonably replace this 'condition' by $[\text{True}_{w'}\,[\lambda w''\,[f_{w''}\,x]]]$; by its equivalence with Instantiate we get:

$=_{df}$ there exists a possible world w' such that an individual x instantiates f in w'

$\equiv^{(o\iota\varphi)\omega}\,[\exists.\lambda w'\,[\text{Instantiate}_{w'}\,x\,f]]$

(The last equivalence might be perhaps read as the definition of PossiblyInstantiate.) Analogously we have:

an individual x *cannot instantiate* a property $f =_{df}$ there does not exist a possible world w' such that an individual x is an f in w'

$[\mathbf{CannotInstantiate}_{|w|}\,x\,f]\equiv^{(o\iota\varphi)\omega}\,\neg[\exists.\lambda w'\,[f_{w'}\,x]]$

The respective equivalences are:

$\equiv^{(o\iota\varphi)\omega}\,\neg[\exists.\lambda w'\,[f_{w'}\,x]]$

$\equiv^{(o\iota\varphi)\omega}\,[\forall.\lambda w'\,\neg[\text{True}_{w'}\,[\lambda w''\,[f_{w''}\,x]]]]$

$=_{df}$ in every possible world w' an individual x does not instantiate f

$\equiv^{(o\iota\varphi)\omega}\,[\forall.\lambda w'\,\neg[\text{Instantiate}_{w'}\,x\,f]]$

On the other hand, it needs to be pointed out that CanLack and CannotLack do not have similar relations to Instantiate because both can be characterized, per definition, by a more complicated 'condition' than Instantiate. Thus it only holds:

$[\,[\text{CanLack}_w\,x\,f]\rightarrow[\exists.\lambda w'\,\neg[\text{Instantiate}_{w'}\,x\,f]]\,]$

$[\,[\forall.\lambda w'\,[\text{Instantiate}_{w'}\,x\,f]]\rightarrow[\text{CannotLack}_w\,x\,f]\,]$

The opposite direction of the first implication would be false for the property undefined in all worlds, which is not desirable. The opposite direction of the second implication would be false for the property undefined in all worlds as well as for the property whose invariant extension is the empty' class (empty' is explained below), which would be better avoided .

Trivial / non-trivial properties

Generally speaking, classification of properties is a rather complicated matter. The main root of its complexity seems to rest in partiality. Tichý often used the term 'trivial property' to name properties (or generally any intensions) with only one stable extension (if any) assigned constantly to all possible worlds. The definition, which states that trivial properties are such properties that in every possible world w' the extension of f in w' is identical with its extension in w (formally $[\text{Trivial}_w\,f]\equiv^{(o\varphi)\omega}\,[\forall.\lambda w'\,[\,[\text{Extension}_{w'}\,f]=[\text{Extension}_w\,f]\,]]$) does not cover, however, the property undefined in all worlds. In order to allow also this property

as a trivial property it would be necessary to expand the just considered preliminary definition in the following way:

being a *trivial* property $f =_{df}$ being a property such that either in every possible world w' it is true in w' that the extension of f (in w) is identical with the extension of f (in w'), or in no world w''' there exists a class s identical with the extension of f in w'''

$$[\textbf{Trivial}_{|w|} f] \equiv^{(o\varphi)\omega} [\ [\forall.\lambda w'\ [\text{True}_{w'}\ [\lambda w''[\ [\text{ExtensionOf}_w f] = [\text{ExtensionOf}_{w''} f]\ ]]]]$$

$$\vee\ [\neg\exists.\lambda w'''\ [\exists.\lambda s\ [s = [\text{ExtensionOf}_{w'''} f]]]]\ ] \qquad\qquad (\lambda w.\lambda f.)$$

Perhaps a simpler definition of Trivial is:

$=_{df}$ being a property such that if there exists a possible world w' in which there exists a class s that is the extension of f, then there exists a class s' which is in every possible world w'' the extension of f

$$\equiv^{(o\varphi)\omega} [\ [\exists.\lambda w'\ [\exists.\lambda s\ [s = [\text{ExtensionOf}_{w'} f]]]] \rightarrow$$

$$[\exists.\lambda s'\ [\forall.\lambda w''\ [s' = [\text{ExtensionOf}_{w''} f]]]]\ ]$$

However, note that one cannot equate trivial properties with those properties that an individual cannot lcak for non-constant properties assigning alternatively an empty class and nothing (s.c. non-trivially void properties, see below) are also those properties that individuals truly cannot lack.

Instead of 'I_1 is not an F' one inclines to say just 'I_1 is a non-F' (compare with 'Yannis is not a smoker' / 'Yannis is a non-smoker'. To identify (the concept) non-F with λw [λx [$\neg[F_w\ x]$]] seems to be adequate, however, only when the property "F" is a total function (it seems not viable to claim truly that Yannis is not the brother of the king of France when there is no king of France; for the question whether Yannis is not his brother 'does not arise', as Strawson would put it). To attribute 'being non-F' more properly, we have to define the denial in a slightly more sophisticated manner than by means of a mere negation.[22]

First, it is necessary to define several auxiliary concepts. It is well known that for any class S there is a class S', which is complementary to S (with regards to the given domain of objects). Accordingly, we will conceive a particular function as complementary to another function under the condition that if an object is in the extension of the first function, then it is not in the extension of the second function, and conversely, if an object is not in the extension

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[22] We do not insist that our proposal is the only reasonable explication of 'being a non-F'. Possibly one can explain 'being a non-F' in the sense that "non-F" is total, thus defined, even if "F" is undefined. We do not adopt this kind of explication for when "non-F" is total there still would be a property "F" which is total (and complementary to "non-F") and certain property "non-F'" which is partial (and complementary to "F"). Perhaps the total sense of 'non-F' originates in our tendency to construe its meaning (for pragmatic reasons) in the sense 'it is not true that (an individual) is an F', which seems to be too far from the more proper (thus semantically more relevant) complement of 'being an F'.

of the first function – for there is no extension of it – then that object is not in the extension of the second function as it has no extension either (it is an implication in both directions which we rebuilt in the definition by xor). We may be tempted to define this concept more precisely as follows:[23]

> a property *f* is *complementary*^{W} to a property *g* $=_{df}$ in every world *w'* for every individual
> *x*, either it is true in *w'* that *x* is in the extension of *f*, or it is true in *w'* that *x* is in
> the extension of *g*, and the proposition that there is no extension of *f* (in *w'*) is
> equivalent with the proposition that there is no extension of *g* (in *w'*)

[**Complementary**^{W} *f g*] $\equiv^{(o\varphi\varphi)}$ [$\forall.\lambda w'$ [$\forall.\lambda x$ [[[True$_{w'}$ [$\lambda w''$ [[ExtensionOf$_{w''}$ *f*] *x*]]]

xor [True$_{w'}$ [$\lambda w'''$ [[ExtensionOf$_{w'''}$ *g*] *x*]]]]

$\wedge$ [$\neg$[$\exists.\lambda s$ [*s*=[ExtensionOf$_{w'}$ *f*]]] $\leftrightarrow$ $\neg$[$\exists.\lambda s'$ [*s'*=[ExtensionOf$_{w'}$ *g*]]]]]]]   (*λfg.*)

It should be noted that this definition of (the concept of) "complementary" is not an ideal one because in some cases it allows more intensions to be complementary to a particular intension. Nevertheless, the group of such complementary intensions consists in very similar intensions; thus we may realize that such a definition of "Complementary^W" is only slightly imprecise (^{W} for 'weak'). To define (the concept of) "complementary" in a strict sense we need some further theoretical notions and findings. For this reason we shall make an attempt at a precise definition later on, in order not to interrupt the present line of explanation (in the nearest definitions we use already the concept Complementary defined below, let the technically oriented reader be patient).

Now, for any property, properties of properties constructed by the variable *h*, "Non" is a mapping assigning to property *h* a property *h'* (of the same type) which is the only property complementary to *h*:

[**Non** *h*] $\equiv^{(((o\varphi)\omega)((o\varphi)\omega))}$ [sing.$\lambda h'$[Complementary *h' h*]]   (*λh.*)

Let us adopt a convention according to which the compound [Non *h*] is equivalent to the single concept Non*h*. Therefore, to provide an exact definition of the property "NonTrivial" we state just (adopting Non of appropriate type):

> a *non-trivial* property of properties $=_{df}$ the only property of properties complementary
> to the property "Trivial"

NonTrivial$_{|w|}$ $\equiv^{(o\varphi)\omega}$ [sing.λh [Complementary *h* Trivial]]$_w$ (*λw.*)

with η-expanded forms of constructions on the both sides:

[NonTrivial$_{|w|}$ *f*] $\equiv^{(o\varphi)\omega}$ [[sing.λh [Complementary *h* Trivial]]$_w$ *f*]   (*λw.λf.*)

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[23] Here we use the exclusive or, xor, because mere disjunction is not sufficient. Let us ignore the imaginary (but not real) dependence of "complementary" on worlds.

Since "trivial" is a total property it holds that:

$$\equiv^{(o\varphi)\omega} \neg [\text{Trivial}_w f]$$

The extension of a trivial property does not vary with respect to distinct worlds. States of worlds are aptly viewed as factors influencing empirical experience. We can thus call trivial properties *non-empirical properties* because, for example, to find out whether I_1 is identical with itself is an *a priori* matter, i.e. the question of self-identity of I_1 is not an empirical business. On the other hand, extensions of non-trivial properties are dependent on worlds. Being a human, for example, happens to an individual in one world but it may not happen in another world. In order to get knowledge about what the actual extension of the property "to be human" is one should go out and detect human beings, i.e. to investigate contingent circumstances occurring in the world (it is in principle not excogitable by means of pure deduction). Thus non-trivial properties can be understood as *empirical properties*. In other words, it means that:

being an *empirical* property $f =_{df}$ being a non-trivial property f $(\lambda w.\lambda f.)$

$[\textbf{Empirical}_{|w|} f] \equiv^{(o\varphi)\omega} [\text{NonTrivial}_w f]$

being a *non-empirical* property $f =_{df}$ being a trivial property f

$[\textbf{NonEmpirical}_{|w|} f] \equiv^{(o\varphi)\omega} [\text{Trivial}_w f]$

Furthermore, it may be even realized that the best construing of trivial and non-trivial properties (and generally of all such intensions), thus replacing Tichý's terminology (influenced by A. Plantinga), is to call non-trivial properties *contingent properties* and to call trivial properties *non-contingent* (or: *constant*) *properties*:

being a *contingent* property $f =_{df}$ being a non-trivial property f

$[\textbf{Contingent}_{|w|} f] \equiv^{(o\varphi)\omega} [\text{NonTrivial}_w f]$

being a *non-contingent* property $f =_{df}$ being a trivial property f

$[\textbf{NonContingent}_{|w|} f] \equiv^{(o\varphi)\omega} [\text{Trivial}_w f]$

Nonetheless, let us leave this suggestion of a terminological change open for future considerations. Further on, we will continue to use only Tichý's notions of trivial / non-trivial.[24]

Empty / universal / undefined, singular properties

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[24] Verbal definitions of Empirical and Trivial have appeared already in Cmorej's papers including intuitively acceptable equality of empirical properties with non-trivial properties (in [Cmorej 2006] the term 'contingent' is used instead of 'empirical'). All his definitions handled only total properties. Of course, one might argue that 'being an extension of' is conceived as a 'totalizing' predicate. But then the phrase 'there is no (there does not exist an) extension of' would be a nonsense, which is clearly not. The only natural conceiving of 'being an extension of' is thus in the partial sense.

At this point, it would be also convenient to determine concepts of several exceptional trivial properties to which we often refer in the present text. It is worth noting that our definition of (the concept of) trivial properties covers also these exceptional properties. First, we can determine *trivial empty property* as the only property which fulfils the following definiens (the classical empty class, $\varnothing$, is the unary total characteristic function assigning F to every ι-argument, it is constructible by [sing.λs [λx [[s x] = F]]]):

being a *trivial empty$^\varnothing$* property $f =_{df}$ being a property such that in all possible worlds w the extension of f is the empty class

$$[\mathbf{Empty}^{\varnothing\mathbf{Tr}}{}_{|w|}\,f] \equiv^{(o\varphi)\omega} [\forall.\lambda w' \ [\ [\mathrm{ExtensionOf}_{w'}\,f] = \varnothing\ ]] \qquad\qquad (\lambda w.\lambda f.)\ ^{25}$$

It is apparent from the definiens that the respective property is trivial, which we indicate by 'Tr' in the superscript. Subsequently we add the concept of *trivial undefined property*, i.e. the only property that fulfils the definiens of:

being a *trivial undefined* property $f =_{df}$ being a property such that in all possible worlds w there does not exist a class s which is identical with the extension of f in w

$$[\mathbf{Undefined}^{\mathbf{Tr}}{}_{|w|}\,f] \equiv^{(o\varphi)\omega} [\forall.\lambda w' \ \neg[\exists.\lambda s \ [s = [\mathrm{ExtensionOf}_{w'}\,f]\ ]]]$$

Below we will use also the concept of the only trivial property which has as its only extension the class of all individuals; thus, we can define the (concept of) *trivial universal property,* i.e. the only property that fulfils the definiens of:[26]

being a *trivial universal* property $f =_{df}$ being a property such that in all possible worlds w the extension of f in w is identical with the only class s which is complementary to the empty class

$$[\mathbf{Universal}^{\mathbf{Tr}}{}_{|w|}\,f] \equiv^{(o\varphi)\omega} [\forall.\lambda w' \ [\ [\mathrm{ExtensionOf}_{w'}\,f] = [\mathrm{sing}.\lambda s\ [\mathrm{Complementary}\ s\ \varnothing]]\ ]]$$

The property such as "being identical with I_1" has in its non-varying extension I_1 and it can be characterized as *trivial singular property*. There are as many trivial singular properties in this sense as there are individuals in our domain (we precise this claim below). They form a class, which is the unique and non-varying extension of the property of properties whose concept is defined by means of:

being a *trivial singular* property $f =_{df}$ being a property such that in all possible worlds w there exists an individual x which is in the extension of f in w, and for every individual y if it is in the extension of f in w, then it is identical with x

$$[\mathbf{Singular}^{\mathbf{Tr}}{}_{|w|}\,f] \equiv^{(o\varphi)\omega} [\forall.\lambda w' \ [\exists.\lambda x \ [\ [[\mathrm{ExtensionOf}_{w'}\,f]\ x]$$

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[25] The only trivial empty$^\varnothing$ property is constructible by the construction [Empty$^{\varnothing\mathrm{Tr}}{}_w\,f$] closed by λw [sing.λf [λf. The same holds also for the trivial undefined and the trivial universal property.

[26] Of course, instead of [sing.λs [Complementary s $\varnothing$]] we can construct the class of all individuals by means of [λx T].

$\wedge \, [\forall.\lambda y \, [\, [[\text{ExtensionOf}_{w'}\,f] \, y] \rightarrow [y{=}x] \,] \,] \,]]]$

(The reader familiar with the problematics of 'numerical quantifiers' can easily rebuild this definition into definitions of trivial properties with constant extension containing two individuals, or three individuals, etc.)

Partial characteristic functions

As a matter of fact, partiality complicates matters from the very bottom of our considerations. It has been already pointed out that mappings assigning truth-values are called *characteristic functions*. Let us consider a collection of just two objects, let us say A and B. Now there are exactly nine characteristic total and partial functions over this collection:

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
A	T	T	T	F	F	F			
B	T	F		T	F		T	F	

A functional system adopting exclusively total functions can recognize only f_1, f_2, f_4, f_5 as characteristic functions, serving thus as representations of the following classes (in this order): {A,B}, {A}, {B}, $\varnothing$. In our system, functions f_1, f_2, f_4, f_5 also serve as a proper representation of these classes. In addition, there are other characteristic functions which are partial. However, it needs to be said that, in a proper sense, those partial characteristic functions cannot be employed as explications of classes since one-to-many way of explication is not allowed. (In our verbal formulations we speak about all characteristic functions as classes only to adhere to the usual set-theoretical custom of speaking.)

Having introduced the notion of a characteristic function, let us classify several kinds of these functions, two kinds of which shall be important in our further consideration. It seems to be awkward to use such terms as '/attribute/ characteristic function' and keep the set-theoretical idea in mind, so we will rather us terms as '/attribute/ class'. Firstly, when we suitably adopt the above definition of "total", we can construct all total (among characteristic) functions, i.e. f_1, f_2, f_4, f_5 in our miniature example, by the construction $\lambda s \, [\forall.\lambda x \, [\exists.\lambda o \, [\, o = [s\,x] \,]]]$ or simply by $\lambda s \, [\forall.\lambda x \, [\, [[s\,x]{=}\text{T}] \vee [[s\,x]{=}\text{F}] \,]]$; we may call such characteristic functions 'dichotomies':

  $[\textbf{Dichotomy}\ s] \equiv^{(o(o\iota))} [\forall.\lambda x \, [\exists.\lambda o \, [\, [s\,x]{=}o \,]]]$ ($\lambda s.$)

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[27] When these definitions of ours are adopted for intensions-offices which are not occupiable by classes of objects but by single objects such as individuals (the case of individual offices) or *n*-tuples of individuals, then their only property is "Undefined"; the other properties (from this section) do not come into account.

Characteristic functions properly dividing the respective collection (i.e. by 'finished bisection process'), compare with f_2, f_4, can be defined as:

$$[\textbf{Dividing } s] \equiv^{(o(o\iota))} [\ [\forall.\lambda x\ [\exists.\lambda o\ [[s\ x]=o]]] \wedge [\exists.\lambda x'\ [[s\ x']=\textsf{T}]] \wedge [\exists.\lambda x''\ [[s\ x'']=\textsf{F}]]\ ]$$

Now let us conceive 'not always finished bisections', which assign $\textsf{T}$ to certain members of the collections (but at least to one of them), however, not assigning $\textsf{F}$ to any of their members, as the (total and partial) *affirmative classes*. In our example, these are f_2, f_3, f_7 (but not f_9):

$$[\textbf{Affirmative } s] \equiv^{(o(o\iota))} [\ [\ [\forall.\lambda x\ [[s\ x]=\textsf{T}]] \vee \neg[\exists.\lambda o\ [o=[s\ x]]]\ ]$$
$$\wedge [\neg\forall.\lambda x'[\neg\exists.\lambda o'\ [o'=[s\ x']]]]\ ]$$

The mates of affirmative classes are *negative classes*, i.e. total and partial functions such that if they assign a truth-value, they assign $\textsf{F}$ (but not $\textsf{T}$). They are all empty, see f_5, f_6, f_8:

$$[\textbf{Negative } s] \equiv^{(o(o\iota))} [\ [\ [\forall.\lambda x\ [[s\ x]=\textsf{F}]] \vee \neg[\exists.\lambda o\ [o=[s\ x]]]\ ]$$
$$\wedge [\neg\forall.\lambda x'[\neg\exists.\lambda o'\ [o'=[s\ x']]]]\ ]$$

More importantly for us, they are *empty' classes* (total and partial), i.e. f_5, f_6, f_8, f_9 in our illustrative example:

$$[\textbf{Empty' } s] \equiv^{(o(o\iota))} [\ [\forall.\lambda x\ [[s\ x]=\textsf{F}]] \vee \neg[\exists.\lambda o\ [o=[s\ x]]]\ ]$$

or just simply:

$$\equiv^{(o(o\iota))} \neg[\exists.\lambda x\ [\ [s\ x]=\textsf{T}]\ ]$$

The second rather important group is formed by total and partial characteristic functions, which assign $\textsf{T}$ to at least one member of a given collection, in our example f_1, f_2, f_3, f_4, f_7 (the variable m constructs $(o(o\iota))$-objects):

$$[\textbf{NonEmpty' } s] \equiv^{(o(o\iota))} [\ [sing.\lambda m\ [Complementary\ m\ Empty']]\ s]$$

Others might better appreciate the following definition:

$$\equiv^{(o(o\iota))} [\exists.\lambda x\ [\ [s\ x]=\textsf{T}]\ ]$$

Non-empty' classes will be viewed as classes containing something.[28]

Turning back to our four kinds of rare properties, we can see that our definition of trivial singular properties is designed not only to deal with dichotomies as extensions of properties but also with those non-empty classes that assign $\textsf{T}$ for just one member, but then they assign $\textsf{F}$ or simply nothing to other members of a given collection. On the other hand, our definition of the trivial empty$^{\varnothing}$ property was designed to allow only the property having just the dichotomy assigning $\textsf{F}$ to all members, i.e. $\varnothing$, as its invariant extension. We can, and we will, define properties which have as its not varying extension any empty' class:

$$[\textbf{Empty}^{\textbf{Tr}}_{|w|}\ f] \equiv^{(o\varphi)\omega} [\exists.\lambda s\ [\ [\forall.\lambda w'\ [s=[Extenze_{w'}f]]] \rightarrow [Prázdná'\ s]\ ] \qquad (\lambda w.\lambda f.)$$

Note that trivially undefined property is not trivially empty.

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[28] For the sake of simplicity we avoid dependency on worlds. The definitions should be carefully modified when adapted for other types of objects (on the other hand, when downgrading Empty^{Tr} defined in the immediately following definition, we must convert it to Empty').

Of course:

$$[\textbf{NonEmpty}^{\text{Tr}}{}_{|w|}f] \equiv^{(o\varphi)\omega} [\;[\text{sing}.\lambda h\;[\text{Complementary } h\; \text{Empty}^{\text{Tr}}]]_w f]$$

Now the last remark that is directly related to the total and partial characteristic functions. The well know relation between classes, "$\subset$", is a total mapping, which is not affected by (couples of) partial classes as arguments. What is relevant for assignment of T is only to check whether the member (of a given collection) assigned by T (if any) in the first one characteristic function is also assigned by T in the second one characteristic function, whereas there can be another member in the second function assigned by T.

Complementary properties

Bearing in mind the previous points, this section shall demonstrate why (the concept of) "complementary^W" that we defined above is somewhat imperfect. A property of individuals, for example, P can have as its extension f_2 – in that case A is in f_2, i.e. A possesses P, but B does not. Our definition of Complementary^W allowed regarding any property which has as its extension f_4 or f_7 (let us say "P^{-1}", "P'$^{-1}$") as complementary to P. (Analogously, this holds for "P'$^{-1}$" having as its extension f^3.) In the ideal sense we would like to deal with Complementary^S ('S' for 'strict'), which collects only those couples of properties such as $\langle P,P^{-1}\rangle$, $\langle P',P'^{-1}\rangle$. Before proposing its definition, it is important to realize that when the values-extensions of certain intensions are not class-like entities but single objects like individuals (the case of individual offices) or couples of individuals (the case of offices occupiable by couples of individuals) the problem of definitional imperfectness does not arise. This means that the definition of Complementary^W can be easily adopted as it is provided that the extensions (of function which should be complementary to some other function) are not of type $(o\xi)$. Hence, the definition of Complementary^S should be such that the (originally conceived) definition of Complementary^W will be improved by addition of a condition determining that when two intensions have (certain) extensions which are of type $(o\xi)$, then these extensions must be mutually complementary in the 'strict' sense (there is a point we are going to explain immediately after the definition):

> being a property *f complementary*^{S} to a property *g* $=_{df}$ being a property such that in every world *w* for every individual *x* either it is true in *w* that *x* is in the extension of *f*, or it is true in *w* that *x* is in the extension of *g*, and the extension of *f* is complementary^W to the extension of *g*, and, at the same time, the proposition that there is no extension of *f* (in *w*) is equivalent with the proposition that there is no extension of *g* (in *w*)

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[**Complementary**^{S} f g] $\equiv^{(o\varphi\varphi)}$ [$\forall.\lambda w$' [$\forall.\lambda x$ [[[True$_{w'}$ [λw'' [[ExtensionOf$_{w''}$ f] x]]]

xor [True$_{w'}$ [λw''' [[ExtensionOf$_{w'''}$ g] x]]]

$\wedge$ [Complementary^W [ExtensionOf$_{w'}$ f] [ExtensionOf$_{w'}$ g]]]

$\wedge$ [$\neg$[$\exists.\lambda s$ [s=[ExtensionOf$_{w'}$ f]]] $\leftrightarrow$ $\neg$[$\exists.\lambda s$' [s'=[ExtensionOf$_{w'}$ g]]]]]]] (λfg.)

Given that extensions of properties of individuals are just classes of individuals, i.e. objects whose extensions are not of type (oξ) but simply of type o, it is sufficient to use in the just above definition (type-theoretically suitable) Complementary^W for the sake of conditioning of the extensions of properties of individuals we desire to reach. This is due to the fact that single objects, e.g. individual I$_1$, cannot be in the logical sense of 'complementary' referred to as 'complementary' to any other object of type ι. If it were not this way, we might have only one strict sense of 'complementary', namely Complementary^S defined with the help of 'type-theoretically lower' Complementary^S. In this connection, it is worth noting, that when we are adopting the definition of Complementary^S for properties of functions assigning, for instance, (oι)-objects to ι-objects (these functions are of type ((oι)ι))), then the extensions of such properties are of type (oξ), namely (oι), hence the extensions of such functions are (oι)-objects and therefore we should use in the respective definition of (type-theoretically suitable) Complementary^S instead of (a type-theoretically suitable) Complementary^W. In those definitions of ours in which we are using Complementary, the reader should interpret it as Complementary^S or Complementary^W dependently on the respective type of extensions that are in the game.

ESSENTIAL / NON-ESSENTIAL PROPERTIES

When discussing Tichý's formulation of antiessentialism ([Cmorej 1996], [Cmorej 2001]) Pavel Cmorej introduced a more detailed classification of properties than Tichý. We will follow his proposals and propose their exact definitions; it will be apparent from the quotations which properties have already been (verbally) defined by Cmorej. Significantly enough, definitions of void properties (mainly of trivially void properties) have to be added to complete the quadruplet of kinds of properties. In addition, we will also provide definitions of accidental properties, which play an important role for philosophers.

Essential for / essential properties

Let us start with:[29]

being a property *f essential for* individual $I_1 =_{df}$ being a property such that in every possible world *w'* the individual I_1 is in extension of *f* in *w'*

$$[\textbf{EssentialFor}_{|w|}\,f\,I_1] \equiv^{(o\varphi)\omega} [\forall.\lambda w' \,[[\text{ExtensionOf}_{w'}\,f]\,I_1]] \qquad (\lambda w.\lambda f.)$$

And equivalently:

$=_{df}$ being a property such that in every possible world *w'* I_1 is an *f*

$$\equiv^{(o\varphi)\omega} [\forall.\lambda w' \,[f_{w'}\,I_1]]$$

$=_{df}$ being a property such that in every world *w'* the individual I_1 instantiates *f*

$$\equiv^{(o\varphi)\omega} [\forall.\lambda w'[\text{Instantiate}_{w'}\,I_1\,f]]$$

It should be noted that the property essential for the specific individual I_1 is such that I_1 cannot lack it. However, this relationship is only implication because any trivial empty property or the trivial undefined property is such that I_1 cannot lack it but it is not essential for it.

$$[\,[\text{EssentialFor}_w\,f\,I_1] \rightarrow [\text{CannotLack}_w\,I_1\,f]\,]$$

Evidently, Cmorej's original definition:

A property *P* is an *essential ... property of an object a* if and only if *P* belongs to the object *a* in every possible world and every time moment. ([Cmorej 2001], p. 104)

is just an analogue of our above definition:

$$\equiv^{(o\varphi)\omega} [\forall.\lambda w' \,[\text{Belong}_{w'}\,f\,I_1]]$$

Consequently, by the obviously valid equivalence:

$$[\textbf{Belong}_w\,f\,I_1] \equiv^{(o\varphi)\omega} [\text{Instantiate}_w\,I_1\,f]$$

we get our EssentialFor, if we convert $[\text{Instantiate}_w\,I_1\,f]$ to $[\text{True}_w\,[\lambda w'\,[f_{w'}\,I_1]]]$ and then to $[\text{True}_w\,[\lambda w'\,[[\text{ExtensionOf}_{w'}\,f]\,I_1]]]$, which can be simplified to $[[\text{ExtensionOf}_w\,f]\,I_1]$.

By means of existential generalization over particular individual I_1 we can state:

being an *essential* property $f =_{df}$ being a property such that there exists *x* for which *f* is essential

$$\equiv^{(o\varphi)\omega} [\exists.\lambda x \,[\text{EssentialFor}_w\,f\,x]]$$

One of the most important equivalence is:

$=_{df}$ being a property such that there exists an individual *x* such that in every world *w'* *x* is in the extension of *f*

$$[\textbf{Essential}_{|w|}\,f] \equiv^{(o\varphi)\omega} [\exists.\lambda x \,[\forall.\lambda w' \,[[\text{ExtensionOf}_{w'}\,f]\,x]]] \qquad (\lambda w.\lambda f.)$$

i.e.:

$=_{df}$ being a property such that there exists an *x* which is an *f* in every world *w'*

---

[29] It is apparent that the concept of property of properties essential for I_j (where *j* is any natural number) can be easily defined by the replacement of I_1 by the chosen I_j.

$$\equiv^{(o\varphi)\omega} [\exists.\lambda x [\forall.\lambda w' [f_{w'} x]]]$$

This is also equivalent to:

> $=_{df}$ being a property such that there exists an individual *x* such that *x* is an *f* and *x* cannot lack *f*

$$\equiv^{(o\varphi)\omega} [\exists.\lambda x [[f_w x] \land [\text{CannotLack}_w x f]]]$$

whereas $[f_w x]$ causes the elimination of the trivial undefined property as well as of any trivial empty property. Due to a certain asymmetry between CannotLack and Instantiate (more precisely NecessarilyInstantiate) we can state simply:

> $=_{df}$ being a property such that there exists an individual *x* such that in every world *w'* *x* instantiates *f*

$$\equiv^{(o\varphi)\omega} [\exists.\lambda x [\forall.\lambda w'[\text{Instantiate}_{w'} x f]]]$$

It is interesting to observe the relationship to trivial properties: whereas there exist trivial empty' properties and the trivial undefined property, essential properties are only those trivial properties that have non-empty' extension.

The last stated equivalence was already expressed by Cmorej:

> A property *P* is an *essential ... property* if and only if there exists an object *a* such that *P* is essential property of the object *a*.   ([Cmorej 2001], p. 104)

whereas the second part of his later definition:

> the property is *essential* if and only if it is an essential property of some object, i.e. when there exists an object to which it belongs necessarily   ([Cmorej 1996], p. 252)

is equivalent to our first definition because 'in every world' we consider it as equivalent with 'it is necessary that ...', or 'necessarily...'.

Note that among essential properties, there are also properties which are essential for some individual but not for other individuals because in some worlds they are in their extension, nevertheless, in other worlds they are not. Now let us add also:

$$[\textbf{NonEssential}_{|w|} f] \equiv^{(o\varphi)\omega} [[\text{sing}.\lambda h [\text{Complementary } h \text{ Essential}]]_w f] \qquad (\lambda w.\lambda f.)$$

Since "NonEssential" is a total property of properties, it holds:

$$\equiv^{(o\varphi)\omega} \neg[\text{Essential}_w f]$$

Accidental for / accidental properties

Having introduced EssentialFor and Essential, it would be convenient to define their mates, namely AccidentalFor and Accidental. It seems reasonable to conceive such a concept of AccidentalFor according to which a) it is possible (i.e. there exists world *w'* in which) for

an individual to have an F, but b) it is also possible to lack F (this other world should be, naturally, distinct from *w'*):

being a property *f accidental for* the individual I_1 $=_{df}$ being a property such that there exists world *w'* in which I_1 is in the extension of *f* and there exists world *w''* in which I_1 lacks *f*

$$[\textbf{AccidentalFor}_{|w|} f\, I_1] \equiv^{(o\varphi)\omega} [\exists.\lambda w' \, [\, [[\text{Extension}_{w'} f]\, I_1] \wedge [\exists.\lambda w'' \, [\text{Lack}_{w''} \, I_1 f]] \,]]$$

$$(\lambda w.\lambda f.)$$

Again, equivalently:

$$\equiv^{(o\varphi)\omega} [\exists.\lambda w' \, [\, [f_{w'} \, I_1] \wedge [\exists.\lambda w'' \, [\text{Lack}_{w''} \, I_1 f]] \,]]$$

It may be noticed that the property with the extension containing I_1 in one world and undefined (or with empty' extension) for all other worlds is accidental for I_1 but it is not accidental for any other individual because such (distinct) individual cannot have it.

By means of existential generalization, we obtain:

being an *accidental* property *f* $=_{df}$ being a property such that there exists an individual *x* such that there exists world *w* in which *x* is an *f* an there exists world *w'* in which *x* lacks *f*

$$[\textbf{Accidental}_{|w|} f] \equiv^{(o\varphi)\omega} [\exists.\lambda x \, [\exists.\lambda w' \, [\, [f_{w'} \, x] \wedge [\exists.\lambda w'' \, [\text{Lack}_{w''} \, x f]] \,]]]$$

Thus there is an important equivalence with:

$=_{df}$ being a property such that there exists an individual *x* that can lack *f*

$$\equiv^{(o\varphi)\omega} [\exists.\lambda x \, [\text{CanLack}_w \, x f]]$$

Of course, we can add also:

$$[\textbf{NonAccidental}_{|w|} f] \equiv^{(o\varphi)\omega} [\, [\text{sing}.\lambda h' [\text{Complementary} \, h' \, \text{Accidental}]]_w f]$$

and it holds that:

$$\equiv^{(o\varphi)\omega} \neg [\text{Accidental}_w f]$$

which is equivalent to:

$=_{df}$ being a property such that there does not exist an individual *x* that can lack *f*

$$\equiv^{(o\varphi)\omega} \neg [\exists.\lambda x \, [\text{CanLack}_w \, x f]]$$

$$\equiv^{(o\varphi)\omega} [\forall.\lambda x \, \neg [\text{True}_w \, [\lambda w' \, [\text{CanLack}_{w'} \, x f]]]]$$

Notice, that "being accidental" does not amount to "being non-trivial" for there exist properties which are non-trivial but they are alternating (any kind of) empty' class with nothing (they are undefined on a given argument) as their values. These non-trivially void properties (see their definition below) cannot be possessed by any individual for analogous reasons as the trivial undefined property and the trivial empty' properties (together: trivially void properties), which are non-accidental too.

Purely essential properties

We will follow Cmorej and distinguish two kinds of essential properties. Since we have already the concept NonEmpty', we are ready to state:

being a *purely essential* property $f =_{df}$ being a property f which is trivial and there exists
a class s which is the non-empty' extension of f

$[\textbf{PurelyEssential}_{|w|}f] \equiv^{(o\varphi)\omega} [\,[Trivial_w f] \wedge [\exists.\lambda s\,[\,[s = [ExtensionOf_w f]]$

$\wedge [NonEmpty'\,s]\,]]\,]$ $\hspace{4cm}(\lambda w.\lambda f.)$

The examples of these properties involve trivial singular properties or the only trivial universal property. Note, that the condition 'and there exists a class which is an extension' excludes from purely essential properties the only trivial undefined property, and the condition of non-emptiness excludes trivial empty' properties.[30] The definition is equivalent to:

$=_{df}$ being a property f which is both trivial and essential

$\equiv^{(o\varphi)\omega} [\,[Trivial_w f] \wedge [Essential_w f]\,]$

since essential properties are such properties that there exists its invariant non-empty' extension(s). The condition 'is trivial' eliminates essential properties with varying extensions like "having the same height as I_1".

Cmorej's definition of (the concept of) "purely essential" uses, in fact, the definition of "trivial" as having a non-varying (the same) extension:

A property P is *purely essential* if and only if in every world ... it has the same non-empty extension. $\hspace{4cm}$([Cmorej 1996], p. 253)

So an apparent equivalence leading from our first definition exactly to the following one is used there:

$\equiv^{(o\varphi)\omega} [\forall.\lambda w'\,[\,[[ExtensionOf_{w'}f] = [ExtensionOf_w f]]$

$\wedge [NonEmpty'\,[ExtensionOf_w f]]\,]]$

To extend (with the help of Accidental) we can also state:

$=_{df}$ being a property f which is essential and not accidental

$\equiv^{(o\varphi)\omega} [\,[Essential_w f] \wedge \neg[Accidental_w f]]$

because being not accidental guarantees here that the extension will not change across the logical space (i.e. collection of possible worlds. Another equivalence to be mentioned is:

[30] It would be wrong (but let us skip here problems with partiality) to equate Trivial with PurelyEssential – 'A property is trivial if and only if it is purely essential' ([Cmorej 2001], p. 107, T7). Because the trivial property(-ies) with empty' extension is (/ are) essential for no object; however, essential properties (including purely essential ones) have non-empty' extension.

$\equiv^{(o\varphi)\omega}$ [$\forall.\lambda x$ [[Instantiate$_w$ x f] $\leftrightarrow$ [CannotLack$_w$ x f]]]

Partly essential properties

The other kind of essential properties is, however, a collection of certain empirical (non-trivial) properties, which can be also called *partly empirical* or *partly contingent properties* (*cf.* [Cmorej 2006], p. 141). The examples of such properties involve those properties like the one denoted by 'having the same height as I$_1$' or 'being identical with I$_1$ or being an animal' (Michael Loux named such properties 'impure properties'). Many individuals possess such properties contingently (the extensions of these properties vary), but I$_1$ has them in all possible worlds, i.e. necessarily. Thus:

being a *partly essential* property f =$_{df}$ being a property f which is non-trivial and essential

[**PartlyEssential**$_{|w|}$ f] $\equiv^{(o\varphi)\omega}$ [[NonTrivial$_w$ f] $\wedge$ [Essential$_w$ f]]   ($\lambda w.\lambda f.$)

i.e.:

=$_{df}$ being a property f which is non-trivial and there exists an individual x which is in the extension of f in every world w'

$\equiv^{(o\varphi)\omega}$ [[NonTrivial$_w$ f] $\wedge$ [$\exists.\lambda x$ [$\forall.\lambda w$' [[ExtensionOf$_{w'}$ f] x]]]]

The following equivalences are in accordance with Cmorej's original proposal (an analogous definition is in [Cmorej 2001], p. 105) whose key verbal definition is not as simple as that of ours:

A property *P* is *partly essential* if and only if it is not purely essential but there exists a non-empty set S which is a [proper; J.R.] subset of every possible extension of P. ([Cmorej 1996], p. 253)

Formally:

$\equiv^{(o\varphi)\omega}$ [$\neg$[PurelyEssential$_w$ f]

$\wedge$ [$\exists.\lambda s$ [[NonEmpty' s] $\wedge$ [$\forall.\lambda w$' [$s \subset$ [ExtensionOf$_{w'}$ f]]]]]]

According to Cmorej's own definition of PurelyEssential, we get:

$\equiv^{(o\varphi)\omega}$ [$\neg$[$\forall.\lambda w$' [[[ExtensionOf$_{w'}$ f] = [ExtensionOf$_w$ f]]

$\wedge$ [NonEmpty' [ExtensionOf$_w$ f]]]]

$\wedge$ [$\exists.\lambda s$ [[NonEmpty' s] $\wedge$ [$\forall.\lambda w$'' [$s \subset$ [ExtensionOf$_{w''}$ f]]]]]]

and by adapted De Morgan's law for quantifiers we obtain:

$\equiv^{(o\varphi)\omega}$ [[$\exists.\lambda w$' $\neg$[True$_{w'}$ [λw'' [[[ExtensionOf$_{w''}$ f] = [ExtensionOf$_w$ f]]

$\wedge$ [NonEmpty' [ExtensionOf$_w$ f]]]]]]

$\wedge$ [$\exists.\lambda s$ [[NonEmpty' s] $\wedge$ [$\forall.\lambda w$''' [$s \subset$ [ExtensionOf$_{w'''}$ f]]]]]]

---

[31] The use of $\rightarrow$ would lead accepting also s.c. void properties (see below) as the trivial undefined property.

The second part of the main conjunction reads: in every world w''' there exists a class s which is non-empty' and it is a subclass of the extension of f (in w'''); the first part of the main conjunction: there exists a world w' such that it is not true that the extension of f in w'' is identical with its extension in w and which is non-empty'. Thus a property f has at least in one world as its extension class s' which has to be larger than s, in order that s is a subclass of s' but not identical with s'.

Another notable equivalence is:

$$\equiv^{(o\varphi)\omega} [\ [\exists.\lambda x\ [CannotLack_w\, x\, f]] \wedge [\exists.\lambda x'\ [CanLack_w\, x'\, f]]\ ]$$

Whereas $[\exists.\lambda x\ [CannotLack_w\, x\, f]]$ constructs T also when f is any trivial empty' or the trivial undefined property, the second part of the conjunction constructs F for them both.

Moreover, we can add one philosophically interesting definition:

$$=_{df} \text{ being a property } f \text{ which is essential and accidental}$$

$$\equiv^{(o\varphi)\omega} [\ [Essential_w\, f] \wedge [Accidental_w\, f]\ ]$$

which discloses the real character of partly essential properties. This breaks the old simply-conceived distinction essential / accidental properties.

We can also state with Cmorej ([Cmorej 1996], p. 253):

$$[NonEssential_{|w|}\, f] \equiv^{(o\varphi)\omega} [\ \neg[PurelyEssential_w\, f] \wedge \neg[PartlyEssential_w\, f]\ ]$$

Let us also highlight a few observations about essential properties, some of which have already been mentioned by Cmorej. Consider a construction of a property (say) "G", of the form:

$$\lambda w\ [\lambda x\ [\ [x = I_1] \vee [x = I_2] \vee ... \vee [x = I_j] \vee [F_w\, x]\ ]]$$

where "F" is an arbitrary non-trivial property. It is apparent, that there is a certain enumeration of individuals $I_1, ..., I_j$, which individuate the class $\{I_1, ..., I_j\}$.[32] The constructed property "G" is not inevitably partly essential – certain conditions must be fulfilled:

a) j is a number lower than the cardinality of the domain of individuals – for if the extension is always the class of all individuals, the property would be, when "F" is total, nothing other than the trivial universal property;

b) "F" must be total – for if it would not have an extension in certain world, than no individual could be in its extension, thus $[F_w\, x]$ would construct no truth-value and the whole disjunction will not return T to any individual constructed by x; or rather, $[True_w\ [\lambda w'\ [F_{w'}\, x]]]$ should be used to avoid possible partiality of "F";

---

[32] This construction has already appeared in [Cmorej 1996], p. 254. Nevertheless, the following conditions are not mentioned there (for Cmorej conceived only total mappings).

c) "F" must be also non-trivial – for otherwise there would be no changes of extensions of "G";

d) "F" must have in at least one world an extension in which there is at least one individual other than $I_1, ..., I_j$ – for total and non-trivial property having in its extensions only individuals from the class $\{I_1, ..., I_j\}$ would not affect the trivialness of "G".

Partly essential and purely essential properties have some invariant non-empty' subclass of each of their possible extensions, a subclass which makes an *essential kernel* of that property (this term was explicitly used by Cmorej in [Cmorej 2006], p. 141) and it is constructible by a construction like $\lambda x\ [\ [x = I_1] \vee [x = I_2] \vee ... \vee [x = I_j]\ ]$. Likewise, let us define:

a class *s* is an essential kernel of a property *f* $=_{df}$ *s* is non-empty' and in every possible
world *w'* it is a subclass of the extension of *f* in *w'*

$$[\textbf{EssentialKernel}_w\ s\ f] \equiv^{(o(o\iota)\varphi)\omega} [\ [\text{NonEmpty' } s] \wedge [\forall.\lambda w'\ [s \subset [\text{ExtensionOf}_{w'}\ f]]]\ ]$$

$$(\lambda w.\lambda s f.)$$

Now the concepts of "essential", "partly essential", and "purely essential" can be defined by means of EssentialKernel:

being an *essential* property *f* $=_{df}$ being a property such that there exists a class *s* which is
an essential kernel of *f*

$$[\text{Essential}_{|w|}\ f] \equiv^{(o\varphi)\omega} [\exists.\lambda s\ [\text{EssentialKernel}_w\ s\ f]]$$

Given the primitive concepts of Cardinality (assigning to classes the number of their members)[33] and the well-known relation (among numbers) ">" we can define:

being a *partly essential* property *f* $=_{df}$ being a property such that there exists a class *s*
which is an essential kernel of *f* and at the same time there does not exist a class
s' which is an essential kernel of *f* and which has greater cardinality than *s*, and at
the same time there exists a world *w'* such that there exists *x* such that *x* is not in *s*
but it is in the extension of *f* in *w'*

$$[\text{PartlyEssential}_{|w|}\ f] \equiv^{(o\varphi)\omega} [\exists.\lambda s\ [\ [\text{EssentialKernel}_w\ s\ f]$$

$$\wedge \neg[\exists.\ \lambda s'\ [\ [\text{EssentialKernel}_w\ s'\ f] \wedge [[\text{Card } s'] > [\text{Card } s]]\ ]\ ]$$

$$\wedge [\exists.\lambda w'\ [\exists.\lambda x\ [\ \neg[\ s\ x] \wedge [[\text{ExtensionOf}_{w'}\ f]\ x]\ ]]]\ ]]$$

---

[33] In a system using $\{\iota,o,\omega,\tau\}$ as its basis, cardinalities of classes are $(\tau(o\iota))$-objects. In our simpler system we must represent numbers by individuals (which is not so much intuitive given that unlike to numbers, successor function is not applicable to individuals, or taking into consideration that individuals are not linearly ordered, etc.).

It means the class *s* should be of the greatest cardinality among the (thinkably more than one) essential kernels of *f*. Nevertheless, in at least one world, there is an individual who is not in *s* but it is in the extension of *f*. Now:

being a *purely essential* property *f* =$_{df}$ being a property such that there exists a class *s* which is an essential kernel of *f*, and at the same time there does not exist a class *s'* which is an essential kernel of *f* and which has greater cardinality than *s*, and at the same time there does not exist a world *w'* such that there exists *x* such that *x* is not in the essential kernel of *f* but it is in the extension of *f* in *w'*

$$[\text{PurelyEssential}_{|w|} f] \equiv^{(o\varphi)\,\omega} [\exists.\lambda s \; [\; [\text{EssentialKernel}_w \; s \; f]$$

$$\wedge \neg[\exists. \lambda s' \; [\; [\text{EssentialKernel}_w \; s' \; f] \wedge [[\text{Card } s'] > [\text{Card } s]] \;]]$$

$$\wedge \neg[\exists.\lambda w' \; [\exists.\lambda x \; [\; \neg[[\text{EssentialKernel}_{w'} f] \; x] \wedge [[\text{ExtensionOf}_{w'} f] \; x] \;]]] \;]] \qquad {}^{34}$$

Void properties

There are trivial properties having either no extension or one kind of empty' class as its stable (non-varying) extension, i.e. the trivial undefined property and trivial empty' properties. They form a special group and can be aptly called *trivially void properties*; no individual can instantiate them:

being a *trivially void* property *f* =$_{df}$ being a property such that either in every world *w'* there does not exist a class *s* which is the extension of *f* in *w'*, or in every world *w'* the extension of *f* in *w'* is empty'

$$[\textbf{TriviallyVoid}_{|w|} f] \equiv^{(o\varphi)\omega} [\; [\forall.\lambda w' \; \neg[\exists.\lambda s \; [s = [\text{ExtensionOf}_{w'} f] \;]]]$$

$$\vee [\forall.\lambda w'' \; [\text{Empty' } [\text{ExtensionOf}_{w''} f] \;]] \;] \qquad (\lambda w.\lambda f.)$$

or simply:

$$\equiv^{(o\varphi)\omega} [\text{Empty}^{\text{Tr}}_{w} f]$$

On the other hand, there is a lot of non-trivial properties assigning alternatively empty' class and nothing at all, for which reason they are also void (as no individual can instantiate them). Thus void properties can be generally defined in the following way:

being a *void* property *f* =$_{df}$ being a property such that in every world *w'* there does not exist an individual *x* which is in the extension of *f* in *w'*

$$[\textbf{Void}_{|w|} f] \equiv^{(o\varphi)\omega} [\forall.\lambda w' \; \neg[\exists.\lambda x \; [[\text{ExtensionOf}_{w'} f] \; x]]]$$

---

[34] Definitions of EssentialFor, Essential, PurelyEssential, and PartlyEssential are generally not convertible for intensions which do not have classes of objects as their values. For example, individual offices do not have as extensions classes of individuals but simply individuals so there is no essential kernel of theirs at all. Individual offices can be total (and thus they can be trivial), they can be partial (and thus they can be undefined, i.e. trivially void), but then they can be conceived – after a careful reworking of the above definitions – as essential for certain individual, thus essential, however, not partly essential.

or just:

> $=_{df}$ being a property such that in every world w' the extension of f is empty'
>
> $\equiv^{(o\phi)\omega} [\forall.\lambda w' \, [\text{Empty}' \, [\text{ExtensionOf}_{w'} f]]]$

What follows is another equivalence:

> $=_{df}$ being a property such that in every world w' there does not exist an individual x which can instantiate f in w'
>
> $\equiv^{(o\phi)\omega} [\forall.\lambda w' \, \neg[\exists.\lambda x \, [\text{CanInstantiate}_{w'} \, x \, f]]]$

which can be simplified to:

> $\equiv^{(o\phi)\omega} \neg[\exists.\lambda x \, [\text{CanInstantiate}_{w} \, x \, f]]$
>
> $\equiv^{(o\phi)\omega} [\forall.\lambda x \, [\text{CannotInstantiate}_{w} \, x \, f]]$
>
> $\equiv^{(o\phi)\omega} [\forall.\lambda x \, \neg[\exists.\lambda w' \, [f_{w'} \, x]]]$ [35]

Now we can define (the concepts of) trivially void and non-trivially void properties just by means of:

> being a trivially void property $f =_{df}$ being a property f which is void and trivial
>
> $[\text{TriviallyVoid}_{|w|} f] \equiv^{(o\phi)\omega} [\, [\text{Void}_{w} f] \wedge [\text{Trivial}_{w} f] \,]$ $(\lambda w.\lambda f.)$
>
> being a *non-trivially void* property $f =_{df}$ being a property f which is void and non-trivial
>
> $[\textbf{NonTriviallyVoid}_{|w|} f] \equiv^{(o\phi)\omega} [\, [\text{Void}_{w} f] \wedge [\text{NonTrivial}_{w} f] \,]$

Note, however, that whereas it holds:

> being a void property $f =_{df}$ being a property f which is not essential and not accidental
>
> $[\text{Void}_{|w|} f] \equiv^{(o\phi)\omega} [\, \neg[\text{Essential}_{w} f] \wedge \neg[\text{Accidental}_{w} f] \,]$,

it does not hold that either solely trivially void properties, or just and only non-trivially void properties are non-essential and non-accidental properties (*cf.* also with the last paragraph of the section 'The Rose of Kinds of Properties').

Purely empirical properties

To suggest a definition of (the concept of) purely empirical properties it would be convenient to directly follow Cmorej's proposal:

> A property *P* is *purely empirical* if and only if *P* is empirical, but it is not essential.
>
> ([Cmorej 2001], p. 106)

Thus we have:

> being a *purely empirical* property $f =_{df}$ being a property f which is empirical and not partly essential

---

[35] In accordance with De Morgan's laws adapted for work with partial functions, the next equivalence should be $[\forall.\lambda x \, [\forall.\lambda w' \, \neg[\text{True}_{w'} \, [\lambda w'' \, [f_{w'} \, x]] \,]]]$.

$$[\textbf{PurelyEmpirical}_{|w|}\,f] \equiv^{(o\varphi)\omega} [\ [\text{Empirical}_w\,f] \wedge \neg[\text{PartlyEssential}_w\,f]\ ] \qquad (\lambda w.\lambda f.)$$

i.e.:

$=_{df}$ being a property f which is not trivial and not partly essential

$$\equiv^{(o\varphi)\omega} [\ \neg[\text{Trivial}_w\,f] \wedge \neg[\text{PartlyEssential}_w\,f]\ ]$$

It should be noted that it would not be sufficient to claim that purely empirical properties are non-essential properties for all void properties are also non-essential. To define purely empirical properties by means of NonEssential we need to exclude trivially void properties:

$=_{df}$ being a property f which is not essential and not trivially void

$$\equiv^{(o\varphi)\omega} [\ \neg[\text{Essential}_w\,f] \wedge \neg[\text{TriviallyVoid}_w\,f]\ ]$$

However, it needs to be pointed out that it does not hold that purely empirical properties are non-essential and accidental, because non-trivially void properties are not accidental, but still they are non-trivial (*cf.* also with the last paragraph of the section 'The Rose of Kinds of Properties ').

THE ROSE OF KINDS OF PROPERTIES

As we have seen, properties can be classified according to several fundamental criteria. We can illustrate the main classifications of properties that we have discussed in the following Rose of Kinds of Properties, which facilitates orientation in the kinds of properties we study. The West-East axis divides properties into essential properties in the South and non-essential properties in the North. All essential properties are total, but non-essential properties are total or partial. The North-South axis divides properties into non-trivial (empirical, contingent) properties in the West and trivial (non-empirical, constant) properties in the East. This results in a *purely empirical / partly essential / purely essential / trivially void quadruplet*. (For the present moment, let us ignore the content between the dotted and non-dotted circle.)

non-essential

essential

(total)
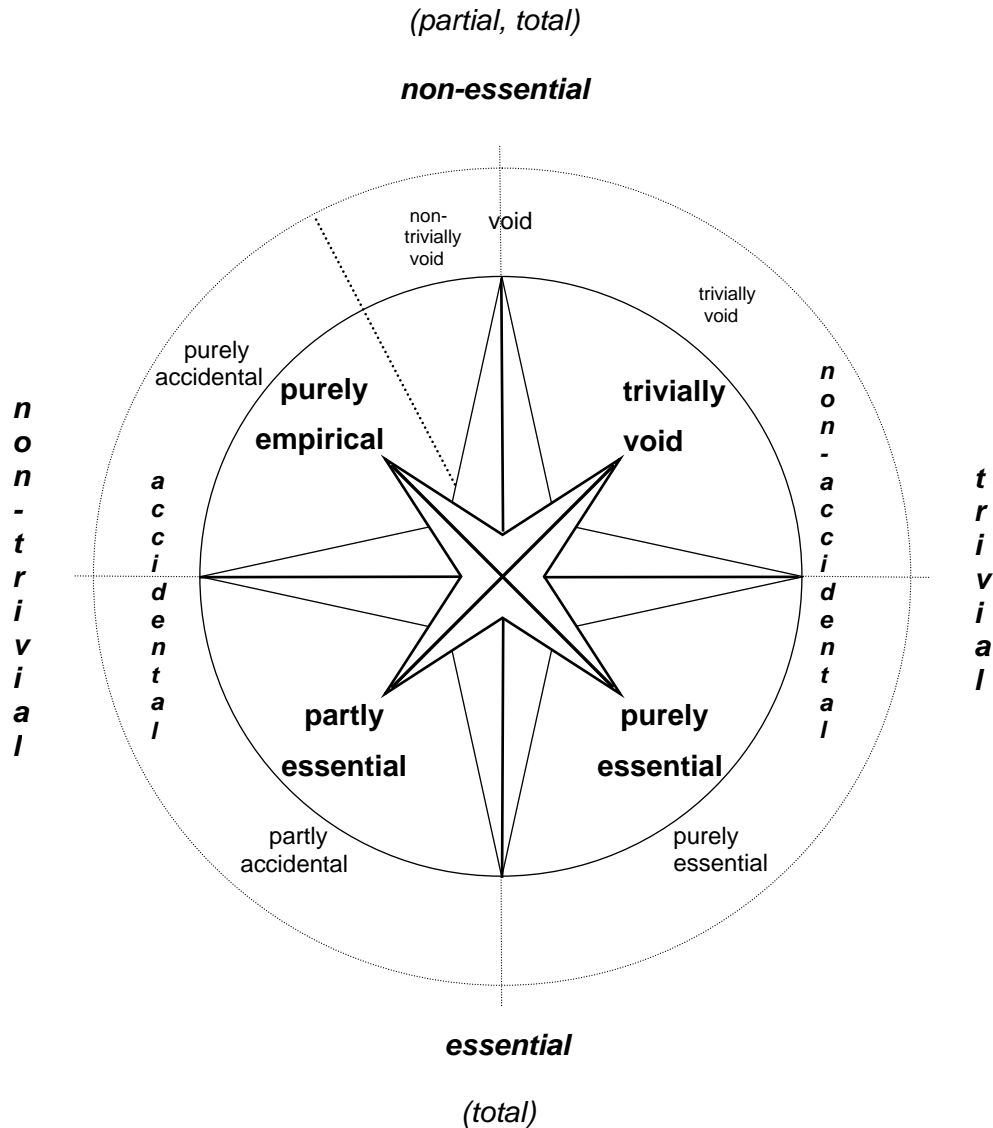
Selected facts diagrammed in the Rose of Kinds of Properties can be formalized as follows (some of them have already been formulated above). It is important to say that the present equivalences may serve in the same way as definitions (btw. we already know that all constructed intensions are trivial):

[Essential$_w$ f]	$\rightarrow$	[Total$_w$ f]	
[NonEssential$_w$ f]	$\rightarrow$	[[Total$_w$ f] $\vee$ [Partial$_w$ f]]	
[NonEssential$_w$ f]	$\leftrightarrow$	[[PurelyEmpirical$_w$ f] $\vee$ [TriviallyVoid$_w$ f]]	[36]
[Essential$_w$ f]	$\leftrightarrow$	[[PartlyEssential$_w$ f] $\vee$ [PurelyEssential$_w$ f]]	[37]
[NonTrivial$_w$ f]	$\leftrightarrow$	[[PurelyEmpirical$_w$ f] $\vee$ [PartlyEssential$_w$ f]]	
[Trivial$_w$ f]	$\leftrightarrow$	[[TriviallyVoid$_w$ f] $\vee$ [PurelyEssential$_w$ f]]	

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[36] It can be adjusted by use of Void instead of TriviallyVoid (but then, there has to be just $\vee$, not xor, in the definition).

[37] This fact was already stated by Cmorej ([Cmorej 2006], p. 141).

[PurelyEmpirical$_w$ f]	$\rightarrow$	[NonTrivial$_w$ f]	
	$\rightarrow$	[NonEssential$_w$ f]	
	$\rightarrow$	[[Total$_w$ f] $\vee$ [Partial$_w$ f]]	
	$\leftrightarrow$	[[NonEssential$_w$ f] $\wedge$ [NonTrivial$_w$ f]]	
	$\leftrightarrow$	[[NonEssential$_w$ f] $\wedge$ ¬[TriviallyVoid$_w$ f]]	
	$\leftrightarrow$	[[NonTrivial$_w$ f] $\wedge$ ¬[PartlyEssential$_w$ f]]	
	$\rightarrow$	[[NonEssential$_w$ f] $\wedge$ [Accidental$_w$ f]]	(not $\leftrightarrow$)[38]
[PartlyEssential$_w$ f]	$\rightarrow$	[NonTrivial$_w$ f]	
	$\rightarrow$	[Essential$_w$ f]	
	$\rightarrow$	[Total$_w$ f]	
	$\leftrightarrow$	[[Essential$_w$ f] $\wedge$ [NonTrivial$_w$ f]]	
	$\leftrightarrow$	[[Essential$_w$ f] $\wedge$ ¬[PurelyEssential$_w$ f]]	
	$\leftrightarrow$	[[NonTrivial$_w$ f] $\wedge$ ¬[PurelyEmpirical$_w$ f]]	
	$\leftrightarrow$	[[Essential$_w$ f] $\wedge$ [Accidental$_w$ f]]	
[TriviallyVoid$_w$ f]	$\rightarrow$	[Trivial$_w$ f]	
	$\rightarrow$	[NonEssential$_w$ f]	
	$\rightarrow$	[[Total$_w$ f] $\vee$ Partial$_w$ f]]	
	$\leftrightarrow$	[[NonEssential$_w$ f] $\wedge$ [Trivial$_w$ f]]	
	$\leftrightarrow$	[[NonEssential$_w$ f] $\wedge$ ¬[PurelyEmpirical$_w$ f]]	
	$\leftrightarrow$	[[Trivial$_w$ f] $\wedge$ ¬[PurelyEssential$_w$ f]]	
	$\rightarrow$	[[NonEssential$_w$ f] $\wedge$ [NonAccidental$_w$ f]]	(not $\leftrightarrow$)
[PurelyEssential$_w$ f]	$\rightarrow$	[Trivial$_w$ f]	
	$\rightarrow$	[Essential$_w$ f]	
	$\rightarrow$	[Total$_w$ f]	
	$\leftrightarrow$	[[Essential$_w$ f] $\wedge$ [Trivial$_w$ f]]	
	$\leftrightarrow$	[[Essential$_w$ f] $\wedge$ ¬[PartlyEssential$_w$ f]]	
	$\leftrightarrow$	[[Trivial$_w$ f] $\wedge$ ¬[TriviallyVoid$_w$ f]]	
	$\leftrightarrow$	[[Essential$_w$ f] $\wedge$ [NonAccidental$_w$ f]]	

Another vital point to be made here is that the Rose of Kinds of Properties is applicable, being adequately type-theoretically adapted, to various properties of properties of individuals, properties of such properties, and so on up. In addition, the Rose of Kinds of Properties is also applicable as far as properties of other objects (from our basis) than individuals are concerned (i.e. properties of truth-values, properties of possible worlds); and, after an extension of the

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[38] Note the asymmetry of equivalences/implications which is not diagrammed in the Rose (*cf.* also the last section of this study).

basis, also to properties of real numbers[39]). Yet, this is not enough: properties of n-tuples of individuals (or n-tuples of truth-values, or worlds, or time-moments) also can be classified according to the Rose. Note also that (first-order, second-order, etc.) constructions can possess properties too.

Every property of properties discussed (i.e. "PurelyEmpirical", "PartlyEssential", "TriviallyVoid", "PurelyEssential") is total$^{(o\varphi)\omega}$, trivial$^{(o\varphi)\omega}$, and purely essential$^{(o\varphi)\omega}$ (these three attributes are properties ascribable to properties of properties):

$[\ [h{=}\text{PurelyEmpirical}] \vee [h{=}\text{PartlyEssential}] \vee [h{=}\text{TriviallyVoid}] \vee [h{=}\text{PurelyEssential}]\ ]$

$\qquad \rightarrow [\text{Total}^{(o\varphi)\omega}{}_w\, h]$ $\hfill (\lambda w.\lambda h.)$

$\qquad \rightarrow [\text{Trivial}^{(o\varphi)\omega}{}_w\, h]$

$\qquad \rightarrow [\text{PurelyEssential}^{(o\varphi)\omega}{}_w\, h]$

(It does not mean, however, that the only properties attributable to various properties of properties are total$^{(o\varphi)\omega}$, trivial$^{(o\varphi)\omega}$, purely essential$^{(o\varphi)\omega}$.)

Note that intensions assigning individuals (i.e. individual offices), or truth-vales (i.e. propositions), etc., do not share all the features of the properties of individuals, etc. Such offices-intensions can be divided only thanks to the fact of being total / partial, trivial / non-trivial, and purely essential / purely empirical / void. In other words, partly essential properties do not come into account here in any sense simply for the fact that, for instance, individual offices cannot have classes as values.

The existence of the non-trivially void properties may move somebody to redraw the Rose of Kinds of Properties to such a Rose, according to which the North-East quadrant contains all void properties. This would, however, distort the understanding of the North-South axis as the border-line between non-trivial properties in the West and trivial properties in the East. Therefore, we suggest to the potential theoretician to save the Rose as it is but to accept the borders marked by dotted-lines plus inscriptions in smaller font. Then there arises a 'clock-face' reading of the Rose where the section 'five minutes to twelve until quarter past' contains all void properties. The section 'quarter to until 5 minutes to twelve' contains those purely empirical properties which can be called *purely accidental properties*:

$[\textbf{PurelyAccidental}_{|w|}\, f] \equiv^{(o\varphi)\omega} [\ [\text{Accidental}_w\, f] \wedge \neg[\text{Essential}_w\, f]\ ]$ $\hfill (\lambda w.\lambda f.)$

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[39] When time-moments are admitted, intensions should be conceived as functions from possible worlds to chronologies of objects (chronology is a mapping from time-moments, i.e. real numbers). Of course, new relations among properties can be investigated and other properties of such properties can be defined. Some of them (namely mildly empirical and persistent properties) were recognized by Cmorej in [Cmorej 2001].

The South-West, i.e. the section 'half past until quarter to', is delimited for partly essential properties which can be now viewed as *partly accidental properties*, i.e. properties that are accidental and essential:

$$[\textbf{PartlyAccidental}_{|w|}f] \equiv^{(o\varphi)\omega} [\ [Accidental_w f] \wedge [Essential_w f]\ ]$$

i.e.:

$$\equiv^{(o\varphi)\omega} [PartlyEssential_w f]$$

(Of course, purely essential properties still remain in the South-East, i.e. in the 'quarter past until half past' section.) Now the following can be easily read from such a modified reading of the Rose:

$$[Accidental_{|w|}f] \equiv^{(o\varphi)\omega} [\ [PurelyAccidental_w f] \vee [PartlyAccidental_w f]\ ]$$

and:

$$[NonAccidental_{|w|}f] \equiv^{(o\varphi)\omega} [\ [PurelyEssential_w f] \vee [Void_w f]\ ]$$

In this conception, an important border between accidental and non-accidental properties is identifiable just with the broken line drawn as watch hands indicating 'five minutes to six o' clock'. In this way we can see the quadruplet of *purely accidental / partly accidental / purely essential / void properties* that may be welcome by some philosophers.[40] [41]

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[42] Reprinted in Cmorej, Pavel (2001): *Na pomedzí logiky a filozofie*. Bratislava: Veda, 125-149.

[43] Originally published as Cmorej, Pavel (1988): Esencializmus versus antiesencializmus, *Philosophica* 26, Univerzita Komenského: Bratislava, 163-202.