Benoit B. Mandelbrot

1924–2010

BIOGRAPHICAL *Memoirs*

A Biographical Memoir by
Michael Frame

NATIONAL ACADEMY OF SCIENCES

BENOIT B. **MANDELBROT**

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By Michael Frame

Benoit Mandelbrot was born in Warsaw, Poland, in 1924. His father was in the clothing trade but was also a well-read and scholarly man, an admirer of Dutch philosopher Baruch Spinoza and German-American mathematician and electrical engineer Charles Steinmetz. Benoit's mother graduated first in her class from the Imperial University of Warsaw Medical School, where she chose dentistry as a specialty because it was more compatible with raising a family. The environment provided by Benoit's parents and extended family encouraged scholarship. Young Benoit was especially fond of maps and chess.

Benoit lived the first 12 years of his life at the culturally rich Warsaw street address of 14 Ulica Muranowska. A good indicator of the atmosphere there was the company at a dinner one night in June of 1930. Standing in the photo are Benoit's Aunt Helena Loterman, her husband (who was Benoit's tutor), and Benoit's Uncle Leon, editor of a Warsaw newspaper. Seated, left to right, are Benoit's Uncle Szolem Mandelbrojt, Benoit's father, Arnaud Denjoy, Jacques Hadamard, Benoit's Grandfather Szlomo, and Paul Montel. Mandelbrojt, Denjoy, Hadamard, and Montel were among the leaders of French mathematics in the early 20th century. Little wonder that the environment that this dinner typified would draw Benoit into a life of science.

In 1936 the family fled the troubles developing in Poland and moved to Paris. But the same troubles followed, and in 1939 the family left Paris for the town of Tulle, in a region Benoit later referred to as the Appalachia of France. In fact, Benoit credited the generosity of the people of Tulle for saving his life. Still, in 1943 Tulle became too dangerous and the family split up; Benoit and his brother Léon went to Lyon, where they survived many close calls. At the Lycée du Parc, Benoit discovered his gift for reformulating and solving algebra problems by geometry. Even at this early stage, Benoit's thinking was predominantly visual: shapes moved fluidly and effortlessly in his mind.

In 1944, the family reunited and returned to Paris. The only student in all of France that year to solve every problem on the university entrance exam, Benoit was admitted both to the École Normale Supérieure (ENS) and to the École Polytechnique. Following uncle Szolem's advice, Benoit enrolled in the ENS, but after a day realized the influence of the mathematics collective Bourbaki[1] was too strong there and the culture too foreign to his visual thinking, so he resigned his position and entered the Polytechnique. There Benoit's teachers included Gaston Julia and Paul Lévy, both of whom introduced him to ideas that would be central to his later work.

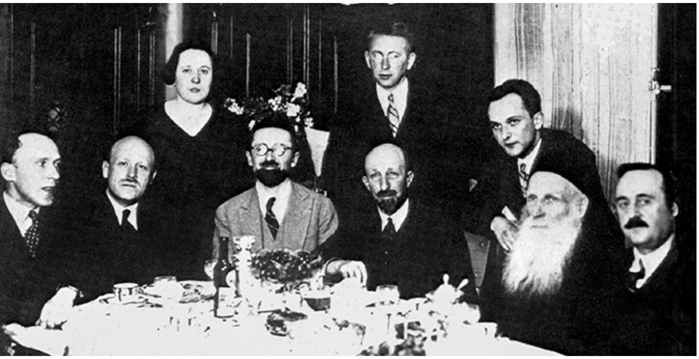

Figure 1: Interesting Dinner Guests.

On the advice of applied mathematician Roger Brard at the Polytechnique, Benoit went to Caltech in 1947 to study aerodynamics, and Richard Tolman's statistical mechanics course there proved especially useful for Benoit's later work. While at Caltech, Benoit met Carlton Gajdusek and Max Delbrück, both of whom later won Nobel Prizes in Physiology and Medicine. Gajdusek became and remained a close friend; he was one of the principal speakers at Benoit's 70th birthday conference. When Benoit was at Caltech, Delbrück was developing molecular biology there, in part by gathering people to work in the field. Later, Benoit's approach to creating fractal geometry was different: he had a crystalclear image of the broad strokes of fractals, and he did much of the early work alone. Had fractal geometry grown in the way Delbrück grew molecular biology, involving many people from the start, likely the field would be much different today. But better? Maybe not. Benoit's early solitary work has left a strong imprint that still contributes greatly to the energy of the field.

Benoit earned his Ph.D. in mathematical sciences from the University of Paris in 1952. His dissertation, *Games of communication*,[2] was a study of Zipf 's law of word frequencies and of statistical thermodynamics. Benoit's work on word frequencies had begun by chance. A visit to his Uncle Szolem in 1951 ended as usual with Benoit's request for something to read on the long Metro ride home. From his wastebasket, Szolem retrieved Joseph Walsh's *Scientific American* review of George Zipf 's book *Human behavior and* *the principle of least effort*. The review mentioned Zipf's law of word frequencies: For any sufficiently long text, denoted by ρ the rank of how often a word occurs in the text, and by P the probability of that word in the text. Then $P \propto 1/\rho$, independently of language or the literary skills of the author. On the Metro ride, Benoit realized that as stated, Zipf's law could not be correct. His improvement was

$$P = F(\rho + V)^{-1/D}$$

Here D is a dimension, the scaling exponent of the language's lexicographic tree, and V is a model parameter. Because Zipf's law is independent of grammar, Benoit linked it to information theory and to statistical thermodynamics. This led him to interpret D as the temperature of discourse, a variable that may distinguish between the literary complexity of text or author.

After receiving his doctorate, Benoit worked at Philips Electronics, where his task was to provide theoretical background for engineers developing color television. In so doing, he applied spectral analysis techniques learned from Uncle Szolem and from turbulence studies at Caltech.

In 1953 Benoit took a postdoctoral position at MIT's Research Laboratory in Electronics, an energetic environment he enjoyed, in part because of Noam Chomsky's early work in linguistics. During 1953–54, Benoit worked at the Institute for Advanced Study as John von Neumann's last postdoc. Benoit's time there included an unusual, and instructive, evening. Invited by J. Robert Oppenheimer to give the first lecture in a series of general interest talks, Benoit panicked as luminaries from math and physics (but not, thank goodness, Albert Einstein) filled the room. Quickly discarding his planned presentation as far too simple for that audience, Benoit tried to construct a more technical lecture in real time, and failed rather miserably. After his talk, and a few polite questions by friends, Benoit was petrified when Otto Neugebauer announced that this was the worst lecture he had ever heard—that it made absolutely no sense at all. But then Oppenheimer gave a brief crystal-clear summary of Benoit's main points and von Neumann added some more detail. Benoit and I discussed this event several times. I believe it was the beginning of his appreciation of pedagogical style.

Benoit returned to Paris in 1954 and assumed the rank of junior research professor at the Centre National de la Recherche Scientifique. There he deepened his understanding of probability theory by attending Paul Lévy's minicourses, where he learned techniques that helped lay the foundations of some of his work on finance.

In 1955, Benoit married Aliette Kagan. In addition to being the mother of their two sons, Laurent and Didier, Aliette was Benoit's constant helper. She proofread his papers, spent many hours in Yale's libraries tracking down often obscure references (some so old that they could not withstand the stress of being photocopied) for the scrupulously careful Benoit, accompanied him on travels, served as his sounding board on thorny issues of scientific politics, and was his constant source of encouragement during the many years he worked in isolation.

Figure 2: Benoit and some students after a lecture at MIT.

In the memorial for Benoit at Yale, Ralph Gomory, Benoit's director at IBM, characterized Benoit's life as courageous; virtually ignored for many years, he dreamed of starting a Keplerian revolution in science, of finding a new way to describe much of the world. Aliette's unwavering support contributed significantly to Benoit's persistence and eventual realization of his dream.

After a postdoc at the University of Geneva, which included working with Jean Piaget for a short time, in 1957 Benoit began teaching as an assistant professor of mathematics at the University of Lille. This position turned out to be a less than perfect match. In his memoir *The Fractalist,*[3] Benoit wrote: "Teaching—even in a university—is a hard profession. One had better start practicing much earlier than I did."

So in 1958, Benoit left France for a summer job at the IBM Thomas J. Watson Research Center in Yorktown Heights, New York. This "summer job" continued for 35 years, though the path from 1958 to 1993 was far from direct and included many detours.

Before continuing, I must mention that although Benoit did not teach often, he had an immense interest in education. For one thing, with retired high-school math teacher Nial Neger and math and music teacher Harlan Brothers, Benoit and I ran a series of NSF funded summer workshops at Yale for high school and college teachers.

But more telling are the picture above and this quotation, both from *The Fractalist*:[4]

> *Words from a charming young lady seemingly representing a group of college students who had packed a lecture I had just given: 'We can't believe that we could actually hear you discuss how part of our school-work had first come to your mind. To shake your hand would be a strange experience...a big event.' Of course, I was glad to shake that young lady's hand. Uncanny forms of flattery! Each lifted me to seventh heaven! Truly and deeply, each marked a very sweet day! Let me put it more strongly: occasions like that make my life.*

Early in his IBM career, Benoit worked with Jay Berger on the distribution of errors in telephone lines—an effort that was important to IBM for getting computers to talk with one another. Standard noise statistics did not fit the data, but Benoit's ideas about scaling were effective and saved IBM from investing in an approach that could not succeed.

But Benoit's intellectual curiosity was free to cover a wide range of subjects. For example, with the programming skill of his IBM colleague Richard Voss, Benoit constructed artificial images of islands and coastlines. The components of fractal geometry were beginning to assemble themselves in Benoit's mind. And a trip to Harvard to lecture on Pareto's law of income distribution led to a surprise discovery that Benoit's Pareto graph was very similar to the cotton-price graphs of his host, Hendrik Houthakker. This was the beginning of Benoit's work on price variations.

In the 1950s and '60s, the standard finance models were based on Louis Bachelier's 1900 thesis that sketched some of the mathematical foundations of Brownian motion, later rediscovered and elaborated by Einstein and Wiener. But without ad hoc fixes, financial data did not support two central properties of Brownian motion increments: statistical independence and normal distribution. Benoit's first paper to present his alternate theory was "On the variation of certain speculative prices" (1963). Though he had not yet developed the language of fractals, Benoit already saw the main point: by erasing the time and price scales, the daily, monthly, and annual records look the same. Yet most of the financial world did not embrace Benoit's work. Based on Lévy stability, it was too complicated; and besides, with enough epicycles, Brownian motion could be made to dance to any tune[5]. There was no perceived need for a model that included large jumps, a so-called long-tailed distribution, when with enough tinkering any large jump could be explained after the fact. It is telling that none of Benoit's critics could predict large jumps before they happened. In Benoit's view, some of these large jumps did not have external causes one could identify: they were a consequence of the scaling distribution of the jumps. Mostly, the inertia of orthodoxy won out against Benoit's models.

One of Benoit's economics lectures reminded an audience member of the variability of river discharges. This led Benoit to Harold Hurst's studies of the flooding of the Nile. Elaborating on Hurst's work, Benoit, together with John Van Ness and James Wallis, developed a version of Brownian motion that included memory. This version was called *fractional Brownian motion* (fBm), not fractal Brownian motion, because the notion of fractal had not yet settled into place in his thinking. Fractional Brownian motion of index α, $0 < \alpha < 1$, is a random process $X(t)$, continuous with probability 1, and having increments $X(t + h) - X(t)$ normally distributed with mean 0 and variance $h^{2\alpha}$. From this, the expected value of the product of two increments

$$\mathbb{E}(X(t) - X(0))(X(t + h) - X(t)) = ((t + h)^{2\alpha} - t^{2\alpha} - h^{2\alpha})/2$$

Then $\alpha = 1/2$ recovers standard Brownian motion with independent increments. For $\alpha > 1/2$, products of increments tend to have the same signs. The graph is smoother and this is persistent fBm. For $\alpha < 1/2$, products of increments tend to have opposite signs. The graph is rougher and this is antipersistent fBm. Like Brownian motion, fBm is scaling, but independent increments are replaced by dependent increments.

After spending 1963–64 in the Harvard economics department, in 1964–65 Benoit moved to applied sciences at Harvard. His course, Topics in Applied Mathematics, covered Benoit's work on hydrology and his collaboration with Berger on clustering of errors in telephone circuits.

Around this time, Benoit began developing a multifractal theory of turbulence. He saw Robert Stewart's research on submarine data and turbulence as related to his work with Berger on error clustering. Back at IBM, he realized that Hausdorff dimension was not only the measure he sought of volatility in financial records but also of roughness in turbulence. The intermittence of turbulence led to multifractals, which Benoit used later in his theory of prices and in many other fields.

A well-received 1973 lecture at the Collége de France in Paris contributed to Benoit's being made an IBM Fellow in 1974. The additional autonomy of this position gave Benoit time to work on his 1975 French book, *Les objets fractals: Forme, hazard, et dimension*, and the more extensive 1977 English version, *Fractals: Form, chance, and dimension*.

Images of the Mandelbrot set are among the most familiar from 20th century mathematics, and they are some of the very few mathematical images that made the jump to popular culture.

In 1979–80 Benoit was back at Harvard, this time in the mathematics department. In the spring semester of 1980 he taught his first—*the* first—fractal geometry course, which included the initial images of what we now call the Mandelbrot set. The first publication of the Mandelbrot set appeared in the proceedings of a 1979 New York Academy of Sciences conference on nonlinear dynamics. When Benoit repeated his New York Academy lecture at Harvard, the audience included David Mumford, who asked if these methods could be adapted to produce pictures of limit sets of Kleinian group actions. In fact, Benoit already had developed this algorithm, in some special cases. Mumford, S. J. Patterson, and David Wright began computer experiments on Kleinian limit sets. One of Mumford's first illustrations is plate 178 of Benoit's 1982 book *The fractal geometry of nature.* With Caroline Series, Mumford and Wright wrote a beautiful book, *Indra's Pearls: The vision of Felix Klein*, on Kleinian limit sets.

Images of the Mandelbrot set are among the most familiar from 20th century mathematics, and they are some of the very few mathematical images that made the jump to popular culture. This transition has not been without a few rough spots. For example, some programmers who did not understand the necessity of starting the quadratic iteration with the critical point of $f(z) = z^2 + c$ simply modified their programs to begin the iterations at some other points. The resulting shapes lacked the pleasing symmetry of the Mandelbrot set, leading Benoit to call them "Mandelbrot roadkill." Despite these hiccups, the complex and surprising images of the Mandelbrot set, produced as they are by a short formula or a few lines of computer code, reinvigorated complex dynamics and brought mathematics into the homes of many high-school students and their families. The importance of this occurrence cannot be overstated.

Underneath these beautiful pictures lie very deep, and equally beautiful, mathematics. From work of Gaston Julia and Pierre Fatou, Benoit knew that quadratic Julia sets are either connected or totally disconnected. Parts of the Mandelbrot set looked like parts of Julia sets, so Benoit asked if the Mandelbrot set is connected. The answer, "Yes," was proved by Adrien Douady and John Hubbard.[6] The more closely one examines the Mandelbrot set boundary, the more filled in it appears. This observation and some calculations led Benoit to conjecture that the Mandelbrot set boundary has Hausdorff dimension 2, later proved by Mitsuhiro Shishikura.[7] The central remaining conjecture, posed by Douady and Hubbard, is that the Mandelbrot set is locally connected. Many other open questions, including the hyperbolicity conjecture and the topological equivalence of the Mandelbrot set with the abstract Mandelbrot set of the Lavaurs algorithm, would be answered by a positive resolution of the local connectivity conjecture. Fields Medalists Jean-Christoph Yoccoz and Curtis McMullen have established local connectivity at many points of the Mandelbrot set, but the full conjecture remains open.

The first book on fractals, Benoit's Fractals: *Form, chance, and dimension*—translated from the French *Les objets fractals*—was expanded in 1982 to *The fractal geometry of nature (FGN)*. This book has had a tremendous influence on many, including the author of this biographical memoir. No standard mathematics or science text, or really any kind of text at all, this book is a meditation, frustrating to some expecting traditional exposition but inspiring to those who approach it as a source of fascinating questions.

On page 243 of *FGN* we find Benoit's conjecture about the dimension of the boundary of a planar "Brownian cluster"—constructed from a Brownian walk by arranging that it end at its starting point. Filling the region bounded by the cluster gave a shape that reminded Benoit of islands natural and artificial. Hard calculations and numerical experiments on this boundary gave a dimension very close to 4/3; Benoit conjectured that the dimension of a Brownian cluster boundary is exactly 4/3. The 4/3 conjecture was proved by Gregory Lawler, Oded Schramm, and Wendelin Werner in 2000. For this work, Werner, the youngest of the three, was awarded the Fields Medal.

In 1982, Dann Passoja, a metallurgist, began discussions with Benoit about using fractal dimension to measure roughness. Other sensations already had their methods of quantification: heat and cold by temperature, loudness by sound amplitude, pitch by sound frequency, heaviness by weight, color by light frequency, brightness by light intensity, sour or sweet by pH. But roughness was not quantified. Root mean square deviation above the average height of a metal fracture was not reproducible. Experiments confirmed that dimension varies directly with perceived roughness—the rougher the surface, the higher its dimension—and dimension was reproducible under repetition of experiments. Published in *Nature* in 1984, the paper of Mandelbrot, Passoja, and Alvin Paullay[8] introduced the idea of fractal dimension as a measure of roughness, a theme Benoit continued to develop throughout his career. Indeed, much of his earlier work had involved just this notion, though it was not emphasized then. In his memoir *The Fractalist*,[9] Benoit wrote, "I realized that something I had long been stating in footnotes should be put on the marquee. I had engaged myself, without realizing it, in undertaking a theory of roughness."

Because it figures so centrally in much of Benoit's work, here we take a moment to explore the notion of dimension. At first glance, the variety of concepts called "dimension" can be quite bewildering. Topological dimension, boxcounting dimension, Hausdorff dimension, packing dimension, not to mention familiar spatial dimension, and more. Good references for careful definitions and relations among these dimensions can be found in the books of Kenneth Falconer.[10] The simplest definition of dimension is that of a self-similar set $A = A_1 \cup \cdots \cup A_N$, where each $A_i = T_i(A)$ for some similarity transformation T_i. If all these transformations have the same scaling factor r, and provided the $T_i(A)$ and $T_j(A)$ don't overlap too much,[11] the Hausdorff dimension d of the fractal A is

$$d = \frac{\log(N)}{\log(1/r)}$$

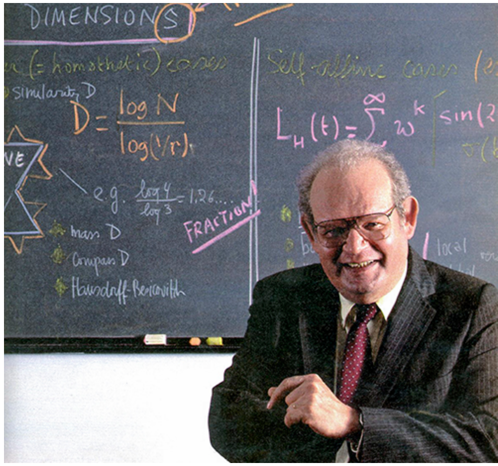

Figure 3. Benoit Lecturing at Harvard.

If different scaling factors r_i occur for the similarity transformations T_i, and again the overlap is not too severe, then the Hausdorff dimension is the solution of the Moran equation,[12]

$$r_1^d + \cdots r_N^d = 1$$

The Moran equation can be extended to many other kinds of fractals, including those for which only some compositions of the T_i are allowed,[13] or the scaling factors are chosen randomly,[14] or with nonlinear transformations, provided the derivatives of the transformations are bounded above and below.[15]

One direction in which it appears that the Moran equation cannot be generalized is self-affine fractals,[16] where different scalings occur in different directions. Some special cases

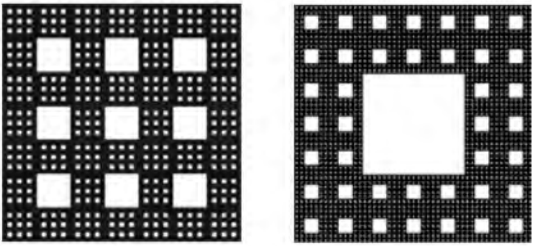

Figure 4. Sierpinski carpets of the same dimension, different lacunarities.

are known: Cartesian products of Cantor sets of different dimensions; the path of Brownian motion; and cases worked out by McMullen, Falconer, Gatzouras, and Lalley,[17] among others.

Fractal dimension is not the whole story. Both Sierpinski carpets of Figure 4 are composed of $N = 40$ copies scaled by $r = 1/7$, so both have dimension log(40)/ log(7). However, their pieces are differently arranged; the gaps, or lacunae, are placed more uniformly on the left, more clumped on the right. To quantify this difference, Benoit introduced *lacunarity*,[18] based on the approach of Minkowski and Bouligand for computing box-counting dimension d. For a set $A \subset \mathbb{R}^n$, A_ε is the ε-thickening of A and vol(A_ε) the n-dimensional volume of A_ε. Minkowski and Bolligand showed that as $\varepsilon \to 0$, vol(A_ε) scales as $\Lambda(\varepsilon)\varepsilon^{n-d}$. The lacunarity is related to the reciprocal of the prefactor $\Lambda(\varepsilon)$, though not in a simple way, for as $\varepsilon \to 0$, the limsup and liminf of $1/\Lambda(\varepsilon)$ differ for both carpets. One approach is logarithmic averaging, which gives the left fractal the lower lacunarity, as expected. Others[19] have explored this notion, but more work remains to be done.

Many, including Benoit, viewed the main strength of fractals to be their applicability to the natural sciences. Because Hausdorff dimension is constructed from a measure, calculating it for experimental data is not straightforward. In these situations, the shape whose dimension we wish to compute is covered with boxes of side length $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_N$. For each i we count $N(\varepsilon_i)$, the minimum number of these boxes needed to cover the shape. The scaling hypothesis is that $N(\varepsilon_i) \propto (1/\varepsilon_i)^d$. We test the scaling hypothesis by plotting the points $\log(1/\varepsilon_i)$, $\log(N(\varepsilon_i))$. If the points lie close to a straight line, this occurrence not only supports the scaling hypothesis but also, with some care, the slope of the line estimates a dimension, called the box-counting dimension. This is the measure of the roughness of coastlines and surfaces; our pulmonary, circulatory, and nervous systems; and clouds, lightning, and all of messy nature that is about equally rough no matter where we look. Also, the dimension d in the Minkowski-Bouligand approach is the box-counting dimension.

If the roughness of a fractal varies with position, a single dimension does not suffice. For this situation, Benoit introduced multifractals.[20] Multifractals characterize not a shape but a measure, μ, supported on the shape. The coarse Hölder exponent of a side length ε box intersecting the support of the measure is $\alpha = \log(\mu(\text{box}))/\log(\varepsilon)$. For given α and ε,

$$f\varepsilon(\alpha) = \frac{\log(N_\varepsilon(\alpha))}{\log(1/\varepsilon)}$$

where $N_\varepsilon(\alpha)$ is the number of side length ε boxes having coarse Hölder exponent α. As $\varepsilon \to 0$, we obtain the $f(\alpha)$ curve, a plot of how the dimension of the iso-α strata of the measure varies with α. Computing the $f(\alpha)$ curve from experimental data is challenging, but several efficient schemes, including a promising approach using wavelets,[21] have been developed. Multifractals contribute to our understanding of fluid turbulence, the distribution of natural resources and of data traffic on the web, the intervals between our heartbeats, and so much more.[22] For the study of nonhomogeneous roughness, multifractals are a more nuanced tool than dimension.

Another application of multifractals, of great interest to Benoit but still at an early stage of development to this day, is the notion of negative dimension.[23] The basic idea is simple, a consequence of the formula for the dimension of intersections of sets. For sets A and B lying in $\mathbb{R}^n$, typical placements of A and B have $d(A \cap B) = d(A) + d(B) - n$, with the usual understanding that a negative result signals disjointness of A and B. Intuition can suggest that a more negative result indicates more room for the sets to miss one another. Some experiments[24] on one-dimensional sections through turbulent fluid flow have given negative dimensions. This project was much on Benoit's mind at the end of his career; he hoped someone would take it up.

In 1987 Benoit began his long association with Yale, half-time because of his continuing ties to IBM, first as Abraham Robinson Adjunct Professor of Mathematical Sciences and then, from 1999, as Sterling Professor of Mathematical Sciences, Yale's highest academic rank. In 2004 he became Sterling Professor Emeritus. Throughout his time at Yale, Benoit maintained his association with IBM, so he and Aliette continued to live in Scarsdale, NY. Usually they were at Yale two days a week—days filled by meetings with visitors, other Yale faculty, and students. Often, early in the fall semester as freshmen walked down the hall past my office, I'd hear a gasp when they passed Benoit's door, followed by some variation of, "What? Mandelbrot works here?" Benoit's public lectures were packed. Once a semester he gave guest lectures in my fractal geometry class, doubling the class enrollment for that one day.

One of our long-term projects at Yale was the teacher-training summer workshop program, originally intended as a vehicle for providing teachers with resources for designing their own courses, or course modules, on fractal geometry. These workshops evolved into curriculum-development sessions in which we found ways to use parts of fractal geometry to motivate topics already in the curriculum. The geometry of plane transformations excites few students, but finding the transformations to construct a given fractal is an addictive game. Why do we need logarithms? Fractal dimension. Complex numbers? The Mandelbrot set.

Benoit spent a day of each workshop with the teachers. When he entered, the room fell silent. Maybe they hadn't really believed Nial Neger and me when we said Mandelbrot himself would be there that day. From the teachers' perspective, here was the world's most famous living mathematician in the room with them. Talking with them. Yet Benoit preferred discussions to lectures, even though some of his answers to questions took a path through many nested footnotes. So to put the teachers at ease, Benoit began with a humorous comment. "Sometimes mathematicians ask me what is the hardest theorem I've proved. Please don't ask me that. I prove only simple theorems." Pause… did he really say that? "But I reserve the right to ask very hard questions." Laughter and energetic discussions followed. These were wonderful experiences for all, and especially for me, watching how easily, how naturally, Benoit interacted with the teachers.

For a long time, Benoit was interested in financial data patterns. His contributions included fat tails,[25] scaling,[26] long-term dependence,[27] the existence of financial bubbles,[28] the notion of trading time,[29] and applications of multifractals[30] to economics. For example, his Trading Time Theorem—that every multifractal cartoon is fractional Brownian motion in multifractal time—has as its basic step a clever application of the Moran equation. In the 1990s he began emphasizing multifractal finance cartoons.[31] Always careful to point out that these cartoons were not an attempt to represent the dynamics underlying the market, Benoit did seek a simple way to reproduce market statistics. In this he succeeded, as shown by the number of seasoned traders who were unable to distinguish real data from fake data generated by Benoit's cartoons. The main lesson of these cartoons, unheeded by many, and at a cost felt by us all, is that market risk scales with time: over longer periods, expect wider swings. At Benoit's memorial at Yale, both Martin Shubik and Nassim Taleb commented that the 2008 crash could have been avoided if financiers had taken to heart Benoit's advice about risk.

Benoit's life was quite busy. His other positions included: from 1997, membership in Yale's Cowles Foundation for Research in Economics; in 1999, appointment as a G. C. Steward Visiting Fellow at Gonville and Caius Colleges, Cambridge, UK; and from 2005 to 2007, Battelle Fellow of the Pacific Northwest National Laboratory.

In addition to about 20 honorary doctorates, Benoit accumulated many other honors and awards. Among them were: election as a fellow of the American Academy of Arts and Sciences (1982); the Barnhard Medal (1985); the Franklin Medal (1986); election as a foreign associate of the U.S. National Academy of Sciences (1987); in 1988, the Steinmetz Medal, the Science for Art Prize, the Caltech Alumni Distinguished Service Award, the Humboldt Preis, and election as an honorary member of the United Mine Workers Union;[32] in 1989, the Harvey Prize and election as a chevalier of the National Legion of Honor, Paris; the Nevada Prize (1991); the Wolf Prize in Physics (1993); the Honda Prize (1994); Medaille de Vermeil de la Ville de Paris (1996); in 1999, the John Scott Award and a Sterling Professorship of Mathematical Sciences at Yale; the Lewis Fry Richardson Award (2000); in 2002, the Sven Berggren Priset and the William Proctor Prize; the Japan Prize (2003); election as a member of the American Philosophical Society (2004); in 2005, the Sierpinski Prize and the Casimir Frank National Science Award; in 2006, the Einstein Public Lecture of the American Mathematical Society and election as an officer in the National Legion of Honor, Paris. Mandelbrot became a U. S. citizen in 2000 and became a member of the National Academy of Sciences at that point.

From very early, Benoit wanted to start his own Keplerian revolution by describing a new way of looking at a significant portion of the world. In this he succeeded. Fractals have been the subject of dozens of scientific conferences, hundreds of books, thousands of papers, and millions (yes, millions) of web pages. And Benoit influenced not just science and finance, but also the music of György Ligeti and Charles Wuorinen and much contemporary art[33] and literature.[34] A substantial portion of how we view the world has been changed by Benoit's extensive work in developing fractal geometry.

Years ago, one of his visitors at Yale remarked that he'd read Benoit's papers in mathematics, physics, finance, hydrology, and linguistics, and he wondered to what field Benoit thought he belonged. How did he see himself ? Without hesitation, Benoit replied that he was a storyteller. It's true that Benoit loved to tell stories, but thinking over his answer, I realized there is another way to understand it. A fractal description of an object is a story about how it grows. The delicate arms of a snowflake are a story of the temperature, pressure, and humidity the snowflake encountered on its dance through the cloud.

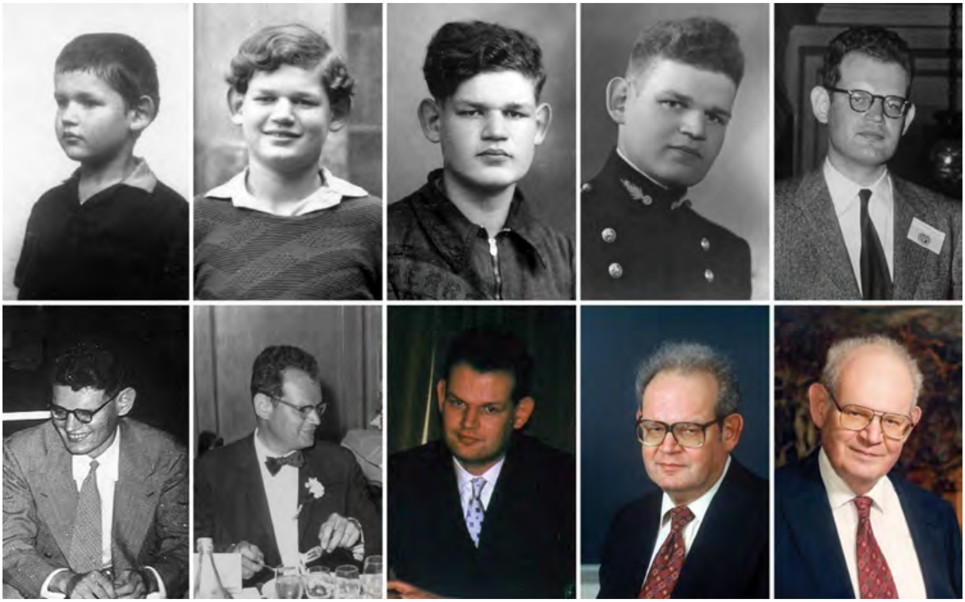

Figure 5. Benoit throughout life.

A coastline is a story of rocks, waves, and tides. Often Benoit remarked that fractals emphasized the importance to science of looking at pictures. Certainly, this is true. But fractals also remind us that science has a narrative component—and that its stories are critical to our lives, though often taken for granted.

I'll end with these pictures, illustrations of the story that was Benoit's complicated life. The very best times I had with Benoit were when he would take a question in a new direction, and we'd be off and running. Hours passed as we explored a multitude of ideas, each branching from those earlier. During our very first conversation of this kind, a vivid image came into my mind. Benoit and I were two little kids, running around in a big field under the bright sky, exploring everything we found, each anxious to share with the other. This image has stayed with me for over 20 years, and has given me some small comfort since Benoit died. Look at these pictures. Do you see that the curious child still was present in the man?

ACKNOWLEDGMENTS

Thanks to Aliette Mandelbrot for permission to use the photographs, and for her comments on the text. Amelia Urry, Jean Maatta, and Roger Howe also contributed many valuable improvements to the text.

NOTES

1. Benoit's Uncle Szolem was a founding member of Bourbaki.

2. Benoit dedicated his dissertation to John von Neumann and Norbert Wiener. The dissertation title references the game theory of von Neumann and the telephone switching circuits of Wiener.

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5. Evidently, the world of finance had not taken up the lessons of Copernicus and Kepler about improvements on Ptolemy's cosmology.

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11. The precise condition for "not too much overlap" is the open set condition. See Hutchinson, J. 1981. Fractals and self-similarity. *Indiana Univ. Math. J.* 30:713–747.

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13. If only some compositions of the Ti are allowed, the Hausdorff dimension is the solution of $\rho[r_i^d m_{ij}] = 1$,, where $m_{ij} = 0$ or 1 according as $T_i \circ T_j$ is forbidden or allowed, and $\rho[M]$ is the spectral radius, the maximum of the magnitudes of the eigenvalues, of M. This was proved in 1988 by R. Mauldin and S. Williams in "Hausdorff dimension in graph-directed constructions." *Trans. Amer. Math. Soc.* 309:811–829.

14. If each of the scaling factors is chosen randomly in some range, restricted so that the open set condition still is satisfied, then the dimension is the solution of $\mathbb{E}(r_i^d + \cdots + r_N^d) = 1$.

15. See Section 5.2 of Falconer's *Techniques in Fractal Geometry.*

16. Some of Mandelbrot's papers on self-affine fractals are collected in *Gaussian self-affinity and fractals*, Springer-Verlag, New York, 2002.

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32. Fractal dimension helped quantify the risk to miners' lungs that is posed by coal dust.

33. Though one of the best examples, Salvador Dali's 1940 *The Visage of War* was painted before the concept of fractal was developed. Independently, Dali discovered the power of scaling symmetry to suggest infinite depth. Much earlier, as seen on the back cover of *Fractal Forms*, edited by E. Guyon and H. Stanley and published by Elsevier (Amsterdam) in 1991, the Sierpinski gasket has been an artistic motif for at least 900 years.

34. Tom Stoppard's *Arcadia*, Peter Høeg's *Smilla's sense of snow*, Richard Powers' *The goldbug variations* and *Plowing the dark*, and John Updike's *Roger's version*, among many others.

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