Critical points and Siegel disks

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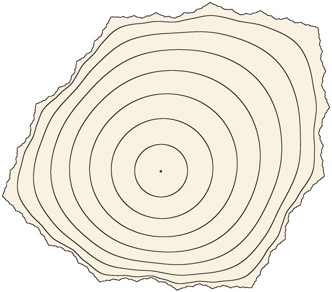
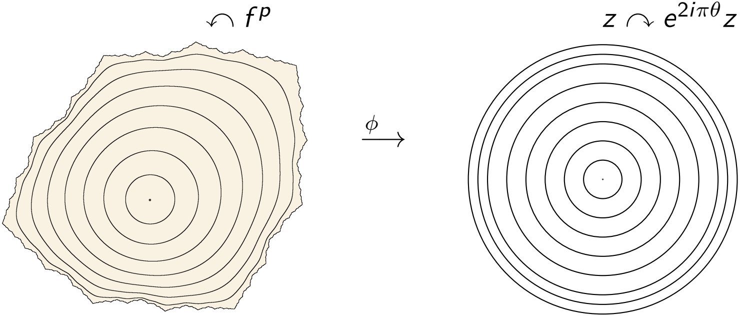
A *Siegel disk* of a rational map f of degree ≥ 2 is a maximal domain on which an iterate of f is conjugated to the rotation

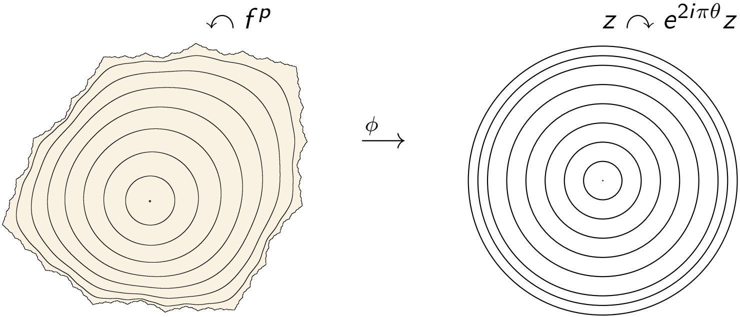
$$R_\theta(z) = e^{2i\pi\theta} z.$$

θ is called the rotation number.



Golden Mean rotation number: $f(z) = e^{2i\pi \frac{\sqrt{5}-1}{2}} z + z^2$.


$\curvearrowleft f^p$
ϕ
$z \curvearrowright e^{2i\pi\theta} z$



Such a domain cannot contain a critical point.

One can wonder which phenomena

at the boundary of a Siegel disk

prevents f from having a larger domain of linearization.

For an attracting k-periodic point:

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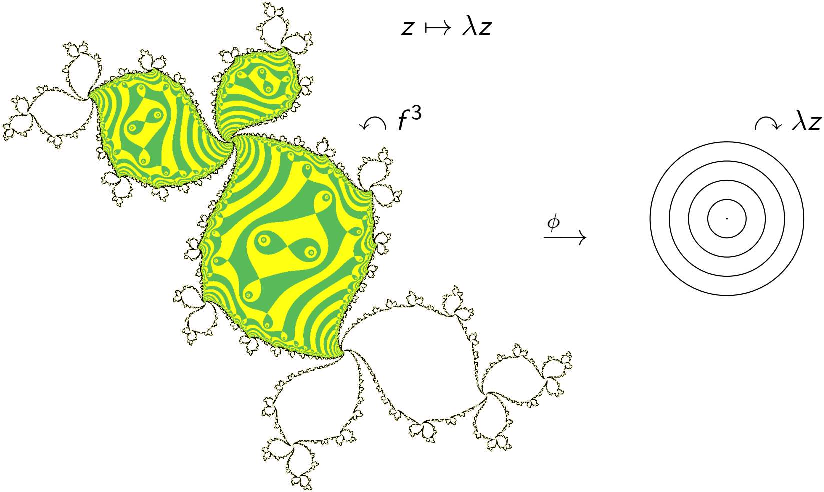

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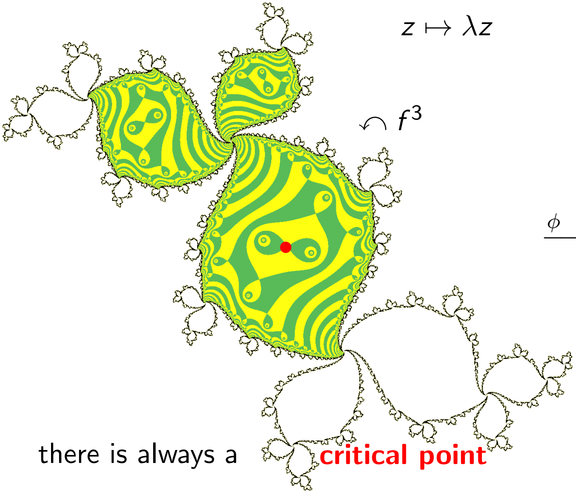
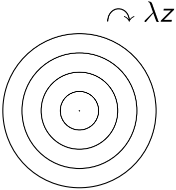


$\xrightarrow{\phi}$



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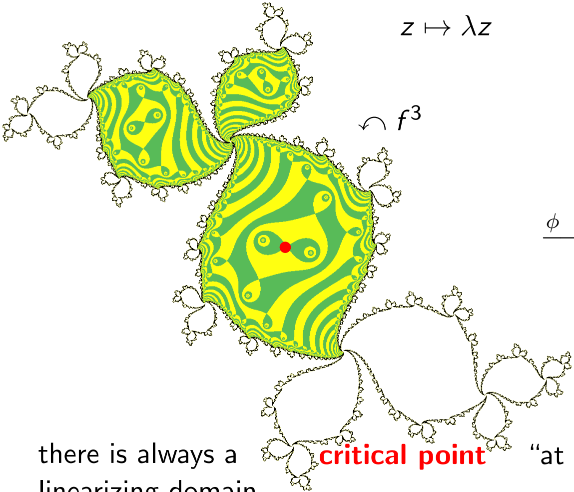
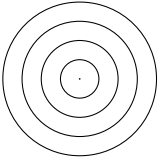


$\curvearrowleft f^3$

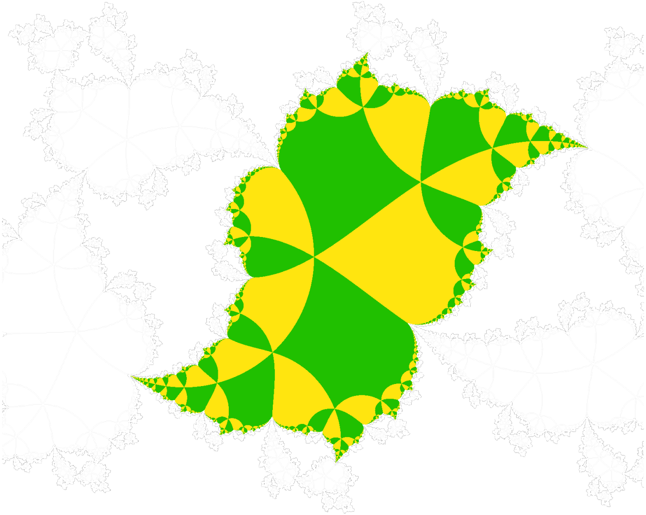
$\xrightarrow{\phi}$



$\curvearrowright \lambda z$

there is always a **critical point** "at the boundary" of a linearizing domain.

Also for a parabolic point: $f'(p) = 1$



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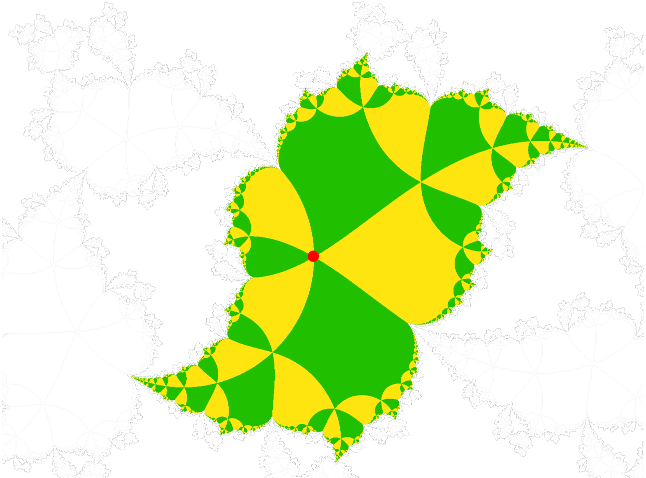

There is a critical point at the boundary of the linearizing domain.

Does the boundary of a Siegel disk always contain a critical point?

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Ghys and Herman gave the first examples of polynomials having a Siegel disk without a critical point on the boundary.

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In fact, from Douady-Sullivan argument we get:

Lemma

If f is a polynomial with a Siegel disk Δ and locally connected Julia set, then there is a critical point on the boundary of Δ.

So,

- either there is a critical point on the boundary of Δ ;
- or the Julia set is not locally connected.

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Here we use

Theorem (Graczyk and Swiatek, 2003)

If a Siegel disk has a bounded type rotation number and is compactly contained in the domain of definition of the map, then its boundary contains a critical point.

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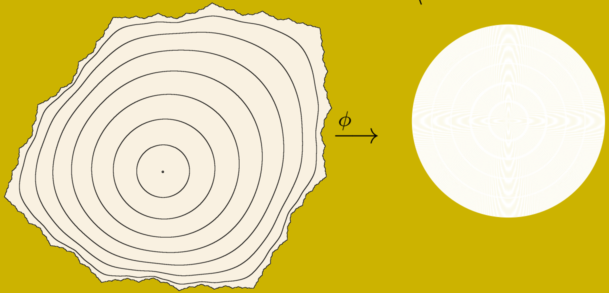
In Ghys-Herman example $\partial\Delta$ is locally connected and there are no critical points on $\partial\Delta$

Assume that $\partial\Delta$ is a Jordan curve

Assume that $\partial\Delta$ is a Jordan curve not containing a critical point**.**

Assume that $\partial\Delta$ is a JORDAN CURVE NOT CONTAINING A CRITICAL POINT. $\mathbf{C} \setminus \overline{\Delta}$ is homeomorphic to $\mathbf{C} \setminus \overline{\mathbf{D}}$.



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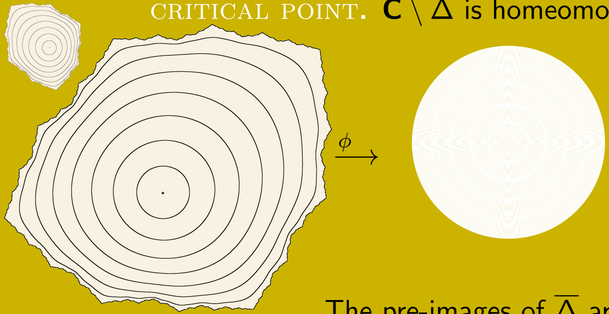

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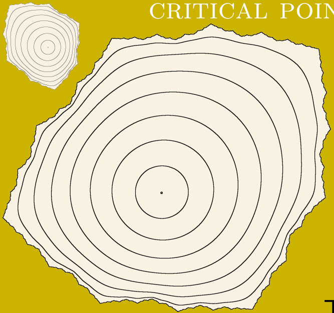

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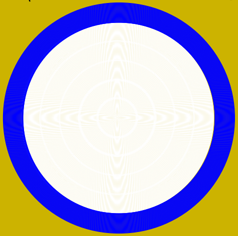

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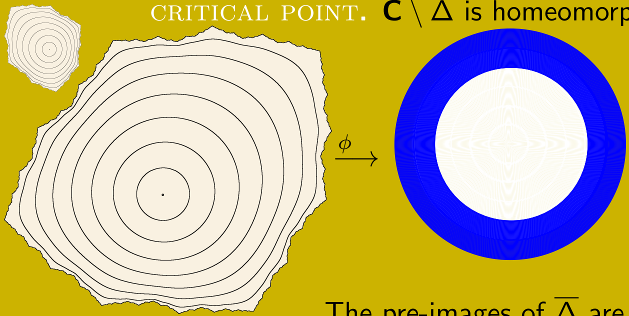

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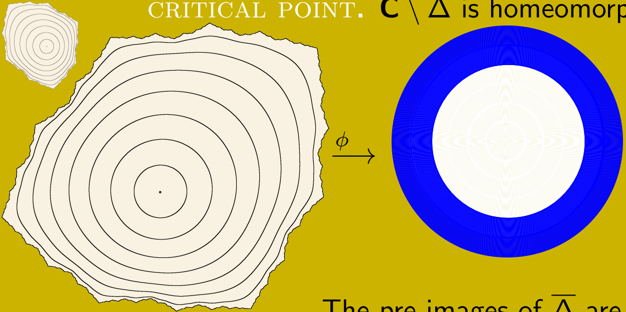

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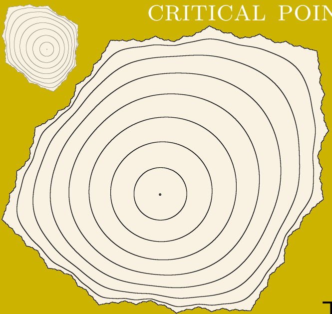

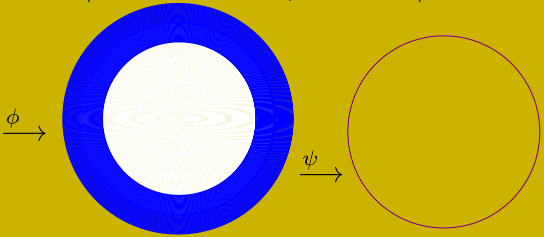

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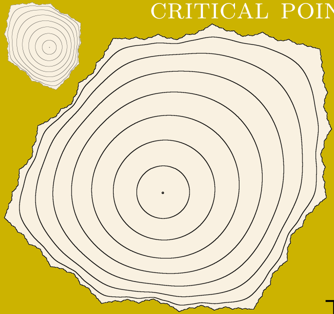

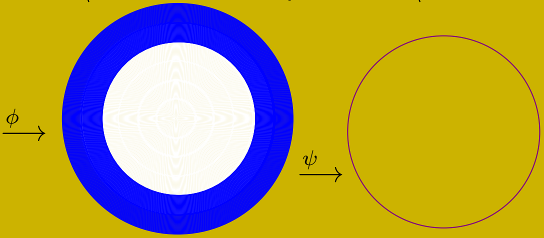

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It is conjugated (by ψ) to the rotation R_θ on the unit circle.
If ψ is analytic then it extends to a neighborhood of the unit circle. Then the rotation domain extends.

Definition

$\mathcal{C}^\omega(\mathbf{S}^1) :=$ {orientation preserving analytic circle diffeomorphisms},

$$\rho(f) := \text{ rotation number of } f,$$

$\mathcal{H} := \{\theta \in \mathbf{R} \mid \forall f \in \mathcal{C}^\omega(\mathbf{S}^1) \text{ with } \rho(f) = \theta \text{ is conjugate to } R_\theta \text{ in } \mathcal{C}^\omega(\mathbf{S}^1)\}.$

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Theorem (Ghys)

Let f be a rational map of degree ≥ 2, Δ a Siegel disk of period one with rotation number in $\mathcal{H}$.
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Theorem (Herman)

The set $\mathcal{H}$ is non empty:

$$\textit{Diophantine} \subset \mathcal{H}.$$

Recall that several people including Petersen, Inou-Shishikura, Zhang... proved that $\partial\Delta$ is locally connected under some hypothesis on the rotation number .

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For the rest of the talk we do not assume $\partial\Delta$ locally connected.

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The Theorem holds also for periodic Siegel disks.

Conjecture

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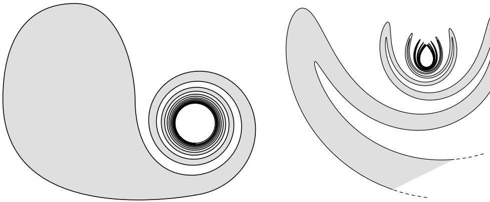
Theorem (Chéritat-R)

rr For all polynomials with two finite critical values, a Siegel disk Δ of arbitrary period and of rotation number in $\mathcal{H}$, there is an element in the cycle of Δ whose boundary contains a critical point.

Important Remark

If the boundary $\partial\Delta$ is not locally connected then $\overline{\Delta}$ is not necessarily full any more.



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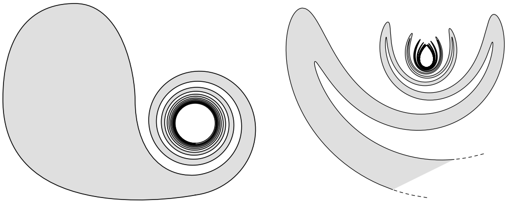

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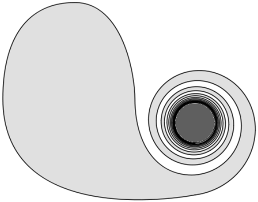

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Otherwise, $P^n(c)$ is in the interior of $\widehat{\Delta}$: in the Fatou set. Then $\omega(c) \cap J(P)$ is finite. Contradiction with the following:

Theorem (Mañé)

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c is recurrent: $c \in \omega(c) = \omega(P^n(c))$
$P(\partial\Delta) = \partial\Delta \implies \omega(P^n(c)) \subset \partial\Delta$.

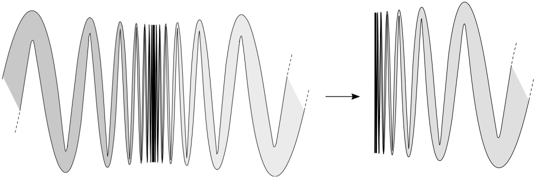
Strategy of the proof

The strategy is to understand the dynamics of P restricted to the filled Siegel disk $\widehat{\Delta}$:

- assume period 1 for simplicity,
- suppose the Julia set is connected: use a polynomial-like map to restrict to the connected component containing $\widehat{\Delta}$,
- prove that $\widehat{\Delta}$ is backward invariant : $\widehat{\Delta}$ is a connected component of $P^{-1}(\widehat{\Delta})$,
- study the external map:
    - if it is a homeomorphism then prove that $\theta \notin \mathcal{H}$,
    - if it has degree > 1
        - prove that it has no non-repelling periodic points,
        - built a quadratic-like map around $\widehat{\Delta}$ since the external map is expanding,
        - reduce then to a uni-critical map and apply Herman's result.

Definition

Let $\widetilde{\Delta}$ be the connected component of $P^{-1}(\widehat{\Delta})$ containing $\widehat{\Delta}$.



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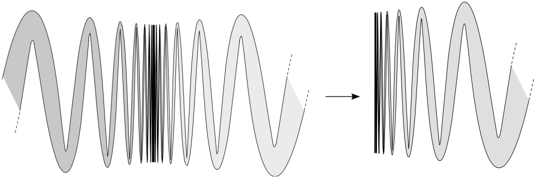

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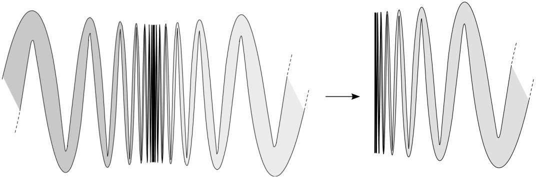

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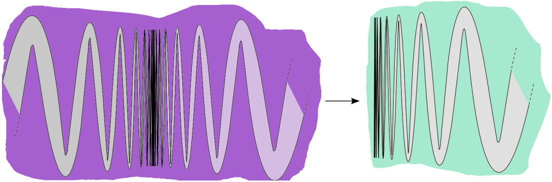
Lemma

There exists U, V topological disks, $\widetilde{\Delta} \subset U$, $\widehat{\Delta} \subset V$ such that $P : U \to V$ is a covering ramified only over $\widehat{\Delta}$.



- $n_1 = 0 \implies n_0 = 0$
  P is a homeomorphism from U to V and $\widehat{\Delta} = \widetilde{\Delta}$,
- $n_1 = 1 \implies n_0 = 0$ or $n_0 = 1$
  since P is a covering over a topological disk ramified over one point (Rieman-Hurwitz formula).

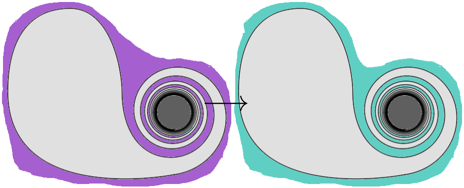
Theorem

1. *If $n_0 = 0$ then $\rho(P_{|_\Delta}) \notin \mathcal{H}$. (similar to Herman's proof)*
2. *If $n_0 = 1$ and P has only two critical values then there is a critical point on $\partial\Delta$.*

This implies the original Theorem since $n_1 \in \{0, 1, 2\}$.

Case 1

$\widehat{\Delta} = \widetilde{\Delta}.$



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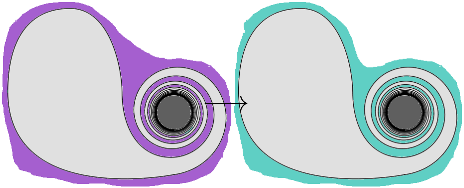

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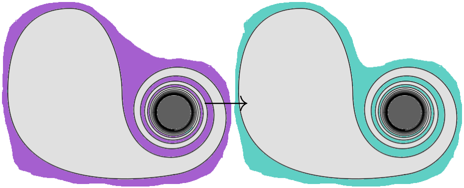

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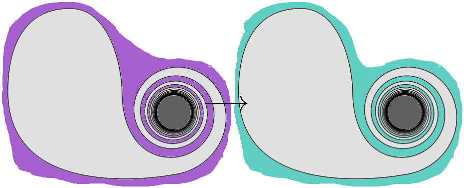

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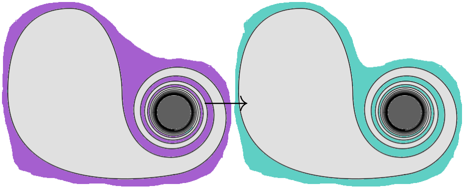

Using the Rieman map $\phi : \mathbf{C} \setminus \widehat{\Delta} \to \mathbf{C} \setminus \overline{\mathbf{D}}$, we can defined the external map by Schwarz reflection :
The map $g = \phi \circ f \circ \phi^{-1}$ is defined on a annulus around the unit circle. The restriction of g to the unit circle is an analytic diffeomorphism with rotation number θ.
If $\theta \in \mathcal{H}$, it is conjugated by a analytic map ψ to the rotation R_θ on the unit circle. Then ψ extends to a neighborhood of the unit circle. By analycity, ψ conjugate to the rotation. The Siegel domain extends.

Case 2

We assume that $n_0 = 1$.
Remark: If $P(c) \in \partial\Delta$ then $c \in \partial\Delta$.

The critical value $P(c) \in V$ has only c as preimage in U ($n_0 = 1$) and $P(\partial\Delta) = \partial\Delta$ so $c \in \partial\Delta$.

Case 2

$P(c)$ belongs to a Fatou component in $\widehat{\Delta} \setminus \Delta$ since $P(c) \in Int(\widehat{\Delta})$

Recall the SKETCH OF THE PROOF

1. $\widehat{\Delta}$ is locally totally invariant by P: $\widetilde{\Delta} = \widehat{\Delta}$ (use Goldberg Milnor Poirier Kiwi separation result).
2. The external map is well defined by Schwarz reflection. It has degree > 1.
3. This circle map is hyperbolic (by Mané's Theorem) since there are no non repelling cycles on the circle.
4. Then P has a polynomial-like restriction which is unicritical.
5. We apply Herman's theorem for uni-critical polynomials.

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Claim: $P(c)$ belongs to a Fatou component W which is eventually mapped to Δ under iteration of P.

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Theorem (Goldberg, Milnor, Poirier, Kiwi)

There exists $m > 0$ such that the union L of the closure of the external rays fixed by P^m cut the plane into regions, each of them containing at most one periodic Fatou component or Cremer point (and never both of them).

But $\partial W \subset \partial\Delta$. Contradiction.

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  since $P^r(W)$ is a preimage of Δ, $P^r(W) = g\Delta$ for some $g \in G$;
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- Using that G is cyclic we can prove that

  $$\forall g \in G \mid g(\Delta) \subset \widehat{\Delta}$$

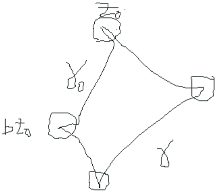
- It "follows" that $\widetilde{\Delta} = \widehat{\Delta}$.

Let H be the stabilizer of $\widehat{\Delta}$: the set of $h \in G$ such that $h(\widehat{\Delta}) = \widehat{\Delta}$.

Claim: $H = G$.

Otherwise, $\rho \notin H$ and neither $\widehat{\Delta} \subset \rho\widehat{\Delta}$ nor $\widehat{\rho}\widehat{\Delta} \subset \widehat{\Delta}$
Then $\rho\widehat{\Delta} \cap \Delta = \emptyset$ and $\widehat{\Delta} \cap \rho\Delta = \emptyset$ since $\rho\Delta \cap \Delta = \emptyset$.

Take a point $z_0 \in \partial\Delta \setminus \rho\widehat{\Delta}$, a ball $B \subset U$, $z_0 \in B$, $B \cap \rho\widehat{\Delta} = \emptyset$, $c \notin B$.
Take a curve in Δ joining a point $z_1 \in B \cap \Delta$ to a point $z_2 \in bB \cap \Delta$ where b is a generator of H.
Complete this curve with a segment in B joining z_0 and z_1, a segment in bB joining bz_0 and z_2. Let γ_0 be this curve.



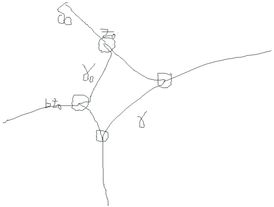
Let $\gamma = \gamma_0 \cup b\gamma_0 \cup b^2\gamma_0 \cup \ldots$

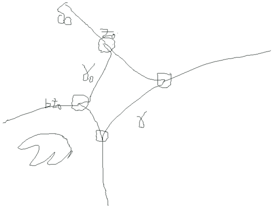
For $z_0' \in \partial\rho\Delta \setminus \widehat{\Delta}$, $z_0' \in B' \subset U$, $B' \cap \widehat{\Delta} = \emptyset$, $c \notin B'$.
So

$$\gamma \subset H\Delta \cup HB, \quad \gamma' \subset \rho H\Delta \cup HB' \implies \gamma \cap \gamma' = \emptyset$$

γ separates c from ∞, γ' separates c from ∞
suppose that γ' is in the unbouded component X of $\mathbf{C} \setminus \gamma$ then $\partial\rho(\Delta) \subset X$.

Join z_0 to the boundary of U by δ_0 disjoint from $\widehat{\rho(\Delta)}$.

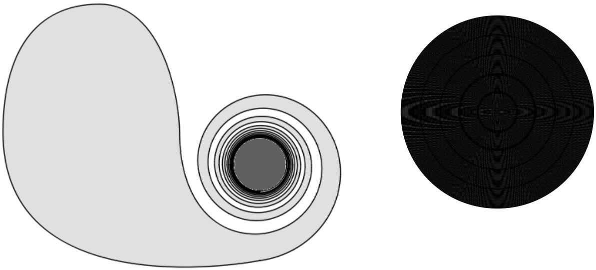




$\partial\rho(\Delta)$ is invariant by H so is in a connected component of $U \setminus C$ invaraint by H : the one containing c. Contradiction.

External map

$\mathbf{C} \setminus \widehat{\Delta}$ is homeomorphic to $\mathbf{C} \setminus \overline{\mathbf{D}}$.



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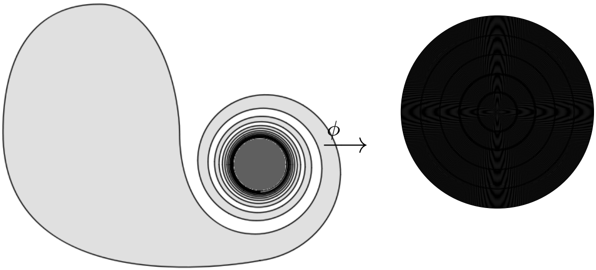

Let $\phi : \mathbf{C} \setminus \widehat{\Delta} \to \mathbf{C} \setminus \overline{\mathbf{D}}$ be the Riemann map.

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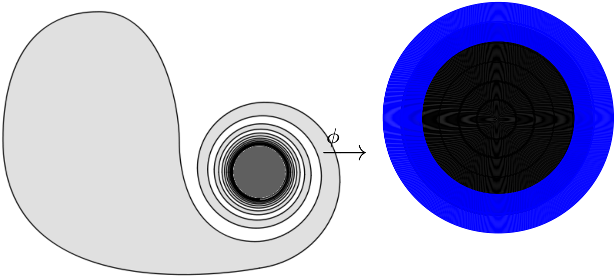

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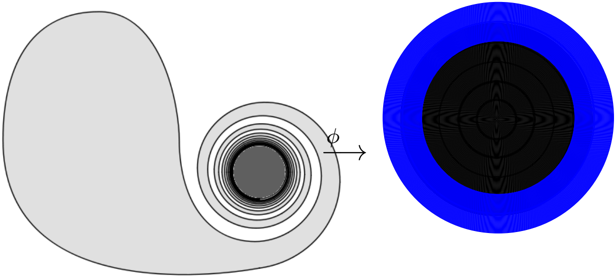

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The map $\tilde{g} = \phi \circ f \circ \phi^{-1}$ is defined on a annulus around the unit circle by Schwarz reflection.

The restriction g of $\tilde{g}$ to the unit circle (the external map) is an analytic covering of degree $m > 1$.

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For an expanding map, we can then construct by hand a polynomial-like restriction. It is uni-critical.

Thank you for your attention