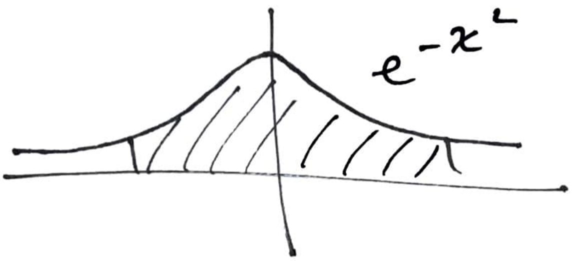
Using Taylor series to integrate

ex: Find $\int e^{-x^2}\,dx$ as a power series and estimate $\int_0^1 e^{-x^2}\,dx$ to w/in .001.

Soln: the integral $\int e^{-x^2}$ famously cannot be written in terms of elementary functions. It computes the area under the "bell curve" e^{-x^2}



But! we can find a power series for e^{-x^2}, and so for $\int e^{-x^2}dx$ too.

→ could find using def'n of Mac series, but instead let's use our Mac series for e^x:

$$e^x = \sum_0^\infty \frac{x^n}{n!} \qquad \text{for all } x$$

$$\Rightarrow e^{-x^2} = \sum_0^\infty \frac{(-x^2)}{n!}$$

$$= \sum_0^\infty \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

Hence:

$$\int e^{-x^2} dx = \int \sum_0^\infty (-1)^n \frac{x^{2n}}{n!} dx$$

$$= \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + C$$

$$= x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots + C$$

So: $$\int_0^1 e^{-x^2} dx = \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots \right]_0^1$$

$$= \left(1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \dots \right)$$

$\frac{1}{9 \cdot 4!}$

$$- 0$$

$$\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216}$$

$$= .7475.$$

by alt. series thm:

$|\text{error}| \leq |\text{next term}|$

$$= \frac{1}{11 \cdot 5!} = \frac{1}{1320} = .00075$$

$$< .001$$

Operations on Power Series

Turnout: we can add, subtract, multiply, divide power series "just like" regular polynomials. But it can get messy.

Until further notice: suppose $\sum_0^\infty c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ converges on $(-R,R)$ and $\sum_0^\infty d_n x^n = d_0 + d_1 x + d_2 x^2 + \cdots$ converges on $(-S,S)$ are power series.

Then: ① if a is a real number: the series:

$$a \sum_0^\infty c_n x^n = a(c_0 + c_1 x + c_2 x^2 + \cdots)$$
$$= ac_0 + ac_1 x + ac_2 x^2 + \cdots$$
$$= \sum_0^\infty a c_n x^n \text{ converges on } (-R,R)$$

ex: Since $\frac{1}{1-x} = \sum_0^\infty x^n$ on $(-1,1)$

we have $\frac{2}{1-x} = 2\sum_0^\infty x^n = \sum_0^\infty 2x^n$

$$= 2 + 2x + 2x^2 + \cdots \text{ on } (-1,1)$$

(2) $x^k \sum_0^\infty c_n x^n = x^k(c_0 + c_1 x + c_2 x^2 + \cdots)$

$= c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \cdots$

$= \sum_0^\infty c_n x^{n+k}$ converges on $(-R, R)$

<u>ex</u>: $e^x = \sum_0^\infty \frac{1}{n!} x^n$ on $(-\infty, \infty)$

so: $x^2 e^x = \sum_0^\infty \frac{1}{n!} x^{n+2}$ on $(-\infty, \infty)$

(3) $\left(\sum_0^\infty c_n x^n\right)\left(\sum_0^\infty d_n x^n\right)$

$= (c_0 + c_1 x + c_2 x^2 + \cdots)(d_0 + d_1 x + d_2 x^2 + \cdots)$

$= c_0 d_0 + c_0 d_1 x + c_1 d_0 x + c_0 d_2 x^2 + c_1 d_1 x^2 + c_2 d_0 x^2 + \cdots$

$= c_0 d_0 + (c_0 d_1 + c_1 d_0) x + (c_0 d_2 + c_1 d_1 + c_2 d_0) x^2 + \cdots$

radius of convergence will be the <u>smaller</u> of R and S.

ex: we know $e^x = \sum_0^\infty \frac{1}{n!} x^n$ on $(-\infty,\infty)$

$$\frac{1}{1-x} = \sum_0^\infty x^n \quad \text{on } (-1,1)$$

so: $$\frac{e^x}{1-x} = e^x\left(\frac{1}{1-x}\right) = \left(\sum_0^\infty \frac{1}{n!} x^n\right)\left(\sum_0^\infty x^n\right)$$

$$= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(1+x+x^2+x^3+\cdots\right)$$

$$= 1+x+x+x^2+x^2+\frac{x^2}{2!}+$$
$$x^3+x^3+\frac{x^3}{2!}+\frac{x^3}{3!}+\cdots$$

$$= 1+2x+\left(\underbrace{2}_{\frac{1}{0!}+\frac{1}{1!}}+\frac{1}{2!}\right)x^2+\left(2+\frac{1}{2!}+\frac{1}{3!}\right)x^3$$

$$+\cdots+\left(2+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)x^n+\cdots$$

$$= \sum_0^\infty\left(\frac{1}{0!}+\frac{1}{1!}+\cdots+\frac{1}{n!}\right)x^n \quad \text{on } (-1,1)$$

take the <u>smaller</u> radius.

④ can also do long division of power series. But it's a mess, so we'll ignore.

E.g. $\dfrac{\sin(x)}{\cos(x)} = \tan(x)$

$$= \frac{\sum_{0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{\sum_{0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

$$= \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots}{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots}$$

$= \dots$ a mess

Taylor polynomials

Def'n Sps that $f(x)$ is a function w/ a Taylor series rep'n @ $x=a$.

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{for } x \text{ in } (a-R, a+R)$$

We define the Taylor polynomials of $f(x)$ as follows:

$$T_0(x) = f(a)$$

$$T_1(x) = f(a) + f'(a)(x-a)$$

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\vdots$$

$$T_N(x) = \sum_{0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\vdots$$

So: Nth Taylor polynomial is sum of first N terms of f's Taylor series.

the point: since f = its Taylor series on $(a-R, a+R)$, the Taylor polys will approximate f on that interval.

the larger N is, and the closer x is to the center a, the better the approx'n $T_N(x)$ will be for $f(x)$.

<u>ex</u>: We know:

$$e^x = \sum_0^\infty \frac{1}{n!} x^n \quad \text{on} \quad (-\infty, \infty)$$

So:

$$T_0(x) = 1 \qquad T_2(x) = 1 + x + \frac{x^2}{2!}$$

$$T_1(x) = 1 + x \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

we expect these poly's to be "reasonably" good approx's to e^x... near $a=0$.

<u>Pics</u>: