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DEPARTAMENTO DE MATEMÁTICA

TOPOLOGICAL OBSTRUCTIONS TO THE EXISTENCE OF METRICS WITH NON-NEGATIVE OR POSITIVE SCALAR CURVATURE AND MEAN CONVEX BOUNDARY

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ORIENTADOR: EZEQUIEL RODRIGUES BARBOSA

BELO HORIZONTE - MG

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Tese de Doutorado submetida ao Programa de Pós-Graduação em Matemática, como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Topological obstructions to the existence of metrics with non-negative or positive scalar curvature and mean convex boundary

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Abstract

In this work we study the geometry of compact and orientable n-dimensional manifolds with non-empty boundary $(M, \partial M)$ such that there is a non-zero degree map $F : (M, \partial M) \to (\Sigma \times T^{n-2}, \partial\Sigma \times T^{n-2})$, where $(\Sigma, \partial\Sigma)$ is a compact, connected and orientable surface with non-empty boundary and $3 \leq n \leq 7$. We show that depending on the topology of Σ, the existence of this non-zero degree map F is a topological obstruction to the existence of a metric in M with positive or non-negative scalar curvature and mean convex boundary. More precisely, we show that

1. If Σ is neither a disk nor a cylinder then M does not admit a metric with non-negative scalar curvature and mean convex boundary.

2. If Σ is not a disk then M does not admit a metric with positive scalar curvature and mean convex boundary. Furthermore, every metric in M with non-negative scalar curvature and mean convex boundary is Ricci-flat with totally geodesic boundary.

Finally, we study the case in which Σ is a disk. In this case we consider a metric g in M with positive scalar curvature and mean convex boundary (i.e., $R_g^M > 0$ and $H_g^{\partial M} \geq 0$) and we define $\mathcal{F}_M$ be the set of all immersed disks in M whose boundaries are curves in ∂M that are homotopically non-trivial in ∂M. We show that

$$\frac{1}{2} \inf R_g^M \mathcal{A}(M, g) + \inf H_g^{\partial M} \mathcal{L}(M, g) \leq 2\pi \tag{1}$$

where

$$\mathcal{A}(M, g) = \inf_{\Sigma \in \mathcal{F}_M} |\Sigma|_g \quad \text{e} \quad \mathcal{L}(M, g) = \inf_{\Sigma \in \mathcal{F}_M} |\partial\Sigma|_g.$$

Moreover, if the boundary ∂M is totally geodesic and the equality holds in (2), then universal covering of (M, g) is isometric to $(\mathbb{R}^n \times \Sigma_0, \delta + g_0)$, where δ is the standard metric

in $\mathbb{R}^n$ and (Σ_0, g_0) is a disk with constant Gaussian curvature $\frac{1}{2}\inf R_g^M$ and $\partial\Sigma_0$ has null geodesic curvature in (Σ_0, g_0).

Keywords: Scalar curvature; Mean convex boundary; Non-zero degree map.

Resumo

Neste trabalho vamos estudar a geometria de variedades n-dimensional orientáveis e compactas com bordo não-vazio $(M, \partial M)$ tais que existe uma aplicação de grau diferente de zero $F : (M, \partial M) \to (\Sigma \times T^{n-2}, \partial\Sigma \times T^{n-2})$, onde $(\Sigma, \partial\Sigma)$ é uma superfície compacta, conexa, orientável com bordo não-vazio e $3 \leq n \leq 7$. Mostramos que dependendo da topologia de Σ, a existência desta aplicação de grau diferente de zero F é uma obstrução topológica para existência de uma métrica em M com curvatura escalar positiva ou não-negativa e bordo mean convexo. Mais precisamente, mostramos que

1. Se Σ não é um disco e nem um cilindro então M não admite uma métrica com curvatura escalar não-negativa e bordo mean convexo.

2. Se Σ não é um disco então M não admite uma métrica com curvatura escalar positiva e bordo mean convexo. Além disso, toda métrica em M com curvatura escalar não-negativa e bordo mean convexo é Ricci-flat com bordo totalmente geodésico..

Por fim, estudamos o caso em que Σ é um disco. Neste caso consideramos uma métrica g em M com curvatura escalar positiva e bordo mean convexo(isto é, $R_g^M > 0$ e $H_g^{\partial M} \geq 0$) e definimos $\mathcal{F}_M$ como sendo o conjunto de todos os discos imersos em M cujos bordos em ∂M são homotopicamente não-triviais em ∂M. Mostramos que

$$\frac{1}{2} \inf R_g^M \mathcal{A}(M, g) + \inf H_g^{\partial M} \mathcal{L}(M, g) \leq 2\pi \tag{2}$$

onde

$$\mathcal{A}(M, g) = \inf_{\Sigma \in \mathcal{F}_M} |\Sigma|_g \quad \text{e} \quad \mathcal{L}(M, g) = \inf_{\Sigma \in \mathcal{F}_M} |\partial\Sigma|_g.$$

Além disso, se ∂M é totalmente geodésico e vale a igualdade em (2), então o recobrimento universal de (M, g) é isométrico a $(\mathbb{R}^n \times \Sigma_0, \delta + g_0)$, onde δ é a métrica canônica de $\mathbb{R}^n$ e (Σ_0, g_0)

é um disco com curvatura Gaussiana constante $\frac{1}{2} \inf R_g^M$ e $\partial\Sigma_0$ tem curvatura geodésica nula em (Σ_0, g_0).

Palavras-Chaves: Curvatura Scalar; Bordo Mean Convexo; Aplicações de Grau Diferente de Zero.

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Chapter 1

Introduction

The relation between minimal hypersurfaces of a Riemannian manifold M and the curvatures of M is a deep connection which was first observed by R. Schoen and S. T. Yau. In this thesis, we will deal with two situations clarifying that link.

The first part of this thesis deal with topological obstruction for the existence of a metric with positive (or non-negative scalar curvature) and mean convex boundary (or strictly mean convex boundary) which is given by the existence of a certain type of hypersurfaces. Let us be more precise. A central problem in modern differential geometry concerns the connection between curvature and topology of a manifold. Especially, if the problem is when a given manifold admits a Riemannian metric with positive or non-negative scalar curvature. We will not go over the case of closed manifolds, instead, our focus here will be on compact manifolds with non-empty boundary. For the case of closed manifolds, see the important works due to Schoen-Yau [30], [31], and Gromov-Lawson [14], [15], [16].

Consider, for instance, the case of surfaces. Let (M^2, g) be an orientable compact two-dimensional Riemannian manifold with non-empty boundary ∂M. The Gauss-Bonnet Theorem states that

$$\int_M K da + \int_{\partial M} k_g ds = 2\pi\chi(M)\,,$$

where K denotes the Gaussian curvature, k_g is the geodesic curvature of the boundary, $\chi(M)$ is the Euler characteristic, da is the element of area and ds is the element of length. Note that the invariant $\chi(M)$ gives a topological obstruction to the existence of certain types of Riemannian metrics on the surface M^2. For instance, a compact surface M^2 with negative (non-positive) Euler characteristic does not admit a Riemannian metric with non-negative (positive) Gaussian curvature and non-negative geodesic curvature.

In higher dimensions, the relationship between curvature and topology is much more complicated. A classical theorem due to Gromov [13], for example, states that every compact manifold with non-empty boundary admits a Riemannian metric of positive sectional curvature.

However, there are topological obstructions if one further imposes geometric restrictions on the boundary. For instance, a result of Gromoll [12] states that a compact Riemannian manifold of positive sectional curvature with non-empty convex boundary is diffeomorphic to the standard disc. Observe, however, that these hypothesis are rather strong because they involve the sectional curvature and not the scalar curvature. Recall that, by the Bonnet-Mayers Theorem, a 3-dimensional manifold with positive Ricci curvature and convex boundary (positive definite second fundamental form) is diffeomorphic to a 3-ball.

The problem of determining topological obstructions for the existence of a metric with non-negative scalar curvature and mean convex boundary (mean curvature of boundary is non-negative) is more subtle. For instance, one such obstruction appears when there exists a compact, orientable and essential surface properly embedded in M which is not a disk or a cylinder (see Theorems 1.1 and 1.2 in [7]). This is the case, for example, if we consider the 3-dimensional manifold $\mathbb{S}^1 \times \Sigma$, where Σ is a compact, connected and orientable surface with non-empty boundary which is neither a disk nor a cylinder. Indeed, the surface $\{1\} \times \Sigma$ is essential in $\mathbb{S}^1 \times \Sigma$, so this manifold carries no metric with non-negative scalar curvature and mean convex boundary. If a 3-dimensional manifold M contains an essential cylinder, then there may exists such a metric on M. This is the case, for example, of the manifold $I \times T^2$, where T^2 denote the torus $\mathbb{S}^1 \times \mathbb{S}^1$. Such manifold contains an essential cylinder and have a flat Riemannian metric with totally geodesic boundary.

From now on, we use the notation $(M, \partial M)$ to represent a compact and orientable manifold with non-empty boundary ∂M. Moreover, R_g^M and $H_g^{\partial M}$ denote the scalar curvature of (M, g) and the mean curvature of the boundary ∂M with respect to the outward unit normal vector field on the boundary, respectively.

Our first result gives a topological obstruction for those 3-dimensional compact manifolds which possess a certain type of surfaces as connected components of their boundaries.

Theorem 1.0.1. *Let $(M, \partial M)$ be a compact 3-dimensional manifold. Assume that the connected components of ∂M are spheres or incompressible tori, but at least one of the components is a torus. Then there is no Riemannian metric on M with positive scalar curvature and mean convex boundary. In particular, if there exists a Riemannian metric g on M with non-negative scalar curvature and mean convex boundary then (M, g) is flat with totally geodesic boundary.*

As a consequence of the theorem above, we obtain that the 3-dimensional manifolds $(\mathbb{S}^1 \times \mathring{T}^2)\#N$ and $(\mathbb{S}^1 \times \mathring{T}^2)\#(I \times \mathbb{S}^2)$ have no metric with non-negative scalar curvature and mean convex boundary, where $\mathring{T}^2$ is a torus minus an open disk, $I = [a, b]$ and N is a closed, connected and orientable 3-dimensional manifold. Moreover, the 3-dimensional manifolds $(I \times T^2)\#(I \times T^2)$, $(\mathbb{S}^1 \times \mathring{T}^2)\#(\mathbb{S}^1 \times \mathring{T}^2)$, $(I \times T^2)\#(S^1 \times \mathring{T}^2)$, $(I \times T^2)\#(I \times \mathbb{S}^2)$, $(\mathbb{S}^1 \times \mathring{T}^2)\#(I \times \mathbb{S}^2)$ have no metric with positive scalar curvature and mean convex boundary. Also, let N be a closed 3-dimensional manifold. Then the manifold $(I \times T^2)\#N$ has no metric with positive scalar curvature and mean convex boundary. If it has a metric with non-negative scalar curvature and mean convex boundary, it is flat with totally geodesic boundary. Thus, from this last claim, we can glue two copies of $(I \times T^2)\#N$ along the boundary and build a flat closed 3-dimensional manifold which is a connected sum of a 3-dimensional torus and a closed 3-dimensional manifold.

With that discussion above, we obtain the following classification result.

Corollary 1.1. *Let $(M, \partial M)$ be a smooth 3-dimensional manifold such that ∂M is the disjoint union of exactly one torus and k spheres, $k \geq 0$. If M has a metric with non-negative scalar curvature and mean convex boundary then*

$$M = N\#(\mathbb{S}^1 \times \mathbb{D}^2)\#^k\mathbb{B}^3,$$

where N is a closed 3-dimensional manifold.

At this point, one should mention two important facts. First, Gromov-Lawson (see Theorem 5.7 in [15]), pointed out that if a compact manifold with boundary possesses metrics with positive scalar curvature and strictly mean convex boundary then its double can be endowed with a metric of positive scalar curvature. Therefore, the problem of characterising the compact manifolds with boundary supporting a metric with positive scalar curvature and strictly mean convex boundary reduces to the problem on theirs doubles manifolds. This was made in a very recent work due to A. Carlotto and C. Li [5]. Second, despite our results are not a complete characterization, they were obtained in a different way and gave us inspiration to deal with the high dimensional case.

We see that the topological condition (the existence of an incompressible torus in the boundary) that we consider here is specifically for dimension 3. For high dimension $3 \leq n \leq 7$, the situation is quite different, the problem is much more delicate and much more involved. However, extending to compact manifolds with boundary some of the ideas developed by Schoen-Yau [31], such as defining a class of manifolds via homology groups and using a descendent argument to recover the 3-dimensional case, we were able to obtain a type of classification result for high dimension.

Theorem 1.0.2. *Let $(M, \partial M)$ be a $(n+2)$-dimensional manifold, , $3 \leq n+2 \leq 7$, such that there is a non-zero degree map $F : (M, \partial M) \to (\Sigma \times T^n, \partial\Sigma \times T^n)$, where $(\Sigma, \partial\Sigma)$ is a connected surface which is not a disk. Then there exists no metric on M with positive scalar curvature and mean convex boundary. However, if Σ is not a disk or a cylinder, then there exists no metric on M with non-negative scalar curvature and mean convex boundary.*

As a consequence of the result above, we conclude that if N is a closed n-dimensional manifold, then $(T^{n-2} \times \mathring{T}^2)\#N$ does not admit a metric of non-negative scalar curvature and mean convex boundary and $(I \times T^{n-1})\#N$ does not admit a metric of positive scalar curvature and mean convex boundary.

The second part of this thesis is devoted to a rigidity result coming from an optimal inequality. We describe it now. In a very recent paper Bray, Brendle and Neves [3] proved an elegant rigidity result concerning to an area-minimizing 2-sphere embedded in a closed 3-dimensional manifold (M^3, g) with positive scalar curvature and $\pi_2(M) \neq 0$. In that work, they showed the following result. Denote by $\mathcal{F}$ the set of all smooth maps $f : \mathbb{S}^2 \to M$ which represent a nontrivial element in $\pi_2(M)$. Define

$$\mathcal{A}(M, g) = \inf\{Area(\mathbb{S}^2, f^*g) \ : \ f \in \mathcal{F}\}\,.$$

If $R_g \geq 2$, the following inequality holds:

$$\mathcal{A}(M, g) \leq 4\pi\,,$$

where R_g denote the scalar curvature of (M, g). Moreover, if the equality holds then the universal cover of (M, g) is isometric to the standard cylinder $\mathbb{S}^2 \times \mathbb{R}$ up to scaling. For more results concerning to rigidity of 3-dimensional closed manifolds coming from area-minimizing surfaces, see [2], [4], [26], [24] and [29]. In [35], J. Zhu showed a version of Bray, Brendle and Neves [3] result for high co-dimension: for $n+2 \leq 7$, let (M^{n+2}, g) be an oriented closed Riemannian manifold with $R_g \geq 2$, which admits a non-zero degree map $F : M \to \mathbb{S}^2 \times T^n$. Then $\mathcal{A}(M, g) \leq 4\pi$. Furthermore, the equality implies that the universal covering of (M^{n+2}, g) is $\mathbb{S}^2 \times \mathbb{R}^n$.

In the same direction, consider a 3-dimensional Riemannian manifold with non-empty boundary $(M^3, \partial M, g)$. Let $\mathcal{F}_M$ be the set of all immersed disks in M whose boundaries are curves in ∂M that are homotopically non-trivial in ∂M. If $\mathcal{F}_M \neq \emptyset$, we define

$$\mathcal{A}(M, g) = \inf_{\Sigma \in \mathcal{F}_M} |\Sigma|_g \ \text{ e } \ \mathcal{L}(M, g) = \inf_{\Sigma \in \mathcal{F}_M} |\partial\Sigma|_g$$

In the paper [1], L. C. Ambrózio proved the following result.

Theorem 1.0.3. *Let (M, g) be a compact Riemannian 3-manifold with mean convex boundary. Assume that $\mathcal{F}_M \neq \emptyset$. Then*

$$\frac{1}{2}\inf R_g^M \mathcal{A}(M,g) + \inf H_g^{\partial M}\mathcal{L}(M,g) \leq 2\pi. \tag{1.1}$$

Moreover, if equality holds, then universal covering of (M, g) is isometric to $(\mathbb{R}\times\Sigma_0, dt^2+g_0)$, where (Σ_0, g_0) is a disk with constant Gaussian curvature $\frac{1}{2}\inf R_g$ and $\partial\Sigma_0$ has constant geodesic curvature $\inf H_g^{\partial M}$ in (Σ_0, g_0).

A question that arises here is the following: *Is it possible to obtain similar result for high co-dimension?* Unfortunately, a general result cannot be true as we can see with the following example. Consider $(M, g) = (\mathbb{S}^2_+(r) \times \mathbb{S}^m(R), h_0 + g_0)$, where $(\mathbb{S}^2_+(r), h_0)$ is the half 2-sphere of radius r with the standard metric, and $(\mathbb{S}^m(R), g_0)$ is the m-sphere of radius R with the standard metric, $m \geq 2$. This case, we have that

$$\frac{1}{2}\inf R_g^M \mathcal{A}(M,g) + \inf H_g^{\partial M}\mathcal{L}(M,g) > 2\pi.$$

On the other hand, consider $(M, g) = (\mathbb{S}^2_+(r) \times T^m, g_0 + \delta)$, where (T^m, δ) is the flat m-torus, $m \geq 2$. Note that the equality holds in (1.1). However, we can see that in this case the universal covering of (M, g) is isometric to $(\mathbb{S}^2_+(r) \times \mathbb{R}^m, g_0 + \delta_0)$, where δ_0 is a standard metric in $\mathbb{R}^m$.

In the first example above, note that there is no map $F : (M, \partial M) \to (\mathbb{D}^2\times T^n, \partial\mathbb{D}^2\times T^n)$ with non-zero degree. However, this is a condition that we need in order to obtain a similar result as in [1]. Our main result of this work is the following.

Theorem 1.0.4. *Let $(M, \partial M, g)$ be a Riemannian $(n + 2)$-manifold, $3 \leq n + 2 \leq 7$, with positive scalar curvature and mean convex boundary. Assume that there is a map $F : (M, \partial M) \to (\mathbb{D}^2 \times T^n, \partial\mathbb{D}^2 \times T^n)$ with non-zero degree. Then,*

$$\frac{1}{2}\inf R_g^M \mathcal{A}(M,g) + \inf H_g^{\partial M}\mathcal{L}(M,g) \leq 2\pi. \tag{1.2}$$

Moreover, if the boundary ∂M is totally geodesic and the equality holds in (1.2), then universal covering of (M, g) is isometric to $(\mathbb{R}^n \times \Sigma_0, \delta + g_0)$, where δ is the standard metric in $\mathbb{R}^n$ and (Σ_0, g_0) is a disk with constant Gaussian curvature $\frac{1}{2}\inf R_g^M$ and $\partial\Sigma_0$ has null geodesic curvature in (Σ_0, g_0).

Organisation of the Thesis

This thesis is organised as follows. In Chapter 1, we present some auxiliaries results to be used in the proof of the main results. In Chapter 2, we present the first part and, in chapter 3, we discuss the second part.

Chapter 2

Preliminaries

In this chapter we will fix some notation and give a short description of the basic concepts necessary for a better understanding of the following chapters. We assume all the manifolds we are working with are compact and orientable.

2.1 Initial concepts

In this work, we denote by $\mathcal{X}(M)$ the set of all smooth vector fields in M and by $\mathcal{T}^k(M)$ the set of all k-covariant tensors in M.

Let (M, g) be a n-dimensional Riemannian manifold. The Levi-Civita theorem states that there is only one map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

(i) $\nabla_{fX+hZ} Y = f\nabla_X Y + h\nabla_Z Y$

(ii) $\nabla_X (Y + Z) = \nabla_X Y + h\nabla_X Z$

(iii) $\nabla_X fY = X(f)Y + f\nabla_X Y$

(iv) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

(v) $\nabla_X Y - \nabla_Y X = [X, Y]$

for every vector field X, Y, $Z \in \mathcal{X}(M)$ and for every function f and $h \in C^\infty(M)$, where $[X, Y] = XY - YX$ denote the Lie bracket of the vector field X and Y. This map is called the *Riemannian connection* of (M, g) and $\nabla_X Y$ is called the *covariant derivative of* X *in the direction of* Y.

Definition 2.1 (Covariant derivative of covariant tensors)**.** *Let (M, g) be a n-dimensional Riemannian manifold. The covariant derivative of a tensor $T \in \mathcal{T}^k(M)$ in the direction of a vector field $X \in \mathcal{X}(M)$ is a tensor $\nabla_X T \in \mathcal{T}^k(M)$ defined by*

$$(\nabla_X T)(X_1, \cdots, X_k) = X(T(X_1, \cdots, X_k)) - \sum_{i=1}^{k} T(X_1, \cdots, \nabla_X X_i, \cdots, X_k)$$

where $X_1, \cdots, X_k \in \mathcal{X}(M)$.

Definition 2.2 (Total covariant derivative of covariant tensors)**.** *Let (M, g) be a n-dimensional Riemannian manifold. The total covariant derivative of a tensor $T \in \mathcal{T}^k(M)$ to be the tensor $\nabla T \in \mathcal{T}^{k+1}(M)$ defined by*

$$(\nabla T)(X_1, \cdots, X_{k+1}) = (\nabla_{X_{k+1}} T)(X_1, \cdots, X_k)$$

where $X_1, \cdots, X_{k+1} \in \mathcal{X}(M)$.

Definition 2.3 (Divergence of vector fields)**.** *Let (M, g) be a n-dimensional Riemannian manifold. We define the divergence of a vector field $X \in \mathcal{X}(M)$ to be a function $div_g(X) \in C^\infty(M)$ given by $div_g(X)(p) = tr(Y(p) \mapsto \nabla_Y X(p))$, $p \in M$.*

In coordinates, the divergence of a vector field $X \in \mathcal{X}(M)$ is

$$div_g(X) = \sum_{i,j=1}^{n} g^{ij} g(\nabla_{\partial_i} X, \partial_j).$$

Definition 2.4 (Divergence of covariant tensors)**.** *Let (M, g) be a n-dimensional Riemannian manifold. The divergence of a tensor $T \in \mathcal{T}^k(M)$ is a tensor $div_g(T) \in \mathcal{T}^{k-1}(M)$ defined by $div_g(T) = tr_g(\nabla T)$.*

Denote by $\Omega^1(M)$ the space of the smooth 1-forms in a n-dimensional Riemannian manifold (M, g). Consider $\omega \in \Omega^1(M)$. We define the *sharp* of ω to be the only vector field $\omega^\# \in \mathcal{X}(M)$ such that $\omega(Y) = g(\omega^\#, Y)$, for every $Y \in \mathcal{X}(M)$. In coordinates,

$$\omega^\# = \sum_{i,j=1}^{n} = g^{ij} \omega_j \partial_i.$$

It is well know that $\mathcal{T}^1(M)$ is isomorphic to $\Omega^1(M)$. Note that $div_g(\omega) = div_g(\omega^\#)$. It follows from Divergence theorem that

$$\int_M div_g(\omega) dv_g = \int_{\partial M} g(\omega^\#, \eta) d\sigma_g,$$

for every $\omega \in \Omega^1(M)$, where η is the outward-pointing unit length normal to ∂M. Here, dv_g and $d\sigma_g$ are the volume forms of (M, g) and $(\partial M, g)$, respectively.

Remark 2.5. *Let (M, g) be a n-dimensional Riemannian manifold. If $T \in \mathcal{T}^2(M)$ is a symmetric tensor then $tr_g(T) \in C^\infty(M)$ and, in coordinates, we have that*

$$div_g(T) = \sum_{i,j,k=1}^{n} g^{ij}(\nabla_i T)_{jk}.$$

Definition 2.6 (Gradient)**.** *Let (M, g) be a n-dimensional Riemannian manifold and a function $f \in C^\infty(M)$. We define the gradient of f to be the only vector field $\nabla_g f \in \mathcal{X}(M)$ such that $df(X) = g(\nabla_g f, X)$, for every $X \in \mathcal{X}(M)$.*

In coordinates,

$$\nabla_g f = \sum_{i,j=1}^{n} g^{ij}\partial_j(f)\partial_i.$$

Definition 2.7 (Hessian)**.** *Let (M, g) be a n-dimensional Riemannian manifold and a function $f \in C^\infty(M)$. The symmetric tensor $\nabla_g^2 f = \nabla\nabla f \in \mathcal{T}^2(M)$ is called hessian of f.*

The hessian of a function $f \in C^\infty(M)$ is given by

$$\nabla_g^2 f(X, Y) = g(\nabla_Y \nabla_g f, X),$$

for every $X, Y \in \mathcal{X}(M)$.

Definition 2.8 (Laplacian)**.** *Let (M, g) be a n-dimensional Riemannian manifold and a function $f \in C^\infty(M)$. We define the laplacian of f by $\Delta_g f = div_g(\nabla_g f) = tr_g(\nabla_g^2 f)$.*

In coordinates,

$$\Delta_g f = \sum_{i,j=1}^{n} \frac{1}{\sqrt{|g|}}\partial_i\left(g^{ij}\partial_j(f)\sqrt{|g|}\right),$$

where $|g| = det\ (g_{ij})$.

Definition 2.9 (Curvatures)**.** *Let (M, g) be a n-dimensional Riemannian manifold. We have the followings definitions of curvature of (M, g):*

(1) The curvature endomorphism of (M, g) is the map

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

where $X, Y, Z \in \mathcal{X}(M)$.

(2) The curvature tensor of (M,g) *is a tensor* $R \in \mathcal{T}^4(M)$ *defined by*

$$R(X,Y,Z,W) = g(R(X,Y)Z,W)$$

where $X,Y,Z,W \in \mathcal{X}(M)$.

(3) Let $p \in M$ *and a* 2*-dimensional subspace* $\sigma \subset T_pM$. *We define the sectional curvature of* σ *by*

$$K(\sigma) = \frac{R(u,v,v,u)}{||u||^2||v||^2 - (g(u,v))^2}$$

where $\{u,v\}$ *is a basis of* σ. *We can show that the sectional curvature of* σ *does not depend on the choice of a basis.*

(4) The Ricci curvature tensor of (M,g) *is a symmetric tensor* $Ric_g \in \mathcal{T}^2(M)$ *defined by*

$$Ric(X,Y) = tr_g(R(.,X,Y,.)).$$

In coordinates,

$$(Ric_g)_{ij} = \sum_{k,l=1}^{n} g^{kl} R_{kijl}.$$

(5) The scalar curvature of (M,g) *is a function* $R_g \in C^\infty(M)$ *defined by*

$$R_g = tr_g(Ric_g).$$

In coordinates,

$$R_g = \sum_{i,j=1}^{n} g^{ij}(Ric_g)_{ij}.$$

For a more detailed discussion of the contents of this section see [6] and [22].

2.2 Geometry of submanifolds

Suppose (M,g) is a m-dimensional Riemannian manifold, Σ is a n-dimensional manifold and $F : \Sigma \to M$ is an immersion. If Σ has the induced Riemannian metric F^*g then F is said to be an *isometric immersion*. If in addition F is injective, so that Σ is an (immersed or embedded) submanifold of M, then Σ is said to be a Riemannian submanifold of M. In all of these situations, M is called the *ambient manifold*.

All the considerations of this section apply to any isometric immersion. Since our computations are all local, and since any immersion is locally an embedding, we may assume Σ is an embedded Riemannian submanifold, possibly after shrinking Σ a bit.

At each $p \in \Sigma$, the ambient tangent space T_pM splits as an orthogonal direct sum $T_pM = T_p\Sigma \oplus (T_p\Sigma)^\perp$, where $(T_p\Sigma)^\perp$ is the orthogonal complement of $T_p\Sigma$ in T_pM. Hence, if $v \in T_pM$, we can write $v = v^\top + v^\perp$, where $v^\top \in T_p\Sigma$ is called tangential component of v and $v^\perp \in (T_p\Sigma)^\perp$ is called normal component of v.

Proposition 2.10. *The Riemannian connection of Σ is*

$$\nabla_X Y = \left(\overline{\nabla}_X Y\right)^\top,$$

for every $X, Y \in \mathcal{X}(\Sigma)$, where $\overline{\nabla}$ is the Riemannian connection of (M, g).

Definition 2.11. *We define the second fundamental form of Σ to be the symmetric $C^\infty(\Sigma)$-bilinear form*

$$B : \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) \to (\mathcal{X}(\Sigma))^\perp$$

given by

$$B(X, Y) = \left(\overline{\nabla}_X Y\right)^\perp,$$

for every $X, Y \in \mathcal{X}(\Sigma)$.

Let $p \in \Sigma$ and $N \in (T_p\Sigma)^\perp$. Define the symmetric bilinear form

$$II_N : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$$

by

$$II_N(X, Y) = g(B(X, Y), N).$$

This bilinear form is associated to a selfadjoint linear operator $S_N : T_p\Sigma \to T_p\Sigma$ which satisfies $g(S_N(u), v) = g(B(u, v), N)$, for every $u, v \in T_p\Sigma$. We can show that

$$S_N(X) = -\left(\overline{\nabla}_X N\right)^\top.$$

Theorem 2.2.1 (Gauss)**.** *Let $e_1, \cdots, e_n$ be a orthonormal frame tangent to Σ. Then*

$$\overline{R}_{ijkl} = R_{ijkl} + B_{ik}B_{jl} - B_{il}B_{jk},$$

where R and $\overline{R}$ are curvature tensors of Σ and M, respectively.

Definition 2.12. *A Riemannian submanifold $\Sigma \subset (M, g)$ is said totally geodesic if $B \equiv 0$.*

Proposition 2.13. *A Riemannian submanifold $\Sigma \subset (M, g)$ is totally geodesic if and only if every geodesic of Σ is a geodesic of M.*

Let $\Sigma \subset (M, g)$ be a Riemannian submanifold and $p \in \Sigma$. Consider $\{E_1, \cdots, E_n\}$ a orthonormal basis $T_p\Sigma$ and $\{N_1, \cdots, N_k\}$ a basis of $(T_p\Sigma)^\perp$, where $k = m - n$. We define the *mean curvature vector* of Σ in p by

$$\vec{H} = \sum_{l=1}^{k} tr(S_{N_l})N_l$$

the *square of the norm of the second fundamental form* of Σ in p by

$$|B|^2 = \sum_{i,j=1}^{n} g(B(E_i, E_j), B(E_i, E_j)).$$

In coordinates, we have that

$$\vec{H} = \sum_{i,j=1}^{n} g^{ij} B_{ij}.$$

If $k = 1$, i.e., $\Sigma \subset (M, g)$ is a hypersurface then the mean curvature vector of Σ in a point $p \in \Sigma$ is $\vec{H}(p) = tr(S_N)N$, where $N \in (T_p\Sigma)^\perp$ is an unit vector. In this case, the number $H(p) := tr(S_N)$ is called *mean curvature* of Σ in p.

Proposition 2.14 (Gauss equation)**.** *Let $\Sigma \subset (M, g)$ be a hypersurface, $p \in \Sigma$ and $N \in (T_p\Sigma)^\perp$ is an unit vector. Then, in p, we have that*

$$2\overline{R}ic_g(N, N) + R_g + |B|^2 = \overline{R}_g + |H|^2$$

where R_g is the scalar curvature of (Σ, g) and $\overline{R}ic_g$, $\overline{R}_g$ are the scalar curvature and Ricci curvature of (M, g), respectively.

For a more detailed discussion of the contents of this section see [6], [8] and [22].

2.3 Stable minimal hypersurfaces with free boundary

Let $(M, \partial M, g)$ be a $(n + 1)$-dimensional Riemannian manifold and $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ be a properly embedded hypersurface, i.e., $\partial\Sigma = \Sigma \cap \partial M$. A *variation* of the hypersurface $\Sigma \subset M$ is a smooth one-parameter family $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ of proper embeddings $F_t : \Sigma \to M$, $t \in (-\epsilon, \epsilon)$, such that F_0 coincides with the inclusion $\Sigma \subset M$. The vector field $X = \frac{d}{dt}F_t\big|_{t=0}$ is called the *variational vector field* associated to $\{F_t\}_{t\in(-\epsilon,\epsilon)}$. The variation $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ is said to be *normal* if the curve $t \in (-\epsilon, \epsilon) \mapsto F_t(x)$ meets Σ orthogonally for each $x \in \Sigma$. Clearly, a variation $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ gives a smooth function $V : (-\epsilon, \epsilon) \to \mathbb{R}$ defined by

$$V(t) = Vol(F_t(\Sigma)) = \int_\Sigma dv_{g_t},$$

where $g_t = F_t^*(g)$ e dv_{g_t} is the volume form of (Σ, g_t).

Definition 2.15. *A properly embedded hypersurface* $(\Sigma, \partial\Sigma) \subset (M, \partial M, g)$ *is minimal with free boundary if* $V'(0) = 0$ *for every variations* $\{F_t\}_{t\in(-\epsilon,\epsilon)}$.

The *first variation of volume* of a properly embedded hypersurface $(\Sigma, \partial\Sigma) \subset (M, \partial M, g)$ with respect to $\{F_t\}_{t\in(-\epsilon,\epsilon)}$ is

$$V'(0) = -\int_\Sigma g(X, H_g^\Sigma) dv_g + \int_{\partial\Sigma} g(X, \eta) d\sigma_g,$$

where X is variational vector field associated to $\{F_t\}_{t\in(-\epsilon,\epsilon)}$, H_g^Σ is a mean curvature of Σ in (M, g) and η is the outward-pointing unit length normal to ∂M. Here, dv_g and $d\sigma_g$ are the volume forms of (Σ, g) and $(\partial\Sigma, g)$, respectively. It follow that the hypersurface Σ is minimal with free boundary if only if $H_g^\Sigma \equiv 0$ and Σ meets ∂M orthogonally along $\partial\Sigma$.

Remark 2.16. *Note that if a properly embedded hypersurface* $(\Sigma, \partial\Sigma) \subset (M, \partial M, g)$ *meets* ∂M *orthogonally along* $\partial\Sigma$ *then*

$$H_g^{\partial\Sigma} = H_g^{\partial M} - B_g^{\partial M}(\nu, \nu),$$

where $H_g^{\partial\Sigma}$ *the mean curvature of* $\partial\Sigma$ *in* (Σ, g) *with respect the outward-pointing unit length normal,* ν *is a globally defined unit normal vector field in* Σ *and* $H_g^{\partial M}$, $B_g^{\partial M}$ *are the mean curvature and second fundamental form of* ∂M *in* (M, g) *with respect to the outward-pointing unit length normal, respectively.*

Definition 2.17. *A properly embedded minimal hypersurface with free boundary* $(\Sigma, \partial\Sigma) \subset (M, \partial M, g)$ *is stable if* $V''(0) \geq 0$ *for every variations* $\{F_t\}_{t\in(-\epsilon,\epsilon)}$. *Otherwise,* Σ *is unstable.*

Remark 2.18. *If a hypersurface* Σ *is minimal with free boundary, then any variational vector field must be parallel to* ν *in* $\partial\Sigma$ *since the variation must go through proper embeddings. Hence, it is enough to consider only normal variations to analyze the second variation of volume.*

Let $(\Sigma, \partial\Sigma) \subset (M, \partial M, g)$ be a properly embedded minimal hypersurface with free boundary. Consider a globally defined unit normal vector field ν on Σ. Any normal vector field on Σ has the form $X = \varphi\nu$ for some $\varphi \in C^\infty(\Sigma)$ and the second variation of volume with respect to $X = \varphi\nu$ is

$$V''(0) = \int_\Sigma (|\nabla^\Sigma \varphi|^2 - \varphi^2(Ric^M(\nu, \nu) + |B^\Sigma|^2) dv - \int_{\partial\Sigma} \varphi^2 B^{\partial M}(\nu, \nu) d\sigma.$$

where ∇^Σ is the gradient operator in (Σ, g), Ric^M is the Ricci curvature of (M, g), and B^Σ is the second fundamental form of Σ in (M, g) with respect to the unit normal ν.

2.4 Conformal Laplacian with minimal boundary conditions

Let $(M, \partial M, g)$ be a Riemannian manifold of dimension $n \geq 3$. Define the following pair of operators acting in $C^{\infty}(M)$:

$$\begin{cases} L_g &= \quad -\Delta_g \varphi + c_n R_g \varphi \text{ in } M \\ T_g &= \quad \dfrac{\partial \varphi}{\partial \eta} + 2c_n H_g^{\partial M} \varphi \text{ on } \partial M, \end{cases}$$

where η denotes the outward unit normal vector of the boundary ∂M in (M, g) and $c_n := \frac{(n-2)}{4(n-1)}$. Consider the first eigenvalue $\lambda_1(M, g)$ of L_g with boundary condition T_g:

$$\begin{cases} L_g(\varphi) &= \quad \lambda_1(M,g)\varphi \text{ in } M \\ T_g(\varphi) &= \quad 0 \text{ on } \partial M \end{cases} \tag{2.1}$$

We have that,

$$\lambda_1(M,g) = \inf_{0 \not\equiv \varphi \in H^1(M)} \frac{\displaystyle\int_M (|\nabla_g \varphi|^2 + c_n R_g \varphi^2) dv_g + 2c_n \int_{\partial M} H_g^{\partial M} \varphi^2 d\sigma_g}{\displaystyle\int_M \varphi^2 dv_g}.$$

We can choose a positive function $\varphi \in C^{\infty}(M)$ solution of (2.1). The conformal metric $h = \varphi^{\frac{4}{n-2}} g$ is such that

$$\begin{cases} R_h &= \quad \lambda_1(M,g)\varphi^{-\frac{4}{n-2}} \text{ in } M \\ H_h^{\partial M} &\equiv \quad 0 \text{ on } \partial M \end{cases}$$

In particular, this implies that if $\lambda_1(M, g) > 0$ then $R_h > 0$ and $H_h^{\partial M} \equiv 0$.

2.5 Topology of 3-dimensional manifolds

In this section we are going to state some topological results and definitions, more specifically, from the topology of 3-dimensinal manifolds, which are useful to better understand this work. For a more detailed discussion of the contents of this section see [18], [19], [20] and [23].

2.5.1 Essential surfaces

Definition 2.19. *Let M be a 3-dimensional manifold. A connected surface Σ properly embedded (or embedded in ∂M) is said incompressible in M if either*

(i) Σ *is a sphere which does not bound a ball in* M*, or*

(ii) Σ *is not a sphere and the homomorphism* $\pi_1(\Sigma) \hookrightarrow \pi_1(M)$ *is injective.*

Otherwise, Σ is *compressible* in M.

Example 2.20. *Consider the solid torus* $M = \mathbb{S}^1 \times \mathbb{D}^2$*.*

(1) Note that ∂M *is a compressible torus in* M *(see Figure 2.1). Note that all the curves in the figure are homotopically non-trivial in* ∂M*, but are homotopically trivial in* M*.*

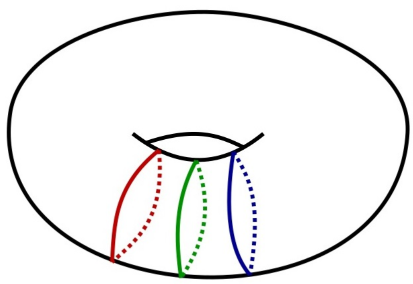

Figure 2.1: Compressible torus in solid torus

(2) Let c_1 *and* c_2 *be two closed curves which represent a non-trivial class in* $\pi_1(\partial M)$ *and bound the "hole" in* ∂M *(see Figure 2.2). Let* C *be a properly embedded cylinder in* M *which has* c_1 *and* c_2 *as boundary. Note that* C *is a incompressible cylinder in* M*.*

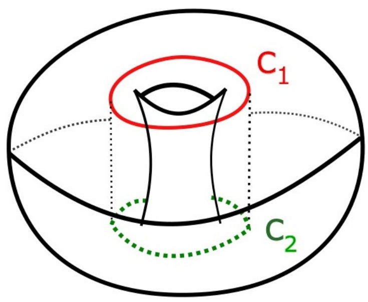

Figure 2.2: Incompressible cylinder in solid torus

Example 2.21. *Generally, the boundary of any handlebody is a compressible surface. Let* M *be a handlybody of genus equal to* 4 *(see Figure 2.3). Note that all the curves in the figure are homotopically non-trivial in* ∂M*, but are homotopically trivial in* M*.*

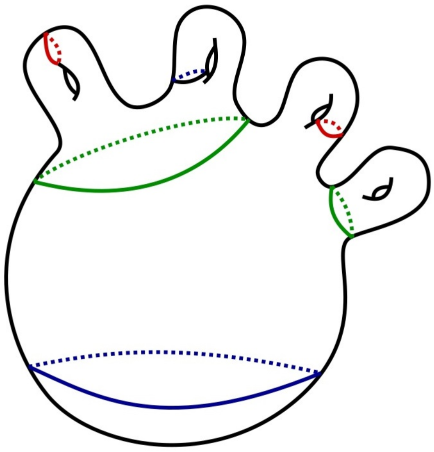

Figure 2.3: Handlebody of genus equal to 4

Definition 2.22. *Let $(M, \partial M)$ be a orientable 3-dimensional manifold and $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ a properly embedded connected and orientable surface.*

(i) We say that Σ is boundary-incompressible if the homomorphism $\pi_1(\Sigma, \partial\Sigma) \hookrightarrow \pi_1(M, \partial M)$ is injective. Otherwise, Σ is boundary-compressible in M.

(ii) We say that Σ is essential in M if it is incompressible and boundary-incompressible.

Example 2.23. *The cylinder from item (2) of Example 2.20 is boundary-compressible in the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$, therefore it is not essential.*

Example 2.24. *Consider the 3-dimensional manifold $M = \mathbb{S}^1 \times \Sigma$, where $(\Sigma, \partial\Sigma)$ is a connected surface which is not a disk. The surface Σ is essential in M.*

Example 2.25. *Consider the 3-dimensional manifold $M = I \times S$, where S is a closed surface with positive genus. Let $\Sigma = I \times \gamma$, where γ is a closed curve which represents a non-trivial class in $\pi_1(S)$ and bounds a "hole" in S. Note that Σ is a properly embedded cylinder in M.*

Claim 1. *Σ is incompressible in M.*

In fact, the curves which represent a non-trivial class of $\pi_1(\Sigma)$ are of the form $\{t\} \times \gamma$, where $t \in I$. Such curves also represent a non-trivial class in $\pi_1(M)$, since γ represents a non-trivial class in $\pi_1(S)$. It follows that the homomorphism $\pi_1(\Sigma) \hookrightarrow \pi_1(M)$ is injective, i.e., Σ is incompressible in M.

Claim 2. *Σ is boundary-incompressible in M.*

In fact, the curves which represent a non-trivial class of $\pi_1(\Sigma, \partial\Sigma)$ are the curves which connect distinct connected components of $\partial\Sigma$. This curves also connect distinct connected components of ∂M, consequently, they represent a non-trivial class in $\pi_1(M, \partial M)$. It follows that the homomorphism $\pi_1(\Sigma\partial\Sigma) \hookrightarrow \pi_1(M, \partial M)$ is injective, i.e., Σ is boundary-incompressible in M.

Therefore, from claims 1 and 2 it follows that the surface Σ is an essential cylinder in M.

Theorem 2.5.1 (See Proposition 9.4.3 in [23])**.** *Let $(M, \partial M)$ be a 3-dimensional manifold. Every non-trivial homology class $\alpha \in H_2(M, \partial M)$ is represented by a properly embedded surface $S \subset M$ such that its connected components are either spheres or essential surfaces.*

Theorem 2.5.2 (See Lemma 6.8 in [19])**.** *Let $(M, \partial M)$ be an 3-dimensional manifold such that ∂M contains a surface of positive genus. Then M contains a properly embedded, connected and incompressible surface $(\Sigma, \partial\Sigma)$ such that $0 \neq [\partial\Sigma] \in H_1(\partial M)$.*

2.5.2 Prime 3-dimensional manifolds

A 3-dimensional manifold M is *prime* if $M = M_1 \# M_2$ implies one of M_1, M_2 is a 3-dimensional sphere. For example, the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$, the 3-dimensional ball $\mathbb{B}^3$ and 3-dimensional sphere $\mathbb{S}^3$ are prime manifolds. On the other hand, the manifold $I \times \mathbb{S}^2$ is not prime, since $I \times \mathbb{S}^2 = \mathbb{B}^3 \# \mathbb{B}^3$.

Remark 2.26. *The solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ is the unique prime 3-dimensional manifold whose boundary is a compressible torus (see proof of Proposition 3.4 in [18]).*

A *prime decomposition* of a 3-dimensional manifold is a decomposition $M = M_1 \# \cdots \# M_k$ with each M_i prime. For example, the decomposition $I \times \mathbb{S}^2 = \mathbb{B}^3 \# \mathbb{B}^3$ is a prime decomposition of $I \times \mathbb{S}^2$.

Theorem 2.5.3 (See Theorem 1.5 in [18])**.** *Let M be a connected 3-dimensional manifold. Then there is a prime decomposition $M = M_1 \# M_2 \# \cdots \# M_k$, and this decomposition is unique up to insertion or deletion of $\mathbb{S}^3$'s.*

Remark 2.27. *Let M be a 3-dimensional manifold such that ∂M has sphere components. We denote by $\hat{M}$ the result of capping off each sphere component of ∂M. We have that $M = \hat{M} \#^k \mathbb{B}^3$, where k is the number of sphere components of ∂M. It follows that, if $\hat{M} = M_1 \# M_2 \# \cdots \# M_k$ is the prime factorization of $\hat{M}$ then $M = M_1 \# M_2 \# \cdots \# M_k \#^k \mathbb{B}^3$ is the prime factorization of M.*

Chapter 3

Topological obstructions to the existence of metrics with non-negative or positive scalar curvature and mean convex boundary

In this chapter we use arguments similar to the ones used by Schoen and Yau in [31] to prove that the existence of a non-zero degree map of a n-dimensional manifold $(M, \partial M)$ to a manifold of the form $\Sigma \times T^n$, where $3 \leq n \leq 7$ and $(\Sigma, \partial\Sigma)$ is a connected surface which is not a disk, is a topological obstruction to the existence of a Riemannian metric in $(M, \partial M)$ with positive scalar curvature and mean convex boundary. Furthermore, if Σ is neither a disk nor a cylinder, then the above condition is a topological obstruction to the existence of a Riemannian metric in $(M, \partial M)$ with non-negative scalar curvature and mean convex boundary.

3.1 Technical results

In this section we are going to state and prove results about Riemannian manifolds with mean convex boundary analogous to the theorems stated below. These theorems by Schoen and Yau play a fundamental role in the article [31].

Theorem 3.1.1 (Schoen and Yau, [31])**.** *Let (M, g) be a closed Riemannian n-dimensional manifold with positive scalar curvature and $n \geq 3$. Then, every embedded stable minimal hypersurface $\Sigma \subset M$ admits a Riemannian metric with positive scalar curvature.*

Theorem 3.1.2 (Kazdan and Warner, [21]). *Let (M,g) be a Riemannian n-dimensional manifold with non-negative scalar curvature and $n \geq 3$. Then either M admits a Riemannian metric with positive scalar curvature or (M,g) is Ricci-flat.*

Let $(M,\partial M,g)$ be a Riemannian manifold of dimension $n\geq 3$. Assume that $R_g, H_g^{\partial M}\geq 0$ and $Vol_g(M)=1$, where H_g denote the mean curvature of ∂M with respect to the outward unit normal vector. For each Riemannian metric $\tilde g$ on M consider $\lambda(\tilde g)\in\mathbb{R}$ and $\Phi_{\tilde g}\in C^\infty(M)$ satisfying:

$$\begin{cases} -\Delta_{\tilde g}\Phi_{\tilde g}+c_nR_{\tilde g}\Phi_{\tilde g} & = & \lambda(\tilde g)\Phi_{\tilde g}\\ \dfrac{\partial\Phi_{\tilde g}}{\partial\eta_{\tilde g}} & = & -2c_nH_{\tilde g}\Phi_{\tilde g}\\ \displaystyle\int_M\Phi_{\tilde g}dv_{\tilde g} & = & 1\end{cases}$$

where $\eta_{\tilde g}$ denote the outward unit normal vector of the boundary ∂M in $(M,\tilde g)$ and $c_n:=\frac{(n-2)}{4(n-1)}$. Note that, as we are considering, we can assume that $\Phi_{\tilde g}>0$.

Moreover, note that

$$\begin{aligned}\lambda(\tilde g) &= -\int_M\Delta_{\tilde g}\Phi_{\tilde g}dv_{\tilde g}+c_n\int_M R_{\tilde g}\Phi_{\tilde g}dv_{\tilde g}\\ &= -\int_{\partial M}\frac{\partial\Phi_{\tilde g}}{\partial\eta_{\tilde g}}d\sigma_{\tilde g}+c_n\int_M R_{\tilde g}\Phi_g dv_{\tilde g}.\end{aligned}$$

Therefore,

$$\lambda(\tilde g)=2c_n\int_{\partial M}\Phi_{\tilde g}H_{\tilde g}d\sigma_{\tilde g}+c_n\int_M R_{\tilde g}\Phi_{\tilde g}dv_{\tilde g}.$$

Lemma 3.1. *Let $(M,\partial M,g)$ be a Riemannian n-dimensional manifold, $n\geq 3$, such that $R_g,H_g\geq 0$ and $Vol_g(M)=1$. If $\lambda(g)=0$ then*

$$D\lambda_g(h)=-c_n\int_{\partial M}\langle h,B_g\rangle d\sigma_g-c_n\int_M\langle h,Ric_g\rangle dv_g,$$

for every 2-covariant symmetric tensor h in M, where B_g and Ric_g is the second fundamental form of ∂M in (M,g) and the Ricci curvature of (M,g), respectively.

Proof. Firstly, note that $\lambda(g)=0$ implies that $R_g\equiv 0,\ H_g\equiv 0$ and $\Phi_g\equiv 1$. Let h be 2-covariant symmetric tensor in M. Consider $g(t)$, for each $t\in(-\epsilon,\epsilon)$ a smooth family of Riemannian metrics on M in a such way that $g(0)=g$ e $g'(0)=h$. Denote by

$$\lambda(t):=\lambda(g(t)),\ h(t):=g'(t),\ R(t):=R_{g(t)}\quad\text{and}\quad H(t):=H_{g(t)}\,.$$

As $R_g\equiv 0,\ H_g\equiv 0$ and $\Phi_g\equiv 1$, we obtain that

$$D\lambda_g(h)=\lambda'(0)=2c_n\int_{\partial M}H'(0)d\sigma_g+c_n\int_M R'(0)dv_g.$$

From Proposition 2.3.9 in [34], we have that

$$R'(t) = -\langle h(t), Ric_{g(t)}\rangle + div_{g(t)}(div_{g(t)}(h(t)) - d(tr_{g(t)}\ h(t))).$$

Hence, from Divergence Theorem, we obtain that

$$\begin{aligned} D\lambda_g(h) &= 2c_n \int_{\partial M} H'(0) d\sigma_g - c_n \int_M \langle h, Ric_g\rangle dv_g + c_n \int_M div_g(div_g(h) - d(tr_g\ h)) dv_g \\ &= c_n \int_{\partial M} \big(2H'(0) + \langle (div_g(h))^{\#} - (d(tr_g\ h))^{\#}, \eta\rangle\big)\, d\sigma_g - c_n \int_M \langle h, Ric_g\rangle dv_g \\ &= c_n \int_{\partial M} (2H'(0) + X) d\sigma_g - c_n \int_M \langle h, Ric_g\rangle dv_g, \end{aligned}$$

where $\eta = \eta_g$ and $X := \langle (div_g(h))^{\#} - (d(tr_g\ h))^{\#}, \eta\rangle$.

Einstein convention and notation:

(1) Without a summation symbol, lower and upper index indicate a summation from 1 to $n-1$.

(2) ∇^t denote the Riemannian connection of $(M, g(t))$, $\nabla := \nabla^0$.

(3) B_t denote the second fundamental form of ∂M in $(M, g(t))$.

Consider $(x_1, \cdots, x_n)$ a local chart on M such that $(x_1, \cdots, x_{n-1})$ is a local chart on ∂M and $\partial_n = \eta$. We divide the proof in some steps.

Step 1: Computation of X in ∂M.

We have that

$$d(tr_g\ h) = \sum_{k=1}^{n} \partial_k \left(\sum_{i,j=1}^{n} g^{ij} h_{ij} \right) dx^k$$

and

$$div_g(h) = \sum_{k=1}^{n} (div_g(h))_k dx^k$$

It follows that

$$(d(tr_g\ h))^{\#} = \sum_{i,j,k,l=1}^{n} g^{lk} \partial_k (g^{ij} h_{ij}) \partial_l,$$

and

$$(div_g(h))^{\#} = \sum_{k,l=1}^{n} g^{lk} (div_g(h))_k \partial_l = \sum_{i,j,k,l=1}^{n} g^{lk} g^{ij} (\nabla_i h)_{jk} \partial_l.$$

Thus,

$$(div_g(h))^{\#} - (d(tr_g\ h))^{\#} = \sum_{l=1}^{n} \left\{ \sum_{i,j,k,=1}^{n} \left(g^{lk} g^{ij} (\nabla_i h)_{jk} - g^{lk} \partial_k (g^{ij} h_{ij}) \right) \right\} \partial_l.$$

In ∂M, we get that $g_{nn} = g^{nn} = 1$ and $g_{ln} = g^{ln} = 0$, for every $l = 1, \cdots n-1$. Hence,

$$\begin{aligned}
X &= \sum_{i,j=1}^{n} \left(g^{ij}(\nabla_i h)_{jn} - \nu(g^{ij}h_{ij})\right) \\
&= g^{ij}(\nabla_i h)_{jn} + \nu(h_{nn}) - \nu(g^{ij}h_{ij}) - \nu(h_{nn}) \\
&= g^{ij}(\nabla_i h)_{jn} - \nu(g^{ij})h_{ij} - g^{ij}\nu(h_{ij}) \\
&= g^{ij}(\nabla_i h)_{jn} + g^{ik}g^{jl}\nu(g_{kl})h_{ij} - g^{ij}\nu(h_{ij}) \\
&= g^{ij}(\nabla_i h)_{jn} + 2g^{ik}g^{jl}(B_g)_{kl}(h)_{ij} - g^{ij}\nu(h_{ij}) \\
&= g^{ij}(\nabla_i h)_{jn} + 2\langle h, B_g\rangle - g^{ij}\nu(h_{ij})
\end{aligned}$$

Step 2: Computation of $H'(0)$:

We have $H(t) = g_t^{ij}(B_t)_{ij}$. Hence,

$$\begin{aligned}
H'(t) &= \frac{d}{dt}(g_t^{ij})(B_t)_{ij} + g_t^{ij}\frac{d}{dt}(B_t)_{ij} \\
&= -g_t^{ik}g_t^{jl}(h_t)_{kl}(B_t)_{ij} + tr\left(\frac{d}{dt}B_t\right) \\
&= -\langle h(t), B(t)\rangle + tr\left(\frac{d}{dt}B_t\right).
\end{aligned}$$

Lets focus our attention on $tr\left(\left.\frac{d}{dt}\right|_{t=0} B_t\right)$. Since,

$$(B_t)_{ij} = -g_t(\eta_t, \nabla_i^t\partial_j),$$

it follows that

$$\left.\frac{d}{dt}\right|_{t=0}(B_t)_{ij} = -h(\eta, \nabla_i\partial_j) - \left\langle \left.\frac{d}{dt}\right|_{t=0}(\eta_t), \nabla_i\partial_j \right\rangle - \left\langle \eta, \left.\frac{d}{dt}\right|_{t=0}(\nabla_i^t\partial_j)\right\rangle.$$

From Proposition 2.3.1 in [34], we obtain that

$$\left.\frac{d}{dt}\right|_{t=0}(B_t)_{ij} = -h(\eta, \nabla_i\partial_j) - \left\langle \left.\frac{d}{dt}\right|_{t=0}(\eta_t), \nabla_i\partial_j \right\rangle - \frac{1}{2}(\nabla_i h)_{jn} - \frac{1}{2}(\nabla_j h)_{in} + \frac{1}{2}(\nabla_\eta h)_{ij}.$$

Claim 3. *In ∂M, we have that*

$$\left.\frac{d}{dt}\right|_{t=0}(\eta_t) = -g^{kl}h_{nk}\partial_l - \frac{1}{2}h_{nn}\eta.$$

Proof. In ∂M, we have $(g_t)_{nk} = 0$ and $(g_t)_{nn} = 1$, for all $k = 1, \cdots n-1$ and $t \in (-\epsilon, \epsilon)$. Thus,

$$\left\langle \left.\frac{d}{dt}\right|_{t=0}(\eta_t), \eta \right\rangle = -\frac{1}{2}h_{nn}$$

and

$$\left\langle \frac{d}{dt}\bigg|_{t=0}(\eta_t), \partial_k \right\rangle = -h_{nk}.$$

Denote by

$$\frac{d}{dt}\bigg|_{t=0}(\eta_t) = \sum_{l=1}^{n} a_l \partial_l.$$

Note that

$$a_n = \left\langle \frac{d}{dt}\bigg|_{t=0}(\eta_t), \eta \right\rangle = -\frac{1}{2}h_{nn}.$$

However, for $k = 1, \cdots, n-1$,

$$-h_{nk} = \left\langle \frac{d}{dt}\bigg|_{t=0}(\eta_t), \partial_k \right\rangle = \sum_{i=1}^{n-1} a_i g_{ki}.$$

It follows that, for $l = 1, \cdots, n-1$, we have that

$$a_l = -g^{lk} h_{nk}.$$

Hence,

$$\frac{d}{dt}\bigg|_{t=0}(\eta_t) = \sum_{l=1}^{n-1} a_l \partial_l + a_n \eta = -g^{lk} h_{nk} \partial_l - \frac{1}{2} h_{nn} \eta.$$

■

It follows from the Claim 3 that

$$\begin{aligned}
\left\langle \frac{d}{dt}\bigg|_{t=0}(\eta_t), \nabla_i \partial_j \right\rangle &= -g^{lk} h_{nk} \langle \nabla_i \partial_j, \partial_l \rangle - \frac{1}{2} h_{nn} \langle \nabla_i \partial_j, \eta \rangle \\
&= -g^{lk} h_{nk} \Gamma_{ij}^m g_{ml} + \frac{1}{2} h_{nn} (B_g)_{ij} \\
&= -h_{nk} \Gamma_{ij}^k + \frac{1}{2} h_{nn} (B_g)_{ij}
\end{aligned}$$

However,

$$-h(\nabla_i \partial_j, \eta) = -h_{nk} \Gamma_{ij}^k - h_{nn} \Gamma_{ij}^n = (B_g)_{ij} h_{nn} - h_{nk} \Gamma_{ij}^k,$$

since

$$\Gamma_{ij}^n = \frac{1}{2} \sum_{k=1}^{n} g^{nk} \{ \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \} = -\frac{1}{2} \eta(g_{ij}) = -(B_g)_{ij}.$$

It implies that

$$\begin{aligned}
\left\langle \frac{d}{dt}\bigg|_{t=0}(\eta_t), \nabla_i \partial_j \right\rangle &= -h(\nabla_i \partial_j, \eta) - (B_g)_{ij} h_{nn} + \frac{1}{2} h_{nn} (B_g)_{ij} \\
&= -h(\nabla_i \partial_j, \eta) - \frac{1}{2} h_{nn} (B_g)_{ij}.
\end{aligned}$$

Hence,

$$\left.\frac{d}{dt}\right|_{t=0}(B_t)_{ij} = -\frac{1}{2}(\nabla_i h)_{jn} - \frac{1}{2}(\nabla_j h)_{in} + \frac{1}{2}(\nabla_\eta h)_{ij} + \frac{1}{2}h_{nn}(B_g)_{ij}.$$

Consequently,

$$tr\left(\left.\frac{d}{dt}\right|_{t=0}(B_t)\right) = -g^{ij}(\nabla_i h)_{jn} + \frac{1}{2}g^{ij}(\nabla_\eta h)_{ij} + \frac{1}{2}h_{nn}H_g.$$

As $H_g = 0$, we obtain that

$$\begin{aligned} 2H'(0) &= -2\langle h, B_g\rangle + 2tr\left(\left.\frac{d}{dt}\right|_{t=0}(B_t)\right) \\ &= -2\langle h, B_g\rangle - 2g^{ij}(\nabla_i h)_{jn} + g^{ij}(\nabla_\eta h)_{ij} \\ &= -2\langle h, B_g\rangle - 2g^{ij}(\nabla_i h)_{jn} + g^{ij}\eta(h_{ij}) - 2g^{ij}h(\nabla_i\eta, \partial_j). \end{aligned}$$

Claim 4. *In ∂M, we have that*

$$g^{ij}h(\nabla_i\eta, \partial_j) = \langle h, B_g\rangle.$$

Proof. Write,

$$\nabla_i\nu = \sum_{k=1}^{n}\Gamma_{in}^k\partial_k.$$

Note that, in ∂M, we have $\Gamma_{in}^n = 0$ e $\Gamma_{in}^k = g^{mk}(B_g)_{im}$, for every $k = 1, \cdots, n-1$. This implies that

$$\nabla_i\eta = g^{mk}(B_g)_{im}\partial_k.$$

Hence, in ∂M, we obtain that

$$g^{ij}h(\nabla_i\eta, \partial_j) = g^{ij}g^{mk}(B_g)_{im}h_{kj} = \langle h, B_g\rangle.$$

∎

It follows from the Claim 4 that

$$2H'(0) = -4\langle h, B_g\rangle - 2g^{ij}(\nabla_i h)_{jn} + g^{ij}\eta(h_{ij}).$$

Therefore,

$$2H'(0) + X|_{\partial M} = -2\langle h, B_g\rangle - g^{ij}(\nabla_i h)_{jn}. \tag{3.1}$$

Claim 5. *In ∂M, we have that*

$$g^{ij}(\nabla_i h)_{jn} = -\langle h, B_g\rangle + div_g^{\partial M}(\omega),$$

for some $\omega \in \Omega^1(\partial M)$.

Proof. It follows from the Claim 4 that, in ∂M,

$$\begin{aligned} g^{ij}(\nabla_i h)_{jn} &= g^{ij}\partial_i(h_{jn}) - g^{ij}h(\nabla_i\partial_j,\eta) - g^{ij}h(\nabla_i\eta,\partial_j) \\ &= g^{ij}\partial_i(h_{jn}) - g^{ij}h(\nabla_i\partial_j,\eta) - \langle h, B_g\rangle. \end{aligned}$$

For $1 \leq i,j \leq n-1$, in ∂M, we can write

$$\nabla_i\partial_j = (B_g)_{ij}\eta + \overline{\nabla}_i\partial_j,$$

where $\overline{\nabla}$ is the Riemannian connection of $(\partial M, g)$.

Hence, since $H_g \equiv 0$, we obtain that

$$g^{ij}h(\nabla_i\partial_j,\eta) = h_{nn}H_g + g^{ij}h(\overline{\nabla}_i\partial_j,\eta) = g^{ij}h(\overline{\nabla}_i\partial_j,\eta).$$

This implies that, in ∂M,

$$g^{ij}(\nabla_i h)_{jn} = g^{ij}\partial_i(h_{jn}) - g^{ij}h(\overline{\nabla}_i\partial_j,\eta) - \langle h, B_g\rangle.$$

Define $\omega \in \Omega^1(\partial M)$ as

$$\omega := h(.,\nu)|_{\partial M}\,.$$

Note that

$$\begin{aligned} div_g^{\partial M}(\omega) &= g^{ij}(\overline{\nabla}_i\omega)_j = g^{ij}\partial_i(\omega_j) - g^{ij}\omega(\overline{\nabla}_i\partial_j) \\ &= g^{ij}\partial_i(h_{jn}) - g^{ij}h(\overline{\nabla}_i\partial_j,\nu). \end{aligned}$$

Therefore, in ∂M,

$$g^{ij}(\nabla_i h)_{jn} = -\langle h, B_g\rangle + div_g^{\partial M}(\omega).$$

■

It follows from equality (3.1) and Claim 5 that

$$2H'(0) + X|_{\partial M} = -\langle h, B_g\rangle - div_g^{\partial M}(\omega).$$

Hence,

$$D\lambda_g(h) = -c_n\int_{\partial M}\langle h, B_g\rangle d\sigma_g - c_n\int_M\langle h, Ric_g\rangle dv_g - c_n\int_{\partial M} div_g^{\partial M}(\omega)d\sigma_g.$$

We conclude, since ∂M is a closed manifold, that

$$D\lambda_g(h) = -c_n\int_{\partial M}\langle h, B_g\rangle d\sigma_g - c_n\int_M\langle h, Ric_g\rangle dv_g.$$

■

Corollary 3.2. *Let $(M, \partial M, g)$ be a Riemannian manifold of dimension $n \geq 3$ such that $R_g, H_g \geq 0$, $Vol_g(M) = 1$ and $\lambda(g) = 0$. The metric g is a critical point of the functional λ if and only if (M, g) is Ricci flat with totally geodesic boundary.*

The following theorems are generalizations of the Theorems 3.1.2 and 3.1.1 for Riemannian manifolds with mean convex boundary, respectively.

Theorem 3.1.3. *Let $(M, \partial M, g)$ be a Riemannian manifold of dimension $n \geq 3$ such that $R_g \geq 0$ and $H_g \geq 0$. Then M admits a metric with positive scalar curvature and minimal boundary or (M, g) is Ricci flat with totally geodesic boundary.*

Proof. We can assume that $Vol_g(M) = 1$. It follows from

$$\lambda(g) = 2c_n \int_{\partial M} \Phi_g H_g d\sigma_g + c_n \int_M R_g \Phi_g dv_g$$

and $R_g \geq 0$, $H_g \geq 0$ that $\lambda(g) \geq 0$. If $\lambda(g) > 0$, then there exists a metric on M with positive scalar curvature and minimal boundary (see Section 2.4).

Then, assume that $\lambda(g) = 0$. If $D\lambda_g \equiv 0$ we have that g is a critical point of the functional λ. It follows from Corollary 3.2 that $Ric_g \equiv 0$ and $B_g \equiv 0$. If $D\lambda_g \not\equiv 0$, there exists a 2-covariant symmetric tensor h_0 in M such that $D\lambda_g(h_0) > 0$. Consider a family of metrics on M, $g(t) = g + th_0$, $t \in (-\epsilon, \epsilon)$. Since $\lambda'(0) = D\lambda_g(h_0) > 0$, we obtain that there exists $\theta \in (0, \epsilon)$ such that the function $t \in (-\theta, \theta) \mapsto \lambda(t) \in \mathbb{R}$ is an increase function. Since $\lambda(0) = \lambda(g) = 0$, we get that $\lambda(t) > 0$ for all $t \in (0, \theta)$. Therefore, for each $t \in (0, \theta)$ there is a metric $\tilde{g}_t$ on M such that $R_{\tilde{g}_t} > 0$ and $H_{\tilde{g}_t} \equiv 0$ (see Section 2.4). ∎

Theorem 3.1.4. *Let $(M, \partial M, g)$ be a Riemannian manifold of dimension $n + 1 \geq 3$ such that $R_g > 0$ and $H_g^{\partial M} \geq 0$. Then every free-boundary stable minimal hypersurface in M has a metric with positive scalar curvature and minimal boundary.*

Proof.

Consider Σ a free-boundary stable minimal in M. It follows from the second variation formula for the volume that

$$\int_\Sigma |\nabla \varphi|^2 dv_g \geq \int_\Sigma \varphi^2 (Ric_g(N, N) + |B_g^\Sigma|^2) dv_g + \int_{\partial \Sigma} \varphi^2 B_g^{\partial M}(N, N) d\sigma_g$$

for every $\varphi \in C^\infty(\Sigma)$, where N denotes a unit vector field on Σ in (M, g). As $R_g > 0$, it follows from the Gauss equation that

$$Ric_g(N, N) + |B_g^\Sigma|^2 = \frac{1}{2}(R_g - R_g^\Sigma + |B_g^\Sigma|^2) > -\frac{1}{2} R_g^\Sigma.$$

Hence,

$$\int_\Sigma |\nabla\varphi|^2 dv_g > -\frac{1}{2}\int_\Sigma \varphi^2 R_g^\Sigma dv_g + \int_{\partial\Sigma} \varphi^2 B_g^{\partial M}(N,N) d\sigma_g\,,$$

for every $\varphi \in C^\infty(\Sigma)$. Since $H_g^{\partial M} \geq 0$, and Σ is a free-boundary hypersurface in (M,g), we obtain

$$B_g^{\partial M}(N,N) = H_g^{\partial M} - H_g^{\partial\Sigma} \geq -H_g^{\partial\Sigma}.$$

Thus,

$$\int_\Sigma |\nabla\varphi|^2 dv_g > -\frac{1}{2}\int_\Sigma \varphi^2 R_g^\Sigma dv_g - \int_{\partial\Sigma} \varphi^2 H_g^{\partial\Sigma} d\sigma_g\,,$$

for every $\varphi \in C^\infty(\Sigma)$. Consequently,

$$\int_\Sigma |\nabla\varphi|^2 dv_g + c_n\int_\Sigma \varphi^2 R_g^\Sigma dv_g + 2c_n\int_{\partial\Sigma} \varphi^2 H_g^{\partial\Sigma} d\sigma_g > (1-2c_n)\int_\Sigma |\nabla\varphi|^2 dv_g\,,$$

for every $0 \not\equiv \varphi \in H^1(\Sigma)$, where $c_n = \frac{n-2}{4(n-1)}$. It follows that

$$\lambda = \inf_{0\not\equiv\varphi\in H^1(\Sigma)} \frac{\displaystyle\int_\Sigma |\nabla\varphi|^2 dv_g + c_n\int_\Sigma \varphi^2 R_g^\Sigma dv_g + 2c_n\int_{\partial\Sigma} \varphi^2 H_g^{\partial\Sigma} d\sigma_g}{\displaystyle\int_\Sigma \varphi^2 dv_g} > 0.$$

Therefore, there exists a metric on Σ with positive scalar curvature and minimal boundary (see Section 2.4). ■

3.2 3-dimensional case

In this section, we are going to find a topological obstruction to the existence of a metric with positive scalar curvature and mean convex boundary in a 3-dimensional manifold (and metric with non-negative scalar curvature and mean convex boundary). The following theorems are important results about stable minimal surfaces with free boundary which play a fundamental role in our investigations.

Theorem 3.2.1 (Chen, Fraser and Pang, [7]). *Let $(M,\partial M,g)$ be a Riemannian 3-dimensional manifold. If $(\Sigma,\partial\Sigma)$ is a connected surface which is not a disk and $f:(\Sigma,\partial\Sigma)\to(M,\partial M)$ is a continuous map such that*

$$f_*:\pi_1(\Sigma)\to\pi_1(M) \quad e \quad f_*^\partial:\pi_1(\Sigma,\partial\Sigma)\to\pi_1(M,\partial M),$$

are injectives, then there exists a free-boundary minimal immersion $F:(\Sigma,\partial\Sigma)\to(M,\partial M)$ and it minimizes area among the maps $h:(\Sigma,\partial\Sigma)\to(M,\partial M)$ such that h_* and h_*^∂ are injectives.

Theorem 3.2.2 (Chen, Fraser and Pang, [7]). *Let $(M, \partial M, g)$ be a Riemannian 3-dimensional manifold with mean convex boundary. If $(\Sigma, \partial\Sigma)$ is a connected surface and $f : (\Sigma, \partial\Sigma) \to (M, \partial M)$ is a free-boundary, minimal and stable immersion, then*

(1) If $R_g^M > 0$, we obtain that Σ is a disk.

(2) If $R_g^M \geq 0$, we obtain that either Σ is a disk or (Σ, g) is a flat cylinder with totally geodesic boundary.

Definition 3.3. *Define $\tilde{\mathcal{C}}_3$ as the set of all smooth 3-dimensional manifolds $(M, \partial M)$ such that there is no continuous map $f : (\Sigma, \partial\Sigma) \to (M, \partial M)$ with f_* and f_*^∂ are injectives, where $(\Sigma, \partial\Sigma)$ is a connected surface which is neither disk nor a cylinder.*

Remark 3.4. *Note that if a 3-dimensional manifold $(M, \partial M)$ has a essential surface which is neither a disk nor a cylinder then $M \notin \tilde{\mathcal{C}}_3$.*

Example 3.5. *Consider the 3-dimensional manifold $M = \mathbb{S}^1 \times \Sigma$, where $(\Sigma, \partial\Sigma)$ is a connected surface which is neither a disk nor a cylinder. Note that Σ is a essential surface in M. Therefore, from Remark 3.4 we have that $M \notin \tilde{\mathcal{C}}_3$.*

Corollary 3.6. *Let $(M, \partial M)$ be a 3-dimensional manifold. If $M \notin \tilde{\mathcal{C}}_3$, then there is no metric on M with non-negative scalar curvature and mean convex boundary.*

Example 3.7. *It follows from Example 3.5 and Corollary 3.6 that the 3-dimensional manifold $M = \mathbb{S}^1 \times \Sigma$, where $(\Sigma, \partial\Sigma)$ is a connected surface which is neither a disk nor a cylinder, admits no metric with non-negative scalar curvature and mean convex boundary.*

Definition 3.8. *Define $\overline{\mathcal{C}}_3$ as the set of all smooth 3-manifolds $(M, \partial M)$ such that there is no continuous map $f : (\Sigma, \partial\Sigma) \to (M, \partial M)$ with f_* and f_*^∂ injectives, where $(\Sigma, \partial\Sigma)$ is a connected surface which is not disk.*

Remark 3.9. *Note that*

1. *$\overline{\mathcal{C}}_3 \subset \tilde{\mathcal{C}}_3$, and*

2. *If a 3-dimensional manifold $(M, \partial M)$ has a essential surface which is not a disk then $M \notin \overline{\mathcal{C}}_3$.*

Example 3.10. *Consider the solid torus $M = \mathbb{S}^1 \times \mathbb{D}^2$. Since $\pi_1(M, \partial M) = 0$, we have that $M \in \overline{\mathcal{C}}_3$.*

Example 3.11. *The 3-dimensional manifold $M = \mathbb{S}^1 \times \Sigma$, where $(\Sigma, \partial\Sigma)$ is a connected surface which is not a disk. Note that Σ is a essential surface in M. It follows from Remark 3.9 that $M \notin \overline{\mathcal{C}}_3$.*

Example 3.12. *Consider the 3-dimensional manifold* $M = I \times \mathbb{S}^2$. *Since* M *is simply connected, we have that* $M \in \overline{\mathcal{C}}_3$.

Example 3.13. *As see in Example 2.25 that the 3-dimensional manifold* $I \times S$, *where* S *is a closed surface with positive genus, has a essential cylinder. Therefore, from Remark 3.9 it follows that* $I \times S \notin \overline{\mathcal{C}}_3$.

Theorem 3.2.3. *Let* $(M, \partial M)$ *be a smooth 3-dimensional manifold. Assume that the connected components of* ∂M *are spheres or incompressible tori, but at least one of the components is a torus. Then* $M \notin \overline{\mathcal{C}}_3$. *However, if the number of the incompressible tori in* ∂M *is exactly one, then* $M \notin \tilde{\mathcal{C}}_3$.

Proof. First, from Theorem 2.5.2, we have that M contains a properly embedded, connected and incompressible surface $(\Sigma, \partial\Sigma)$ such that $0 \neq [\partial\Sigma] \in H_1(\partial M)$. If Σ is a disk we have that $\partial\Sigma$ represents a non-trivial class in $\pi_1(\partial M)$, since $0 \neq [\partial\Sigma] \in H_1(\partial M)$. It follows that $\partial\Sigma$ is in a connected component T of ∂M which is a torus (see Figure 3.1).

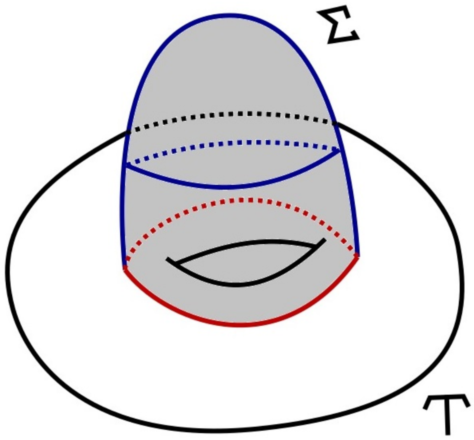

Figure 3.1: Properly embedded disk Σ in the torus T

So $\partial\Sigma$ is a non-trivial curve in the torus T which is trivial in M. But this is a contradiction, since T is incompressible. Therefore, Σ is not a disk.

As Σ is an incompressible surface, which is not a disk, we have that each connected component of $\partial\Sigma$ represents a non-trivial class in $\pi_1(\partial M)$. This implies that $\partial\Sigma$ is contained in the union of the tori of ∂M. Hence, either Σ is boundary-incompressible or it is a cylinder boundary-compressible (see Lemma 2.1 in [17]). If Σ is a boundary-compressible cylinder, the connected components c_1 and c_2 of $\partial\Sigma$ are contained in a same torus of ∂M. Consequently, we have only two possible situation for the circles c_1 and c_2, as we can see in the figures 3.2 and 3.3.

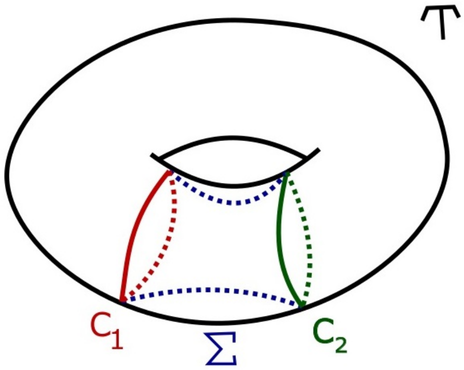

Figure 3.2: Possibility 1

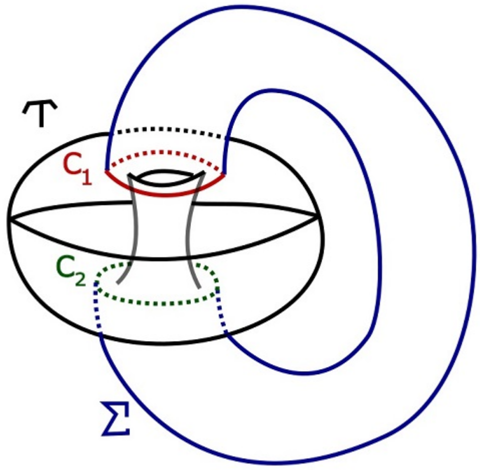

Figure 3.3: Possibility 2

Note that in both situation we have that c_1 and c_2 are homologous in ∂M. This implies that $\partial\Sigma$ represent the trivial class in $H_1(\partial M)$. But this is a contradiction. It follows that Σ is not a boundary-compressible cylinder. Hence, Σ is an essential surface in M which is not a disk. Therefore, $M \notin \overline{\mathcal{C}}_3$. However, note that if the number of the incompressible tori in ∂M is exactly one, then the essential surface Σ can not be a cylinder. In this case, we have that $M \notin \tilde{\mathcal{C}}_3$. ∎

Remark 3.14. *The incompressibility condition of at least one torus of ∂M in the proposition above is necessary. Actually, just consider the 3-dimensional manifold $M = \mathbb{S}^1 \times \mathbb{D}^2$. Note that the connected component of ∂M is a compressible torus and $M \in \overline{\mathcal{C}}_3$ (see Example 3.10).*

Corollary 3.15. *Let $(M, \partial M)$ be a smooth 3-dimensional manifold such that ∂M is the disjoint union of exactly one torus and k spheres, $k \geq 0$. If M has a metric with non-negative scalar curvature and mean convex boundary then*

$$M = N\#(\mathbb{S}^1 \times \mathbb{D}^2)\#^k\mathbb{B}^3,$$

where N is a closed 3-dimensional manifold.

Proof. The prime factorization of M is

$$M = N_1 \# \cdots \# N_s \# N' \#^k \mathbb{B}^3,$$

where $N_1, \cdots, N_s$ are closed and prime 3-dimensional manifolds and N' is a prime 3-dimensional manifold such that $\partial N'$ is a torus. If M has a metric with non-negative scalar curvature and mean convex boundary, it follows from Theorem 3.2.3 that $\partial N'$ is a compressible torus in N'. Since the solid torus is the unique prime 3-dimensional manifold whose boundary is a compressible torus, we have that $N' = \mathbb{S}^1 \times \mathbb{D}^2$. Therefore,

$$M = N \# (\mathbb{S}^1 \times \mathbb{D}^2) \#^k \mathbb{B}^3,$$

where $N = N_1 \# \cdots \# N_s$. ∎

Corollary 3.16. *Let $(M_1, \partial M_1), \cdots, (M_k, \partial M_k)$ be 3-dimensional manifolds as in proposition 4.15, and $N_1, \cdots, N_s$ closed 3-dimensional manifolds. For every integer $l \geq 0$, we have that*

1. $M_1 \# \cdots \# M_k \#^l \mathbb{B}^3 \notin \overline{\mathcal{C}}_3$,
2. $M_1 \# \cdots \# M_k \# N_1 \# \cdots \# N_s \#^l \mathbb{B}^3 \notin \overline{\mathcal{C}}_3$.

Moreover, if the number of the incompressible tori in ∂M_1 is exactly one then

3. $M_1 \#^l \mathbb{B}^3 \notin \tilde{\mathcal{C}}_3$,
4. $M_1 \# N_1 \# \cdots \# N_s \#^l \mathbb{B}^3 \notin \tilde{\mathcal{C}}_3$.

Example 3.17. *Define the 3-dimensional manifolds $M_1 = (\mathbb{S}^1 \times \mathring{T}^2) \# N$ and $M_2 = (\mathbb{S}^1 \times \mathring{T}^2) \# (I \times \mathbb{S}^2)$, where $\mathring{T}^2$ is a torus minus an open disk and N is a closed 3-dimensional manifold. It follows from the corollary 3.16 that $M_1, M_2 \notin \tilde{\mathcal{C}}_3$. Therefore, from Corollary 3.6, we have that M_1 and M_2 have no metric with non-negative scalar curvature and mean convex boundary.*

Lemma 3.18. *Let $(M, \partial M, g)$ be a connected Riemannian 3-dimensional manifold such that g is flat with totally geodesic boundary. Then, M is covered by $I \times T^2$. In particular, $M \notin \overline{\mathcal{C}}_3$.*

Proof. It follows from the Theorem 5 in [25] that either M is diffeomorphic to a 3-dimensional handlebody or M is covered by $I \times T^2$. Since (M, g) is flat with totally geodesic boundary, from Gauss Equation, we have that $(\partial M, g)$ is a flat surface. Assume M is a 3-dimensional handlebody. In this case, we have that ∂M is connected. It follows from the Gauss-Bonnet theorem that ∂M is a 2-dimensional torus. This implies that $M = \mathbb{S}^1 \times \mathbb{D}^2$. It follows from second variation of area ∂M is a stable minimal flat torus in (M, g). But, this is a contradiction (see Theorem 8 in ([25]). Therefore, M is covered by $I \times T^2$. Consider then $p : I \times T^2 \to M$ a covering map. It follows from Example 2.25 that there is an essential cylinder C which is properly embedded in $I \times T^2$. Define $f = p \circ i : (C, \partial C) \to (M, \partial M)$, where $i : C \to I \times T^2$ is the inclusion map. We have that $f_* = p_* \circ i_*$ and $f_*^\partial = p_*^\partial \circ i_*^\partial$. Since p is a covering map, we have that p_* and p_*^∂ are injectives. Furthermore, since C is essential in $I \times T^2$, we have that i_* and i_*^∂ are injectives. Consequently, f_* and f_*^∂ are injectives. Therefore, $M \notin \overline{\mathcal{C}}_3$. ∎

Theorem 3.2.4. *Let $(M, \partial M, g)$ be a 3-dimensional Riemannian manifold such that $R_g^M \geq 0$ and $H_g^{\partial M} \geq 0$. Then either $M \in \overline{\mathcal{C}}_3$ or (M, g) is flat with totally geodesic boundary.*

Proof. Note that as $R_g^M \geq 0$ and $H_g^{\partial M} \geq 0$, it follows from Corollary 3.6 that $M \in \tilde{\mathcal{C}}_3$. Assume that $M \notin \overline{\mathcal{C}}_3$ and g is not flat or $B_g^{\partial M} \not\equiv 0$. Since $M \in \tilde{\mathcal{C}}_3$ and $M \notin \overline{\mathcal{C}}_3$, we have that there is a continuous map $f : (C, \partial C) \to (M, \partial M)$ such that f_* and f_*^∂ are injectives, where C is a cylinder. As g is not flat or $B_g^{\partial M} \not\equiv 0$, it follows from the Proposition 3.1.3 there exists a Riemannian metric h on M such that $R_h^M > 0$ and $H_h^{\partial M} \equiv 0$. It follows from the Theorem 3.2.1 that there exists a stable free-boundary minimal immersion $F : (C, \partial C) \to (M, \partial M)$ with respect to the metric h. Hence, from Theorem 3.2.2, we have a contradiction. This implies that $M \in \overline{\mathcal{C}}_3$ or (M, g) is flat with totally geodesic boundary. It follows from Lemma 3.18 that either $M \in \overline{\mathcal{C}}_3$ or (M, g) is flat with totally geodesic boundary. ∎

Corollary 3.19. *If a 3-dimensional Riemannian manifold $(M, \partial M)$ admits a metric with positive scalar curvature and mean convex boundary then $M \in \overline{\mathcal{C}}_3$.*

Example 3.20. *Consider the 3-dimensional manifold $I \times S$, where S is a closed surface with positive genus. From Example 3.13, we have that $I \times S \notin \overline{\mathcal{C}}_3$. It follows from the Corollary 3.19 that there is no metric on $I \times S$ with positive scalar curvature and mean convex boundary. In particular, there is no such metric on $I \times T^2$.*

Example 3.21. *It follows from the Corollary 3.16 that the 3-dimensional manifolds bellow are not in the set $\overline{\mathcal{C}}_3$.*

(1) $(I \times T^2)\#(I \times T^2)$
(2) $(\mathbb{S}^1 \times \mathring{T}^2)\#(\mathbb{S}^1 \times \mathring{T}^2)$
(3) $(I \times T^2)\#(S^1 \times \mathring{T}^2)$
(4) $(I \times T^2)\#(I \times \mathbb{S}^2)$

(5) $(\mathbb{S}^1 \times \mathring{T}^2)\#(I \times \mathbb{S}^2)$

(6) $(I \times T^2)\#N$*, where* N *is a closed 3-dimensional manifold.*

Therefore, from the Theorem 3.2.4 that these manifolds have no metric with positive scalar curvature and mean convex boundary. Furthermore, every metric in these manifolds with non-negative scalar curvature and mean convex boundary are flat with totally geodesic boundary.

3.3 n-dimensional case, $3 \leq n \leq 7$

In this section, we are going to study possible generalizations of some results on the existence of certain metrics in 3-dimensional manifolds to manifolds with dimension not greater than seven and we are going to prove the main theorem of this chapter. The following theorem is a very important result from geometric measure theory which plays a fundamental role in our investigations.

Theorem 3.3.1 (See Chapter 8 in [27] and Theorem 5.4.15 in [10])**.** *Let* $(M, \partial M, g)$ *be a Riemannian n-dimensional manifold,* $3 \leq n \leq 7$*. Assume that* $\alpha \in H_{n-1}(M, \partial M)$ *is a non-trivial class. Then there exists a free-boundary, minimal and stable hypersurface* Σ *properly embedded in* (M, g) *which represents the class* α*.*

For $n \geq 4$, we define inductively the set $\tilde{\mathcal{C}}_n$ as the set of all smooth n-dimensional manifolds $(M, \partial M)$ such that every non-trivial homology class $\alpha \in H_{n-1}(M, \partial M)$ can be represented by a hypersurface $(\Sigma, \partial\Sigma)$ such that $\Sigma \in \tilde{\mathcal{C}}_{n-1}$.

Example 3.22. *Consider the n-dimensional manifold* $M^n = T^{n-2} \times \Sigma$*, where* $(\Sigma, \partial\Sigma)$ *is a connected surface which is neither a disk nor a cylinder. We have that* $M^n \notin \tilde{\mathcal{C}}_n$*, for every* $n \geq 3$*. In fact, it follows from Example 3.5 that this claim is true for* $n = 3$*. Assume this claim is valid for* $n-1$*. Consider the hypersurface* $M^{n-1} \subset M^n$*. It is well know that* M^{n-1} *represents a non-trivial homology class* $\alpha \in H_{n-1}(M^n, \partial M^n)$ *and every hypersuface of* M^n *which represents the homology class* α *is homeomorphic to* M^{n-1}*. From the induction hypothesis we have that* $M^{n-1} \notin \tilde{\mathcal{C}}_{n-1}$*. Therefore,* $M^n \notin \tilde{\mathcal{C}}_n$*.*

Theorem 3.3.2. *Let* $(M, \partial M)$ *be a n-dimensional manifold such that* $3 \leq n \leq 7$ *and* $M \notin \tilde{\mathcal{C}}_n$*. Then there is no metric on* M *with non-negative scalar curvature and mean convex boundary.*

Proof. We note that it follows from a Corollary 3.6 the result is true for $n = 3$. We proof by induction on n. Assume the result is valid for $n-1$. Assume there exists a metric g on M such that $R_g^M \geq 0$ and $H_g^{\partial M} \geq 0$. It follows from Theorem 3.1.3 that two cases can occurs.

Case 1: There exists a metric h on M such that $R_h > 0$ and $H_h^{\partial M} \equiv 0$.

In this case, since $M \notin \tilde{\mathcal{C}}_n$, from the Theorem 3.3.1 we have that there exists a free-boundary, minimal and stable hypersurface Σ properly embedded in (M, h) such that $\Sigma \notin \tilde{\mathcal{C}}_{n-1}$. From the Theorem 3.1.4 there exists a metric on Σ with positive scalar curvature and minimal boundary. However, this is a contradiction since $\Sigma \notin \tilde{\mathcal{C}}_{n-1}$ and from the induction hypothesis does not exists such metric.

Case 2: $Ric_g^M \equiv 0$ and $B_g^{\partial M} \equiv 0$.

Arguing as in the case 1, there exists a free-boundary, minimal and stable hypersurface Σ properly embedded in (M, g) such that $\Sigma \notin \tilde{\mathcal{C}}_{n-1}$. Since Σ is free-boundary in (M, g), we have

$$H_g^{\partial\Sigma} = H_g^{\partial M} - B_g^{\partial M}(\nu, \nu) \equiv 0,$$

where ν is a unit vector field of Σ em (M, g). Also, it follows from the Gauss Equation and of the stability of Σ that $R_g^{\Sigma} \equiv 0$. However, this is a contradiction, since $\Sigma \notin \tilde{\mathcal{C}}_{n-1}$, from the induction hypothesis, does not exists a metric on Σ with null scalar curvature and minimal boundary.

Therefore, there is no metric on M with non-negative scalar curvature and mean convex boundary.

■

Example 3.23. *Consider the n-dimensional manifold $M^n = T^{n-2} \times \Sigma$, where $(\Sigma, \partial\Sigma)$ is a connected surface which is neither a disk nor a cylinder. we showed in the Example 3.22 that $M^n \notin \tilde{\mathcal{C}}_n$, for every $n \geq 3$. It follows from Theorem 3.3.2 that there is no metric on M^n with non-negative scalar curvature and mean convex boundary, if $3 \leq n \leq 7$.*

For $n \geq 4$, we define inductively $\overline{\mathcal{C}}_n$ as the set of all smooth n-dimensional manifolds $(M, \partial M)$ such that every non-trivial class $\alpha \in H_{n-1}(M, \partial M)$ can be represented by a hypersurface $(\Sigma, \partial\Sigma)$ such that $\Sigma \in \overline{\mathcal{C}}_{n-1}$. Note that $\overline{\mathcal{C}}_n \subset \tilde{\mathcal{C}}_n$, for every $n \geq 3$.

Theorem 3.3.3. *Let $(M, \partial M, g)$ be a Riemannian n-dimensional manifold, $3 \leq n \leq 7$, such that $R_g^M \geq 0$ and $H_g^{\partial M} \geq 0$. Then $M \in \overline{\mathcal{C}}_n$ or (M, g) is Ricci-flat with totally geodesic boundary.*

Proof. It follows from Theorem 3.2.4 that the result is valid for $n = 3$. Let us do it by induction on n. Assume the result is valid for $n - 1$. Suppose that $Ric_g \not\equiv 0$ or $B_g \not\equiv 0$ and $M \notin \overline{\mathcal{C}}_n$. It follows from the Theorem 3.1.3 that there exists a metric h on M such that $R_h > 0$ and $H_h \equiv 0$. Since $M \notin \overline{\mathcal{C}}_n$, from the Theorem 3.3.1 we have that there exists a free-boundary, minimal and stable hypersurface Σ properly embedded in (M, h) such that

$\Sigma \notin \overline{\mathcal{C}}_{n-1}$. From the induction hypothesis we have that Σ does not admit a metric with positive scalar curvature and minimal boundary. This is a contradiction with the proposition 3.1.4. Therefore, $M \in \overline{\mathcal{C}}_n$ or (M, g) is Ricci-flat with totally geodesic boundary. ■

Corollary 3.24. *If a n-dimensional Riemannian manifold $(M, \partial M)$, $3 \leq n \leq 7$, admits a metric with positive scalar curvature and mean convex boundary then $M \in \overline{\mathcal{C}}_n$.*

Example 3.25. *Consider the n-dimensional manifold $M^n = I \times T^{n-1}$. Arguing as in the Example 3.22, we can show that $M^n \notin \overline{\mathcal{C}}_n$, for every $n \geq 3$. Hence, from the Corollary 3.24, there exists no metric on M^n with positive scalar curvature and mean convex boundary, if $3 \leq n \leq 7$.*

Denote by $\mathcal{M}_n$ the set of all n-dimensional manifolds with non-empty boundary. We have that $\overline{\mathcal{C}}_n \subset \tilde{\mathcal{C}}_n \subset \mathcal{M}_n$. Consider $3 \leq n \leq 7$. Putting together what we have done so far:

(1) The n-dimensional manifolds of $\mathcal{M}_n \setminus \tilde{\mathcal{C}}_n$ do not admit a metric with non-negative scalar curvature and mean convex boundary (Theorem 3.3.2).

(2) The n-dimensional manifolds of $\mathcal{M}_n \setminus \overline{\mathcal{C}}_n$ do not admit a metric with positive scalar curvature and mean convex boundary (Corollary 3.24).

(3) The metrics with non-negative scalar curvature and mean convex boundary in n-dimensional manifolds of $\tilde{\mathcal{C}}_n \setminus \overline{\mathcal{C}}_n$ are Ricci-flat with totally geodesic boundary (Theorem 3.3.3).

Lemma 3.26. *Let $(M, \partial M)$ be a n-dimensional manifold such that there is a non-zero degree map $F : (M, \partial M) \to (\Sigma \times T^{n-2}, \partial\Sigma \times T^{n-2})$, where $(\Sigma, \partial\Sigma)$ is a connected surface and $n \geq 3$. Then there exists a properly embedded hypersurface $(\Sigma_{n-1}, \partial\Sigma_{n-1}) \subset (M, \partial M)$ such that*

1. $0 \neq [\Sigma_{n-1}] \in H_{n-1}(M, \partial M)$;

2. *The map $F|_{\Sigma_{n-1}} : (\Sigma_{n-1}, \partial\Sigma_{n-1}) \to (\Sigma \times T^{n-3}, \partial\Sigma \times T^{n-3})$ has non-zero degree.*

Proof. Without loss of generality, we assume that F is a smooth function. Consider the projection $p : \Sigma \times T^{n-2} \to S^1$ given by $p(x, (t_1, \cdots, t_{n-2})) = t_{n-2}$, for $x \in \Sigma$ and $(t_1, \cdots, t_{n-2}) \in T^{n-2} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. Define $f = p \circ F : M \to \mathbb{S}^1$. It follows from the Sard's Theorem that there is $\theta \in S^1$ which is a regular value of f and $f|_{\partial M}$. Define

$$\Sigma_{n-1} = f^{-1}(\theta) = F^{-1}(\Sigma \times T^{n-3} \times \{\theta\}).$$

Note that $\Sigma_{n-1} \subset M$ is a properly embedded hypersurface which represents a non-trivial class in $H_{n-1}(M, \partial M)$ and the map $F|_{\Sigma_{n-1}} : (\Sigma_{n-1}, \partial\Sigma_{n-1}) \to (\Sigma \times T^{n-3}, \partial\Sigma \times T^{n-3})$ has non-zero degree. ■

Theorem 3.3.4. *Let $(M, \partial M)$ be a n-dimensional manifold, , $3 \leq n \leq 7$, such that there is a non-zero degree map $F : (M, \partial M) \to (\Sigma \times T^{n-2}, \partial\Sigma \times T^{n-2})$, where $(\Sigma, \partial\Sigma)$ is a connected surface which is not a disk. Then there exists no metric on M with positive scalar curvature and mean convex boundary. However, if Σ is neither a disk nor a cylinder, then there exists no metric on M with non-negative scalar curvature and mean convex boundary.*

Proof. Firstly, we are going to prove the following claim.

Claim 6. *Every n-dimensional manifold which admits a non-zero degree map to the manifold $\Sigma \times T^{n-2}$, where $n \geq 3$ and $(\Sigma, \partial\Sigma)$ is a connected surface which is not a disk, is not in the set $\overline{\mathcal{C}}_n$.*

We proof this claim by induction on n. Assume $n = 3$. Consider $(M, \partial M)$ be a 3-dimensional manifold such that there is a non-zero degree map $F : (M, \partial M) \to (\Sigma \times S^1, \partial\Sigma \times S^1)$. It follows from Lemma 3.26 that there exists a properly embedded surface $(\Sigma_2, \partial\Sigma_2) \subset (M, \partial M)$ which represents a non-trivial homology class $\alpha \in H_2(M, \partial M)$ and the map $F|_{\Sigma_2} : (\Sigma_2, \partial\Sigma_2) \to (\Sigma, \partial\Sigma)$ has non-zero degree. It follows from Theorem 2.5.1 that there is a properly embedded surface $S_2 \subset M$ which represents the homology class α such that its connected components are either spheres or essential surfaces. Since S_2 and Σ_2 represent the same homology class in $H_2(M, \partial M)$ and $deg(F|_{\Sigma_2}) \neq 0$, we have that $F(S_2) \subset \Sigma$ and the map $F|_{S_2} : (S_2, \partial S_2) \to (\Sigma, \partial\Sigma)$ has non-zero degree. Since the degree of a map is the sum of the degree of such a map restricted to each connected component, it follows that there is a connected component $(S_2', \partial S_2')$ of S_2 such that $F_3|_{S_2'} : (S_2', \partial S_2') \to (\Sigma, \partial\Sigma)$ has non-zero degree. Consequently, the first betti number of S_2' is greater than or equal to the first betti number of Σ. This implies that $\chi(S_2') \leq \chi(\Sigma)$. Since Σ is not a disk, we have that $\chi(S_2') \leq 0$. This implies that S_2' is not a disk. It follows that S_2' is an essential surface in M which is not a disk. Hence, $M \notin \overline{\mathcal{C}_3}$.

Assume this claim is true for $n-1$. Consider $(M, \partial M)$ be a n-dimensional manifold such that there is a non-zero degree map $F : (M, \partial M) \to (\Sigma \times T^{n-2}, \partial\Sigma \times T^{n-2})$. It follows from Lemma 3.26 that there exists a properly embedded hypersurface $(\Sigma_{n-1}, \partial\Sigma_{n-1}) \subset (M, \partial M)$ which represents a non-trivial homology class $\alpha \in H_{n-1}(M, \partial M)$ and the map $F|_{\Sigma_{n-1}}$: $(\Sigma_{n-1}, \partial\Sigma_{n-1}) \to (\Sigma \times T^{n-3}, \partial\Sigma \times T^{n-2})$ has non-zero degree. From induction hypothesis we have that $\Sigma_{n-1} \notin \overline{\mathcal{C}}_{n-1}$. Consider a hypersurface $S_{n-1} \subset M$ which represents the homology class α. Since S_{n-1} and Σ_{n-1} represent the same homology class in $H_{n-1}(M, \partial M)$ and $deg(F|_{\Sigma_{n-1}}) \neq 0$, we have that $F(S_{n-1}) \subset \Sigma \times T^{n-3}$ and the map $F|_{S_{n-1}} : (S_{n-1}, \partial S_{n-1}) \to (\Sigma \times T^{n-3}, \partial\Sigma \times T^{n-3})$ has non-zero degree. From induction hypothesis we have that $S_{n-1} \notin \overline{\mathcal{C}}_{n-1}$. Hence, $M \notin \overline{\mathcal{C}_n}$. Therefore, it follows the claim.

From Claim 6 we obtain that $M \notin \overline{\mathcal{C}_n}$. Therefore, it follows from Corollary 3.24 that there exists no metric on M with positive scalar curvature and mean convex boundary. However, note that if Σ is neither a disk nor a cylinder, we can replace $\overline{\mathcal{C}}$ by $\tilde{\mathcal{C}}$ in the Claim 6 and conclude that $M \notin \tilde{\mathcal{C}}_n$. Consequently, from Theorem 3.3.2, we have that there exists no metric on M with non-negative scalar curvature and mean convex boundary. ■

Corollary 3.27. *We have that*

1. *The manifold $(I \times T^{n-1})\#N$ admits no metric with positive scalar curvature and mean convex boundary.*
2. *The manifold $(\mathring{T}^2 \times T^{n-2})\#N$ admits no metric with non-negative scalar curvature and mean convex boundary*

where N is a closed manifold of dimension $3 \leq n \leq 7$.

Chapter 4

Disks area-minimizing in mean convex n-dimensional Riemannian manifold

Consider $(M, \partial M, g)$ a n-dimensional Riemannian manifold. Let $\mathcal{F}_M$ be the set of all immersed disks in M whose boundaries are curves in ∂M that are homotopically non-trivial in ∂M. If $\mathcal{F}_M \neq \emptyset$, we define

$$\mathcal{A}(M,g) = \inf_{\Sigma \in \mathcal{F}_M} |\Sigma|_g \text{ e } \mathcal{L}(M,g) = \inf_{\Sigma \in \mathcal{F}_M} |\partial\Sigma|_g.$$

The goal of this chapter is to prove the following theorem.

Theorem 4.0.1. *Let $(M, \partial M, g)$ be a $(n+2)$-dimensional Riemannian manifold, $3 \leq n+2 \leq 7$, with positive scalar curvature and mean convex boundary. Assume that there is a non-zero degree map $F : (M, \partial M) \to (\mathbb{D}^2 \times T^n, \partial\mathbb{D}^2 \times T^n)$. Then,*

$$\frac{1}{2} \inf R_g^M \mathcal{A}(M,g) + \inf H_g^{\partial M} \mathcal{L}(M,g) \leq 2\pi.$$

Moreover, if the boundary ∂M is totally geodesic and the equality holds above, then the universal covering of (M, g) is isometric to $(\mathbb{R}^n \times \Sigma_0, \delta + g_0)$, where δ is the standard metric in $\mathbb{R}^n$ and (Σ_0, g_0) is a disk with constant Gaussian curvature $\frac{1}{2} \inf R_g^M$ and $\partial\Sigma_0$ has null geodesic curvature in (Σ_0, g_0).

4.1 Warped product

In this section, we are going to study the geometry of special warped products that will allow us to better understand the content of this chapter, namely

$$\left(M \times T^k, g + \sum_{p=1}^{k} f_p^2 dt_p^2\right)$$

where (M, g) is a Riemannian manifold and $f_1, \cdots, f_k \in C^\infty(M)$ are positive functions.

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and let $f \in C^\infty(M_1)$ be a positive function. On the manifold $M_1 \times M_2$ consider the warped metric $g_1 + f^2 g_2$. Denote by ∇^1 and ∇^2 the Riemannian connections of (M_1, g_1) and (M_2, g_2), respectively. The Riemannian connection ∇ of $(M_1 \times M_2, g_1 + f^2 g_2)$ is

$$\nabla_{X_1+X_2}(Y_1 + Y_2) = \nabla^1_{X_1} Y_1 + \frac{X_1(f)}{f} Y_2 + \frac{Y_1(f)}{f} X_2 + \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) \nabla_{g_1} f,$$

for every $X_i, Y_i \in \mathcal{X}(M_i)$, $i = 1, 2$.

The curvature endomorphism R of $(M_1 \times M_2, g_1 + f^2 g_2)$ satisfies:

(1) $R(X_1, Y_1)Z_1 = R^1(X_1, Y_1)Z_1$;

(2) $R(X_1, Y_2)Z_2 = -f g_2(Y_2, Z_2) \nabla^1_{X_1} \nabla_{g_1} f$;

(3) $R(X_1, Y_1)Z_2 = 0$;

(4) $R(X_2, Y_2)Z_1 = 0$;

(5) $R(X_2, Y_1)Z_1 = -\frac{1}{f} \left(\nabla^2_{g_1} f\right)(Y_1, Z_1)X_2$;

(6) $R(X_2, Y_2)Z_2 = R^2(X_2, Y_2)Z_2 + g_1(\nabla_{g_1} f, \nabla_{g_1} f)\left(g_2(X_2, Z_2)Y_2 - g_2(Z_2, Y_2)X_2\right)$

for every $X_i, Y_i, Z_i \in \mathcal{X}(M_i)$, $i = 1, 2$, where $\nabla^2_{g_1} f$ is the hessian of f and R^1, R^2 are the curvature tensors of (M_1, g_1) and (M_2, g_2), respectively.

Proposition 4.1. *Let (M, g) be a m-dimensional Riemannian manifold and $f \in C^\infty(M)$ a positive function. Then the Ricci curvature of $(M \times \mathbb{S}^1, g + f^2 dt^2)$ is*

$$Ric^{M \times \mathbb{S}^1} = Ric^M - \frac{1}{f}\left(\nabla^2_g f\right) - f \Delta_g f dt^2,$$

where Ric^M is the Ricci curvature of (M, g).

Proof. Consider $(x_1, \cdots, x_m, t = x_{m+1})$ a local chart in $M \times \mathbb{S}^1$ such that $(x_1, \cdots, x_m)$ is a local chart in M. Denote by ∇^M the Riemannian connection of (M, g) and R, R^M the curvature tensors of $(M \times \mathbb{S}^1, h = g + f^2 dt^2)$ and (M, g), respectively. Note that

$$Ric^{M \times \mathbb{S}^1}_{ij} = \sum_{k,l=1}^{m+1} h^{kl} R_{kijl} = \sum_{k,l=1}^{m} h^{kl} R_{kijl} + \frac{1}{f^2} R_{tijt}.$$

For $i, j = 1, \cdots, m$, we have that

$$
\begin{aligned}
Ric_{ij}^{M\times\mathbb{S}^1} &= \sum_{k,l=1}^{m} g^{kl} R_{kijl}^{M} + \frac{1}{f^2} R_{tijt} \\
&= Ric_{ij}^{M} - \frac{1}{f^3}\left(\nabla_g^2 f\right)_{ij} h_{tt} \\
&= Ric_{ij}^{M} - \frac{1}{f}\left(\nabla_g^2 f\right)_{ij}
\end{aligned}
$$

Furthermore,

$$
Ric_{it}^{M\times\mathbb{S}^1} = \sum_{k,l=1}^{m} h^{kl} R_{kitl} + \frac{1}{f^2} R_{titt} = 0,
$$

and

$$
\begin{aligned}
Ric_{tt}^{M\times\mathbb{S}^1} &= \sum_{k,l=1}^{m} h^{kl} R_{kttl} \\
&= -f \sum_{k,l=1}^{m} g^{kl} g\left(\nabla_{\partial_k}^{M} \nabla_g f, \partial_l\right) \\
&= -f \sum_{k,l=1}^{m} g^{kl}\left(\nabla_g^2 f\right)_{kl} \\
&= -f \Delta_g f
\end{aligned}
$$

Therefore, we have that

$$
\left\{
\begin{array}{rcll}
Ric_{ij}^{M\times\mathbb{S}^1} &=& Ric_{ij}^{M} & - \frac{1}{f}\left(\nabla_g^2 f\right)_{ij} \\
Ric_{tt}^{M\times\mathbb{S}^1} &=& -f\Delta_g f & \\
Ric_{it}^{M\times\mathbb{S}^1} &=& 0 &
\end{array}
\right.
$$

for every $i, j = 1, \cdots, m$. ∎

Proposition 4.2. *Let (M, g) be a m-dimensional Rieamannian manifold and $f \in C^\infty(M)$ a positive function. Then the scalar curvature of $(M \times \mathbb{S}^1, g + f^2 dt^2)$ is*

$$
R^{M\times\mathbb{S}^1} = R_g^M - \frac{2}{f}\Delta_g f,
$$

where R_g^M is the scalar curvature of (M, g).

Proof. Consider $(x_1, \cdots, x_m, t = x_{m+1})$ a local chart in $M \times \mathbb{S}^1$ such that $(x_1, \cdots, x_m)$ is a local chart in M. Denote the metric $h = g + f^2dt^2$. From Proposition 4.1, we have that

$$\begin{aligned} R^{M\times\mathbb{S}^1} &= \sum_{i,j=1}^{m+1} h^{ij} Ric_{ij}^{M\times\mathbb{S}^1} \\ &= \sum_{i,j=1}^{m} h^{ij} Ric_{ij}^{M\times\mathbb{S}^1} + \frac{1}{f^2} Ric_{tt}^{M\times\mathbb{S}^1} \\ &= \sum_{i,j=1}^{m} g^{ij} Ric_{ij}^{M} - \frac{1}{f}\sum_{i,j=1}^{m} g^{ij}\left(\nabla_g^2 f\right)_{ij} - \frac{1}{f}\Delta_g f \\ &= R_g^M - \frac{2}{f}\Delta_g f. \end{aligned}$$

■

Proposition 4.3. *Let (M,g) be a m-dimensional Riemannian manifold, $\Sigma \subset M$ be a hypersurface and $f \in C^\infty(M)$ be a positive function. Then, the second fundamental form of $\Sigma \times \mathbb{S}^1$ in $(M\times\mathbb{S}^1, g + f^2dt^2)$ is*

$$B^{\Sigma\times\mathbb{S}^1} = B^\Sigma - f\nu(f)dt^2,$$

where ν is a globally defined unit normal vector field in Σ and B^Σ is the second fundamental form of Σ in (M,g).

Proof. Consider $(x_1,\cdots,x_{m-1},t=x_m)$ a local chart in $\Sigma\times\mathbb{S}^1$ such that $(x_1,\cdots,x_{m-1})$ is a local chart in Σ. Denote by ∇ and ∇^M the Riemannian connections of $(M\times\mathbb{S}^1, h = g+f^2dt^2)$ and (M,g), respectively. For $i,j = 1,\cdots,m-1$, we have that

$$\nabla_{\partial_i}\partial_j = \nabla^M_{\partial_i}\partial_j, \quad \nabla_{\partial_i}\partial_t = \frac{\partial_i(f)}{f}\partial_t \ \text{ and } \ \nabla_{\partial_t}\partial_t = -f\nabla_g f.$$

It follows that,

$$B_{ij}^{\Sigma\times\mathbb{S}^1} = h(\nabla_{\partial_i}\partial_j,\nu) = h(\nabla^M_{\partial_i}\partial_j,\nu) = g(\nabla^M_{\partial_i}\partial_j,\nu) = B_{ij}^\Sigma$$

and

$$B_{im}^{\Sigma\times\mathbb{S}^1} = h(\nabla_{\partial_i}\partial_t,\nu) = \frac{\partial_i(f)}{f}h(\partial_t,\nu) = 0.$$

Furthermore,

$$B_{mm}^{\Sigma\times\mathbb{S}^1} = h(\nabla_{\partial_t}\partial_t,\nu) = h(-f\nabla_g f,\nu) = -fg(\nabla_g f,\nu) = -f\nu(f).$$

Therefore,

$$\begin{cases} B_{ij}^{\Sigma\times\mathbb{S}^1} &= \quad B_{ij}^\Sigma \\ B_{mm}^{\Sigma\times\mathbb{S}^1} &= \quad -f\nu(f) \\ B_{im}^{\Sigma\times\mathbb{S}^1} &= \quad 0 \end{cases}$$

for every $i,j = 1,\cdots,m-1$. ■

Proposition 4.4. *Let (M,g) be a m-dimensional Riemannian manifold and $f_1,\cdots,f_k\in C^\infty(M)$ positive functions. Then scalar curvature of $\left(M\times T^k, g+\sum_{p=1}^k f_p^2dt_p^2\right)$ is*

$$R^{M\times T^k}=R_g^M-2\sum_{p=1}^k\frac{1}{f_p}\Delta_g f_p-2\sum_{1\le p<q\le k}g(\nabla_g\log f_p,\nabla_g\log f_q),$$

where R_g^M is the scalar curvature of (M,g).

Proof. We proof by induction on k. From Proposition 4.2, this result is valid for $k=1$. Assume the result is valid for $k-1$, i.e., the scalar curvature of $\left(M\times T^{k-1}, g+\sum_{p=1}^{k-1} f_p^2dt_p^2\right)$ is

$$R^{M\times T^{k-1}}=R_g^M-2\sum_{p=1}^{k-1}\frac{1}{f_p}\Delta_g f_p-2\sum_{1\le p<q\le k-1}g(\nabla_g\log f_p,\nabla_g\log f_q).\tag{4.1}$$

Note that,

$$\left(M\times T^k, g+\sum_{p=1}^k f_p^2dt_p^2\right)=\left((M\times T^{k-1})\times\mathbb{S}^1, g+\sum_{p=1}^{k-1} f_p^2dt_p^2+f_k^2dt_k^2\right).$$

It follows from Proposition 4.2 that

$$R^{M\times T^k}=R^{M\times T^{k-1}}-\frac{2}{f_k}\Delta_h f_k\tag{4.2}$$

where Δ_h is the laplacian in $(N,h)=\left(M\times T^{k-1},\sum_{p=1}^{k-1} f_p^2dt_p^2\right)$.

Claim 7. *We have that,*

$$\Delta_h f_k=\Delta_g f_k+\sum_{p=1}^{k-1}\frac{1}{f_p}g(\nabla_g f_p,\nabla_g f_k).$$

In fact, consider $(x_1,\cdots,x_m,t_1=x_{m+1},\cdots,t_{k-1}=x_{m+k-1})$ a local chart in N such that $(x_1,\cdots,x_m)$ is a local chart in M. Denote by ∇ and ∇_M the Riemannian connection of (N,h) and (M,g), repectively. Note that

$$\begin{aligned}\Delta_h f_k &= \sum_{i,j=1}^{m+k-1}h^{ij}h(\nabla_{\partial_i}\nabla_h f_k,\partial_j)\\ &= \sum_{i,j=1}^{m}h^{ij}h(\nabla_{\partial_i}\nabla_h f_k,\partial_j)+\sum_{p=m+1}^{m+k-1}h^{pp}h(\nabla_{\partial_p}\nabla_h f_k,\partial_p)\\ &= \sum_{i,j=1}^{m}g^{ij}h(\nabla_{\partial_i}\nabla_h f_k,\partial_j)+\sum_{p=1}^{k-1}\frac{1}{f_p^2}h(\nabla_{\partial_{t_p}}\nabla_h f_k,\partial_{t_p}).\end{aligned}$$

Since f_k is a function defined in M, we have that $\nabla_h f_k = \nabla_g f_k$. Consequently,

$$\begin{aligned}
\Delta_h f_k &= \sum_{i,j=1}^{m} g^{ij} h(\nabla_{\partial_i}\nabla_g f_k, \partial_j) + \sum_{p=1}^{k-1} \frac{1}{f_p^2} h(\nabla_{\partial_{t_p}}\nabla_g f_k, \partial_{t_p}) \\
&= \sum_{i,j=1}^{m} g^{ij} g(\nabla^M_{\partial_i}\nabla_g f_k, \partial_j) + \sum_{p=1}^{k-1} \frac{1}{f_p^2} h\left(\frac{\nabla_g f_k(f_p)}{f_p}\partial_{t_p}, \partial_{t_p}\right) \\
&= \Delta_g f_k + \sum_{p=1}^{k-1} \frac{\nabla_g f_k(f_p)}{f_p} \\
&= \Delta_g f_k + \sum_{p=1}^{k-1} \frac{1}{f_p} g(\nabla_g f_p, \nabla_g f_k).
\end{aligned}$$

Hence, it follows the claim.

It follow from (4.2), (4.1) and Claim 7 that

$$\begin{aligned}
R^{M\times T^k} &= R_g^M - 2\sum_{p=1}^{k-1} \frac{1}{f_p}\Delta_g f_p - 2\sum_{1\le p<q\le k-1} g(\nabla_g \log f_p, \nabla_g \log f_q) \\
&- \frac{2}{f_k}\Delta_g f_k - 2\sum_{p=1}^{k-1} \frac{1}{f_p f_k} g(\nabla_g f_p, \nabla_g f_k) \\
&= R_g^M - 2\sum_{p=1}^{k} \frac{1}{f_p}\Delta_g f_p - 2\sum_{1\le p<q\le k-1} g(\nabla_g \log f_p, \nabla_g \log f_q) \\
&- 2\sum_{p=1}^{k-1} g(\nabla_g \log f_p, \nabla_g \log f_k) \\
&= R_g^M - 2\sum_{p=1}^{k} \frac{1}{f_p}\Delta_g f_p - 2\sum_{1\le p<q\le k} g(\nabla_g \log f_p, \nabla_g \log f_q).
\end{aligned}$$

■

Proposition 4.5. *Let (M,g) be a $(m+1)$-dimensional Riemannian manifold, $\Sigma \subset M$ a hypersurface and let $f_1, \cdots, f_k \in C^\infty(M)$ be positive functions. Then, the second fundamental form of $\Sigma \times T^k$ in $\left(M \times T^k, g + \sum_{p=1}^{k} f_p^2 dt_p^2\right)$ is*

$$B^{\Sigma\times T^k} = B^\Sigma - \sum_{p=1}^{k} f_p \nu(f_p) dt_p^2 \tag{4.3}$$

where ν is a globally defined unit normal vector field in Σ and B^Σ is a second fundamental form of Σ in (M,g). In particular,

$$\left|B^{\Sigma\times T^k}\right|^2 = |B^\Sigma|^2 + \sum_{p=1}^{k} (\nu(\log u_p))^2. \tag{4.4}$$

Proof. We are going to proof the equality (4.3) by induction on k. From Proposition 4.3, the equality (4.3) is valid for $k=1$. Assume this is valid for $k-1$, i.e., the second fundamental form of $\Sigma \times T^{k-1}$ in $\left(M \times T^{k-1}, g + \sum_{p=1}^{k-1} f_p^2 dt_p^2\right)$ is

$$B^{\Sigma \times T^{k-1}} = B^{\Sigma} - \sum_{p=1}^{k-1} f_p \nu(f_p) dt_p^2. \tag{4.5}$$

Note that

$$\begin{aligned} \Sigma \times T^k = (\Sigma \times T^{k-1}) \times \mathbb{S}^1 \quad &\subset \quad \left((M \times T^{k-1}) \times \mathbb{S}^1, g + \sum_{p=1}^{k-1} f_p^2 dt_p^2 + f_k^2 dt_k^2\right) \\ &= \quad \left(M \times T^k, g + \sum_{p=1}^{k} f_p^2 dt_p^2\right). \end{aligned}$$

It follows from Proposition 4.3 that

$$B^{\Sigma \times T^k} = B^{\Sigma \times T^{k-1}} - f_k \nu(f_k) dt_k^2.$$

Therefore, from (4.5) we have that

$$B^{\Sigma \times T^k} = B^{\Sigma} - \sum_{p=1}^{k} f_p \nu(f_p) dt_p^2.$$

For (4.4), consider a orthonormal basis $\{E_1, \cdots, E_m\}$ of $T\Sigma$ with respect to metric g. For each $m+1 \leq l \leq m+k$, define $E_l = f_{l-m}^{-1} \partial_{t_{l-m}}$. Note that $\{E_1, \cdots, E_{m+k}\}$ is a orthonormal basis of $T(\Sigma \times T^k)$ with respect to the metric $g + \sum_{p=1}^{k} f_p^2 dt_p^2$. It follows that

$$\left|B^{\Sigma \times T^k}\right|^2 = \sum_{i,j=1}^{m+k} \left(B_{ij}^{\Sigma \times T^k}\right)^2.$$

From (4.3) we have that

$$\begin{cases} B_{ij}^{\Sigma \times T^k} &= & B_{ij}^{\Sigma} \\ B_{il}^{\Sigma \times T^k} &= & 0 \\ B_{rl}^{\Sigma \times T^k} &= & 0 \\ B_{ll}^{\Sigma \times T^k} &= & -\nu(\log f_{l-m}) \end{cases}$$

for every $i, j = 1, \cdots, m$ and $l, r = m+1, \cdots, k+l$, where $r \neq l$. This implies that

$$\begin{aligned} \left|B^{\Sigma \times T^k}\right|^2 &= \sum_{i,j=1}^{m} \left(B_{ij}^{\Sigma}\right)^2 + \sum_{l=m+1}^{m+k} (\nu(\log f_{l-m}))^2 \\ &= \left|B^{\Sigma}\right|^2 + \sum_{p=1}^{k} (\nu(\log f_p))^2. \end{aligned}$$

■

4.2 Free boundary minimal k-slicings

4.2.1 Definition and Examples

Let $(M, \partial M, g)$ be a n-dimensional Riemannian manifold. Assume there is a properly embedded free-boundary hypersurface $\Sigma_{n-1} \subset M$ which minimizes volume in (M, g). Choose $u_{n-1} > 0$ a first eigenfunction for the second variation S_{n-1} of the volume of Σ_{n-1} in (M, g). Define $\rho_{n-1} = u_{n-1}$ and the weighted volume functional $V_{\rho_{n-1}}$ for hypersurfaces of Σ_{n-1},

$$V_{\rho_{n-1}}(\Sigma) = \int_{\Sigma} \rho_{n-1} dv_{\Sigma},$$

where dv_{Σ} is the volume form in (Σ, g). Assume there is a properly embedded free-boundary hypersurface $\Sigma_{n-2} \subset \Sigma_{n-1}$ which minimizes the weighted volume functional $V_{\rho_{n-1}}$. Choose a first eigenfunction $u_{n-2} > 0$ for the second variation S_{n-2} of the weighted volume functional $V_{\rho_{n-1}}$ in Σ_{n-2}. Define $\rho_{n-2} = \rho_{n-1} u_{n-2}$. Assume that we can keep doing this, inductively. Hence, we obtain a family of free-boundary minimal submanifolds

$$\Sigma_k \subset \Sigma_{k+1} \subset \cdots \subset \Sigma_{n-1} \subset (\Sigma_n, g) := (M, g),$$

which was constructed by choosing, for each $j \in \{k, \cdots, n-1\}$, a properly embedded free-boundary hypersurface $\Sigma_j \subset \Sigma_{j+1}$ which minimizes the weighted volume functional $V_{\rho_{j+1}}$, where $\rho_{j+1} := \rho_{j+2} u_{j+1} = u_{j+1} u_{j+2} \cdots u_{n-1}$. We call such family of free-boundary minimal hypersurfaces a *free-boundary minimal k-slicing* in (M, g).

Example 4.6. *Let $(N, \partial N, g)$ be a k-dimensional Riemannian manifold. Consider the following n-dimensional Riemannian manifold $(N \times T^{n-k}, g + \delta)$, where δ is the flat metric on the torus T^{n-k}. The family of hypersurfaces*

$$N \subset N \times S^1 \subset N \times T^2 \subset \cdots \subset N \times T^{n-k-1} \subset (N \times T^{n-k}, g + \delta),$$

where $\rho_j \equiv u_j \equiv 1$, for every $j = k, \cdots, n-1$, is a free-boundary minimal k-slicing in $(N \times T^{n-k}, g + \delta)$.

4.2.2 Geometric formulas for free-boundary minimal k-slincing

Let $(M, \partial M, g)$ be a n-dimensional Riemannian manifold. Consider a free-boundary k-slicing in M:

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset (\Sigma_n, g) := (M, g).$$

Notation:

- R_j:= Scalar curvature of (Σ_j, g).
- ν_j:= Unit vector field of Σ_j in (Σ_{j+1}, g).
- B_j:= Second fundamental form of Σ_j in (Σ_{j+1}, g).
- H_j:= Mean curvature of Σ_j in (Σ_{j+1}, g)
- η_j:= Outward unit normal smooth vector field on the boundary $\partial\Sigma_j$ in (Σ_j, g).
- $B^{\partial\Sigma_j}$:= Second fundamental form of $\partial\Sigma_j$ in (Σ_j, g) with respect to η_j.
- $H^{\partial\Sigma_j}$:= Mean curvature of $\partial\Sigma_j$ in (Σ_j, g).

Remark 4.7. *Since Σ_j is a free-boundary hypersurface in (Σ_{j+1}, g), for every $j = k, \cdots, n-1$, we have that*

1. *$\eta_j = \eta_p$ in $\partial\Sigma_j$, for every $p \geq j$.*
2. *$H^{\partial\Sigma_j} = H^{\partial\Sigma_{j+1}} - B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j) = H^{\partial M} - \sum_{p=j}^{n-1} B^{\partial\Sigma_{p+1}}(\nu_p, \nu_p)$.*

For each $j \in \{k, \cdots, n-1\}$, define on $\Sigma_j \times T^{n-j}$ a metric

$$\hat{g}_j = g + \sum_{p=j}^{n-1} u_p^2 dt_p^2.$$

Note that, for every hypersurface $\Sigma \subset \Sigma_{j+1}$, we obtain

$$Vol(\Sigma \times T^{n-j-1}, \hat{g}_{j+1}) = \int_\Sigma \rho_{j+1} dv_j = V_{\rho_{j+1}}(\Sigma). \tag{4.6}$$

Since Σ_j is a free-boundary hypersurface of Σ_{j+1} which minimizes the weight volume functional $V_{\rho_{j+1}}$, we have that $\Sigma_j \times T^{n-j-1}$ is a free-boundary hypersurface which minimizes volume in $(\Sigma_{j+1} \times T^{n-j-1}, \hat{g}_{j+1})$. We define

$$\hat{\Sigma}_j = \Sigma_j \times T^{n-j} \text{ e } \tilde{\Sigma}_j = \Sigma_j \times T^{n-j-1}.$$

Notation:

- $\tilde{B}_j$:= Second fundamental form of $\tilde{\Sigma}_j$ in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$.
- $\tilde{R}_j$:= Scalar curvature of $(\tilde{\Sigma}_j, \hat{g}_{j+1})$.
- $\hat{R}_j$:= Scalar curvature of $(\hat{\Sigma}_j, \hat{g}_j)$
- $\hat{B}_j$:= Second fundamental form of $\partial\hat{\Sigma}_j$ in $(\hat{\Sigma}_j, \hat{g}_j)$.

- $\hat{H}_j$:= Mean curvature of $\partial\hat{\Sigma}_j$ in $(\hat{\Sigma}_j, \hat{g}_j)$.

Lemma 4.8. *For every $j = k, \cdots, n-1$, we have that*

$$\tilde{B}_j = B_j - \sum_{p=j+1}^{n-1} u_p \nu_j(u_p) dt_p^2. \tag{4.7}$$

In particular,

$$|\tilde{B}_j|^2 = |B_j|^2 + \sum_{p=j+1}^{n-1} (\nu_j(\log u_p))^2.$$

Proof. It follows from Proposition 4.5. ∎

Lemma 4.9. *We have that*

$$\hat{B}_j = B^{\partial\Sigma_j} - \sum_{p=j}^{n-1} u_p \eta_j(u_p) dt_p^2.$$

In particular,

$$\hat{B}_{j+1}(\nu_j, \nu_j) = B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j).$$

Proof. It follows from Proposition 4.5. ∎

Denote by S_j the second variation for weight volume functional $V_{\rho_{j+1}}$ on Σ_j and $\tilde{S}_j$ the second variation for volume functional of $\tilde{\Sigma}_j$ in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$. It follows from (4.6) that $S_j(\varphi) = \tilde{S}_j(\varphi)$, for every $\varphi \in C^\infty(\Sigma_j)$. This implies that

$$\begin{aligned} S_j(\varphi) &= \int_{\Sigma_j} (|\nabla_j \varphi|^2 - c_j \varphi^2)\rho_{j+1} dv_j - \int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j)\rho_{j+1} d\sigma_j \\ &= -\int_{\Sigma_j} \varphi \tilde{L}_j(\varphi)\rho_{j+1} dv_j + \int_{\partial\Sigma_j} \varphi \left(\frac{\partial\varphi}{\partial\eta_j} - \varphi B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j) \right) \rho_{j+1} d\sigma_j \end{aligned}$$

for every $\varphi \in C^\infty(\Sigma_j)$, where $\tilde{L}_j : C^\infty(\Sigma_j) \to C^\infty(\Sigma_j)$ is a differential operator given by

$$\tilde{L}(\varphi) = \tilde{\Delta}_j \varphi + c_j \varphi$$

where $\tilde{\Delta}_j$ denote the Laplacian operator of $(\tilde{\Sigma}_j, \hat{g}_{j+1})$ and $c_j = \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_j + |\tilde{B}_j|^2)$. Here, dv_j and $d\sigma_j$ are the volume forms of (Σ_j, g) and $(\partial\Sigma_j, g)$, respectively.

Consider λ_j the first eigenvalue of S_j associated the first eigenfunction u_j. We have that,

$$\begin{cases} \tilde{L}_j(u_j) = -\lambda_j u_j & \text{on } \Sigma_j \\ \dfrac{\partial u_j}{\partial \eta_j} = u_j B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j) & \text{on } \partial\Sigma_j \end{cases} \tag{4.8}$$

Lemma 4.10. *For every $j \le p \le n-1$, we have that, in $\partial\Sigma_j$,*

$$B^{\partial\Sigma_{p+1}}(\nu_p, \nu_p) = \langle \nabla_j \log u_p, \eta_j \rangle.$$

Proof. It follows from (4.8) that, in $\partial\Sigma_p$,

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p)=\frac{1}{u_p}\frac{\partial u_p}{\partial\eta_p}=\langle\nabla_p\log u_p,\eta_p\rangle,$$

for every $p=k,\cdots,n-1$. Consider $j\leq p\leq n-1$. Note that, in $\partial\Sigma_j$,

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p)=\langle\nabla_p\log u_p,\eta_j\rangle,$$

because we have $\eta_p=\eta_j$ in $\partial\Sigma_j$ (see remark 4.7). In Σ_j, we can write

$$\nabla_p\log u_p=\nabla_j\log u_p+\sum_{l=j}^{p-1}\langle\nabla_p\log u_p,\nu_l\rangle\nu_l.$$

Hence, in $\partial\Sigma_j$, we have that

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p)=\langle\nabla_j\log u_p,\eta_j\rangle+\sum_{l=j}^{p-1}\langle\nabla_p\log u_p,\nu_l\rangle\langle\nu_l,\eta_j\rangle.$$

However, we have $\eta_j\perp\nu_l$ in $\partial\Sigma_j$, for every $j\leq l\leq n-1$. Therefore,

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p)=\langle\nabla_j\log u_p,\eta_j\rangle$$

■

Lemma 4.11. *For $k\leq j\leq n-1$, we have that*

$$\begin{aligned}\tilde{R}_j &= R_j-2\sum_{p=j+1}^{n-1}u_p^{-1}\Delta_j u_p-2\sum_{j+1\leq p<q\leq n-1}\langle\nabla_j\log u_p,\nabla_j\log u_q\rangle && (4.9)\\ &= R_j-4\rho_{j+1}^{-\frac{1}{2}}\Delta_j(\rho_{j+1}^{\frac{1}{2}})-\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2. && (4.10)\end{aligned}$$

Proof. The equality (4.9) follows from proposition 4.4. For the equality (4.10), note that

$$\begin{aligned}\left|\sum_{p=j+1}^{n-1}\nabla_j\log u_p\right|^2 &= \left\langle\sum_{p=j+1}^{n-1}\nabla_j\log u_p,\sum_{q=j+1}^{n-1}\nabla_j\log u_q\right\rangle+\sum_{p,q=j+1}^{n-1}\langle\nabla_j\log u_p,\nabla_j\log u_q\rangle\\ &= \sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2+2\sum_{j+1\leq p<q\leq n-1}\langle\nabla_j\log u_p,\nabla_j\log u_q\rangle\end{aligned}$$

It follows from (4.9) that

$$\tilde{R}_j=R_j-2\sum_{p=j+1}^{n-1}u_p^{-1}\Delta_j u_p-\left|\sum_{p=j+1}^{n-1}\nabla_j\log u_p\right|^2+\sum_{p=j+1}^{n-1}|\nabla_j\log u_p|^2.$$

Since

$$-2\Delta_j \log u_p = -\frac{2}{u_p}\Delta_j u_p + 2|\nabla_j \log u_p|^2$$

we have that

$$\begin{aligned}
\tilde{R}_j &= R_j - 2\sum_{p=j+1}^{n-1}(\Delta_j \log u_p + |\nabla_j \log u_p|^2) - \left|\sum_{p=j+1}^{n-1}\nabla_j \log u_p\right|^2 + \sum_{p=j+1}^{n-1}|\nabla_j \log u_p|^2 \\
&= R_j - \sum_{p=j+1}^{n-1}|\nabla_j \log u_p|^2 - 2\Delta_j\left(\sum_{p=j+1}^{n-1}\log u_p\right) - \left|\nabla_j\left(\sum_{p=j+1}^{n-1}\log u_p\right)\right|^2
\end{aligned}$$

Since,

$$\sum_{p=j+1}^{n-1}\log u_p = \log(u_{j+1}u_{j+2}\cdots u_{n-1}) = \log \rho_{j+1}$$

we obtain that

$$\begin{aligned}
\tilde{R}_j &= R_j - \sum_{p=j+1}^{n-1}|\nabla_j \log u_p|^2 - 2\Delta_j \log \rho_{j+1} - |\nabla_j \log \rho_{j+1}|^2 \\
&= R_j - 4\rho_{j+1}^{-\frac{1}{2}}\Delta_j(\rho_{j+1}^{\frac{1}{2}}) - \sum_{p=j+1}^{n-1}|\nabla_j \log u_p|^2.
\end{aligned}$$

■

Lemma 4.12. *For $k \le j \le n-1$, we have that*

$$\begin{aligned}
\hat{R}_j &= R_j - 2\sum_{p=j}^{n-1}u_p^{-1}\Delta_j u_p - 2\sum_{j\le p<q\le n-1}\langle \nabla_j \log u_p, \nabla_j \log u_q\rangle && (4.11)\\
&= \hat{R}_{j+1} + |\tilde{B}_j|^2 + 2\lambda_j && (4.12)\\
&= R^M + \sum_{p=j}^{n-1}|\tilde{B}_p|^2 + 2\sum_{p=j}^{n-1}\lambda_p. && (4.13)
\end{aligned}$$

Proof. The equality (4.11) follows from proposition 4.4. For the equality (4.12), note that from proposition 4.2 that scalar curvature $\hat{R}_j$ of

$$\left(\Sigma_j \times T^{n-j}, \hat{g}_j = g + \sum_{p=j}^{n-1}u_p^2 dt_p^2\right) = \left(\tilde{\Sigma}_j \times \mathbb{S}^1, \hat{g}_j = \hat{g}_{j+1} + u_j^2 dt_j^2\right)$$

is given by

$$\hat{R}_j = \tilde{R}_j - \frac{2}{u_j}\tilde{\Delta}_j u_j.$$

So, from (4.8)

$$\begin{aligned}
2\lambda_j &= -\frac{2}{u_j}\tilde{\Delta}_j u_j - \hat{R}_{j+1} + \tilde{R}_j - |\tilde{B}_j|^2 \\
&= \hat{R}_j - \hat{R}_{j+1} - |\tilde{B}_j|^2.
\end{aligned}$$

Hence, it follows the equality (4.12). To get (4.13) we iterate (4.12) $n-j$ times. ■

Proposition 4.13. *If $R_g^M > 0$ and $H_g^{\partial M} \geq 0$ then*

$$4\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j > -2\int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} d\sigma_j - \int_{\Sigma_j} \varphi^2 R_j dv_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$ and $j = k, \cdots, n-1$.

Proof. Since Σ_j minimizes the weighted volume functional $V_{\rho_{j+1}}$, we have that $S_j(\varphi) \geq 0$, for every $\varphi \in C^\infty(\Sigma_j)$. It follows that,

$$4\int_{\Sigma_j} |\nabla_j \varphi|^2 \rho_{j+1} dv_j \geq 2\int_{\Sigma_j} c_j \varphi^2 \rho_{j+1} dv_j + 2\int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j)\rho_{j+1} d\sigma_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$. Since $R_g^M > 0$, from lemma 4.12, we have that $\hat{R}_i > 0$, for every $k \leq i \leq n-1$. It follows from the lemma 4.11 that

$$2c_j > -R_j + 4\rho_{j+1}^{-\frac{1}{2}} \Delta_j(\rho_{j+1}^{\frac{1}{2}})$$

Thus,

$$\begin{aligned} 4\int_{\Sigma_j} |\nabla_j \varphi|^2 \rho_{j+1} dv_j \quad &> \quad -\int_{\Sigma_j} R_j \varphi^2 \rho_{j+1} dv_j + 4\int_{\Sigma_j} \rho_{j+1}^{\frac{1}{2}} \Delta_j(\rho_{j+1}^{\frac{1}{2}})\varphi^2 dv_j \\ &\quad +2\int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j)\rho_{j+1} d\sigma_j, \end{aligned}$$

for every $\varphi \in C^\infty(\Sigma_j)$. Replacing φ by $\varphi\rho_{j+1}^{-\frac{1}{2}}$ at the last inequality, we obtain that

$$\begin{aligned} 4\int_{\Sigma_j} |\nabla_j (\varphi\rho_{j+1}^{-\frac{1}{2}})|^2 \rho_{j+1} dv_j \quad &> \quad -\int_{\Sigma_j} R_j \varphi^2 dv_j + 4\int_{\Sigma_j} \rho_{j+1}^{-\frac{1}{2}} \Delta_j(\rho_{j+1}^{\frac{1}{2}})\varphi^2 dv_j \\ &\quad +2\int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j) d\sigma_j. \end{aligned}$$

Observe that

$$\nabla_j(\varphi\rho_{j+1}^{-\frac{1}{2}}) = \varphi\nabla_j\rho_{j+1}^{-\frac{1}{2}} + \rho_{j+1}^{-\frac{1}{2}}\nabla_j\varphi$$

This implies that,

$$|\nabla_j(\varphi\rho_{j+1}^{-\frac{1}{2}})|^2 = \rho_{j+1}^{-1}|\nabla_j\varphi|^2 + \varphi^2|\nabla_j\rho_{j+1}^{-\frac{1}{2}}|^2 + 2\varphi\rho_{j+1}^{-\frac{1}{2}}\langle\nabla_j\rho_{j+1}^{-\frac{1}{2}}, \nabla_j\varphi\rangle$$

Thus,

$$\rho_{j+1}|\nabla_j(\varphi\rho_{j+1}^{-\frac{1}{2}})|^2 = |\nabla_j\varphi|^2 + \varphi^2\rho_{j+1}|\nabla_j\rho_{j+1}^{-\frac{1}{2}}|^2 + \langle\nabla_j \log\rho_{j+1}^{-\frac{1}{2}}, \nabla_j(\varphi^2)\rangle$$

Using integration by parts, we have that

$$\begin{aligned}
\int_{\Sigma_j} \langle \nabla_j \log \rho_{j+1}^{-\frac{1}{2}}, \nabla_j(\varphi^2)\rangle dv_j &= -\int_{\Sigma_j} \varphi^2 \Delta_j \log \rho_{j+1}^{-\frac{1}{2}} dv_j + \int_{\partial\Sigma_j} \varphi^2 \frac{\partial(\log \rho_{j+1}^{-\frac{1}{2}})}{\partial \eta_j} d\sigma_j \\
&= + \int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-\frac{1}{2}} \Delta_j \rho_{j+1}^{\frac{1}{2}} dv_j - \int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_{j+1}^{\frac{1}{2}}|^2) dv_j \\
&- \frac{1}{2} \int_{\partial\Sigma_j} \varphi^2 \langle \nabla_j \log \rho_{j+1}, \eta_j \rangle d\sigma_j \\
&= -\int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_{j+1}^{\frac{1}{2}}|^2 dv_j + \int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-\frac{1}{2}} \Delta_j \rho_{j+1}^{\frac{1}{2}} dv_j \\
&- \frac{1}{2} \int_{\partial\Sigma_j} \varphi^2 \langle \nabla_j \log \rho_{j+1}, \eta_j \rangle d\sigma_j
\end{aligned}$$

Then,

$$\begin{aligned}
4\int_{\Sigma_j} \rho_{j+1} |\nabla_j(\varphi \rho_{j+1}^{-\frac{1}{2}})|^2 dv_j &= 4\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j + 4\int_{\Sigma_j} \varphi^2 \rho_{j+1} |\nabla_j \rho_{j+1}^{-\frac{1}{2}}|^2 dv_j \\
&- 4\int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_{j+1}^{\frac{1}{2}}|^2 dv_j + 4\int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-\frac{1}{2}} \Delta_j \rho_{j+1}^{\frac{1}{2}} dv_j \\
&- 2\int_{\partial\Sigma_j} \varphi^2 \langle \nabla_j \log \rho_{j+1}, \eta_j \rangle d\sigma_j
\end{aligned}$$

Since,

$$\nabla_j \rho_{j+1}^{-\frac{1}{2}} = -\rho_{j+1}^{-1} \nabla_j \rho_{j+1}^{\frac{1}{2}},$$

we obtain that

$$\rho_{j+1} |\nabla_j \rho_{j+1}^{-\frac{1}{2}}|^2 = |\nabla_j \log \rho_{j+1}^{\frac{1}{2}}|^2.$$

This implies that

$$\begin{aligned}
4\int_{\Sigma_j} \rho_{j+1} |\nabla_j(\varphi \rho_{j+1}^{-\frac{1}{2}})|^2 dv_j &= 4\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j + 4\int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-\frac{1}{2}} \Delta_j \rho_{j+1}^{\frac{1}{2}} dv_j \\
&- 2\int_{\partial\Sigma_j} \varphi^2 \langle \nabla_j \log \rho_{j+1}, \eta_j \rangle d\sigma_j
\end{aligned}$$

Consequently,

$$4\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j > 2\int_{\partial\Sigma_j} \varphi^2 \left(B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j) + \langle \nabla_j \log \rho_{j+1}, \eta_j \rangle \right) d\sigma_j - \int_{\Sigma_j} R_j \varphi^2 dv_j.$$

Since $H_g^{\partial M} \geq 0$, from the remark 4.7 and lemma 4.10 that

$$\begin{aligned}
4\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j &> 2\int_{\partial\Sigma_j} \varphi^2 \left(\sum_{p=j}^{n-1} B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p)\right) d\sigma_j - \int_{\Sigma_j} R_j \varphi^2 dv_j \\
&= 2\int_{\partial\Sigma_j} \varphi^2 \left(H_g^{\partial M} - H^{\partial\Sigma_j}\right) d\sigma_j - \int_{\Sigma_j} R_j \varphi^2 dv_j \\
&\geq -2\int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} d\sigma_j - \int_{\Sigma_j} R_j \varphi^2 dv_j
\end{aligned}$$

Therefore,

$$4\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j > -2\int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} d\sigma_j - \int_{\Sigma_j} \varphi^2 R_j dv_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$. ■

Teorem a 4.14. *Let $(M, \partial M, g)$ be a n-dimensional Riemannian manifold such that $R_g^M > 0$ and $H_g^{\partial M} \geq 0$. Consider the free boundary minimal k-slicing in (M, g)*

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset \Sigma_n = M.$$

Then:

(1) The manifold Σ_j has a metric with positive scalar curvature and minimal boundary, for every $3 \leq k \leq j \leq n-1$.

(2) If $k = 2$, then the connected components of Σ_2 are disks.

Proof.

(1) Consider $j \in \{k, \cdots, n-1\}$, here $k \geq 3$. It follows from Proposition 4.13 that

$$-4k_j \int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j < 2k_j \int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} d\sigma_j + k_j \int_{\Sigma_j} \varphi^2 R_j dv_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$ such that $\varphi \not\equiv 0$ and $k_j = \frac{j-2}{4(j-1)} > 0$. This implies that

$$\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j + 2k_j \int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} d\sigma_j + k_j \int_{\Sigma_j} \varphi^2 R_j dv_j > (1 - 4k_j) \int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j,$$

for every $\varphi \in H^1(\Sigma_j)$ such that $\varphi \not\equiv 0$. It follows that

$$\lambda = \inf_{0 \not\equiv \varphi \in H^1(\Sigma_j)} \frac{\displaystyle\int_{\Sigma_j} |\nabla_j \varphi|^2 dv_j + 2k_j \int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} d\sigma_j + k_j \int_{\Sigma_j} \varphi^2 R_j dv_j}{\displaystyle\int_{\Sigma_j} \varphi^2 dv_j} > 0.$$

Therefore, there exists a metric in Σ with positive scalar curvature and minimal boundary (see section 2.4).

(2) From proposition 4.13 we have that

$$4\int_{\Sigma_2}|\nabla_2\varphi|^2 dv_2 > -2\int_{\partial\Sigma_2}\varphi^2 H^{\partial\Sigma_2}d\sigma_2 - 2\int_{\Sigma_2}\varphi^2 K dv_2,$$

for every $\varphi \in C^\infty(\Sigma_2)$ such that $\varphi \not\equiv 0$, because $R_2 = 2K_2$, where K_2 is the Gaussian curvature of (Σ_2, g). In particular, for $\varphi \equiv 1$ we have that

$$\int_{\partial\Sigma_2} H^{\partial\Sigma_2}d\sigma_2 + \int_{\Sigma_2} K dv_2 > 0. \tag{4.14}$$

Let S be a connected component of Σ_2. From inequality (4.14) and from Gauss-Bonnet theorem, we have that $\chi(S) > 0$. Therefore S is a disk.

■

4.3 Proof of the main theorem

Proposition 4.15. *There is a free boundary minimal* 2*-slicing*

$$\Sigma_2 \subset \Sigma_3 \subset \cdots \subset \Sigma_{n+1} \subset (M, g),$$

such that Σ_k *is connected and the map* $F_k := F|_{\Sigma_k} : (\Sigma_k, \partial\Sigma_k) \to (\mathbb{D}^2 \times T^{k-2}, \partial\mathbb{D}^2 \times T^{k-2})$ *has non-zero degree, for every* $k = 2, \cdots, n+1$.

Proof. Without loss of generality, we assume that F is a smooth function. Consider the projection $p_j : \mathbb{D}^2 \times T^j \to S^1$ given by

$$p_j(x, (t_1, \cdots, t_j)) = t_j,$$

for every $x \in \Sigma$ and $(t_1, \cdots, t_j) \in T^j = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

We will start constructing the manifold Σ_{n+1}. For this, define $f_n = p_n \circ F$. It follows from the Sard's Theorem that there is $\theta_n \in S^1$ which is a regular value of f_n and $f_n|_{\partial M}$. Define

$$S_{n+1} := f_n^{-1}(\theta_n) = F^{-1}(\mathbb{D}^2 \times T^{n-1} \times \{\theta_n\}).$$

Note that $S_{n+1} \subset M$ is a properly embedded hypersurface which represents a non-trivial homology class in $H_{n+1}(M, \partial M)$ and $F|_{S_{n+1}} : (S_{n+1}, \partial S_{n+1}) \to (\mathbb{D}^2 \times T^{n-1}, \partial\mathbb{D}^2 \times T^{n-1})$ is a non-zero degree map. It follows from Theorem 3.3.1 that there is a properly embedded free-boundary hypersuface $\Sigma'_{n+1} \subset M$ which minimizes volume in (M, g) and represents the homology class $[S_{n+1}] \in H_{n+1}(M, \partial M)$. Since Σ'_{n+1} and S_{n+1} represent the same homology class in $H_{n+1}(M, \partial M)$, we have that $F|_{\Sigma'_{n+1}} : (\Sigma'_{n+1}, \partial\Sigma'_{n+1}) \to (\mathbb{D}^2 \times T^{n-1}, \partial\mathbb{D}^2 \times T^{n-1})$

has non-zero degree. Since the degree of a map is the sum of the degree of such a map restricted to each connected component, we have that there is Σ_{n+1} a connected component of Σ'_{n+1} such that $F_{n+1} := F|_{\Sigma_{n+1}} : (\Sigma_{n+1}, \partial\Sigma_{n+1}) \to (\mathbb{D}^2 \times T^{n-1}, \partial\mathbb{D}^2 \times T^{n-1})$ has non-zero degree. It follows from Lemma 33.4 in [33] that Σ_{n+1} is still a properly embedded free-boundary hypersurface which minimizes volume in (M, g). Consider $u_{n+1} \in C^\infty(\Sigma_{n+1})$ a positive first eigenfunction for the second variation S_{n+1} of the volume of Σ_{n+1} in (M, g). Define $\rho_{n+1} = u_{n+1}$.

By a similar reasoning used to construct Σ_{n+1}, we obtain a properly embedded free boundary connected smooth hypersurface $\Sigma_n \subset \Sigma_{n+1}$ which minimizes the weighted volume functional $V_{\rho_{n+1}}$ and $F_n := F|_{\Sigma_n} : (\Sigma_n, \partial\Sigma_n) \to (\mathbb{D}^2 \times T^{n-2}, \partial\mathbb{D}^2 \times T^{n-2})$ has non-zero degree. Consider $u_n \in C^\infty(\Sigma_{n+1})$ a positive first eigenfunction for the second variation S_n of $V_{\rho_{n+1}}$ on Σ_n. We then define $\rho_n = u_n\rho_{n+1}$ and we continue this process. ∎

Lemma 4.16. *We have that $\Sigma_2 \in \mathcal{F}_M$.*

Proof. Since $R_g^M > 0$ and $H_g^{\partial M} \geq 0$, it follows from Theorem 4.14 that Σ_2 is a disk. Since there is a non-zero degree map $F_2 : (\Sigma_2, \partial\Sigma_2) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$, then the map $F_2|_{\partial\Sigma_2} : \partial\Sigma_2 \to \partial\mathbb{D}^2$ has non-zero degree. It follows that $\partial\Sigma_2$ is a curve homotopically non-trivial in ∂M. Therefore, $\Sigma_2 \in \mathcal{F}_M$. ∎

Lemma 4.17. *We have that,*

$$\frac{1}{2}\inf R_g^M |\Sigma_2|_g + \inf H_g^{\partial M} |\Sigma_2|_g \leq 2\pi.$$

Moreover, if equality holds then $R_2 = \inf R_g^M$, $H_g^{\partial\Sigma_2} = \inf H_g^{\partial M}$ and $u_k|_{\Sigma_2}$ are positive constants, for every $k = 2, \cdots, n+1$.

Proof. From the Remark 4.7 and Lemma 4.10

$$\inf H_g^{\partial M} \leq \sum_{p=2}^{n+1} \langle \nabla_2 \log u_p, \eta_2 \rangle + H^{\partial\Sigma_2}.$$

This implies that

$$\inf H_g^{\partial M} |\partial\Sigma_2|_g \leq \sum_{p=2}^{n+1} \int_{\partial\Sigma_2} \langle \nabla_2 \log u_p, \eta_2 \rangle d\sigma_2 + \int_{\partial\Sigma_2} H^{\partial\Sigma_2} d\sigma_2. \tag{4.15}$$

From Lemma 4.12, we have that

$$\begin{aligned} \hat{R}_2 &= R_2 - 2\sum_{p=2}^{n+1} u_p^{-1}\Delta_2 u_p - 2\sum_{2\leq p<q\leq n+1} \langle \nabla_2 \log u_p, \nabla_2 \log u_q \rangle \\ &= R_2 - 2\sum_{p=2}^{n+1} u_p^{-1}\Delta_2 u_p - \left|\sum_{p=2}^{n+1} X_p\right|^2 + \sum_{p=2}^{n+1} |X_p|^2, \end{aligned}$$

where $X_p := \nabla_2 \log u_p$. Since

$$u_p^{-1}\Delta_2 u_p = \Delta_2 \log u_p + |X_p|^2,$$

we have that

$$\hat{R}_2 = R_2 - 2\sum_{p=2}^{n+1}\Delta_2 \log u_p - \left|\sum_{p=2}^{n+1} X_p\right|^2 - \sum_{p=2}^{n+1}|X_p|^2.$$

Since $\hat{R}_2 \geq \inf R_g^M$, we obtain

$$\begin{aligned} \frac{1}{2}\inf R_g^M |\Sigma_2|_g &\leq \frac{1}{2}\int_{\Sigma_2}\hat{R}_2 dv_2 \\ &= \frac{1}{2}\int_{\Sigma_2} R_2 dv_2 - \sum_{p=2}^{n+1}\int_{\Sigma_2}\Delta_2 \log u_p dv_2 \\ &\quad -\frac{1}{2}\int_{\Sigma_2}\left|\sum_{p=2}^{n+1} X_p\right|^2 dv_2 - \frac{1}{2}\sum_{p=2}^{n+1}\int_{\Sigma_2}|X_p|^2 dv_2 \\ &\leq \frac{1}{2}\int_{\Sigma_2} R_2 dv_2 - \sum_{p=2}^{n+1}\int_{\Sigma_2}\Delta_2 \log u_p dv_2. \end{aligned}$$

It follows from Divergence Theorem that

$$\frac{1}{2}\inf R^M |\Sigma_2|_g \leq \frac{1}{2}\int_{\Sigma_2} R_2 dv_2 - \sum_{p=2}^{n+1}\int_{\partial\Sigma_2}\langle \nabla_2 \log u_p, \eta_2\rangle d\sigma_2. \tag{4.16}$$

By inequalities (4.15) and (4.16), we have that

$$\frac{1}{2}\inf R^M |\Sigma_2|_g + \inf H^{\partial M}|\partial\Sigma_2|_g \leq \frac{1}{2}\int_{\Sigma_2} R_2 dv_2 + \int_{\partial\Sigma_2} H^{\partial\Sigma_2} d\sigma_2.$$

Therefore, from Gauss-Bonnet Theorem, we obtain

$$\frac{1}{2}\inf R^M |\Sigma_2|_g + \inf H^{\partial M}|\partial\Sigma_2|_g \leq 2\pi\chi(\Sigma_2) = 2\pi.$$

However, note that if holds equality then the field $X_p = 0$ for every $p = 2, \cdots, n+1$. It follows that $u_p|_{\Sigma_2}$ are positive constants for every $p = 2, \cdots, n+1$. Consequently, $R_2 = \hat{R}_2 \geq \inf R_g^M$ and $H^{\partial\Sigma_2} \geq \inf H_g^{\partial M}$. Therefore, from Gauss-Bonnet theorem, we have that $R_2 = \inf R_g^M$ and $H^{\partial\Sigma_2} = \inf H_g^{\partial M}$. ∎

Corollary 4.18. *We have that,*

$$\frac{1}{2}\inf R_g^M \mathcal{A}(M,g) + \inf H_g^{\partial M}\mathcal{L}(M,g) \leq 2\pi.$$

Moreover, if equality holds then $R_2 = \inf R_g^M$, $H^{\partial\Sigma_2} = \inf H_g^{\partial M}$ and $u_k|_{\Sigma_2}$ are positive constants, for every $k = 2, \cdots, n+1$.

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