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A Mathematical Model for an Epidemic without Immunity

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Abstract

In this paper we introduce a model for some diseases which have temporary immunity. It means after recovery, there is immunity but it is not permanent. In this disease, the people are divided into some groups, susceptible, infective, immune and dead people. It is to be noted that we pay attention to people who are born or die because of any reasons except of the disease too. Then we get the equilibrium point and prove it is unstable.

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1 Introduction

The diseases and any studying about them can be very helpful and important for human beings. Specially, mathematical modeling can be a good and suitable instrument for having a better life. The basic SIR model has a long history. At first Kermak and Mckendrich introduced SIR model in 1927 [6]. Now, SIR model is developing more and more that you can even find it discussed in some introductory calculus text books [5]. SIR model can be very useful and helpful in characterizing some disease. More numbers of these mathematical models can be seen in [2,3,7,8,9]. We concentrate on some disease which are very famous because of their properties meaning temporary immunity. It means every body who get the disease, after the recovery, get immunity for a short time, not for ever and ever. Also, it is notable that the disease is contagious where the disease is transmitted from the infective people to susceptible people. Also we assume new born people are not infective, they are susceptible people. As we try our model to be more and more real, we consider to this reality that, always, there are some people who die because of any reasons except of the contagious disease. In section 3, we get equilibrium points and see they are unstable.

2 The Model

In this model the population is divided into some groups. The first group is susceptible people, denoting them by $S(t)$ at time t. We have one assumption: Every member of this group have the same chance to get disease. It means the disease is well-stirred or every individual has an equal chance to meet the other members of the population.
The second group is infective people which is denoted by $I(t)$ at the time t. They are source of infectious. Also, there is another group which is called immune people. It is denoted by $R(t)$ at time t. The immune people are not immune for all the times, because the disease dose not have permanent immunity. $D(t)$ is denoted for people who die because of the disease at time t. We consider there are some people who die because of any reasons except of the disease, let $f_1(t), f_2(t)$ and $f_3(t)$ be the number of susceptible people, infective people and immune people who die because of any reasons except of the disease, respectively at time t. Then $f_1(t)+f_2(t)+f_3(t)+D(t)$ is the total number of people who die at time t. There is a positive quantity which called infection rate $r > 0$. There is another positive quantity which called recovery rate $b > 0$. c is another quantity which called dead rate because of disease. Then differential equation for the infective people is

$$\dot{I} = rSI - bI - \dot{f}_2 - cI$$

Finally $a > 0$ is the last positive quantity which called remove rate. This quantity distinguishes the rate of entering of immune people into susceptible group.$g(t)$ is denoted for the number of people who are born at time t. We assume the people who are born, are just, susceptible people. Hence, the differential equations for the susceptible people is

$$\dot{S} = -rSI + aR - \dot{f}_1 + \dot{g}$$

Now the differential equations for immune people and dead people because of disease are

$$\dot{R} = bI - \dot{f}_3 - aR$$

$$\dot{D} = cI$$

Finally the differential equations system for the disease is

$$\dot{S} = -rSI + aR - \dot{f}_1 + \dot{g} \tag{1}$$

$$\dot{I} = rSI - bI - cI - \dot{f}_2 \tag{2}$$

$$\dot{R} = bI - \dot{f}_3 - aR \tag{3}$$

$$\dot{D} = cI \tag{4}$$

It is to be noted that in this model there is not latent period for the illness; A susceptible person who has contracted the disease becomes infective immediately. If incubation is short, this observation may be accepted. The initial conditions attached to the system is $S_0 = S(0) > 0$, $I_0 = I(0) > 0$ and $R_0 = R(0) = 0$.
Usually, we assume that the disease starts with a small number of infectives. It means I_0 is small with respect to S_0. We have an epidemic when the number of infective people is increasing faster than the number of people who recovers. The threshold parameter $\frac{b}{r}$ is called the relative recovery rate, which is the percentage of those recovered in unit time divided by the percentage of those infected by a single infective in unit time [1,4]. Now by (3) and (4) we get

$$R(t+\frac{1}{b}) + f_3(t+\frac{1}{b}) - R(t) - f_3(t) =$$

$$R(t+\frac{1}{b}) - R(t) + f_3(t+\frac{1}{b}) - f_3(t) \simeq$$

$$\dot{R}(t)\frac{1}{b} + \dot{f}_3(t)\frac{1}{b} =$$

$$(\dot{R}(t) + \dot{f}_3(t))\frac{1}{b} =$$

$$(bI - Ra)\frac{1}{b} =$$

$$I(t) - R(t)\frac{a}{b}$$

also

$$D(t+\frac{1}{c}) - D(t) \simeq \dot{D}\frac{1}{c} = I(t)$$

The second equation means the number of people who die in interval t and $t + \frac{1}{c}$ is equal to the number people who are infective at time t.

If the difference between the numbers of newborn people and the numbers of people who die because of any reasons except of the disease, is fixed meaning $\dot{g}-(\dot{f}_1+\dot{f}_2+\dot{f}_3)=0$ or $g-(f_1+f_2+f_3)=k$ where k is fixed then $\dot{D}+\dot{R}+\dot{I}+\dot{S}=0$ or $D+R+I+S$ is fixed: it means the summation of the numbers of people who are susceptible or infective or immune or die because of disease is fixed.

3 Equilibrium points

Now the first question is what are the equilibrium points?
By making the right-hand sides of the differential equations equal to zero we have

$$-rSI+aR-\dot{f}_1+\dot{g}=0 \tag{5}$$

$$rSI-bI-cI-\dot{f}_2=0 \tag{6}$$

$$bI-\dot{f}_3-aR=0 \tag{7}$$

$$\dot{D}=cI=0 \tag{8}$$

With respect to this reality $I(t)\neq 0$ then

$$c=0 \tag{9}$$

or $D(t)=k$, f or all t where k is fixed

it means the numbers of people who die because of disease are fixed. By (7) we have

$$I=\frac{aR+\dot{f}_3}{b} \tag{10}$$

As (6), (9) and (10) we have

$$S=\frac{b\dot{f}_2}{r(aR+\dot{f}_3)}+\frac{b}{r} \tag{11}$$

Finally by (5), (11) and (10) we have $\dot{f}_1+\dot{f}_2+\dot{f}_3=\dot{g}$. Then by the condition $\dot{f}_1+\dot{f}_2+\dot{f}_3=\dot{g}$ the point

$$(S,I,R,D)=(\frac{b\dot{f}_2}{r(aR+\dot{f}_3)}+\frac{b}{r},\frac{aR+\dot{f}_3}{b},R,k)$$

is equilibrium point. The condition says the total number who die because of diseases is as the same as the number of newborn people.

The second question is that the equilibrium point is stable or unstable?
To provide an answer to this question, we write the jacobi matrix of right-hand sides of the differential equations and determine characteristic polynomial [4]. Let
$f : N \times N \times N \times N \longrightarrow N \times N \times N \times N$
$(S, I, R, D) \mapsto (-rSI + aR - \dot{f}_1 + \dot{g}, rSI - bI - cI - \dot{f}_2, bI - \dot{f}_3 - aR, cI)$
We get the Jacobi matrix of above function

$$J = \begin{bmatrix} -rI & -rS & a & 0 \\ rI & rS - b - c & 0 & 0 \\ 0 & b & -a & 0 \\ 0 & c & 0 & 0 \end{bmatrix}$$

$$det(J - \lambda I) = (-\lambda)\begin{vmatrix} -rI - \lambda & -rS & a \\ rI & rS - b - c - \lambda & 0 \\ 0 & b & -a - \lambda \end{vmatrix} =$$

$$(-\lambda)[(-rI - \lambda)\begin{vmatrix} rS - b - c - \lambda & 0 \\ b & -a - \lambda \end{vmatrix} -$$

$$(rI)\begin{vmatrix} -rS & a \\ b & -a - \lambda \end{vmatrix}] =$$

$(-\lambda)((-rI - \lambda)((rS - b - c - \lambda)(-a - \lambda)) - (rI)((-rS)(-a - \lambda) - ab) =$

$(-\lambda)((-rI-\lambda)(-rSa+ab+ac+a\lambda-rS\lambda+b\lambda+c\lambda+\lambda^2)-rI(raS+r\lambda S-ab)) =$

$(-\lambda)(r^2SaI + \lambda rSa - rabI - \lambda ab - racI - ac\lambda - ra\lambda I - a\lambda^2 + r^2SI\lambda + rS\lambda^2$

$-brI\lambda - b\lambda^2 - rcI\lambda - c\lambda^2 - r\lambda^2 I - \lambda^3) - rI(raS + r\lambda S - ab)) =$

$\lambda^4 - (-a + rS - b - c - rI)\lambda^3 - (rSa - ab - ac - raI + r^2SI - brI - rcI$

$-r^2IS)\lambda^2 - (r^2SaI - rabI - r^2SaI + rabI - racI)\lambda =$

$\lambda^4 + (a - rS + b + c + rI)\lambda^3 + (arI + brI + rcI - rSa + ab + ac)\lambda^2 + racI\lambda$

Then the characteristic polynomial is

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$$

where

$$a_4 = 0$$

$$a_3 = racI$$

$$a_2 = arI + brI + rcI - rSa + ab + ac$$

$$a_1 = a - rS + b + c + rI$$

Where by (9),(10) and (11) we have

$a_1 = a - \frac{rb\dot{f}_2}{r(aR+\dot{f}_3)} - b + b + 0 + \frac{r(aR+\dot{f}_3)}{b} = a - \frac{b\dot{f}_2}{aR+\dot{f}_3} + \frac{r(aR+\dot{f}_3)}{b}$

$a_2 = \frac{ar(aR+\dot{f}_3)}{b} + r(aR+\dot{f}_3) + 0 - \frac{ab\dot{f}_2}{(aR+\dot{f}_3)} - ab + ab + 0 =$

$\frac{ar(aR+\dot{f}_3)}{b} + r(aR+\dot{f}_3) - \frac{ab\dot{f}_2}{(aR+\dot{f}_3)}$

$a_3 = 0,$

$a_4 = 0.$

Now we consider the following Theorem [10]

Theorem The polynomial $P_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_1\lambda + a_0$ is stable if and only if its coefficients are positive and all the principal diagonal minors are positive in its Hurwitz matrix.

Remark. By the above theorem

$$(\frac{b\dot{f}_2}{r(aR+\dot{f}_3)} + \frac{b}{r}, \frac{aR+\dot{f}_3}{b}, R, k) = (S, I, R, D)$$

With the condition $\dot{f}_1 + \dot{f}_2 + \dot{f}_3 = \dot{g}$ is unstable equilibrium point for this disease. Because a_0 and a_1 are not positive.

4 Conclusion

In this paper mathematical modeling was introduced which helps us to distinguish the epidemics which does not permanent immunity like flu. Also newborns and dead people are important as effective groups. After studying the model, we got the equilibrium points and researched about stability of them. It can be very helpful in some programs in the society for preventing from some epidemics without immunity or control them which can be a topic for further research.

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