THE CHARACTERISTIC POLYNOMIAL OF A CERTAIN MATRIX OF BINOMIAL COEFFICIENTS

L. CARLITZ
Duke University, North Carolina

1. Put

(1.1) $$A_{n+1} = \left[\binom{r}{n-s}\right] \quad (r,\ s = 0,\ 1,\ \ldots,\ n) \quad ,$$

a matrix of order $n+1$; for example

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} .$$

Let

(1.2) $$f_{n+1}(x) = \det(xI - A_{n+1})$$

denote the characteristic polynomial of A_{n+1}. Hoggatt has communicated the following result to the writer.

Let $$F_o = 0,\ F_1 = 1,$$

$$F_{n+1} = F_n + F_{n-1} \quad (n \geq 1)$$

denote the Fibonacci numbers. Define

(1.3) $$F_{n,r} = \frac{F_n F_{n-1} \cdots F_{n-r+1}}{F_1 F_2 \cdots F_r} \quad (r \geq 1), \quad F_{n,o} = 1.$$

Then we have

(1.4) $$f_{n+1}(x) = \sum_{r=o}^{n+1} (-1)^{r(r+1)/2} F_{n+1,r}\, x^{n+1-r} \quad .$$

In the present paper we prove the truth of (1.4). Moreover we show that

(1.5) $$f_{n+1}(x) = \prod_{j=o}^{n} (x - \alpha^j \beta^{n-j}) \quad ,$$

where

(1.6) $$\alpha = (1+\sqrt{5})/2\,, \quad \beta = (1-\sqrt{5})/2 \quad .$$

Thus the characteristic values of A_{n+1} are

(1.7) $$\alpha^n, \ \alpha^{n-1}\beta, \ \ldots, \ \alpha\beta^{n-1}, \ \beta^n \quad .$$

Since they are distinct it follows that A_{n+1} is similar to a diagonal matrix.

2. We recall first that for any matrix A of the nth order with characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

(2.1) $$\mathrm{tr}(A^k) = \lambda_1^k + \ldots + \lambda_n^k \quad (k = 0, 1, \ldots) \quad ,$$

where $\mathrm{tr}(A^k)$ denotes the trace of A^k. Moreover once these traces are known it is a simple matter to get the characteristic polynomial. We shall accordingly attempt to evaluate

(2.2) $$\mathrm{tr}(A_{n+1}^k) \quad (k = 0, 1, \ldots) \quad .$$

For $k = 1$ it is evident from (1.1) that

(2.3) $$\mathrm{tr}(A_{n+1}) = \sum_r \binom{r}{n-r} = F_{n+1} \quad .$$

For $k = 2$ we have

$$\mathrm{tr}(A_{n+1}^2) = \sum_{r,s} \binom{r}{n-s}\binom{s}{n-r} = \sum_{r,s} \binom{n-r}{s}\binom{n-s}{r}$$

$$= \sum_{r,s} \frac{(n-r)!\,(n-s)!}{r!\,s!\,((n-r-s)!)^2} = \sum_k \frac{n!}{k!\,(n-k)!} \sum_r \frac{(-k)_r(n-k+1)_r}{r!\,(-n)_r},$$

where

$$(a)_r = a(a+1)\ldots(a+r-1) \quad .$$

Since [1] page 37

$$\sum_r \frac{(-n)_r(a)_r}{r!\,(c)_r} = \frac{(c-a)_n}{(c)_n} \,,$$

we get

$$\operatorname{tr}(A_{n+1}^2) = \sum_k \frac{n!}{k!\,(n-k)!}\,\frac{(-2n+k-1)_k}{(-n)_k}$$

$$= \sum_k \frac{n!}{k!\,(n-k)!}\,\frac{(n-k)!}{n!}\,\frac{(2n-k+1)!}{(2n-2k+1)}$$

$$= \sum_k \binom{2n-k+1}{k} ,$$

so that

$$\operatorname{tr}(A_{n+1}^2) = F_{2n+2} . \tag{2.4}$$

In the next place we have

$$\operatorname{tr}(A_{n+1}^3) = \sum_{r,s,t} \binom{r}{n-s}\binom{s}{n-t}\binom{t}{n-r} = \sum_{r,s,t} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{r} ,$$

but it does not seem possible to evaluate this sum by the above method. We shall instead employ the method used in [2] .

Starting with the identity

$$x^r(1+x)^{n-r} = \sum_s \binom{n-r}{s} x^{n-s} \tag{2.5}$$

replace x by $1+x^{-1}$. We get

$$(1+x)^r(1+2x)^{n-r} = \sum_{s,t} \binom{n-r}{s}\binom{n-s}{t} x^{n-t} . \tag{2.6}$$

Next multiply both sides by x^r and sum over r. This gives

$$\sum_{r=o}^{n} x^r(1+x)^r(1+2x)^{n-r} = \sum_{r,s,t} \binom{n-r}{s}\binom{n-s}{t} x^{n+r-t} .$$

The coefficient of x^n on the right is equal to

$$\sum_{r,s} \binom{n-r}{s}\binom{n-s}{r} = \sum_{r,s} \binom{r}{n-s}\binom{s}{n-r} = \operatorname{tr}(A_{n+1}^2) .$$

On the left we get

$$\sum_{r+s+t=n} \binom{r}{s}\binom{n-r}{s} 2^t = \sum_{r+s\leq n} \binom{r}{s}\binom{n-r}{s} 2^{n-r-s} = u_n ,$$

say. Then

$$\sum_{n=o}^{\infty} u_n x^n = \sum_{r,s=o}^{\infty} \binom{r}{s} x^{r+s} \sum_{n=r+s}^{\infty} \binom{n-r}{s} (2x)^{n-r-s}$$

$$= \sum_{r,s=o}^{\infty} \binom{r}{s} x^{r+s} (1-2x)^{-s-1}$$

$$= \sum_{s=o}^{\infty} x^{2s}(1-x)^{-s-1}(1-2x)^{-s-1}$$

$$= \frac{1}{1-3x+x^2} = \frac{1}{\alpha^2-\beta^2}\left(\frac{\alpha^2}{1-\alpha^2 x} - \frac{\beta^2}{1-\beta^2 x}\right)$$

where α, β are defined by (1.6). We have therefore

$$u_n = \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha^2 - \beta^2} = \frac{F_{2n+2}}{F_2} = F_{2n+2}$$

in agreement with (2.4).

Returning to (2.6), again replace x by $1+x^{-1}$. We find that

$$(2.7) \qquad (1+2x)^r(2+3x)^{n-r} = \sum_{s,t,j} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{j} x^{n-j} .$$

Multiply by x^r and sum over r. We get

$$(2.8) \qquad \sum_{r=o}^{n} x^r(1+2x)^r(2+3x)^{n-r} = \sum_{r,s,t,j} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{j} x^{n+r-j} .$$

The coefficient of x^n on the right of (2.8) is evidently

$$\sum_{r,s,t} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{r} = \operatorname{tr}(A_{n+1}^3) .$$

On the left we get

$$\sum_{r+s+t=n} \binom{r}{s}\binom{n-r}{t} 2^s 2^{n-r-t} 3^t = \sum_{r+s\leq n} \binom{r}{s}\binom{n-r}{s} 2^{2s} 3^{n-r-s} = u_n ,$$

say. Then as above

$$\sum_{n=o}^{\infty} u_n x^n = \sum_{r,\, s=o}^{\infty} \binom{r}{s} 2^{2s} x^{r+s} (1-3x)^{-s-1}$$

$$= \sum_{s=o}^{\infty} (2x)^{2s} (1-x)^{-s-1} (1-3x)^{-s-1} = \frac{1}{(1-x)(1-3x)-4x^2}$$

$$= \frac{1}{1-4x-x^2} = \frac{1}{(1-\alpha^3 x)(1-\beta^3 x)}$$

$$= \frac{1}{\alpha^3-\beta^3}\left(\frac{\alpha^3}{1-\alpha^3 x} - \frac{\beta^3}{1-\beta^3 x}\right),$$

so that

$$u_n = \frac{\alpha^{3n+3} - \beta^{3n+3}}{\alpha^3 - \beta^3} = \frac{F_{3n+3}}{F_3} .$$

It follows that

$$\text{(2.9)} \qquad tr(A^3_{n+1}) = \frac{F_{3n+3}}{F_3} .$$

3. We are now able to handle the general case. In (2.6) replace x by $1+x^{-1}$ and we get

$$\text{(3.1)} \qquad (2+3x)^r (3+5x)^{n-r} = \sum_{s,\, t,\, j,\, k} \binom{n-r}{s}\binom{n-s}{t}\binom{n-t}{j}\binom{n-j}{k} x^{n-k} .$$

The general formula of this type is

$$\text{(3.2)} \quad (F_{k-1}+xF_k)^r (F_k+xF_{k+1})^{n-r}$$

$$= \sum_{r_1,\ldots,r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} x^{n-r_k} (k=1, 2, 3, \ldots) .$$

Indeed for $k = 1, 2, 3, 4$, (3.2) reduces to (2.5), (2.6), (2.7), (3.1), respectively. Assuming that (3.2) holds for the value k we replace x by $1+x^{-1}$ and multiply the result by x^n. The left member becomes

$$(xF_{k-1} + xF_k + F_k)^r(xF_k + xF_{k+1} + F_{k+1})^{n-r}$$
$$= (F_k + xF_{k+1})^r(F_{k+1} + xF_{k+2})^{n-r} \quad ,$$

while the right member becomes

$$\sum_{r_1,\ldots,r_{k+1}} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k}\binom{n-r_k}{r_{k+1}} x^{n-r_{k+1}} .$$

This evidently completes the proof of (3.2).

Next multiply (3.2) by x^r and sum over r. This gives

$$(3.3) \qquad \sum_{r=o}^{n} x^r(F_{k-1}+xF_k)^r(F_k+xF_{k+1})^{n-r}$$
$$= \sum_{r,r_1,\ldots,r_k} \binom{n-r}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_{k-1}}{r_k} x^{n+r-r_k} .$$

The coefficient of x^n on the right of (3.3) is equal to

$$\sum_{r+s+t=n} \binom{r}{s}\binom{n-r}{t} F_{k-1}^{r-s} F_k^s F_k^{n-r-t} F_{k+1}^t$$
$$= \sum_{r+s\leq n} \binom{r}{s}\binom{n-r}{s} F_{k-1}^{r-s} F_k^{2s} F_{k-1}^{n-r-s} = u_n^{(k)} \quad ,$$

say. Then

$$\sum_{n=o}^{\infty} u_n^{(k)}x^n = \sum_{r,s=o}^{\infty} \binom{r}{s}F_{k-1}^{r-s} F_k^{2s} x^{r+s}(1-F_{k+1}x)^{-s-1}$$
$$= \sum_{s=o}^{\infty} F_k^{2s} x^{2s}(1-F_{k-1}x)^{-s-1}(1-F_{k+1}x)^{-s-1}$$
$$= \frac{1}{(1-F_{k-1}x)(1-F_{k+1}x)-F_k^2x^2} = \frac{1}{1-(F_{k-1}+F_{k+1})x+(-1)^kx^2} .$$

But

$$1-(F_{k-1}+F_{k+1})x+(-1)^k x^2 = 1-(\alpha^k+\beta^k)x+(\alpha\beta)^k x^2 = (1-\alpha^k x)(1-\beta^k x)$$

and

$$\frac{1}{(1-\alpha^k x)(1-\beta^k x)} = \frac{1}{\alpha^k-\beta^k}\left(\frac{\alpha^k}{1-\alpha^k x} - \frac{\beta^k}{1-\beta^k x}\right).$$

It follows that

$$u_n^{(k)} = \frac{\alpha^{nk+k} - \beta^{nk+k}}{\alpha^k - \beta^k} = \frac{F_{nk+k}}{F_k} .$$

Comparison with (3.4) yields

$$\text{(3.5)} \qquad \operatorname{tr}(A_{n+1}^k) = \frac{F_{nk+k}}{F_k} .$$

4. We now return to the characteristic polynomial

$$f_{n+1}(x) = \det(xI - A_{n+1}) .$$

If we denote the characteristic values by $\lambda_o, \lambda_1, \ldots, \lambda_n$, we have

$$\frac{f'_{n+1}(x)}{f_{n+1}(x)} = \sum_{j=o}^{n} \frac{1}{x-\lambda_j} = \sum_{k=o}^{\infty} x^{-k-1} \sum_{j=o}^{n} \lambda_j^k = \sum_{k=o}^{\infty} x^{-k-1} \operatorname{tr}(A_{n+1}^k)$$

$$= \sum_{k=o}^{\infty} x^{-k-1} \frac{\alpha^{nk+k}-\beta^{nk+k}}{\alpha^k - \beta^k} = \sum_{k=o}^{\infty} x^{-k-1} \sum_{j=o}^{n} \alpha^{jk}\beta^{(n-j)k}$$

$$= \sum_{j=o}^{n} \frac{1}{x-\alpha^j\beta^{n-j}} .$$

It follows that

$$\text{(4.1)} \qquad f_{n+1}(x) = \prod_{j=o}^{n} (x-\alpha^j\beta^{n-j})$$

and therefore the characteristic values of A_{n-1} are the numbers

$$\text{(4.2)} \qquad \alpha^n, \ \alpha^{n-1}\beta, \ \ldots, \ \alpha\beta^{n-1}, \ \beta^n .$$

We shall now show that

$$(4.3) \qquad \prod_{j=o}^{n} (x-\alpha^j\beta^{n-j}) = \sum_{r=o}^{n+1} (-1)^{r(r+1)/2} F_{n+1,r}\, x^{n+1-r} ,$$

with $F_{n+1,r}$ defined by (1.3).

To prove (4.3) we make use of the familiar identity

$$(4.4) \qquad \prod_{j=o}^{n-1} (1-q^jx) = \sum_{r=o}^{n} (-1)^r q^{r(r-1)/2} \left[{n \atop r}\right] x^r ,$$

where

$$(4.5) \qquad \left[{n \atop r}\right] = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-r+1})}{(1-q)(1-q^2)\ldots(1-q^r)} .$$

If we replace q by β/α we find that

$$\left[{n \atop r}\right] \rightarrow \alpha^{r^2-nr} F_{nr} .$$

Thus (4.4) becomes

$$\prod_{j=o}^{n-1} (1-\alpha^{-j}\beta^jx) = \sum_{r=o}^{n} (-1)^r \alpha^{r(r+1)/2\ -nr} \beta^{r(r-1)/2} F_{n,r}\, x^r .$$

Now replace x by $\alpha^{n-1}x$. Then

$$\prod_{j=o}^{n-1} (1-\alpha^{n-j-1}\beta^jx) = \sum_{r=o}^{n} (-1)^r(\alpha\beta)^{r(r-1)/2} F_{n,r}\, x^r$$

$$= \sum_{r=o}^{n} (-1)^{r(r+1)/2} F_{n,r}\, x^r .$$

Replacing x by x^{-1} we get

$$\prod_{j=o}^{n-1} (x-\alpha^{n-j-1}\beta^j) = \sum_{r=o}^{n} (-1)^{r(r+1)/2} F_{n,r}\, x^{n-r} .$$

This evidently proves (4.3).

Incidentally we have proved the stronger result that (4.3) holds when α, β are any numbers such that $\alpha\beta = -1$ and

$$F_{n,r} = \frac{(\alpha^n-\beta^n)(\alpha^{n-1}-\beta^{n-1})\ldots(\alpha^{n-r+1}-\beta^{n-r+1})}{(\alpha-\beta)(\alpha^2-\beta^2)\ldots(\alpha^r-\beta^r)} .$$

If we now compare (4.3) with (4.1) it is clear that we have proved (1.4).

5. It is of interest to note that the particular characteristic values α^n, β^n can be predicted directly as follows. We have

$$\sum_s \binom{r}{n-s} \alpha^s = \alpha^n \sum_s \binom{r}{n-s} \alpha^{s-n}$$

$$= \alpha^n(1+\alpha^{-1})^r = \alpha^{n-r}(\alpha+1)^r = \alpha^{n+r} .$$

This shows that $[1, \alpha, \ldots, \alpha^n]$ is the characteristic vector corresponding to α^n. Similarly $[1, \beta, \ldots, \beta^n]$ is the characteristic vector corresponding to β^n.

However it is not evident how to find the remaining characteristic vectors when $n > 1$. We can for example show that there are no other characteristic vectors of the type $[1, \gamma, \ldots, \gamma^n]$. Indeed assume that

$$\sum_s \binom{r}{n-s} \gamma^s = \lambda\gamma^r \quad (r = 0, 1, \ldots, n) . \tag{5.1}$$

Then since

$$\sum_s \binom{r}{n-s} \gamma^s = \gamma^n(1+\gamma^{-1})^r = \gamma^{n-r}(\gamma+1)^r ,$$

it follows from (5.1) that

$$\gamma^{-2r}(\gamma+1)^r = \lambda\gamma^{-n} \quad (r = 0, 1, \ldots, n) .$$

Since the right side is independent of r we must have $\gamma + 1 = \gamma^2$, so that $\gamma = \alpha$ or β.

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A RECURSIVE OPERATION ON TWO-DIGIT INTEGERS

CHARLES W. TRIGG
San Diego, California

Let d be the sum of the digits of the integer S, and $S_1 + d_1 = S_2$, $S_2 + d_2 = S_3$, ..., $S_k + d_k = S_{k+1}$, with each S being reduced modulo 100. Since there are only 100 two-digit integers (with initial zeros being permitted), this operation must eventually repeat one or more integers with the production of one or more closures of the series into loops. Indeed, the 100 integers fall into four groups of branched chains of 2, 11, 25 and 62 integers terminating in loops of 1, 11, 13 and 20 members, respectively. The loops consist of the integers between the asterisks (**) in the following series.

(1) 86*00*00

(2) *52 59 73 83 94 07 14 19 29 40 44*52

(3) 20 22 26 34 41 46 56 67 80 88 }
75 87*02 } 04 08 16 23 28

38 49 62 70 77 91 01*02

(4)
64 74 85 98 }
31 35 43 50 55 65 76 89 } 06 12 } 15 21
53 61 68 82 92 03 }

24 30 33 39 51 57 69 84 96 } 11 }
42 48 60 66 78 93 05 10 } 97 } 13*17 25 32

37 47 58 71 79 95 09 18 27 36 45 54 63 72
81 90 99*17.

Thus all the numbers can be generated from the eleven integers 86, 52, 20, 75, 64, 31, 53, 24, 42, 97 and 47.

If the S's are reduced modulo 10, two groups are formed — a loop of one with one attachment, 5*0*0, and a loop of four with four attachments, $\begin{matrix}1-2-4-7\\ 3-6-8-9\end{matrix}$, so that it takes five digits to generate the set.

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THE CHARACTERISTIC POLYNOMIAL OF THE GENERALIZED SHIFT MATRIX

V.E. HOGGATT, JR. and A.P. HILLMAN
San Jose State College and University of Santa Clara

T. A. Brennan [3] obtained the characteristic polynomial for the k by k matrix $P_k = [P_{ij}]$ with the binomial coefficient $\binom{i-1}{k-j}$ as the element P_{ij} in the i-th row and j-th column. See [6] and [7] for special cases. L. Carlitz [5] used another method involving some very interesting identities to achieve the same result. In this paper we find the characteristic polynomial for a generalization of the P_k.

Let F be a field of characteristic zero, let p and q be in F, and let

$$(1) \qquad y_{n+2} = qy_n + py_{n+1}, \quad q \neq 0$$

be a second order homogeneous linear difference equation over F. We restrict n to be an integer in (1). Let a and b be the zeros of the auxiliary polynomial

$$x^2 - px - q = (x - a)(x - b)$$

of (1). We deal only with the case in which (1) is <u>ordinary</u> in the sense of R. F. Torretto and J. A. Fuchs [4], i.e., we assume that either $a = b$ or $a^n \neq b^n$ for all positive integers n. Using the notation of E. Lucas [1] we let U_n be the solution $(a^n - b^n)/(a - b)$ of (1). Also we use the notation of [3] and [4] for the generalized binomial coefficient

$$\begin{bmatrix} m \\ j \end{bmatrix} = \frac{U_m U_{m-1} \cdots U_{m-j+1}}{U_1 U_2 \cdots U_j}, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1$$

of D. Jarden [2].

Jarden showed that the product z_n of the n-th terms of k-1 solutions of (1) satisfies

$$(2) \qquad \sum_{h=0}^{k} (-1)^h \begin{bmatrix} k \\ h \end{bmatrix} (-q)^{h(h-1)/2} z_{n-h} = 0 .$$

Torretto and Fuchs showed that (1) is ordinary if and only if the "sequences" (i.e., functions of the integral variable n)

$$z_n(i, k) = U_n^{k-i} U_{n+1}^{i-1} \; ; \; i = 1, 2, \ldots, k \tag{3}$$

form a basis for the vector space of all sequences satisfying (2).

Let $C_n = C_n(k)$ be the k-dimensional column vector with $z_n(i, k)$ the element in the i-th row and let $S = S(k)$ be the k by k matrix $[s_{ij}]$ with

$$s_{ij} = \binom{i-1}{k-1} q^{k-j} p^{i+j-k-1} \quad . \tag{4}$$

We show below that S has the shifting property $SC_n = C_{n+1}$ and that the characteristic polynomial of S is the auxiliary polynomial

$$f(X) = \sum_{h=0}^{k} (-1)^h \left[{k \atop h}\right] (-q)^{h(h-1)/2} X^{n-h} \tag{5}$$

of the difference equation (2).

Using (3) and (1) we have,

$$z_{n+1}(i, k) = U_{n+1}^{k-i} (qU_n + pU_{n+1})^{i-1} = \sum_{h=0}^{i-1} \binom{i-1}{h} q^h p^{i-1-h} U_n^h U_{n+1}^{k-1-h} \quad . \tag{6}$$

Letting $h = k - j$ in (6) and reversing the order of the terms leads to

$$z_{n+1}(i, k) = \sum_{j=k+1-i}^{k} \binom{i-1}{k-j} q^{k-j} p^{i+j-k-1} U_n^{k-j} U_{n+1}^{j-1} \quad . \tag{7}$$

Using (4) and the fact that $\binom{m}{r} = 0$ for $m < r$, we can rewrite (7) as

$$z_{n+1}(i, k) = \sum_{j=1}^{k} s_{ij} z_n(j, k) \quad . \tag{8}$$

Let $T = [t_{ij}]$ be the matrix $f(S)$, where $f(X)$ is as defined in (5). In matrix notation (8) is $SC_n = C_{n+1}$. By induction it follows that $S^i C_n = C_{n+i}$. Since the elements of the C_n in a fixed position satisfy the difference equation (2), so do the vectors C_n. This is equivalent to $TC_n = 0$ for all integers n, i.e.,

(9) $$t_{i1} z_n(1, k) + t_{i2} z_n(2, k) + \ldots + t_{ik} z_n(k, k) = 0$$

for all n. Since it was proved in [3] that the sequences

$$z_n(1, k), \ \ldots, \ z_n(k, k)$$

are linearly independent, (9) implies that each $t_{ij} = 0$. Hence $T \equiv 0$ and we have shown that S satisfies $f(X) = 0$. Let $g(X) = 0$ be the monic polynomial equation of least degree over K for which $g(S) = 0$. Then $g(X)$ divides $f(X)$.

Clearly the last column of S is C_1. Since only the last column of S^n is involved in finding the last column of S^{n+1} by the formula $S \cdot S^n = S^{n+1}$ and since $SC_n = C_{n+1}$, it follows by induction that the last column of S^n is C_n. In particular, the element in the first row and k-th column of S^n is $z_n(1, k)$, which we shorten to z_n in what follows. By definition

$$z_n = U_n^{k-1} = \left[(a^n - b^n)/(a - b)\right]^{k-1} \quad .$$

Expanding the binomial $(a^n - b^n)^{k-1}$ we see that

(10) $$z_n = c_1(a^{k-1})^n + c_2(a^{k-2}b)^n + \ldots + c_k(b^{k-1})^n$$

with each c_h different from zero.

Since $g(S) = 0$, the elements in the S^n in a fixed position, and in particular the z_n, satisfies the difference equation for which $g(x)$ is the auxiliary polynomial. Jarden showed in [5] that the zeros of $f(x)$ are

(11) $$a^{k-1}, \ a^{k-2}b, \ a^{k-3}b^2, \ \ldots, \ b^{k-1} \quad .$$

The zeros of $g(x)$ thus are some or all of these zeros of $f(x)$. If $f(x) \neq g(x)$, then $g(x)$ has lower degree than $f(x)$ and so

$$z_n = d_1 r_1^n + d_2 r_2^n + \ldots + d_m r_m^n$$

with $m < k$, the d_i in F, and each r_i one of the elements of (11). Since no c_h in (10) is zero, this would mean that (10) is not unique and hence that the sequences $(a^h b^{k-1-h})^n$, $0 \leq h \leq k-1$, are linearly dependent. As in [4], this would contradict the fact that (1) is ordinary. Hence $f(X) \equiv g(X)$. Since the characteristic polynomial $\phi(X)$ of S is monic, of degree k, and a multiple of $g(X)$, $\phi(X)$ must also be $f(X)$ and (11) gives the characteristic values of S. This completes the proof.

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(Continued from page 134)

A more extensive analysis of the generated compositions which yield Fibonacci numbers will be jointly attempted by Dr. Hoggatt and the author in a subsequent paper. In addition, the author is planning to submit some papers in the future, which will furnish some original models and theorems connected with Fibonacci numbers and their properties. These models and theorems have been incorporated in part in the author's doctoral thesis, which has been cited as a reference in this article.

SUMMATION FORMULAE FOR MULTINOMIAL COEFFICIENTS

SELMO TAUBER
Portland State College, Portland, Oregon

1. INTRODUCTION

In [1] we have given some historical background to the multinomial coefficients and proved some of the basic summation formulae. More of the summation formulae can be found in [2]. In this paper we shall prove additional relations involving multinomial coefficients. Some of these can be considered as generalizations of corresponding formulae for binomial coefficients. We shall refer to [3] for these formulae.

2. FIRST SET OF FORMULAE

In order to simplify the notation used in [1], at least for the proof, we shall write

$$N!/\prod_{s=1}^{n} k_s! = \binom{N}{k_1, k_2, \ldots, k_n}, \quad \text{with,} \quad \sum_{s=1}^{n} k_s = N ,$$

and, for, $k_1+k_2+\ldots+k_n = N+1$, we shall have the simplified notation

$$\binom{N}{k_1, k_2, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_n} = \left[N, (k_j-1), k_n\right] .$$

Under these conditions equation (6) of [1] can be written

$$(1) \qquad \sum_{j=1}^{n} \left[N, (k_j-1), k_n\right] = \left[N+1, k_n\right] .$$

For $0 \leqq p \leqq n$, we can write (1) in the form

$$\sum_{j=1}^{p-1} \left[N, (k_j-1), k_p, k_n\right] + \left[N, k_p-1, k_n\right] + \sum_{j=p+1}^{n} \left[N, k_p, (k_j-1), k_n\right]$$
$$= \left[N+1, k_p, k_n\right] .$$

and similar relations for $N-1, N-2, \ldots, N-q, \ldots, N-k_p$, thus,

$$\sum_{j=1}^{p-1} \left[N-1, (k_j-1), k_p-1, k_n\right] + \left[N-1, k_p-2, k_n\right] + \sum_{j=p+1}^{n} \left[N-1, k_p-1, (k_j-1), k_n\right]$$
$$= \left[N, k_p-1, k_n\right]$$

. . . .

$$\sum_{j=1}^{p-1}\left[N-q, (k_j-1), k_p-q, k_n\right]+\left[N-q, k_p-q-1, k_n\right]+\sum_{j=p+1}^{n}\left[N-q, k_p-q, (k_j-1), k_n\right]$$

$$= \left[N-q+1, k_p-q, k_n\right]$$

. . . .

$$\sum_{j=1}^{p-1}\left[N-k_p, (k_j-1), 0, k_n\right]+\sum_{j=p+1}^{n}\left[N-k_p, 0, (k_j-1), k_n\right] = \left[N-k_p+1, 0, k_n\right] .$$

By adding the first q equations and simplifying we obtain

$$\sum_{\alpha=0}^{q}\left[\sum_{j=1}^{p-1}\left[N-\alpha, (k_j-1), k_p-\alpha, k_p\right]+\sum_{j=p+1}^{n}\left[N-\alpha, k_p-\alpha, (k_j-1), k_n\right]\right] =$$

$$= \left[N+1, k_p, k_n\right] - \left[N-q, k_p-q-1, k_n\right] ,$$

or, using the classical notation,

$$(2)\quad \sum_{\alpha=0}^{q}\left[\sum_{j=1}^{p-1}\binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_p-\alpha, \ldots, k_n} + \sum_{j=p+1}^{n}\binom{N-\alpha}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n}\right] =$$

$$= \binom{N+1}{k_1, k_2, \ldots, k_p, \ldots, k_n} - \binom{N-q}{k_1, k_2, \ldots, k_p-q, \ldots, k_n} .$$

For $q = k_p$, we obtain

$$(3)\quad \sum_{\alpha=0}^{k_p}\left[\sum_{j=1}^{p-1}\binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_p-\alpha, \ldots, k_n} + \sum_{j=p+1}^{n}\binom{N-\alpha}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n}\right] =$$

$$= \binom{N+1}{k_1, k_2, \ldots, k_p, \ldots, k_n} .$$

It will be noted that in both (2) and (3) the sum is independent of p, thus by summing on p we obtain

$$(4)\quad \sum_{p=1}^{n}\ \sum_{\alpha=0}^{k_p}\left[\sum_{j=1}^{p-1}\binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_p-\alpha, \ldots, k_n} + \sum_{j=p+1}^{n}\binom{N-\alpha}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n}\right] =$$

$$= n\binom{N+1}{k_1, k_2, \ldots, k_s, \ldots, k_n} .$$

For n = 2, (2) and (3) reduce to (3) and (4) of [3], p. 246.

3. SECOND SET OF FORMULAE

Consider the formulae leading to (2) and (3). If we multiply the first relation by +1, the second by -1, ..., the (q+1)-th relation by $(-1)^q$, etc., ... we obtain

$$(5)\quad \sum_{\alpha=0}^{q}\left[(-1)^{\alpha}\sum_{j=1}^{p-1}\binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_p-\alpha, \ldots, k_n} + \sum_{j=p+1}^{n}\binom{N-\alpha}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n}\right] =$$

$$2\sum_{\alpha=1}^{q}(-1)^{\alpha}\binom{N-\alpha+1}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_n} + \binom{N-1}{k_1, k_2, \ldots, k_p, \ldots, k_n} +$$

$$(-1)^{q+1}\binom{N-q}{k_1, k_2, \ldots, k_p-q-1, \ldots, k_n} ,$$

and,

$$(6)\quad \sum_{\alpha=0}^{k_p}\left[(-1)^{\alpha}\sum_{j=1}^{p-1}\binom{N-\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_p-\alpha, \ldots k_n} + \sum_{j=p+1}^{n}\binom{N-\alpha}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n}\right] =$$

$$2\sum_{\alpha=1}^{k_p}\binom{N-\alpha+1}{k_1, k_2, \ldots, k_p-\alpha, \ldots, k_n} + \binom{N+1}{k_1, k_2, \ldots, k_p, \ldots, k_n} .$$

(5) and (6) for $n = 2$ reduce to (7) of [3], p. 247.

4. THIRD SET OF FORMULAE

Using the notation of section 2 we write for

$$\sum_{s=1}^{n} k_s = N+1, \quad k_p = 0 ,$$

$$\sum_{j=1}^{p-1} \left[N, (k_j-1), 0, k_n\right] + \sum_{j=p+1}^{n} \left[N, 0, (k_j-1), k_n\right] = \left[N+1, 0, k_n\right]$$

$$\sum_{j=1}^{p-1} \left[N+1, (k_j-1), 1, k_n\right] + \left[N+1, 0, k_n\right] + \sum_{j=p+1}^{n} \left[N+1, 1, (k_j-1), k_n\right] = \left[N+2, 1, k_n\right]$$

. . . .

$$\sum_{j=1}^{p-1} \left[N+q, (k_j-1), q, k_n\right] + \left[N+q, q-1, k_n\right] + \sum_{j=p+1}^{n} \left[N+q, q, (k_j-1), k_n\right] = \left[N+q+1, q, k_n\right]$$

. . . .

$$\sum_{j=1}^{p-1} \left[N+h, (k_j-1), h, k_n\right] + \left[N+h, h-1, k_n\right] + \sum_{j=p+1}^{b} \left[N+h, h, (k_j-1), k_n\right] = \left[N+h+1, h, k_n\right].$$

By adding and simplifying we obtain

$$\sum_{\alpha=0}^{q} \left[\sum_{j=1}^{p-1} \left[N+\alpha, (k_j-1), \alpha, k_n\right] + \sum_{j=p+1}^{n} \left[N+\alpha, \alpha, (k_j-1), k_n\right] \right] = \left[N+q, q, k_n\right] ,$$

or, using the classical notation

$$\text{(7)} \qquad \sum_{\alpha=0}^{q} \left[\sum_{j=1}^{p-1} \binom{N+\alpha}{k_1, k_2, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, \alpha, \ldots, k_n} + \sum_{j=p+1}^{n} \binom{N+\alpha}{k_1, k_2, \ldots, \alpha, \ldots, k_{j-1}, k_j-1, k_{j+1}, \ldots, k_n} \right] =$$

$$\binom{N+q+1}{k_1, k_2, \ldots, k_{p-1}, q+1, k_{p+1}, \ldots, k_n} ,$$

and similarly

$$(8)\quad \sum_{\alpha=q}^{h}\left[\sum_{j=1}^{p-1}\binom{N+\alpha}{k_1, k_2, \dots, k_{j-1}, k_j-1, k_{j+1}, \dots, \alpha, \dots, k_n} + \sum_{j=p+1}^{n}\binom{N+\alpha}{k_1, k_2, \dots, \alpha, \dots, k_{j-1}, k_j-1, k_{j+1}, \dots, k_n}\right] =$$

$$\binom{n+h+1}{k_1, k_2, \dots, k_{p-1}, h+1, k_{p+1}, \dots, k_n} - \binom{n+q-1}{k_1, k_2, \dots, k_{p-1}, q-1, k_{p+1}, \dots, k_n} .$$

For $n = 2$ (8) reduces to (11) of [3] p. 248.

5. FOURTH SET OF FORMULAE

(8) of [1] can be simplified in writing by introducing the notation

$$(9)\quad \sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2}\cdots\sum_{j_{n-1}=0}^{k_{n-1}} = \left(\prod_{s=1}^{n-1}\cdot\sum_{j_s=0}^{k_s}\right) ,$$

where Π operates on the operator Σ. Under these conditions (8) of [1] can be written for,

$$\sum_{s=1}^{n} k_s = p+q, \quad \sum_{s=1}^{n} j_s = p \quad ,$$

$$(10)\quad \left(\prod_{s=1}^{n-1}\cdot\sum_{j_s=0}^{k_s}\right)\binom{p}{j_1, j_2, \dots, j_n}\binom{q}{k_1-j_1, k_2-j_2, \dots, k_n-j_n} = \binom{p+q}{k_1, k_2, \dots, k_n} .$$

Let us substitute $p+r$ for p in (10), we obtain for $\binom{p}{j_1, j_2, \dots, j_n}\cdot$

$$\binom{p+r}{j_1, j_2, \dots, j_n} = \left(\prod_{t=1}^{n-1}\cdot\sum_{h_t=0}^{j_t}\right)\binom{p}{j_1-h_1, j_2-h_2, \dots, j_n-h_n}\binom{r}{h_1, h_2, \dots, h_n} ,$$

with

$$\sum_{i=1}^{n} h_i = r, \quad \sum_{i=1}^{n} j_i = p+r, \quad \sum_{i=1}^{n} k_i = p+q+r \quad ,$$

so that substituting into (10) we obtain

$$(11) \quad \left(\prod_{s=1}^{n-1} \cdot \sum_{j_s=0}^{k_s}\right)\left(\prod_{t=1}^{n-1} \cdot \sum_{h_t=0}^{j_t}\right)\binom{p}{j_1-k_1, j_2-k_2, \ldots, j_n-k_n}\binom{q}{k_1-j_1, \ldots, k_n-j_n} \cdot$$

$$\cdot \binom{r}{h_1, h_2, \ldots, h_n} = \binom{p+q+r}{k_1, k_2, \ldots, k_n} \quad .$$

More generally as can be proved by induction we can write

$$(12) \quad \prod_{j=1}^{m-1} \cdot \left(\prod_{i=1}^{n-1} \cdot \sum_{k_{j+1,i}=0}^{k_{i,j}}\right) \prod_{j=1}^{m-1} \binom{q_j}{k_{j,1}-k_{j+1,1}, k_{j,2}-k_{j+1,2}, \ldots, k_{j,n}-k_{j+1,n}} .$$

$$\circ \binom{q_m}{k_{m,1}, k_{m,2}, \ldots, k_{m,n}} = \binom{q_1+q_2+\ldots+q_m}{k_{11}, k_{12}, \ldots, k_{1n}} \quad ,$$

where,

$$\sum_{t=1}^{n} (k_{j,t} - k_{j+1,t}) = q_j, \qquad \text{for,} \qquad j=1, 2, \ldots, n \quad .$$

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EXPANSION OF ANALYTIC FUNCTIONS IN TERMS INVOLVING LUCAS NUMBERS OR SIMILAR NUMBER SEQUENCES

PAUL F. BYRD
San Jose State College, San Jose, California

1. INTRODUCTION

In a previous article [1], certain available results concerning polynomial expansions were applied in order to illustrate a simple general technique for obtaining the coefficients $\beta_n(\alpha)$ in the series

$$(1.1) \qquad f(\alpha) = \sum_{n=0}^{\infty} \beta_n(\alpha) F_{n+1} ,$$

where $f(\alpha)$ is an "arbitrary" analytic function of α, and F_n are Fibonacci numbers. The same method may also be applied to develop series expansions of the form

$$(1.2) \qquad f(\alpha) = \frac{1}{2} A_o(\alpha) L_o + \sum_{n=1}^{\infty} A_n(\alpha) L_n ,$$

where L_n are the Lucas numbers ($L_o = 2$, $L_1 = 1$; $L_{n+2} = L_{n+1} + L_n$ for $n = 0, 1, \ldots$). Such series, which can be derived as special cases of more general expansions, are of use when one desires to make some given function f serve as a generating function* of the Fibonacci or Lucas sequence — two famous sequences whose many number-theoretical properties are of primary concern to this journal.

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*In general, any infinite series of the form

$$f(\alpha) = G\left[\alpha; \{y_n\}\right] = \sum_{n=0}^{\infty} g_n(\alpha)\, y_n$$

is called a generating function of a number sequence $\{y_n\}$ if $g_n(\alpha)$ are linearly independent functions of α. The familiar type, when $g_n(\alpha)$ is taken to be α^n or $\alpha^n/n!$, is a special case of the more general definition.

The main purpose of the present article is to review one technique for finding the coefficients of (1.1) or (1.2), and to give explicit expansions of a variety of transcendental functions in terms involving Lucas numbers. Another objective is to point out how certain extensions might be made to considerations involving similar sequences of integers.

2. EXPANSIONS IN TERMS OF GEGENBAUER POLYNOMIALS

We begin by first seeking a formal series expansion expressed by

$$(2.1)\qquad f(2ax) = D_{o,k}(2a)\, C_{o,k}(x) + \sum_{m=1}^{\infty} D_{m,k}(2a)\, C_{m,k}(x) \quad ,$$

where the functions* $C_{m,k}(x)$, the well-known Gegenbauer polynomials [2], are given by $C_{o,k}(x) = 1$, and

$$(2.2)\quad C_{m,k}(x) = \frac{1}{\Gamma(k)} \sum_{r=o}^{[m/2]} (-1)^r \frac{\Gamma(m-r+k)}{\Gamma(m-r+1)} \binom{m-r}{r} (2x)^{m-2r}, \qquad \text{(for } k > -1/2,\ k \neq 0)$$

$$= \sum_{r=o}^{[m/2]} \frac{(-1)^r}{m-r} \binom{m-r}{r} (2x)^{m-2r}, \qquad \text{(for } k = 0,\ m > 0)$$

These polynomials satisfy the orthogonality relation

$$(2.3)\quad \int_{-1}^{1} \frac{(1-x^2)^k}{\sqrt{1-x^2}}\, C_{m,k}(x) C_{p,k}(x) dx = \frac{2\pi \Gamma(2k+m)\,\delta_{mp}}{4^k (m+k)\Gamma(m+1)[\Gamma(k)]^2} \qquad (k \neq 0),$$

$$= 2\pi\delta_{mp}/m^2 \quad \text{for } k = 0,\ m \neq 0,$$

with δ_{mp} being equal to 0 when $m \neq p$ and to 1 when $m = p$. If we multiply both sides of (2.1) by $(1 - x^2)^{k-1/2} C_{p,k}(x)$ and then integrate from -1 to 1, we obtain (upon setting $x = \cos \gamma$ and making use of (2.3)) the coefficient formulas

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*We find it convenient to write $C_{m,k}(x)$ instead of using the standard notation $C_m^k(x)$.

$$(2.4)\begin{cases} D_{m,k}(2a) = \dfrac{4^k(m+k)\,\Gamma(m+1)[\Gamma(k)]^2}{2\,\pi\,(2k+m)} \displaystyle\int_0^\pi \sin^{2k}\gamma\, f(2a\cos\gamma)C_{m,k}(\cos\gamma)d\gamma \\ \qquad\qquad (\text{for } m = 0, 1, \ldots;\ k \neq 0) \\ D_{m,o}(2a) = \dfrac{m^2}{2\pi} \displaystyle\int_0^\pi f(2a\cos\gamma)C_{m,o}(\cos\gamma)d\gamma (\text{for } m \neq 0,\ k = 0)\ , \\ D_{o,k}(2a) = \dfrac{\Gamma(k+1)}{\sqrt{\pi}\,\Gamma\left(k+\frac{1}{2}\right)} \displaystyle\int_0^\pi \sin^{2k}\gamma\ f(2a\cos\gamma)d\gamma\ . \end{cases}$$

Now it will be seen that both (1.1) and (1.2) can easily be obtained as special cases of the more general series (2.1).

3. RELATIONSHIP BETWEEN GEGENBAUER POLYNOMIALS AND THE SO-CALLED FIBONACCI AND LUCAS POLYNOMIALS

The Gegenbauer (ultraspherical) polynomials $C_{m,k}(x)$ of degree m and order k satisfy the recurrence formula, given in reference [2],

$$(3.1)\quad (m+2)C_{m+2,k}(x) = 2(m+1+k)xC_{m+1,k}(x) - (m+2k)C_{m,k}(x)\ ,$$

which reduces to

$$(3.2)\qquad C_{m+2,1}(x) - 2xC_{m+1,1}(x) + C_{m,1}(x) = 0$$

when $k = 1$. Relation (3.2), with conditions $C_{0,1}(x) = 1$ and $C_{1,1}(x) = 2x$, is the well-known recurrence formula defining the Chebyshev polynomials $U_m(x)$ of the second kind, and one may thus write

$$(3.3)\quad C_{m,1}(x) = U_m(x) = U_m(\cos\gamma) = \frac{\sin(m+1)\gamma}{\sin\gamma}\ , \quad (x = \cos\gamma)\ .$$

When $k = 0$, formula (3.1) becomes

$$(3.4)\quad (m+2)C_{m+2,0}(x) - 2(m+1)xC_{m+1,0}(x) - mC_{m,0}(x) = 0\ ,$$

so that, since $C_{o,o}(x) = 1$, $C_{1,o}(x) = 2x$, and

$$(3.5)\quad C_{m,o}(x) = \frac{2}{m}\,T_m(x) = \frac{2}{m}\,T_m(\cos\gamma) = \frac{2\cos m\gamma}{m}\ , \ (x = \cos\gamma\ ,\ m \neq 0),$$

we have

$$T_{m+2}(x) - 2x\,T_{m+1}(x) + T_m(x) = 0 \quad , \tag{3.6}$$

which is the relation satisfied by Chebyshev polynomials $T_m(x)$ of the first kind.

Now, as pointed out in references [1] and [3], the Fibonacci polynomials $\phi_m(x)$ and the Lucas polynomials $\lambda_m(x)$ are simply modified Chebyshev polynomials having the relationship

$$\phi_{m+1}(x) = (-i)^m U_m(ix), \quad \lambda_m(x) = 2(-i)^m T_m(ix), \quad (i = \sqrt{-1}) \quad . \tag{3.7}$$

In view of (3.3) and (3.5), we have

$$\phi_{m+1}(x) = (-i)^m C_{m,1}(ix), \quad \lambda_m(x) = m(-i)^m C_{m,o}(ix), \quad (m \geq 1) , \tag{3.8}$$

thus showing that the Fibonacci and Lucas polynomials are related to modified Gegenbauer polynomials for the special cases of $k = 1$ and $k = 0$.

Moreover, the Fibonacci and Lucas numbers are particular values of (3.8) when $x \equiv 1/2$; that is

$$\begin{cases} F_1 = C_{o,1}(i/2) = 1, & F_{m+1} = (-i)^m C_{m,1}(i/2) \\ L_o = 2C_{o,o}(i/2) = 2, & L_m = m(-i)^m C_{m,o}(i/2) \end{cases} \quad (m \geq 1) \tag{3.9}$$

With the above relationships, the series expansions (1.1) or (1.2) for a given function f in terms involving Fibonacci or Lucas numbers can be obtained from (2.1) by taking

$$x = i/2 \quad \text{and} \quad 2a = -2i\alpha \quad . \tag{3.10}$$

Thus, we have the series

$$\begin{cases} f(\alpha) = \frac{1}{2} D_{o,o}(-2i\alpha) L_o + \sum_{m=1}^{\infty} \frac{i^m}{m} D_{m,o}(-2i\alpha) L_m \quad , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = \sqrt{-1}) \\ f(\alpha) = \sum_{m=o}^{\infty} i^m D_{m,1}(-2i\alpha) F_{m+1} \end{cases} \tag{3.11}$$

where, from (2.4), (3.3), and (3.5), the coefficients may be expressed by the definite integrals

$$(3.12)\quad \begin{cases} D_{o,o}(-2i\alpha) = \dfrac{1}{\pi}\displaystyle\int_o^{\pi} f(-2i\alpha\cos\gamma)d\gamma = A_o(\alpha)\ , \\[2ex] D_{m,o}(-2i\alpha) = \dfrac{m}{\pi}\displaystyle\int_o^{\pi} f(-2i\alpha\cos\gamma)\cos m\gamma d\gamma = (-i)^m mA_m(\alpha), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (m \geq 1) \\[2ex] D_{m,1}(-2i\alpha) = \dfrac{2}{\pi}\displaystyle\int_o^{\pi} f(-2i\alpha\cos\gamma)\sin\gamma\,\sin(m+1)\gamma d\gamma = (-i)^m\beta_m(\alpha) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (m \geq 0)\ . \end{cases}$$

4. EXAMPLES

Since many specific examples were presented in reference [1] for certain series in terms of Fibonacci numbers, we shall now only give some explicit expansions in terms involving Lucas numbers.

Consider first the function

$$(4.1)\qquad f(\alpha) = e^{\alpha}\ ,$$

so that from (3.12) we have

$$(4.2)\qquad D_{o,o}(-2i\alpha) = \frac{1}{\pi}\int_o^{\pi} e^{-2i\alpha\cos\gamma}\,d\gamma = J_o(-2\alpha) = J_o(2\alpha)\ ,$$

and

$$(4.3)\qquad D_{m,o}(-2i\alpha) = \frac{m}{\pi}\int_o^{\pi} e^{-2i\alpha\cos\gamma}\cos m\gamma\,d\gamma = m(-i)^m J_m(2\alpha)\ ,$$

where J_m are Bessel functions of order m [4]. (Evaluation of the above integrals, as well as others to follow, was made by use of tables and formulas in [2], [4], and [5].) Substituting the values of (4.2) and (4.3) into (3.11) then yields the expansion

$$(4.4)\qquad e^{\alpha} = \frac{1}{2}J_o(2\alpha)L_o + \sum_{m=1}^{\infty} J_m(2\alpha)L_m\ ,$$

which converges for $0 \leq |a| < \infty$, since

$$\lim_{m \to \infty} \frac{J_{m+1}(2a)L_{m+1}}{J_m(2a)L_m} = \lim_{m \to \infty} \frac{a}{m+1} \frac{1+\sqrt{5}}{2} = 0 \tag{4.5}$$

for all finite values of a. If application is now made of the familiar relations

$$\begin{cases} \cosh a = (e^a + e^{-a})/2, & \sinh a = (e^a - e^{-a})/2, \\ \cos a = (e^{ia} + e^{-ia})/2, & \sin a = i(e^{-ia} - e^{ia})/2, \\ J_m(ia) = i^m I_m(a), & J_m(-a) = (-1)^m J_m(a), \end{cases} \tag{4.6}$$

the following series expansions* can be easily derived from (4.4):

$$\begin{cases} \sin a = \sum_{m=1}^{\infty} (-1)^m I_{2m-1}(2a) L_{2m-1}, \\ \cos a = I_o(2a) + \sum_{m=1}^{\infty} (-1)^m I_{2m}(2a) L_{2m}, \end{cases} \tag{4.7}$$

where I_m are modified Bessel functions of order m, and

$$\begin{cases} \sinh a = \sum_{m=1}^{\infty} J_{2m-1}(2a) L_{2m-1}, \\ \cosh a = J_o(2a) + \sum_{m=1}^{\infty} J_{2m}(2a) L_{2m}, \end{cases} \tag{4.8}$$

The four examples in (4.7) and (4.8) all converge for $0 \leq |a| < \infty$.

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*Although these series are apparently not found in the literature in the specific form we have given for our purposes, they are modified cases of some expansions due to Gegenbauer (e.g., see [4], pp. 368-369). Series for such functions in terms involving certain powers of Lucas numbers may also be obtained and will be presented in a later article.

Next, consider the odd function

$$f(\alpha) = \arctan \alpha \, . \tag{4.9}$$

Now

$$D_{2m,o}(-2i\alpha) = 0, \quad \text{for} \quad m = 0, 1, \ldots; \tag{4.10}$$

but

$$D_{2m-1,o}(-2i\alpha) = \frac{2m-1}{\pi} \int_0^\pi \arctan(-2i\alpha \cos\gamma)\cos(2m-1)\gamma \, d\gamma \, , \tag{4.11}$$

which can be integrated by parts to give

$$D_{2m-1,o}(-2i\alpha) = -\frac{2i\alpha}{\pi} \int_0^\pi \frac{\sin(2m-1)\gamma \sin\gamma}{1-4\alpha^2 \cos^2\gamma} \, d\gamma \quad (m = 1, 2, \ldots), \tag{4.12}$$

$$= \frac{i\alpha}{\pi} \int_0^\pi \frac{\cos 2m\gamma - \cos(2m-2)\gamma}{1-4\alpha^2 \cos^2\gamma} \, d\gamma \, ,$$

or, finally,

$$D_{2m-1,o}(-2i\alpha) = i \left(\frac{\sqrt{1-4\alpha^2} - 1}{2\alpha} \right)^{2m-1} . \tag{4.13}$$

Use of (4.10) and (4.13) in the first equation of (3.11) yields the series expansions

$$\arctan \alpha = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} \left(\frac{1 - \sqrt{1-4\alpha^2}}{2\alpha} \right)^{2m-1} \cdot L_{2m-1} \quad (\alpha \neq 0) \quad , \tag{4.14}$$

which will converge* for $0 \leq |\alpha| \leq 1/\sqrt{5}$. If we now take $\alpha = \sqrt{2} - 1$, we obtain, since $\arctan(\sqrt{2} - 1) = \pi/8$, the interesting equation

$$\pi = 8 \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} \left[\frac{(\sqrt{2} + 1)(1 - \sqrt{8\sqrt{2} - 11})}{2} \right]^{2m-1} \cdot L_{2m-1} \, . \tag{4.15}$$

---

*For $\alpha = 0$, the right-hand side of (4.14) becomes an indeterminate form, but the correct result is obtained in the limit.

Certain higher transcendental functions such as the Bessel function $J_o(a)$ can also be easily expanded in terms involving Lucas numbers. Thus, if $f(a) = J_o(a)$, we have

$$(4.16)\qquad D_{o,o}(-2ia) = \frac{1}{\pi}\int_o^{\pi} J_o(-2ia\cos\gamma)\,d\gamma = \left[I_o(a)\right]^2 ;$$

and

$$(4.17)\qquad D_{2m,o}(-2ia) = \frac{2m}{\pi}\int_o^{\pi} J_o(-2ia\cos\gamma)\cos 2m\gamma\, d\gamma = 2m\left[I_m(a)\right]^2,$$

and hence from (3.11) we obtain, since for an even function $D_{2m-1,o} = 0$, the series

$$(4.18)\qquad J_o(a) = \frac{1}{2}\left[I_o(a)\right]^2 L_o + \sum_{m=1}^{\infty} (-1)^m \left[I_m(a)\right]^2 L_{2m} \quad .$$

It can be shown in a similar manner that the expansions of Bessel functions for all even orders are given by

$$(4.19)\qquad J_{2n}(a) = \frac{1}{2}\left[I_n(a)\right]^2 L_o + \sum_{m=1}^{\infty} (-1)^{m-n} I_{m+n}(a) I_{m-n}(a)\, L_{2m} , \qquad (n = 0, 1, 2, \ldots)$$

and are convergent for $0 \leq |a| < \infty$.

A proposed problem for the reader is to show that the Bessel function $J_1(a)$ may be expressed in the form

$$(4.20)\qquad J_1(a) = \sum_{m=1}^{\infty} (-1)^m I_m(a) I_{m-1}(a)\, L_{2m-1} \quad .$$

The reader may also use the last equations in (3.11) and (3.12) to show that, in terms of Fibonacci numbers F_{2m},

$$(4.21)\qquad \arctan a = \sum_{m=1}^{\infty} (-1)^{m-1}\left[\frac{1}{2m-1} - \frac{b^2}{2m+1}\right] b^{2m-1}\, F_{2m} \quad ,$$

and whence

(4.22) $$\pi = 8 \sum_{m=1}^{\infty} (-1)^{m-1}\left[\frac{1}{2m-1} - \frac{d^2}{2m+1}\right] d^{2m-1} F_{2m} ,$$

where

(4.23) $$b = \frac{1}{2a}\left[1 - \sqrt{1-4a^2}\right], \quad d = \frac{1}{2}\left[(\sqrt{2}+1)(1 - \sqrt{8\sqrt{2}-11})\right] .$$

The results (4.21) and (4.22) can actually be obtained more readily, as indicated in the following remarks.

5. REMARKS

If one has already found the coefficients $A_n(a)$ in (1.2) for an expansion in terms of Lucas numbers, it is not necessary to carry out the integration in the last equation of (3.12) in order to obtain the coefficients $\beta_n(a)$ in (1.1) for a series in terms involving Fibonacci numbers. For, since $F_o = 0$, $L_o = 2$, and $L_n = F_{n+1} + F_{n-1}$, it is easy to show that

(5.1) $$\beta_n(a) = A_{n+1}(a) + A_{n-1}(a) ,$$

and thus that

(5.2) $$f(a) = \sum_{n=o}^{\infty} \left[A_n(a) + A_{n+2}(a)\right] F_{n+1} .$$

Expansions in terms of Fibonacci numbers or of Lucas numbers, however, are not very efficient for computing approximate values of a function. For example, to compute π correctly to 6 places using formula (4.15) requires 36 terms in comparison to 9 terms using the series

(5.3) $$\pi = 8 \sum_{m=o}^{\infty} \frac{(-1)^m}{2m+1} (\sqrt{2} - 1)^{2m+1} ,$$

which is based on a slowly convergent Maclaurin expansion. But the series

$$\pi = 16 \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \left[(\sqrt{2}+1)(\sqrt{4-2\sqrt{2}}-1) \right]^{2m+1} ,$$

based on a more rapidly convergent expansion in terms of Chebyshev polynomials, yields 6-place accuracy with only 5 terms.

As pointed out by Gould [6], if $f(\alpha)$ has a power series expansion

$$f(\alpha) = \sum_{n=0}^{\infty} \gamma_n \alpha^n , \tag{5.4}$$

then the series

$$G(\alpha) = f(a_1\alpha) + f(a_2\alpha) = \sum_{n=0}^{\infty} \gamma_n \alpha^n L_n , \tag{5.5}$$

where

$$a_1 = \frac{1+\sqrt{5}}{2}, \quad a_2 = \frac{1-\sqrt{5}}{2} , \tag{5.6}$$

will furnish a whole special class of generating functions for the Lucas sequence. Similarly, the series

$$H(\alpha) = f(a_1\alpha) - f(a_2\alpha) = \sum_{n=0}^{\infty} \sqrt{5}\, \gamma_n \alpha^n F_n \tag{5.7}$$

yields a class of generating functions for the Fibonacci sequence. It is to be noted, however, that this technique of Gould for obtaining generating functions for Lucas or Fibonacci numbers is not intended to accomplish our purpose of making some given function f serve as the generating function by the expansion (1.1) or (1.2). Clearly the functions $G(\alpha)$ and $H(\alpha)$ in (5.5) and (5.7) are not the same as the given function f.

6. CERTAIN EXTENSIONS

In reference [1], we considered the expansion of functions in terms involving numbers (those of Fibonacci) associated with modified

Gegenbauer polynomials for the special case when $k = 1$, and in the present article in terms of numbers (those of Lucas) for the case when $k = 0$. Now from the general class, there appear to be other special cases which may also prove of interest to students "devoted to the study of integers with special properties."

For instance, upon taking $k = 1/2$, one may consider the set of polynomials $R_m(x)$ defined by

$$(6.1) \quad R_m(x) = 4^m(-i)^m C_{m,1/2}(ix) = 4^m(-i)^m P_m(ix), \quad (m = 0, 1, \ldots),$$

where P_m are the Legendre polynomials. From equation (3.1) we then have the recurrence relation

$$(6.2) \qquad (m+2)R_{m+2}(x) = 4(2m+3)xR_{m+1}(x) + 16(m+1)R_m(x)$$

with $R_o(x) = 1$ and $R_1(x) = 4x$; or, more explicitly, from (2.2), we can write

$$(6.3) \qquad R_m(x) = 2^m \sum_{j=o}^{[m/2]} \binom{m}{j} \binom{2m-2j}{m} x^{m-2j},$$

which has a generating function expressed by

$$(6.4) \qquad \frac{1}{\sqrt{1-8xz-16z^2}} = \sum_{n=o}^{\infty} R_n(x)z^n, \quad (|8xz| + |16z^2| < 1).$$

Now let H_m be the sequence of numbers (which we shall call "H-numbers") obtained from $R_m(x)$ by taking $x \equiv 1/2$. Thus

$$(6.5) \qquad H_m = \sum_{j=o}^{[m/2]} \binom{m}{j} \binom{2m-2j}{m} 4^j, \quad (m \geq 0),$$

with $H_o = 1$, $H_1 = 2$, $H_2 = 14$, $H_3 = 68$, ...), and one may investigate what particular properties* these numbers might have.

---

*What, for instance, is $\lim_{m \to \infty} (H_m/H_{m+1})$? Are there any interesting identities, etc.?

By the procedure we have illustrated, one may also find expansions which would make a given function serve as a generating function for such numbers. Thus, from (2.1), (2.4), (3.10), and (6.1), we obtain the series

$$(6.6)\qquad f(a) = D_{o,1/2}(-2ia)\,H_o + \sum_{m=1}^{\infty} \frac{i^m}{4^m}\, D_{m,1/2}(-2ia)\,H_m \;,$$

where the coefficients are expressed by

$$(6.7)\quad D_{m,1/2}(-2ia) = \frac{2m+1}{2}\int_o^{\pi} \sin\gamma\; f(-2ia\cos\gamma)P_m(\cos\gamma)d\gamma (m = 0, 1, \ldots).$$

For example, if we take the analytic function

$$(6.8)\qquad f(a) = e^a$$

then

$$(6.9)\quad D_{m,1/2}(-2ia) = \frac{2m+1}{2}\int_o^{\pi} \sin\gamma\, e^{-2ia\cos\gamma}\, P_m(\cos\gamma)d\gamma$$

$$= \frac{2m+1}{2}\int_{-1}^{1} e^{-2iaz}P_m(z)dz = \frac{2m+1}{2}(-i)^m\sqrt{\frac{\pi}{a}}\,J_{m+1/2}(2a),$$

$$(m = 0, 1, 2, \ldots)$$

and hence from (6.6) we have the expansion

$$(6.10)\quad e^a = \frac{1}{2}\sqrt{\frac{\pi}{a}}\left[J_{1/2}(2a)H_o + \sum_{m=1}^{\infty}\frac{2m+1}{4^m}\, J_{m+1/2}(2a)H_m\right], \quad (a \neq 0)$$

where $J_{m+1/2}$ are Bessel functions of order half an odd integer. Other functions f may be expanded in a similar way.

The Lucas numbers, the Fibonacci numbers, and our so-called H-numbers, in terms of which we have expanded a given analytic function, are all seen to be mere special cases of a more general sequence $\{V_{m,k}\}$, where

$$(6.11)\qquad V_{m,k} = \frac{q(m,k)}{\Gamma(k)}\sum_{r=o}^{[m/2]} \frac{\Gamma(m-r+k)}{\Gamma(m-r+1)}\binom{m-r}{r} \quad (k > -1/2) \;.$$

Our three particular cases of these may be summarized as follows:

$$(6.12)\begin{cases} k = 1, & q(m,k) = 1, & \text{then } V_{m,1} = F_{m+1}, \\ k = 0, & \dfrac{q(m,k)}{\Gamma(k)} = m, & \text{then } V_{m,o} = L_m, \quad (m = 0, 1, 2, \ldots) \\ k = 1/2, & q(m,k) = 4^m, & \text{then } V_{m,1/2} = H_m . \end{cases}$$

In the family* (6.11), however, there may be many other interesting sets of integers worthy of consideration. For example, if k is any integer > 1, then

$$(6.13) \qquad V_{m,k} = \frac{q(m,k)}{(k-1)!} \sum_{r=o}^{[m/2]} \frac{(m-r+k-1)!}{(m-r)!} \binom{m-r}{r}$$

will obviously lead to various sequences of integers whenever $q(m,k)/(k-1)!$ is any arbitrarily chosen function yielding a positive integer. Expansion of a given function $f(\alpha)$ in terms involving the numbers $V_{m,k}$ may easily be made by the familiar procedure already described.

Besides the Gegenbauer polynomials, there are of course other well-known families of orthogonal polynomials which may be modified to furnish still other sources of integer-sequences. A given function could be expanded in terms of such numbers by a technique similar to the one presented in reference [1], or in this article.

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*It can be easily shown that

$$\lim_{m \to \infty} \frac{V_{m,k}}{V_{m+1,k}} = \left(\frac{\sqrt{5}-1}{2}\right) \left[\lim_{m \to \infty} \frac{q(m,k)}{q(m+1,k)}\right] ,$$

and thus that the value of this limit for all sequences of the general family has the common factor $(\sqrt{5}-1)/2$, which is the classical "golden mean" for the Fibonacci or Lucas sequence. (Of course, an appropriate choice of $q(m,k)$ should be made so that the limit on the right-hand side exists.)

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INFORMATION AND AN OMISSION

Reference related to H-37.

References by R. E. Greenwood: Problem 4047, Amer. Math. Mon. (issue of Feb. 1944, pp. 102-104), proposed by T. R. Running, solved by E. P. Starke. Problem #65, Nat. Math. Mag. (now just Math. Mag.) issue of November 1934, p. 63.

Omission H-37. Also solved by J. A. H. Hunter.

CORRECTION

H-28 Let

$$S_n(r, a, b) = \sum_{j=0}^{\infty} C_j(r, n)\, a^j b^{rn-n-j} = b^{(r-1)n} \sum_{N=0}^{r^{n-1}} \left(\frac{a}{b}\right)^{N_o+N_1+\ldots N_{n-1}}$$

ADVANCED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, JR.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-59 *Proposed by D.W. Robinson, Brigham Young University, Provo, Utah*

Show that, if $m > 2$, then the period of the Fibonacci sequence $0, 1, 1, 2, 3, \ldots, F_n, \ldots$ reduced modulo m is twice the least positive integer n such that $F_{n+1} \equiv (-1)^n F_{n-1} \pmod{m}$.

H-60 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

It is well known that if p_k is the least integer such that $F_{n+p_k} \equiv F_n \bmod 10^k$, then $p_1 = 60$, $p_2 = 300$ and $p_k = 1.5 \times 10^k$ for $k \geq 3$. If $Q(n, k)$ is the kth digit of the nth Fibonacci, then for fixed k, $Q(n, k)$ is periodic, that is q_k is the least integer such that $Q(n+q_k, k) \equiv Q(n, k) \bmod 10$. Find an explicit expression for q_k.

H-61 *Proposed by P.F. Byrd, San Jose State College, San Jose, California*

Let $f_{n,k} = 0$ for $0 \leq n \leq k-2$, $f_{k-1,k} = 1$ and

$$f_{n,k} = \sum_{j=1}^{k} f_{n-j,k} \quad \text{for} \quad n \geq k \ .$$

Show that

$$\frac{1}{2} < \frac{f_{n,k}}{f_{n+1,k}} < \frac{1}{2} + \frac{1}{2k} \quad \text{for} \quad n \geq 1 \ .$$

Hence

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{f_{n,k}}{f_{n+1,k}} = \frac{1}{2}$$

See E. P. Miles, "Generalized Fibonacci Numbers and their Associated Matrices," The American Mathematical Monthly, Vol. 67, No. 8.

H-62 *Proposed by H.W. Gould, West Virginia University, Morgantown, West Va.*

Find all polynomials $f(x)$ and $g(x)$, of the form

$$f(x+1) = \sum_{j=o}^{r} a_j x^j, \quad a_j \text{ an integer}$$

$$g(x) = \sum_{j=o}^{s} b_j x^j, \quad b_j \text{ an integer}$$

such that

$$2\left\{x^2 f^3(x+1) - (x+1)^2 g^3(x)\right\} + 3\left\{x^2 f^2(x+1) - (x+1)^2 g^2(x)\right\} + 2(x+1)\left\{xf(x+1) - (x+1)g(x)\right\} = 0 \quad .$$

H-63 *Proposed by Stephen Jerbic, San Jose State College, San Jose, California*

Let

$$F(m,o) = 1 \text{ and } F(m,n) = \frac{F_m F_{m-1} \cdots F_{m-n+1}}{F_n F_{n-1} \cdots F_1} \quad o < n \le m \quad ,$$

be the Fibonomial coefficients, where F_n is the nth Fibonacci number. Show

$$\sum_{n=o}^{2m-1} F(2m-1, n) = \prod_{i=o}^{m-1} L_{2i}, \quad m \ge 1 \quad .$$

H-64 *Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.*

Show

$$F_{n+1} = \prod_{j=1}^{n} \left(1 - 2i \cos \frac{j\pi}{n+1}\right) \quad ,$$

where F_n is the nth Fibonacci number.

ALL THE SOLUTIONS

H-30 *Proposed by J.A.H. Hunter, Toronto, Ontario, Canada*

Find all non-zero integral solutions to the two Diophantine equations,

(a) $$x^2 + xy + x - y^2 = 0$$

(b) $$x^2 - xy - x - y^2 = 0 \quad .$$

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

We first observe that (x_o, y_o) is a solution of (a), if and only $(-x_o, y_o)$ is a solution of (b). Thus we may limit our considerations to just one of the equations, say (b).

Equation (b) has the form

$$x^2 - (y+1)x - y^2 = 0$$

which, considering y as a parameter, has solutions

$$x = \frac{(y+1) \pm \sqrt{(y+1)^2 + 4y^2}}{2}$$

For x to be an integer, it is clearly necessary and sufficient that $(y+1)^2 + 4y^2$ be a perfect square, that is, there exists an integer z such that

$$(y+1)^2 + 4y^2 = z^2 \quad ,$$

or,

$$\boxed{(y+1)^2 + (2y)^2 = z^2 \; .}$$

Let us look first for solutions with $y > 0$. Note that $2y/d$ and $(y+1)/d$ are relatively prime integers, where $d \geq 1$ is the greatest common divisor of $2y$ and $y+1$, so that, by the well-known theorem on solutions of $x^2 + y^2 = z^2$, there exist two relatively prime <u>positive</u> integers r and s of different parity, with $r > s$, such that either

(1) $$\begin{cases} y+1 = d(r^2 - s^2) \\ 2y = d(2rs) \end{cases}$$

or

$$(2)\qquad \begin{cases} y+1 = d(2rs) \\ 2y = d(r^2-s^2) \end{cases} .$$

For case (1), it follows easily that $d = 1$, while in case (2), $d = 2$. Hence, solving case (1) is equivalent to finding relative prime positive integers r and s of different parity satisfying

$$(3)\qquad \boxed{r^2 - rs - s^2 = 1 .}$$

Now, in case (2), let

$$r' = r+s$$

$$s' = r-s .$$

Then, recalling that $d = 2$ in case (2), we have

$$(4)\qquad \begin{cases} y+1 = r'^2-s'^2 \\ 2y = 2r's' , \end{cases}$$

which has formally the same appearance as case (1) and implies

$$r'^2-r's'-s'^2 = 1 .$$

Thus, since

$$r = \frac{r'+s'}{2} \quad \text{and} \quad s = \frac{r'-s'}{2} ,$$

solving case (2) is equivalent to finding odd positive integers r' and s' satisfying (3).

In either case, we see that every solution of (b) with $y > 0$ is generated by an appropriate solution of the diophantine equation:

$$(*)\qquad \boxed{r^2 - rs - s^2 = 1 .}$$

Note that any solution (r, s) of (*) in positive integers has r and s relatively prime and $r > s$. Note that the case (r even, s even) cannot occur as a solution of (*).

Now, if (r, s) is a solution of (*) with positive integers r and s of different parity, then case (1) is indicated with $y = rs$ and either

$x = r^2$ or $x = -s^2$. Thus, we obtain two solutions (x, y) of (b), namely (r^2, rs) and $(-s^2, rs)$.

If (r', s') is a solution of (*) with odd positive integers r' and s', then we have case (2) and $y = r's'$ with both $x = r'^2$ and $x = -s'^2$, again giving two solutions of (b).

Thus, every positive solution (r, s) of (*) leads to two solutions of equation (b) having positive values for y, namely (r^2, rs) and $(-s^2, rs)$.

It remains to consider solutions of (b) having $y < 0$.

If $y < 0$, let $y = -|y|$; then, from (b),

$$x = \frac{(-|y| + 1) \pm \sqrt{(|y| - 1)^2 + 4y^2}}{2},$$

so that $(|y| - 1)^2 + 4|y|^2$ must be a perfect square, or equivalently, there exists an integer z such that

$$\boxed{(|y| - 1)^2 + (2|y|)^2 = z^2 .}$$

As before, letting d = the greatest common divisor of $|y| - 1$ and $2|y|$, we deduce the existence of two relatively prime positive integers r and s of different parity, with $r > s$, such that either

(1)* $$|y|-1 = d(r^2-s^2)$$
$$2|y| = d(2rs)$$

or

(2)* $$|y|-1 = d(2rs)$$
$$2|y| = d(r^2-s^2) .$$

Clearly, $d = 1$ in case (1)* and $d = 2$ for case (2)*. In case (1)*, we find that r and s must satisfy

(**) $$\boxed{r^2-s^2-rs = -1 ,}$$

while in case (2)*, the substitution $r' = r+s$, $s' = r-s$ yields (using $d = 1$ for case (1)* and $d = 2$ for case (2)*)

$$|y|-1 = r'^2-s'^2$$
$$2|y| = 2r's' ,$$

which shows that (r', s') is also a solution in positive integers of (**).

Note that any solution (r, s) of (**) in positive integers has r and s relatively prime and $r > s$ if we exclude the solution $r = s = 1$. Also the case (r even, s even) cannot occur as a solution of (**). Thus, every solution of (**) in positive integers either has both terms odd or r even and s odd. The latter case gives a solution of (b) with $|y| = rs$ and both $x = -r^2$ and $x = s^2$, so that the two generated solutions of (b) are $(-r^2, -rs)$ and $(s^2, -rs)$.

Similarly, if (r', s') is a solution of (**) with r' and s' both odd and $r' > s'$, then $|y| = r's'$ with $x = -r'^2$ and $x = s'^2$.

Thus, every solution of (**) in positive integers (r, s) (including $(1, 1)$) yields two solutions of (b) with negative y, namely $(-r^2, -rs)$ and $(s^2, -rs)$.

To find the actual solutions, we recall that every solution of $r^2 - rs - s^2 = 1$ in positive integers r, s has the form $r = F_{2k+1}$ and $s = F_{2k}$ for some integer $k \geq 1$. (See solution of H-31). The corresponding solutions of (b) are $(F_{2k+1}^2, F_{2k} F_{2k+1})$ and $(-F_{2k}^2, F_{2k} F_{2k+1})$ for $k = 1, 2, 3, \ldots$.

The other equation $r^2 - rs - s^2 = -1$ may be transformed to $r'^2 - r's' - s'^2 = 1$ by the change of variable, $r' = r+s$, $s' = r$; it follows that every solution of $r^2 - rs - s^2 = -1$ in positive integers (r, s) has the form $r = F_{2k}$, $s = F_{2k-1}$ for some integer $k \geq 1$. The corresponding solutions of (b) are $(-F_{2k}^2, -F_{2k}F_{2k-1})$ and $(F_{2k-1}^2, -F_{2k} F_{2k-1})$ for $k = 1, 2, 3, \ldots$.

Summarizing, the set of solutions, $(F_{2k+1}^2, F_{2k} F_{2k+1})$, $(-F_{2k}^2, F_{2k} F_{2k+1})$, $(-F_{2k}^2, -F_{2k} F_{2k-1})$, $(F_{2k-1}^2, -F_{2k} F_{2k-1})$ for $k = 1, 2, 3, \ldots,$ constitute all the non-zero integral solutions of $x^2 - xy - x - y^2 = 0$, and the set

$$(-F_{2k+1}^2, F_{2k} F_{2k+1}), (F_{2k}^2, F_{2k} F_{2k+1}), (F_{2k}^2, -F_{2k} F_{2k-1}),$$
$$(-F_{2k-1}^2, -F_{2k} F_{2k-1}) \text{ for } k = 1, 2, 3, \ldots$$

constitute all non-zero integral solutions of $x^2 + xy + x - y^2 = 0$.

AN OLD PROBLEM

H-41 *Proposed by Robert A. Laird, New Orleans, La.*

Find rational integers, x, and positive integers, m, so that

$$N = x^2 - m \quad \text{and} \quad M = x^2 + m$$

are rational squares.

Solution by Joseph Arkin, Spring Valley, New York

Professor Oystein Ore, Sterling Professor of Mathematics at Yale University, in his book, Number Theory and Its History, 1st ed., 1948, gives the complete solution to this problem on pages 188-193.

Also solved by Maxey Brooke, Sweeny, Texas

COMMENTS ON THE HISTORICAL CASE

Solved by Robert A. Laird

A solution to the historical problem submitted to Fibonacci (Leonardo of Pisa) by John of Palermo, an imperial notary of Emperor Frederick II, about 1220 A. D. (see page 124, Cajori's "History of Mathematics" for reference). The problem: Find a number x, such that $x^2 + 5$ and $x^2 - 5$ are each square numbers. In other words, find the square which increased or decreased by 5, remains a square. Leonardo solved the problem by a method (not known to me) of building squares by the summation of odd numbers.

Solution to this problem was published in the "Mathematics Teacher" in December 1952.

I offer it here for your interest and pleasure. Let

x = side of the desired square

x + b = side of a larger square

x - a = side of a smaller square

a and b are positive, rational numbers

(1) $$(x + b)^2 = x^2 + 5$$

(2) $$(x^2 - a)^2 = x^2 - 5$$

Solving (1) and (2)

(3) $$x = \frac{5 + a^2}{2a}$$

(4) $$x = \frac{5 - b^2}{2b}$$

Equating (3) and (4)

$$\frac{5 + a^2}{2a} = \frac{5 - b^2}{2b}$$

Solving for b in terms of a, we have

$$b = \frac{-(5+a^2) \pm \sqrt{a^4+30a^2+25}}{2a} \tag{5}$$

In order for b to be a rational number, the radical must clear. So find value of a that will do this.

We can find a by trial substitution or by factoring. Let's take factoring:

$$a^4 + 30a^2 + 25$$

$$a^4 + \underbrace{26a^2 + 4a^2}{} + 25$$

$$a^4 + 26a^2 + 169 + 4a^2 - 144$$

$$(a^2 + 13)^2 + 4(a^2 - 36)$$

If $a^2 = 36$ or $a = 6$, the radical will clear. For immediate result, substitute $a = 6$ in (3)

$$x = \frac{5 + a^2}{2a} = \frac{5 + 36}{12} = \frac{41}{12} \qquad \text{Q. E. D.}$$

Generally, find the square which if increased or decreased by m will remain a square (m = positive integer). Strangely, when m = 6, a solution can be found, but not for m = 1, or 2, or 3, or 4.

FROM BEST SET OF K TO BEST SET OF K+1 ?

H-42 *Proposed by J.D.E. Konhauser, State College, Pa.*

A set of nine integers having the property that no two pairs have the same sum is the set consisting of the nine consecutive Fibonacci numbers, 1, 2, 3, 5, 8, 13, 21, 34, 55 with total sum 142. Starting with 1, and annexing at each step the smallest positive integer which produces a set with the stated property yields the set 1, 2, 3, 5, 8, 13, 21, 30, 39 with sum 122. Is this the best result? Can a set with lower total sum be found?

Partial solution by the proposer.

Partial answer. The set 1, 2, 4, 5, 9, 14, 20, 26, 35 has total sum 116. For eight numbers the best set appears to be 1, 2, 3, 5, 9, 15, 20, 25 with sum 80. Annexing the lowest possible integer to extend the set to nine members requires annexing 38 which produces a set with sum 118. It is not clear (to me, at least) how to progress from a best set of k integers to a best set for k + 1 integers.

H-43 *(Corrected) Proposed by H.W. Gould, West Virginia University, Morgantown, West Va.*

Let

$$\varphi(x) = \sum_{n=1}^{\infty} x^{F_{mn}} ,$$

where F_j is the j-th Fibonacci number, find

$$\lim_{x \to 1} \frac{\varphi(x)}{-\log(1-x)}$$

See special case m = 2 in Revista Matematica Hispano-Americana (2) 9 (1934) 223-225 problem 115.

A FAVORABLE RESPONSE

H-44 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let $u_o = q$ and $u_1 = p$, and $u_{n+2} = u_{n+1} + u_n$, then the u_n are called generalized Fibonacci numbers.

(1) Show $$u_n = pF_n + qF_{n-1}$$

(2) Show that if

$$V_{2n+1} = u_n^2 + u_{n+1}^2 \quad \text{and} \quad V_{2n} = u_{n+1}^2 - u_{n-1}^2 ,$$

then V_n are also generalized Fibonacci numbers.

Solution by Lucile R. Morton, San Jose State College, San Jose, California

We prove formula (1) by induction on n. It is obvious that

$$u_1 = p = pF_1 + qF_0 \quad \text{and} \quad u_2 = p + q = pF_2 + qF_1 \quad .$$

Now let us assume formula (1) holds for $n = k$ and $n = k+1$. Thus

$$u_k = pF_k + qF_{k-1}$$

and

$$u_{k+1} = pF_{k+1} + qF_k \quad .$$

Adding we get

$$u_{k+1} + u_k = p(F_{k+1} + F_k) + q(F_k + F_{k-1}) \quad ,$$

or

$$u_{k+2} = pF_{k+2} + qF_{k+1} \quad ,$$

which was to be proved.

We prove V_n are generalized Fibonacci numbers by showing they satisfy the recursion formula $V_{n+2} = V_{n+1} + V_n$, where $V_0 = 2pq - q^2$ and $V_1 = p^2 + q^2$. We can do this by showing

(3) $$V_{2n+1} = V_{2n} + V_{2n-1}$$

(4) $$V_{2n+2} = V_{2n+1} + V_{2n} \quad .$$

From formulas (2)

$$\begin{aligned} V_{2n} + V_{2n-1} &= (u_{n+1}^2 - u_{n-1}^2) + (u_{n-1}^2 + u_n^2) \\ &= u_{n+1}^2 + u_n^2 = V_{2n+1} \quad , \end{aligned}$$

and

$$\begin{aligned} V_{2n+1} + V_{2n} &= (u_n^2 + u_{n+1}^2) + (u_{n+1}^2 - u_{n-1}^2) \\ &= u_n^2 - u_{n-1}^2 + 2u_{n+1}^2 \\ &= (u_{n+1})(u_{n-2}) + (u_{n+1})(u_{n+1} + u_n + u_{n-1}) \\ &= u_{n+1}(u_{n-2} + u_{n-1} + u_{n+2}) \\ &= (u_{n+2} - u_n)(u_{n+2} + u_n) = u_{n+2}^2 - u_n^2 \\ &= V_{2n+2} \quad . \qquad \text{Q.E.D.} \end{aligned}$$

Now let us carry our problem a little further. Let m be a fixed integer, and let $V_n = u_{n+m}$. Are there any restrictions on p and q? Since V_n and u_n are generalized Fibonacci numbers

$$V_{n+1} = V_0F_n + V_1F_{n+1} = (2pq-q^2)F_n + (p^2+q^2)F_{n+1}$$

and

$$u_{n+m+1} = u_mF_n + u_{m+1}F_{n+1} = (pF_m + qF_{m-1})F_n + (pF_{m+1} + qF_m(F_{n+1}.$$

Thus we have

$$(5) \qquad 2pq-q^2 = pF_m + qF_{m-1}$$

$$(6) \qquad p^2 + q^2 = pF_{m+1} + qF_m$$

Our question becomes: For what integral values p and q do equations (5) and (6) hold? Obviously $p = q = 0$ is a solution. Then $V_n = u_n = 0$. Let

$$p = \frac{x + F_{m+1}}{2} \quad \text{and} \quad q = \frac{y + F_m}{2},$$

substituting into equations (5) and (6) we have

$$(7) \qquad 2xy - y^2 = F_{m+1}^2 - F_{m-1}^2 = F_{2m} \quad \text{and}$$

$$(8) \qquad x^2 + y^2 = F_{m+1}^2 + F_m^2 = F_{2m+1} .$$

Eliminating x and simplifying

$$5y^4 + 2(F_{2m} - 2F_{2m+1})y^2 + F_{2m}^2 = 0 ,$$

or

$$5y^4 - 2L_{2m}y^2 + F_{2m}^2 = 0 .$$

Thus

$$y^2 = \frac{L_{2m} \pm \sqrt{4L_{2m}^2 - 20F_{2m}^2}}{10} .$$

$$= \frac{L_{2m} \pm \sqrt{L_{2m}^2 - 5F_{2m}^2}}{5} .$$

Then

$$y^2 = \frac{L_{2m} \pm \sqrt{4(-1)^{2m}}}{5} = \frac{L_{2m} \pm 2}{5} = \frac{L_m^2 - 2(-1)^m \pm 2}{5}$$

and we have $5y^2 = L_m^2$, which has no integral solutions, or

(9) $$5y^2 = L_m^2 \pm 4 = 5F_m^2 + 4(-1)^m \pm 4 \quad .$$

Now $5y^2 = 5F_m^2 \pm 8$, which has no integral solutions, or $5y^2 = 5F_m^2$, and $y = \pm F_m$. Therefore the equations (7) and (8) have the solutions $x = F_{m+1}$, $y = F_m$ and $x = -F_{m+1}$, $y = -F_m$ for all m, and $x = -F_{m+1}$, $y = F_m$ and $x = F_{m+1}$, $y = -F_m$ for $m = 0, -1$.

Thus

$$\begin{aligned} p &= F_{m+1} \\ q &= F_m \end{aligned} \quad \text{or} \quad \begin{aligned} p &= 0 \\ q &= 0 \end{aligned}$$

are solutions of (5) and (6) for all m, and

$$\begin{aligned} p &= 0 \\ q &= F_m \end{aligned} \quad \text{or} \quad \begin{aligned} p &= F_{m+1} \\ q &= 0 \end{aligned}$$

are solutions of (5) and (6) for $m = 0, -1$.

Therefore $V_n = u_{m+n} = F_{2m+n}$ when $p = F_{m+1}$ and $q = F_m$ for all m, or $V_n = u_{m+n} = 0$ when $p = q = 0$.

If we consider nonintegral solutions, from (7) and (8) we had

$$5y^2 = L_m^2$$

which gives us

$$y = \pm \frac{L_m}{\sqrt{5}} \quad \text{and} \quad x = \pm \frac{L_{m+1}}{\sqrt{5}} \quad .$$

Thus the solutions of (7) and (8) are

$$x = \frac{L_{m+1}}{\sqrt{5}}, \quad y = \frac{L_m}{\sqrt{5}} \quad \text{and} \quad x = -\frac{L_{m+1}}{\sqrt{5}}, \quad y = -\frac{L_m}{\sqrt{5}}$$

for all m. Therefore

$$p = \frac{\dfrac{L_{m+1}}{\sqrt{5}} + F_{m+1}}{2} = \frac{\alpha^{m+1}}{\sqrt{5}}$$

$$q = \frac{\dfrac{L_m}{\sqrt{5}} + F_m}{2} = \frac{\alpha^m}{\sqrt{5}}$$

and

$$p = \frac{-\dfrac{L_{m+1}}{\sqrt{5}} + F_{m+1}}{\;} = -\frac{\beta^{m+1}}{\sqrt{5}}$$

$$q = \frac{-\dfrac{L_m}{\sqrt{5}} + F_m}{2} = -\frac{\beta^m}{\sqrt{5}} \quad .$$

Also solved by Clifton T. Whyburn, Douglas Lind, Clyde A. Bridger, Charles R. Wall, John L. Brown, Jr., Joseph Arkin, Raymond E. Whitney, John Wessner, W.A. Al-Slalm and A. A. Gioia (jointly), Charles Ziegenfus and L. Carlitz.

ITERATED SUMS OF SQUARES

H-45 *Proposed by R.L. Graham, Bell Telephone Labs., Murray Hill, N.J.*

Prove

$$\sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{r=0}^{q} \sum_{s=0}^{r} F_s^2 = F_{n+2}^2 - \frac{1}{8}\left(2n^2+8n+11-3(-1)^n\right) \quad ,$$

where F_n is the nth Fibonacci number.

Solution by Charles R. Wall, Texas Christian University, Ft. Worth, Texas

Using the identities

$$\sum_{n=0}^{k} F_n^2 = \sum_{n=1}^{k} F_n^2 = F_k F_{k+1} \quad ,$$

$$\sum_{n=0}^{k} F_n F_{n+1} = F_{k+1}^2 - \frac{1 + (-1)^k}{2} \quad ,$$

we have

$$\sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{r=0}^{q} \sum_{s=0}^{r} F_s^2 = \sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{r=0}^{q} F_r F_{r+1}$$

$$= \sum_{p=0}^{n} \sum_{q=0}^{p} \left(F_{q+1}^2 - \frac{1+(-1)^q}{2} \right)$$

$$= \sum_{p=0}^{n} \left\{ F_{p+1} F_{p+2} - \frac{p+1}{2} - \frac{1+(-1)^p}{4} \right\}$$

$$= \sum_{p=0}^{n} \left\{ F_{p+1} F_{p+2} - \frac{p}{2} - \frac{3}{4} - \frac{(-1)^p}{4} \right\}$$

$$= F_{n+2}^2 - \frac{1}{2} - \frac{(-1)^n}{2} - \frac{n(n+1)}{4} - \frac{3(n+1)}{4} - \frac{1+(-1)^n}{8}$$

$$= F_{n+2}^2 - \frac{1}{8} (2n^2 + 8n + 11 - 3(-1)^n) \quad .$$

Also solved by Douglas Lind, L. Carlitz, and Al-Slalm and A. A. Gioia (jointly).

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HAVE YOU SEEN?

J. Arkin, "An Extension of the Fibonacci Numbers," American Mathematical Monthly, Vol. 72, No. 5, March 1965, pp. 275-279.

Marvin Wunderlich, "Another Proof of the Infinite Primes Theorem," American Mathematical Monthly, Vol. 72, No. 5, March 1965, p. 305. This is an extremely neat proof for the Fibonacci Fan!

Benjamin B. Sharpe, Problem 561, Mathematics Magazine, Vol. 28, No. 2, March 1965, pp. 121-122.

SEEKING THE LOST GOLD MINE OR EXPLORING FOR FIBONACCI FACTORIZATIONS

BROTHER ALFRED
Saint Marys College

Now that summer is coming on, everybody is looking for a good way to waste time. Seek no farther. The search for factors of Fibonacci numbers is the perfect answer.

And first, some ground rules. People with computers who program their machines and then sit idly by while they grind out answers should not be considered in the class of working Fibonacci factorizers. The challenge is to be able with the available tables and the mathematical bow-and-arrow — the calculator — to find some method or methods that facilitate the determination of factors in Fibonacci sequences.

Just to get away from the well-worn path we start in virgin territory with a sequence 1, 4, 5, 9, 14, 23, etc. We discover very soon that this has a prolongation to the left of ....-19, 12, -7, 5, -2, 3, 1, 4, 5, 9....... and since the factors of both portions of the sequence are the same we might call the sequence 2, 5, 7, 12, etc., the conjugate sequence to 1, 4, 5, 9, 14, etc. This is a first help in factoring the initial hundred terms of each portion of our sequence — a not too modest goal.

Next, we can determine the primes that do not divide the members of our sequence. Taking the square of any term minus the product of the two adjacent terms gives ± 11. Thus

$$5^2 - 4 \cdot 9 = -11$$

In general, if we designate the terms of the sequence T_n,

$$T_n^2 - T_{n-1} T_{n+1} = \pm 11$$

Hence if a prime divides T_{n-1}, for example, it would follow that

$$T_n^2 \equiv \pm 11 \pmod p$$

Thus if neither +11 nor -11 is a quadratic residue of a given prime p, then this prime cannot be a factor of the sequence. We can eliminate from consideration in this way: 11, 13, 17, 29, 41, 61, etc.

The smaller quantities in our sequence can be factored either by inspection or factor tables. Next, apart from 11, the primes have the same period in as in the Fibonacci sequence. Hence we can have some idea of when they should be entering the sequence by looking at the size of the period and more specifically the entry point in the Fibonacci sequence. For the spacing of the members of our sequence that are divisible by the given prime is the same as in the Fibonacci sequence should it be a factor of the sequence at all. For small spacings we can then extend the factor to other members of our sequence by using this information regarding the period and entry point of the prime in the Fibonacci sequence.

But how should we organize a systematic and convenient method of factoring using previous information on the Fibonacci and Lucas sequences? The following approach was tried. Since

$$T_1 = 1 = F_0 + L_1$$

$$T_2 = 4 = F_1 + L_2$$

it follows in general that $T_n = F_{n-1} + L_n$. Thus if we know the Fibonacci numbers modulo p and the Lucas numbers modulo p, it is simply necessary to check and see whether the sum of the residues of F_{n-1} and L_n is congruent to zero modulo p. Another dividend comes from the fact that if we call the members of the conjugate sequence R_n then

$$R_1 = 2 = F_2 + L_1$$

$$R_2 = 5 = F_3 + L_2$$

so that in general $R_n = F_{n+1} + L_n$. Thus the residues can be used in two ways. The original thought was that once these residues are on hand, it would be possible to use them for factoring many Fibonacci sequences.

The method works. But — as we get to larger and larger primes the periods increase and so likewise do the entry points so that the probability that the prime will be a factor between T_{100} and R_{100} gets less. Also, with large primes such as 911 with an entry point of 70 in the Fibonacci sequence (1, 1, 2, 3...) the probability that this will be a factor of a Fibonacci sequence chosen at random is relatively small, being only 7.6%. This same pattern applies to all large primes with relatively small entry points.

Again the sequence of primes that factor all Fibonacci sequences have the maximum period, 2p + 2 and hence tend to have a small probability of factoring our sequences within the limited range from T_{100} to R_{100}.

All in all, the high hopes entertained for this method were not realized. Does some one have a better way of attacking this problem?

As a byproduct, it would appear to be a worthwhile goal to have available factorizations of the first hundred terms of a few Fibonacci sequences such as (1, 4) and (2, 5) — even if somebody does it on a computer.

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TIME GENERATED COMPOSITIONS YIELD FIBONACCI NUMBERS

HENRY WINTHROP
University of South Florida, Tampa, Florida

1. INTRODUCTION

Imagine a particle the number of whose collisions with other particles during the t^{th} time interval is given by $\phi(t)$. Assume that this particle possesses a property, p, which it can transmit by collision to every particle with which it collides. Further suppose that every particle that has received property p by collision can also transmit it by collision. Assume that for an indefinite period of time every particle possessing property, p, collides only with those particles not possessing this property. The number of new particles to which property, p, has been imparted is given by the following model.

2. THE MODEL

Let Δ_i be the number of collisions with new particles in the time interval $i < t \leq i + 1$ by particles possessing property, p, at $t = i$. The new particles do not start their private times until the end of the time interval of their initial collision.

(1)
$$\begin{aligned}
\Delta_0 &= 1 \\
\Delta_1 &= \phi(1) \\
\Delta_2 &= \phi(2) + \phi^2(1) \\
\Delta_3 &= \phi(3) + 2\phi(2)\phi(1) + \phi^3(1) \\
\Delta_4 &= \phi(4) + [2\phi(3)\phi(1) + \phi^2(2)] + 3\phi(2)\phi^2(1) + \phi^4(1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\Delta_i &= F(h_i, \phi)
\end{aligned}$$

The model is obtained as follows:

Up to $t = 1$, Δ_0 generates the increment Δ_1, whose magnitude is $\phi(1)$, the number of particles with which Δ_0 collided in the first time interval.

At the time $t = 2$, Δ_0 has collided with $\phi(2)$ more new particles during the second time interval and Δ_1 has collided with $\phi(1)$ new particles, since its collisions are subject to the phase rule constraint of its own private time. Therefore when t = 2 in public time, $\Delta_2 = \phi(2) + \phi(1)\phi(1) = \phi(2) + \phi^2(1)$.

When t = 3, Δ_0 has collided with $\phi(3)$ more new particles during the third time interval for it is in phase 3 of its private time, each particle of $\Delta_1 = \phi(1)$ has collided with $\phi(2)$ more new particles, producing $\phi(1)\phi(2)$ new particles altogether, because Δ_1 is in the second phase of its private time. Each particle of Δ_2 collides with $\phi(1)$ new particles since it is in the first phase of its private time, thus producing

$$\Delta_2\phi(1) = (\phi(2) + \phi^2(1))\phi(1) = \phi(2)\phi(1) + \phi^3(1)$$

particles. Therefore when t = 3, we have

$$\begin{aligned}\Delta_3 &= [\phi(3)] + [\phi(1)\phi(2)] + [\phi(2)\phi(1) + \phi^3(1)] \\ &= \phi(3) + 2\phi(2)\phi(1) + \phi^3(1) \quad .\end{aligned}$$

Now if we substitute $\phi(t) = t$ into the model display (1), we obtain

(2)
$$\begin{aligned}\Delta_0 &= 1 \\ \Delta_1 &= 1 \\ \Delta_2 &= 2 + 1^2 = 3 \\ \Delta_3 &= 3 + 2^2 \cdot 1 + 1^3 = 8 \\ \Delta_4 &= 4 + 2 \cdot 3 \cdot 1 + 2^2 + 3 \cdot 2 \cdot 1^2 + 1^4 = 21\end{aligned}$$

Neglecting Δ_0, one observes that the numbers 1, 3, 8, 21, 55, ..., $U_{n+2} = 3U_{n+1} - U_n$ are the alternate terms of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots, F_{n+2} = F_{n+1} + F_n$$

so that the sequence of cumulative sums (including Δ_0) is

$$1,\ 1 + 1 = 2,\ 1 + 1 + 3 = 5,\ 1 + 1 + 3 + 8 = 13,\ \ldots,$$

$U_{n+2} = 3\,U_{n+1} - U_n$ which is the other set of alternate Fibonacci numbers. The proof of these statements will follow as a special case of the theorem in the following section.

3. ANOTHER SPECIAL MODEL

If we assume that the time generator is $\phi(t) = kt$ (k a positive integer), the same model display (1) yields

(3)
$$\begin{aligned}
\Delta_0 &= 1\\
\Delta_1 &= k\\
\Delta_2 &= k^2 + 2k\\
\Delta_3 &= k^3 + 4k^2 + 3k\\
\Delta_4 &= k^4 + 6k^3 + 10k^2 + 4k\\
&\cdots\cdots\cdots\cdots\\
\Delta_i &= P_i(k)
\end{aligned}$$

Note: The coefficient of k^m in the polynomial $P_n(k)$ is the number of distinct compositions of integer n in m positive integers. The coefficients are also the alternate rising diagonals of Pascal's arithmetic triangle upward from left to right.

We now prove the following theorem.

Theorem: If $\phi(t) = kt$, then model display (3) has as its nth row a polynomial $P_n(k)$ satisfying the recursion relation:

$$P_{n+2}(k) = (k + 2)P_{n+1}(k) - P_n(k) \quad ,$$

where $P_1(k) = k$ and $P_2(k) = k^2 + 2k$.

4. PROOF OF THE THEOREM

Let $T_n(k)$ be the total number of particles possessing property, p, at time $t = n$. Clearly $T_{n+1}(k) = T_n(k) + \Delta_{n+1}$, while collectively the $T_n(k)$ particles collide with Δ_{n+1} new particles during the next time interval, each particle collides with k more new particles than during the previous time interval so that

(4) $$\Delta_{n+2} = k(T_n(k) + \Delta_{n+1}) + \Delta_{n+1} = kT_{n+1}(k) + \Delta_{n+1} \quad .$$

Thus, since $\Delta_{n+1} = T_{n+1}(k) - T_n(k)$ equation (4) yields

(5) $$T_{n+2}(k) = (k+2)\, T_{n+1}(k) - T_n(k) \quad .$$

But, since $\Delta_{n+1} = T_{n+1}(k) - T_n(k)$ is the difference of two solutions of (5), it is also a solution of (5). Now, $\Delta_1 = k = P_1(k)$ and $\Delta_2 = k^2 + 2k = P_2(k)$ and the proof is complete. If $k = 1$, then (5) becomes

(6) $$U_{n+2} = 3U_{n+1} - U_n$$

If $U_1 = P_1(1) = 1$, and $U_2 = P_2(1) = 3$, then the numbers generated are the alternate Fibonacci numbers promised after (2), while

$$U_0 = T_0(1) = \Delta_0 = 1, \text{ and } U_1 = T_1(1) = \Delta_0 + \Delta_1 = 1 + 1 = 2 \quad ,$$

recursion relation (6) yields the other set of alternate Fibonacci numbers as the sequence of cumulative sums, the total particle count.

5. CONCLUDING REMARKS

One is directed to advanced problem H-50 December 1964, Fibonacci Quarterly, for the partitioning interpretation of the integer n of the model for $\phi(t) = kt$.

Suppose one defines two sets of Morgan-Voyce polynomials

$$b_0(x) = 1,\ b_1(x) = 1 + x;\ B_0(x) = 1,\ B_1(x) = 2 + x \quad ,$$

both sets satisfying

$$P_{n+2}(x) = (x + 2)\, P_{n+1}(x) - P_n(x), \quad n \geq 0 \quad . \tag{7}$$

It is easy to establish that

$$P_n(k) = \Delta_n = k\, B_{n-1}(k)$$

$$T_n(k) = \Delta_0 + \Delta_1 + \dots + \Delta_n = b_n(k) \quad .$$

Thus for $k = 1$, we again find $B_{n-1}(1) = F_{2n}$ and $b_n(1) = F_{2n+1}$. See corrected problem B-26 with solution by Douglas Lind in the Elementary Problem Section of this issue, where the binomial coefficient relation mentioned in the note of Section 3 is shown. A future paper by Prof. M. N. S. Swamy dealing extensively with Morgan-Voyce polynomials will appear in an early issue of the Fibonacci Quarterly.

Acknowledgment: The author is completely indebted to Dr. V. E. Hoggatt, Jr., for bringing to his attention the theorem and its proof.

Additional references to work along the lines of generated compositions — some of which yield numbers with Fibonacci properties — will be found in the references at the end of this paper. (See note, page 94)

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H. Winthrop "A Theory of Behavioral Diffusion" A contribution to the Mathematical Biology of Social Phenomena. Unpublished thesis submitted to the Faculty of The New School for Social Research, 1953.

H. Winthrop, "Open Problems of Interest in Applied Mathematics," Mathematics Magazine, 1964, Vol. 37, pp. 112-118.

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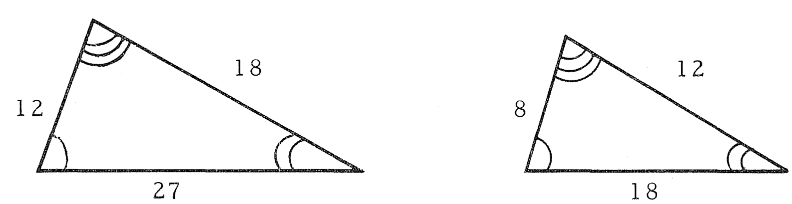
MYSTERY PUZZLER AND PHI

MARVIN H. HOLT
Wayzata, Minnesota

A problem proposed by Professor Hoggatt is as follows: Does there exist a pair of triangles which have five of their six parts equal but which are not congruent? (Here the six parts are the three sides and the three angles.) The initial impulsive answer is no! The problem also appears in [1] as well as in the MATH LOG.

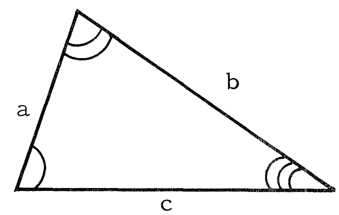
I have taken some time to work on the problem you suggested. I think you will agree that the solution I have is interesting. One problem, as you have stated it, is posed in a high school geometry text entitled, "Geometry" by Moise and Downs, published by Addison Wesley Company, (page 369).

In their solution key, they gave one possible pair of triangles that work:



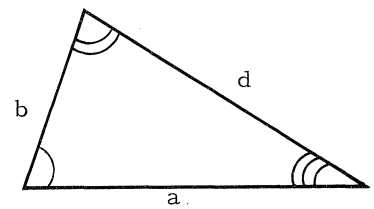
I discovered this after I solved the problem myself. But the above solution does not do justice to the problem at all, since my old friend τ is really the key to the solution. Note: Golden Mean = ϕ = τ in what follows.

I attacked the problem as follows: First, the five congruent parts cannot contain all three sides, since the triangles would then be congruent. Therefore, the five parts must be three angles and two sides which means that the two triangles are similar. But, the two sides cannot be in corresponding order, or the triangles would be congruent either by ASA or SAS. So, the situation must be one of two possibilities as I have sketched below: (My sketches are not to scale.)

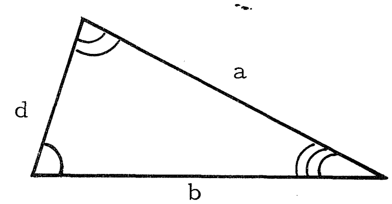


triangle 1

triangle 2



possibility 1



possibility 2

In both cases, by using relationships from similar triangles, it follows that $\frac{a}{b} = \frac{b}{c}$ or $b = ka$ and $c = kb = k^2a$ from possibility 2 and $\frac{a}{b} = \frac{b}{d}$ or $b = ka$ and $d = kb = k^2a$ from possibility 1.

So, the three sides of the triangle must be three consecutive members of a geometric series: a, ak, ak^2, where k is a proportionality constant and $k > 0$ and $k \neq 1$. If $k = 1$, the triangles would both be equilateral and thus congruent. Therefore, $k \neq 1$.

From my previous article on the Golden Section (Pentagon, Spring 1964) I worked out two problems on right triangles where the sides formed a geometric progression and the constants turned out to be $\sqrt{\tau}$ and $\sqrt{\frac{1}{\tau}}$. So, I knew of two more situations where the original problem could be solved. Then I began to consider various other values of k and I began to wonder what values of "k" will work. In other words, for what values of k will the numbers a, ak, and ak^2 be sides of a triangle. Once we know this, then another triangle with sides $\frac{a}{k}$, a, ak or ak, ak^2, ak^3 will have five parts congruent but the triangles would not be congruent.

In order for a, ak and ak^2 to be sides of a triangle, three statements must be true:

These are instances of the strict triangle inequality.

1. $a + ak > ak^2$ $(a + b > c)$

2. $a + ak^2 > ak$ $(a + c > b)$

3. $ak + ak^2 > a$ $(b + c > a)$

$[a > 0, \quad k > 0, \quad k \neq 1]$

For Case 1, consider $k > 1$

(a) $k > 1 \rightarrow k^2 > k \rightarrow 1 + k^2 > k$

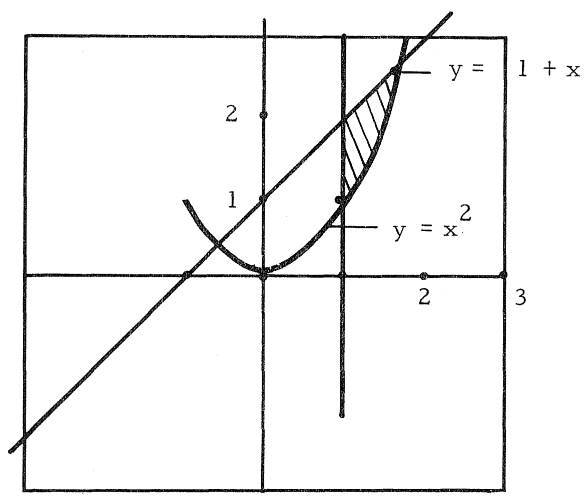
therefore, $a + ak^2 > ak$ (condition 2 above)

(b) $k > 1 \rightarrow k + 1 > 1 \rightarrow k^2 + k > 1$

therefore, $ak^2 + ak > a$ (condition 3 above)

(c) if $k > 1$ show $a + ak > ak^2$ (condition 1 above).

This part revolves around the problem of finding out when $1 + k > k^2$, or, graphically: For what $x > 1$ will $\underline{1 + x = y}$ be above $\underline{y = x^2}$?



Solving this problem produces the result that

$$k < \frac{1 + \sqrt{5}}{2} \quad \underline{\text{or}} \quad k < r:$$

So, if $1 < k < r$ then the numbers a, ak, ak^2 are the sides of the triangle that can be matched with $\frac{a}{k}$, a, ak or ak, ak^2, ak^3 to solve the original problem. (Incidentally: $1 < \sqrt{r} < r$. So this fits in here.)

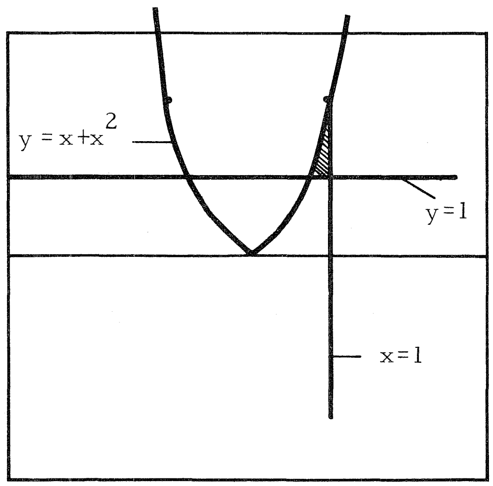
For Case 2, consider $k < 1$

(a) if $k < 1 \rightarrow k^2 < k \rightarrow k^2 < k + 1$ Therefore $ak^2 < ak + a$ (condition 1)

(b) if $k < 1 \rightarrow 1 + k > 1 \rightarrow a + ak^2 > ak$ (condition 2)

(c) Now, if $k < 1$ show $ak + ak^2 > a$. This is, essentially, finding what values of k make $k + k^2 > 1$.

Again, graphically, for what $x < 1$ will the parabola $y = x + x^2$ be above the line $y = 1$?



Solving this problem produces the result that $k > \frac{-1 + \sqrt{5}}{2}$. If you will follow this closely, $\frac{-1 + \sqrt{5}}{2}$ is the additive inverse of the conjugate of r. (i.e., $r = \frac{1 + \sqrt{5}}{2}$. Therefore, the conjugate of r is $\frac{1 - \sqrt{5}}{2}$ and its additive inverse is $\frac{-1 + \sqrt{5}}{2}$.) So, if $\frac{-1 + \sqrt{5}}{2} < k < 1$ the problem is again solved. (Again, $\frac{-1 + \sqrt{5}}{2} < \sqrt{\frac{1}{r}} < 1$, so my second problem fits here.)

Therefore, the complete solution can be summed up as follows, if k is a number such that $1 < k < \frac{1 + \sqrt{5}}{2} = r$ or $\frac{-1 + \sqrt{5}}{2} < k < 1$. Then the three sets of triangles with sides $\frac{a}{b}$, a, ak or a, ak, ak^2 or ak, ak^2, or ak^3 can be used to produce two triangles with five parts equal and the triangles themselves not congruent.

So, there are an infinite number of pairs of triangles that solve this problem and once again, r proves to be an interesting number and a key to the solution of interesting problems.

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1. Moise and Downs, Geometry, Addison-Wesley, p. 369.

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LADDER NETWORK ANALYSIS USING POLYNOMIALS

JOSEPH ARKIN
Spring Valley, New York

In this paper we develop some ideas with the recurring series

(1) $$B_n = k_1 B_{n-1} + k_2 B_{n-2}, \quad B_0 = 1, \ (k_1 \text{ and } k_2 \neq 0) \quad ,$$

and show a relationship between this sequence and the simple network of resistors known as a ladder-network.

The ladder-network in Figure 1 is an important network in communication systems. The m-L sections in cascade that make up this network can be characterized by defining:

(2) a) the attenuation (input voltage/output voltage) = A,

b) the output impedance = z_0,

c) the input impedance = z_1.

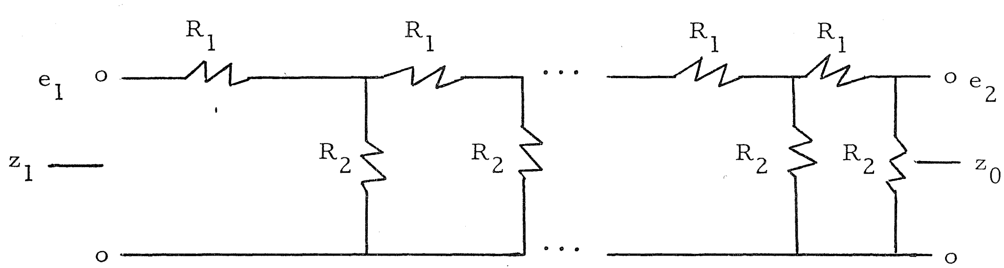

Figure 1

A result obtained by applying Kirchhoff's and Ohm's Laws to ladder-networks with $m = 1, 2, 3, \ldots, R_1 = R_2 k_1$, was tabulated with the results in Table 1, where setting $k_1 = 1$, $R_2 = 1$ ohm, the network in Figure 1 was analyzed by inspection [1] .

m	z_0	A	z_1
1	R_2	(k_1+1)	$(k_1+1)R_2$
2	$\left(\frac{k_1+1}{k+2}\right)R_2$	$(k_1^2+3k_1+1)$	$\left(\frac{k_1^2+3k_1+1}{k_1+2}\right)R_2$
3	$\left(\frac{k_1^2+3k_1+1}{k_1^2+4k_1+3}\right)R_2$	$(k_1^3+5k_1^2+6k_1+1)$	$\left(\frac{k_1^3+5k_1^2+6k_1+1}{k_1^2+4k_1+3}\right)R_2$
⋮	⋮	⋮	⋮

Table 1

We observe that the nth row in Table 1, may be written

m	z_0	A	z_1
n	$(C_{2n-2}/y_{2n-1})R_2$	C_{2n}	$(C_{2n}/y_{2n-1})R_2$

where,

(3) a) $C_n = k_1^{1/2}C_{n-1} + C_{n-2},\ C_0 = 1$,

b) $y_n = k_1^{1/2}y_{n-1} + y_{n-2},\ y_0 = 1/k_1^{1/2}$.

It then remains to solve for y_n and C_n in (3), to be able to analyze (Figure 1) by inspection for any value of k_1 ($k_1 \neq 0$), where $R_2 = 1$ ohm. So that, in (1), we let

(4) a) $w = (k_1 + (k_1^2 + 4k_2)^{1/2})/2$,

b) $v = (k_1 - (k^2 + 4k_2)^{1/2})/2$,

where it is evident,

c) $k_1 = w + v$,

and

d) $k_2 = -wv$.

Then, combining (c) and (d) with (1), leads to

(5)
$$B_n = ((w^2 - v^2)B_{n-1} - wv(w-v)B_{n-2})/(w-v) ,$$
$$B_n = ((w^3 - v^3)B_{n-2} - wv(w^2 - v^2)B_{n-3})/(w-v) ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot ,$$
$$B_n = ((w^n - v^n)(w+v) - wv(w^{n-1} - v^{n-1})B_0)/(w-v) ,$$

and we have

(6) $$B_n = \frac{w^{n+1} - v^{n+1}}{w - v} .$$

Where, in (1) we replace k_1 with $k_1^{1/2}$ and k_2 with 1, and combining this result with (3) and (6), leads to

(7) a) $$C_n = \frac{(k_1^{1/2} + (k_1+4)^{1/2})^{n+1} - (k_1^{1/2} - (k_1+4)^{1/2})^{n+1}}{((k_1+4)^{1/2})2^{n+1}} = \phi(k_1),$$

and

b) $$y_n = \phi(k_1)/k_1^{1/2} .$$

(8) Theorem.

The attenuation (input voltage/output voltage = A) of m-L sections in cascade in a ladder-network is given by

$$A^2 = \sum_{r=0}^{2m-2} C_r((-C_{2m-1})/C_{2m-2})^r) .$$

The proof of the theorem rests on the following

(9) Lemma.

The power series

$$(-1)^n \sum_{r=0}^{n} B_r x^r ,$$

is always a square, where B_r is defined in (1).

Proof of lemma.

Let

(10) $$1 = (1-k_1x - k_2x^2)(\sum_{r=0}^{n} B_r x^r) ,$$

then, by comparing coefficients and by (1), we have

(11) $$x = \frac{-(B_nk_1 + B_{n-1}k_2)}{B_nk_2} = \frac{-B_{n+1}}{B_nk_2} ,$$

and replacing x with $(-B_{n+1})/(B_nk_2)$ in $(1-k_1x-k_2x^2)$, leads to

(12) $$1-k_1x-k_2x^2 = (B_n^2k_2+B_nB_{n+1}k_1-B_{n+1}^2)/(B_n^2k_2) .$$

By (4, d) and (6) it is easily verified

(13) $$B_n^2 - B_{n+1}B_{n-1} = (-k_2)^n ,$$

so that

(14) $$B_n^2 k_2 + B_n B_{n+1} k_1 - B_{n+1}^2 = (-1)^n k_2^{n+1} .$$

Then, replacing the numerator in (12) by the result in (14) leads to

(15) $$1 - k_1 x - k_2 x^2 = ((-1)^n k_2^n)/B_n^2 ,$$

so that (10) may be written as

(16) $$(-1)^n B_n^2 = \sum_{r=0}^{n} B_r x^r ,$$

which completes the proof of the lemma.

(17) The proof of the theorem is immediate, when in (11) and (16), we replace n with 2m-2, k_1 with $k_1^{1/2}$, k_2 with 1, and combine the result with (7, a) and the values of the attenuation in Table 1.

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1. a) S. L. Basin, "The Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory," Math Mag., 36(1963) pp. 84-97.

b) S. L. Basin, "The Fibonacci Sequence as it Appears in Nature," Fibonacci Quarterly, 1(1963) pp. 54-55.

The author expresses his gratitude and thanks to Professor L. Carlitz, Duke University; Professor V. E. Hoggatt, Jr., San Jose State College; and the referee.

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CONCERNING LATTICE PATHS AND FIBONACCI NUMBERS

DOUGLAS R. STOCKS, JR.
Arlington State College, Arlington, Texas

R. E. Greenwood [1] has investigated plane lattice paths from (0, 0) to (n, n) and has found a relationship between the number of paths in a certain restricted subclass of such paths and the Fibonacci sequence. Considering such paths and using a method of enumeration different from that used by Greenwood, an unusual representation of Fibonacci's sequence is suggested.

The paths considered here are comprised of steps of three types: (i) horizontal from (x, y) to (x + 1, y); (ii) vertical from (x, y) to (x, y + 1); and (iii) diagonal from (x, y) to (x + 1, y + 1).

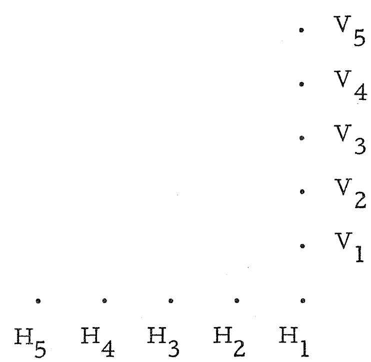

Figure 1

In the interest of simplicity of representation, we will here consider the paths from H_i to V_i, for each positive integer i. Note that the number of paths from H_i to V_i is the number of paths from (0, 0) to (i, i). However, instead of considering the total number of paths from H_i to V_i as was done by Greenwood, we will count only the number of paths from H_i to V_i which do not contain as subpaths any of the paths from H_j to V_j, for $j < i$. This number plus the number of paths from H_{i-1} to V_{i-1} is the total number of paths from H_i to V_i. The use of this counting device suggest the

Theorem:

Let

$$1_D = 1$$

$$2_D = \left[\frac{D-1}{2}\right], \text{ where } [\ ] \text{ denotes the greatest integer function}$$

$$3_D = 3_{D-1} + 2_{D-1}$$

$$4_D = 4_{D-2} + 3_{D-2}$$

$$\cdots$$

$$(2n)_D = (2n)_{D-2} + (2n-1)_{D-2}$$

$$(2n+1)_D = (2n+1)_{D-1} + (2n)_{D-1}$$

$$\cdots$$

with the restriction that $k_D = 0$ if $k > D$. For each positive integer D, let

$$f(D) = \sum_{k=1}^{D} k_D \ .$$

The sequence $\{f(D) \mid D = 1, 2, 3, \ldots\}$ is the Fibonacci sequence.

The proof is direct and is therefore omitted.

The geometric interpretation of the numbers k_D and f(D) mentioned in the theorem is interesting. However, before considering this interpretation it is necessary to define a section of a path. For this purpose we will now consider a path as the point set to which p belongs if and only if for some step ((x, y), (u, v)) of the path, p belongs to the line interval whose end points are (x, y) and (u, v). A section of a path is a line interval which is a subset of the path and which is not a subset of any other line interval each of whose points is a point of the path.

The above mentioned geometric interpretation follows: By definition f(1) = 1. For each positive integer $D \geq 2$, let L_D denote the set of paths from H_D to V_D which do not contain as subpaths any of the paths from H_j to V_j, for $j < D$. f(D) is the number of paths belonging to the set L_D. k_D is the number of paths in the subset X of L_D such that x belongs to X if and only if x contains as subsets exactly k diagonal sections.

Figure 2 portrays the five paths which belong to L_5. In Figure 2a appears the one path of L_5 which contains only one diagonal section ($1_5 = 1$). The two paths of L_5 which contain exactly two diagonal sections appear in Figure 2b ($2_5 = 2$). In Figure 2c the two paths of L_5 which contain exactly three diagonal sections are shown ($3_5 = 2$). It is noted that $4_5 = 5_5 = 0$.

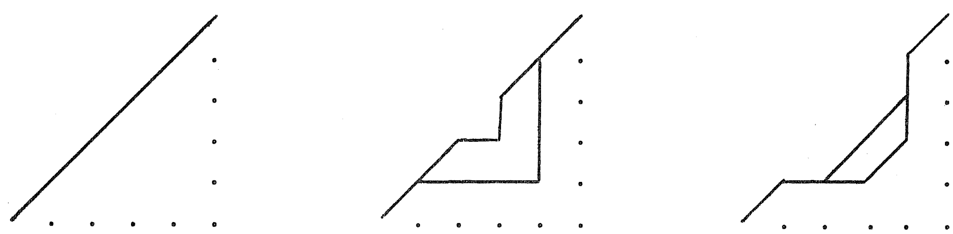

Fig. 2a $1_5 = 1$ Fig. 2b $2_5 = 2$ Fig. 2c $3_5 = 2$

$$f(5) = 1 + 2 + 2 + 0 + 0 = 5$$

Figure 2

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1. R. E. Greenwood, "Lattice Paths and Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2, No. 1, pp. 13-14.

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REPLY TO EXPLORING FIBONACCI MAGIC SQUARES*

JOHN L. BROWN, JR.
Pennsylvania State University, State College, Pennsylvania

Problem. For $n \geq 2$, show that there do not exist any nxn magic squares with distinct entries chosen from the set of Fibonacci numbers, $u_1 = 1$, $u_2 = 2$, $u_{n+2} = u_{n+1} + u_n$ for $n \geq 1$.

Proof. Trivial for $n = 2$.

If an nxn magic square existed for some $n \geq 3$ with distinct Fibonacci entries, then the requirement that the first three columns add to the same number would yield the equalities:

$$(*) \quad F_{i_1} + F_{i_2} + \ldots + F_{i_n} = F_{j_1} + F_{j_2} + \ldots + F_{j_n} = F_{k_1} + F_{k_2} + \ldots + F_{k_n} .$$

Since the entries are distinct, we may assume without loss of generality that $F_{i_1} > F_{i_2} > \ldots > F_{i_n}$, $F_{j_1} > F_{j_2} > \ldots > F_{j_n}$ and $F_{k_1} > F_{k_2} > \ldots > F_{k_n}$.

Noting that the columns contain no common elements, and by rearrangement if necessary, we assume $F_{i_1} > F_{j_1} > F_{k_1}$, again without losing generality; thus, $F_{i_1} \geq F_{k_1} + 2$.

Now

$$F_{i_1} + F_{i_2} + \ldots + F_{i_n} > F_{i_1} \geq F_{k_1+2} ,$$

while

$$F_{k_1} + F_{k_2} + \ldots + F_{k_n} \leq \sum_{1}^{k_1} F_i = F_{k_1+2} - 1 .$$

This contradicts the equality postulated in (*), and we conclude no magic squares in distinct Fibonacci numbers are possible.

*The Fibonacci Quarterly, October 1964, Page 216.

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THE FIBONACCI NUMBER F_u WHERE u IS NOT AN INTEGER

ERIC HALSEY
Redlands, California

INTRODUCTION

Fibonacci numbers, like factorials, are not naturally defined for any values except integer values. However the gamma function extends the concept of factorial to numbers that are not integers. Thus we find that $(1/2)! = \sqrt{\pi}/2$. This article develops a function which will give F_n for any integer n but which will furthermore give F_u for any rational number u. The article also defines a quantity $n\not\Delta^m$ and develops a function $f(x, y) = x\not\Delta^y$ where x and y need not be integers.

(1) DEFINITIONS

Let $n\not\Delta^0 = 1$ (Definitions (1) hold for all $n \in N$)

Let

$$n\not\Delta^1 \text{ (read "n cardinal") } = \sum_{k=1}^{n} k\not\Delta^0 = \sum_{k=1}^{n} 1 = n \quad .$$

This gives the cardinal numbers 1, 2, 3, ...

Let

$$n\not\Delta^2 \text{ (read "n triangular") } = \sum_{k=1}^{n} k\not\Delta^1 = \sum_{k=1}^{n} k \quad .$$

This gives the triangular numbers 1, 3, 6, 10, ...

Let

$$n\not\Delta^3 \text{ (read "n tetrahedral") } = \sum_{k=1}^{n} k\not\Delta^2 \quad .$$

This gives the tetrahedral numbers 1, 4, 10, 20, ...

In general, let

$$n\not\Delta^m \text{ (read "n delta-slash m") } = \sum_{k=1}^{n} k\not\Delta^{m-1} \quad .$$

This gives a figurate number series which can be assigned to the m-dimensional analog of the tetrahedron (which is the 3-dimensional analog of the triangle, etc.).

Let us construct an array $(a_{i,j})$, where we assign to each $a_{i,j}$ an appropriate coefficient of Pascal's triangle.

$$(a_{i,j}) = \begin{matrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & & & & & \end{matrix}$$

It is clear that in this arrangement the usual rule for forming Pascal's triangle is just

$$a_{i,j} = a_{i,j-1} + a_{i-1,j} \quad . \tag{2}$$

But a comparison of this rule with the definitions (1) shows that Pascal's triangle can be written:

$$\begin{matrix} 1\Delta^0 & 1\Delta^1 & 1\Delta^2 & \cdots & 1\Delta^m & \cdots \\ 2\Delta^0 & 2\Delta^1 & 2\Delta^2 & \cdots & 2\Delta^m & \cdots \\ 3\Delta^0 & 3\Delta^1 & 3\Delta^2 & \cdots & 3\Delta^m & \cdots \\ \vdots & & & \cdots & & \cdots \\ n\Delta^0 & n\Delta^1 & n\Delta^2 & \cdots & n\Delta^m & \cdots \\ \vdots & & & & & \end{matrix}$$

where $a_{i,j} = i\Delta^{j-1}$. From the symmetry of Pascal's triangle, $a_{i,j} = a_{j,i}$. Therefore

$$i\Delta^{j-i} = j\Delta^{i-1}; \quad n\Delta^m = (m+1)\Delta^{n-1} \quad . \tag{3}$$

Pascal's triangle is a well-known generator of Fibonacci numbers in the way shown in the following diagram.

1	1	1	1	1	...	$1 = 1 = F_1$
1	2	3	4	5	...	$1 = 1 = F_2$
1	3	6	10	15	...	$1+1 = 2 = F_3$
1	4	10	20	35	...	$1+2 = 3 = F_4$
1	5	15	35	70	...	$1+3+1 = 5 = F_5$
⋮						⋮

We can apply the same course to our abstracted Pascal's triangle.

$1\not\Delta^0$	$1\not\Delta^1$	$1\not\Delta^2$	...	$F_1 = 1\not\Delta^0$
$2\not\Delta^0$	$2\not\Delta^1$	$2\not\Delta^2$	...	$F_2 = 2\not\Delta^0$
$3\not\Delta^0$	$3\not\Delta^1$	$3\not\Delta^2$	...	$F_3 = 3\not\Delta^0 + 1\not\Delta^1$
⋮				$F_4 = 4\not\Delta^0 + 2\not\Delta^1$
				⋮

It is clear that, if we keep forming Fibonacci numbers from Pascal's triangle in this way, $F_n = n\not\Delta^0 + (n-2)\not\Delta^1 + (n-4)\not\Delta^2 + \ldots + (n-2m)\not\Delta^m$, or

(4) $$F_n = \sum_{k=0}^{m} (n-2k)\not\Delta^k \quad ,$$

where we require that m be an integer and that $0 < n-2m \leq 2$, or in other words that $n/2 - 1 \leq m < n/2$. Now let us prove

(5) Theorem 1 $$n\not\Delta^m = \binom{n+m-1}{m}$$

Proof: It is sufficient to perform induction on n. Let the theorem be $E(n)$. Then if $n = 1$, $E(1)$ states

$$\binom{n+m-1}{m} = \binom{1+m-1}{m} = \frac{m!}{m!} = 1 \quad .$$

But by definition (1), $(m+1)\Delta^0 = 1$ for any $(m+1) \in N$. Then by equation (3) $1\Delta^m = 1$ for $m = 0, 1, 2, 3, \ldots$ and E(1) is true. Now let us assume that, for arbitrary $m \in N$, E(n) is true. Then

$$n\Delta^m = \binom{n+m-1}{m} .$$

From the definitions (1) it can be seen that

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + n\Delta^{m-1} = n\Delta^m .$$

Therefore the induction hypothesis can be restated

(6) $$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} = \binom{n+m-1}{m} .$$

Add $\binom{n+m-1}{m-1}$ to both sides of equation (6) to obtain

(7) $$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{n+m-1}{m-1}$$
$$= \binom{n+m-1}{m} + \binom{n+m-1}{m-1}$$

The right-hand side of equation (7) is $\binom{n+m}{m}$ by the standard identity for combinations, so we have

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{n+m-1}{m-1} = \binom{n+m}{m} ,$$

or

$$1\Delta^{m-1} + 2\Delta^{m-1} + \ldots + \binom{n+m-2}{m-1} + \binom{(n+1)+m-2}{m-1}$$
$$= \binom{(n+1)+m-1}{m} ,$$

which is E(n+1). Therefore E(n) implies E(n+1) and Theorem 1 is true by mathematical induction.

Now let us prove

(8) Theorem 2 $$n\Delta^m = \left[(n+m)\int_0^1 x^{n-1}(1-x)^m dx\right]^{-1}$$

Proof: $\Gamma(n) = (n-1)!$ (gamma function)

$$B(m, n) = B(n, m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{(beta function)}$$

Therefore

$$\frac{1}{B(m,n)} = \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} ,$$

and

$$\frac{1}{B(m+1,\,n-m+1)} = \frac{\Gamma(n+2)}{\Gamma(m+1)\,\Gamma(n-m+1)} = \frac{(n+1)!}{m!\,(n-m)!}$$

$$= \frac{(n+1)n!}{m!\,(n-m)!} = (n+1)\binom{n}{m} .$$

Then

$$(9) \qquad \binom{n}{m} = \frac{1}{(n+1)B(m+1,\,n-m+1)} = \left[(n+1)B(m+1,\,n-m+1)\right]^{-1} .$$

We can now substitute the right-hand side of equation (5) into equation (9) to obtain

$$n\not\Delta^{m} = \binom{n+m-1}{m} = \left[(n+m)B(m+1,\,n)\right]^{-1} ,$$

where

$$B(m+1,\,n) = B(n,\,m+1) = \int_0^1 x^{n-1}(1-x)^m dx .$$

Therefore

$$n\not\Delta^{m} = \left[(n+m)\int_0^1 x^{n-1}(1-x)^m dx\right]^{-1} .$$

Both equations (5) and (8) assert that $n\not\Delta^{m} = (m+1)\not\Delta^{n-1}$. Some interesting special cases of equation (5) are

$$n\not\Delta^{0} = \binom{n-1}{0} = \frac{(n-1)!}{(n-1)!} = 1 ,$$

$$n\not\Delta^{1} = \binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n ,$$

and

$$\sum_{k=1}^{n} k = n\not\Delta^{2} = \binom{n+1}{2} = \frac{(n+1)!}{(n-1)!\,2!} = \frac{(n)(n+1)}{2} .$$

Now we can put equation (8) into equation (4) to obtain

$$F_n = \sum_{k=0}^{m} \left[(n-k) \int_0^1 x^{n-2k-1}(1-x)^k dx\right]^{-1} , \tag{10}$$

where m is an integer, $n/2 - 1 \leq m < n/2$. But whereas equations (4) and (5) have meaning only for integer arguments, equations (8) and (10) can be used to find $x\Delta^y$ and F_u, where x, y, and u are any rational numbers.

In particular

$$F_u = \sum_{k=0}^{m} \left[(u-k) \int_0^1 x^{u-2k-1}(1-x)^k dx\right]^{-1} , \tag{11}$$

where m is an integer, $u/2 - 1 \leq m < u/2$. The equation (11), and the definite integral in it, are easily programmed for solution on a digital computer. A few values of F_u follow.

u	F_u
4.1000000	3.1550000
4.2000000	3.3200000
4.3000000	3.4950000
4.4000000	3.6800000
4.5000000	3.8750000
4.6000000	4.0800000
4.7000000	4.2950000
4.8000000	4.5200000
4.9000000	4.7550000
5.0000000	5.0000000
5.1000000	5.2550000
5.2000000	5.5200000
5.3000000	5.7950000
5.4000000	6.0800000
5.5000000	6.3750000
5.6000000	6.6800000
5.7000000	6.9950000
5.8000000	7.3200000
5.9000000	7.6550000
6.0000000	8.0000000

u	F_u
0.1	1.0
0.2	1.0
⋮	⋮
2.0	1.0
2.1	1.1
2.2	1.2
⋮	⋮
3.0	2.0
3.1	2.1
⋮	⋮
4.0	3.0

XXXXXXXXXXXXXXX

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems should be submitted on separate sheets in the format used below within two months of publication.

B-64 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $L_n L_{n+1} = L_{2n+1} + (-1)^n$, where L_n is the n-th Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$.

B-65 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Let u_n and v_n be sequences satisfying $u_{n+2}+au_{n+1}+bu_n=0$ and $v_{n+2}+cv_{n+1}+dv_n=0$ where a, b, c, and d are constants and let $(E^2+aE+b)(E^2+cE+d) = E^4+pE^3+qE^2+rE+s$. Show that $y_n=u_n+v_n$ satisfies

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0 \quad .$$

B-66 *Proposed by D.G. Mead, University of Santa Clara, Santa Clara, California*

Find constants p, q, r, and s such that

$$y_{n+4}+py_{n+3}+qy_{n+2}+ry_{n+1}+sy_n = 0$$

is a 4th order recursion relation for the term-by-term products $y_n=u_nv_n$ of solutions of $u_{n+2}-u_{n+1}-u_n = 0$ and $v_{n+2}-2v_{n+1}-v_n = 0$.

B-67 *Proposed by D.G. Mead, University of Santa Clara, Santa Clara, California*

Find the sum $1\cdot 1+1\cdot 2+2\cdot 5+3\cdot 12+\ldots+F_nG_n$, where $F_{n+2}=F_{n+1}+F_n$ and $G_{n+2}=2G_{n+1}+G_n$.

B-68 *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania*

Find expressions in terms of Fibonacci numbers which will generate integers for the dimensions and diagonal of a rectangular parallelopiped, i.e., solutions of

$$a^2+b^2+c^2 = d^2 \quad .$$

B-69 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the system of simultaneous equations:

$$xF_{n+1}+yF_n = x^2+y^2$$

$$xF_{n+2}+yF_{n+1} = x^2+2xy$$

where F_n is the n-th Fibonacci number.

SOLUTIONS

CHEBYSHEV POLYNOMIALS

B-27 *Proposed by D.C. Cross, Exeter, England*

Corrected and restated from Vol. 1, No. 4: The Chebyshev Polynomials $P_n(x)$ are defined by $P_n(x) = \cos(n\text{Arccos } x)$. Letting $\phi = \text{Arccos } x$, we have

$$\cos \phi = x = P_1(x),$$

$$\cos (2\phi) = 2\cos^2\phi - 1 = 2x^2 - 1 = P_2(x),$$

$$\cos (3\phi) = 4\cos^3\phi - 3\cos\phi = 4x^3 - 3x = P_3(x),$$

$$\cos (4\phi) = 8\cos^4\phi - 8\cos^2\phi + 1 = 8x^4 - 8x^2 + 1 = P_4(x), \quad \text{etc.}$$

It is well known that

$$P_{n+2}(x) = 2xP_{n+1}(x) - P_n(x) \quad .$$

Show that

$$P_n(x) = \sum_{j=0}^{m} B_{jn} x^{n-2j}$$

where

$$m = \left[n/2 \right] ,$$

the greatest integer not exceeding $n/2$, and

(1) $B_{on} = 2^{n-1}$

(2) $B_{j+1, n+1} = 2B_{j+1, n} - B_{j, n-1}$

(3) If $S_n = |B_{on}| + |B_{1n}| + \ldots + |B_{mn}|$, then $S_{n+2} = 2S_{n+1} + S_n$.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

By De Moivre's Theorem,

$$(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi \quad .$$

Letting $x = \cos \phi$, and expanding the left side,

$$\cos n\phi + i \sin n\phi = (x + i \sqrt{1 - x^2})^n$$

$$= \sum_{j=0}^{n} (-1)^{j/2} \binom{n}{j} x^{n-j} (1 - x^2)^{j/2} \quad .$$

We equate real parts, noting that only the even terms of the sum are real,

$$\cos n\phi = P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} x^{n-2k}(1 - x^2)^k \quad .$$

We may prove from this (cf. Formula (22), p. 185, Higher Transcendtal Functions, Vol. 2 by Erdelyi et al; R. G. Buschman, "Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations," Fibonacci Quarterly, Vol. 1, No. 4, p. 2) that

(*) $$B_{j, n} = \frac{n (-1)^j 2^{n-2j-1} (n-j-1)!}{j! (n-2j)!} \quad .$$

From this, we have

(1) $$B_{o, n} = 2^{n-1} \quad .$$

It is also easy to show from (*) that

(2) $$B_{j+1, n+1} = 2\,B_{j+1, n} - B_{j, n-1} \quad .$$

Now (*) implies

$$B_{j, n} = (-1)^j \left| B_{j, n} \right| \quad ,$$

so that (2) becomes

$$(-1)^{j+1} \left| B_{j+1, n+1} \right| = 2\,(-1)^{j+1} \left| B_{j+1, n} \right| + (-1)^{j+1} \left| B_{j, n-1} \right| \quad ,$$

or

$$\left| B_{j+1, n+1} \right| = 2 \left| B_{j+1, n} \right| + \left| B_{j, n-1} \right| \quad .$$

Summing both sides for j to $\left[\frac{n+1}{2} \right]$, we have

(3) $$S_{n+1} = 2\,S_n + S_{n-1} \quad .$$

Also solved by the proposer.

A SPECIAL CASE

B-52 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that $F_{n-2}F_{n+2} - F_n^2 = (-1)^{n+1}$, where F_n is the n-th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Identity XXII (Fibonacci Quarterly, Vol. 1, No. 2, April 1963, p. 68) states:

$$F_nF_m - F_{n-k}F_{m+k} = (-1)^{n-k} F_kF_{m+k-n} \quad .$$

The proposed identity is immediate on taking $m = n$ and $k = 2$.

More generally, we have

$$F_n^2 - F_{n-k}F_{n+k} = (-1)^{n-k}F_k^2 \quad \text{for} \quad 0 \le k \le n \quad .$$

Also solved by Marjorie Bicknell, Herta T. Freitag, John E. Homer, Jr., J.A.H. Hunter, Douglas Lind, Gary C. MacDonald, Robert McGee, C.B.A. Peck, Howard Walton, John Wessner, Charles Ziegenfus, and the proposer.

SUMMING MULTIPLES OF SQUARES

B-53 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Show that

$$(2n - 1)F_1^2 + (2n - 2)F_2^2 + \ldots + F_{2n-1}^2 = F_{2n}^2 \quad .$$

Solution by James D. Mooney, University of Notre Dame, Notre Dame, Indiana

Remembering that

$$\sum_{k=0}^{n} F_k^2 = F_n F_{n+1} \quad ,$$

we may proceed by induction. Clearly for $n = 1$, $F_1^2 = 1 = F_2^2$. Assume

$$\left[2(n-1) - 1\right] F_1^2 + \left[2(n-1) - 2\right] F_2^2 + \ldots + F_{2(n-1)-1} =$$

$$= (2n-3)F_1^2 + (2n-4)F_2^2 + \ldots + F_{2n-3} = F_{2n-2}^2 \quad .$$

Then

$$(2n-1)F_1^2 + \ldots + F_{2n-1} = \left[(2n-3)F_1^2 + \ldots + F_{2n-3}\right] +$$

$$2(F_1^2 + \ldots + F_{2n-2}^2) + F_{2n-1}^2 = F_{2n-2}^2 + \sum_{k=0}^{2n-2} F_k^2 + \sum_{k=0}^{2n-1} F_k^2 =$$

$$F_{2n-2}^2 + F_{2n-2}F_{2n-1} + F_{2n-1}F_{2n} = F_{2n-2}^2 + F_{2n-2}F_{2n-1} +$$

$$+ F_{2n-1}(F_{2n-2} + F_{2n-1}) = F_{2n-2}^2 + 2F_{2n-2}F_{2n-1} + F_{2n-1}^2 =$$

$$(F_{2n-2} + F_{2n-1})^2 = F_{2n}^2 \quad . \quad \text{Q.E.D.}$$

Also solved by Marjorie Bicknell, J.L. Brown, Jr., Douglas Lind, John E. Homer, Jr., Robert McGee, C.B.A. Peck, Howard Walton, David Zeitlin, Charles Ziegenfus, and the proposer.

RECURRENCE RELATION FOR DETERMINANTS

B-54 *Proposed by C.A. Church, Jr., Duke University, Durham, N. Carolina*

Show that the n-th order determinant

$$f(n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & & 0 & 0 \\ -1 & a_2 & 1 & 0 & & 0 & 0 \\ 0 & -1 & a_3 & 1 & & 0 & 0 \\ 0 & 0 & -1 & a_4 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ \cdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & a_n \end{vmatrix}$$

satisfies the recurrence $f(n) = a_n f(n-1) + f(n-2)$ for $n > 2$.

Solution by John E. Homer, Jr., La Crosse, Wisconsin

Expanding by elements of the n-th column yields the desired relation immediately.

Also solved by Marjorie Bicknell, Douglas Lind, Robert McGee, C.B.A. Peck, Charles Ziegenfus, and the proposer.

AN EQUATION FOR THE GOLDEN MEAN

B-55 *From a proposal by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that $x^n - xF_n - F_{n-1} = 0$ has no solution greater than a, where $a = (1 + \sqrt{5})/2$, F_n is the n-th Fibonacci number, and $n > 1$.

Solution by G.L. Alexanderson, University of Santa Clara, California

For $n > 1$ let $p(x, n) = x^n - xF_n - F_{n-1}$, $g(x) = x^2 - x - 1$, and $h(x, n) = x^{n-2} + x^{n-3} + 2x^{n-4} + \ldots + F_k x^{n-k-1} + \ldots + F_{n-2}x + F_{n-1}$. It is easily seen that $p(x, n) = g(x)h(x, n)$, $g(x) < 0$ for $-1/a < x < a$, $g(a) = 0$, $g(x) > 0$ for $x > a$, and $h(x, n) > 0$ for $x \geq 0$. Hence $x = a$ is the unique positive root of $p(x, n) = 0$.

Also solved by J.L. Brown, Jr., Douglas Lind, C.B.A. Peck, and the proposer.

GOLDEN MEAN AS A LIMIT

B-56 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let F_n be the n-th Fibonacci number. Let $x_0 \geq 0$ and define $x_1, x_2, \ldots$ by $x_{k+1} = f(x_k)$ where

$$f(x) = \sqrt[n]{F_{n-1} + xF_n} .$$

For $n > 1$, prove that the limit of x_k as k goes to infinity exists and find the limit. (See B-43 and B-55.)

Solution by G.L. Alexanderson, University of Santa Clara, Santa Clara, California

For $n > 1$ let $p(x) = x^n - xF_n - F_{n-1}$. Let $a = (1 + \sqrt{5})/2$. As in the proof of B-55, one sees that $p(x) > 0$ for $x > a$ and that $p(x) < 0$ for $0 \leq x < a$. If $x_k > a$, we then have

$$(x_k)^n > x_kF_n + F_{n-1} = (x_{k+1})^n$$

and so $x_k > x_{k+1}$. It is also clear that $x_k > a$ implies

$$(x_{k+1})^n = x_kF_n + F_{n-1} > aF_n + F_{n-1} = a^n$$

and hence $x_{k+1} > a$. Thus $x_o > a$ implies $x_o > x_1 > x_2 > \ldots > a$. Similarly, $0 \leq x_o < a$ implies $0 \leq x_o < x_1 < x_2 < \ldots < a$. In both cases the sequence $x_o, x_1, \ldots$ is monotonic and bounded. Hence x_k has a limit $L > 0$ as k goes to infinity. Since L satisfies

$$L = \sqrt[n]{F_{n-1} + LF_n} ,$$

L must be the unique positive solution of $p(x) = 0$.

Also solved by Douglas Lind and the proposer.

A FIBONACCI-LUCAS INEQUALITY

B-57 *Proposed by G.L. Alexanderson, University of Santa Clara, Santa Clara, California*

Let F_n and L_n be the n-th Fibonacci and n-th Lucas number respectively. Prove that

$$(F_{4n}/n)^n > L_2L_6L_{10} \cdots L_{4n-2}$$

for all integers $n > 2$.

Solution by David Zeitlin, Minneapolis, Minnesota

Using mathematical induction, one may show that

$$F_{4n} = \sum_{k=1}^{n} L_{4k-2}, \quad n = 1, 2, \ldots \quad .$$

If we apply the well-known arithmetic-geometric inequality to the unequal positive numbers $L_2, L_6, L_{10}, \ldots, L_{4n-2}$, we obtain for $n = 2, 3, \ldots$,

$$\frac{F_{4n}}{n} = \frac{\sum_{k=1}^{n} L_{4k-2}}{n} = \sqrt[n]{L_2 L_6 L_{10} \cdots L_{4n-2}} \quad ,$$

which is the desired inequality.

Also solved by Douglas Lind and the proposer.

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ACKNOWLEDGMENT

It is a pleasure to acknowledge the assistance furnished by Prof. Verner E. Hoggatt, Jr. concerning the essential idea of "Maximal Sets" and the line of proof suggested in the latter part of my article "On the Representations of Integers as Distinct Sums of Fibonacci Numbers." The article appeared in Feb., 1965. H. H. Ferns

CORRECTION Volume 3, Number 1

Page 26, line 10 from bottom of page

$$V_{7,3} + V_{7,4} + V_{7,5} = F_8 - F_7 = F_6 = 8$$

Page 27, lines 4 and 5

$$F_2 + F_4 + F_6 + \ldots + F_n = F_{n+1} - 1 \quad \text{(n even)}$$

$$F_3 + F_5 + F_7 + \ldots + F_n = F_{n+1} - 1 \quad \text{(n odd)}$$

ACKNOWLEDGMENT

Both the papers "Fibonacci Residues" and "On a General Fibonacci Identity," by John H. Halton, were supported in part by NSF grant GP2163.

CORRECTION Volume 3, Number 1

Page 40, Equation (81), the R. H. S. should have an additional term

$$- v^2 F_{v+2}$$