BINOMIAL COEFFICIENTS, THE BRACKET FUNCTION, AND COMPOSITIONS WITH RELATIVELY PRIME SUMMANDS

H.W. GOULD
West Virginia University, Morgantown, West Virginia*

To Professor Alfred T. Brauer on the occasion of his 70th birthday.

It is known [5] that a necessary and sufficient condition for p to be prime is that for every natural number n

$$\binom{n}{p} \equiv \left[\frac{n}{p}\right] \pmod{p}, \tag{1}$$

where $[x]$ denotes the greatest integer less than or equal to x.

Indeed this result is equivalent to the congruence

$$(1 - x)^k \equiv 1 - x^k \pmod{p}$$

as is evident from the generating functions

$$\sum_{n=k}^{\infty} \binom{n}{k} x^{n-k} = (1 - x)^{-k-1}, \quad |x| < 1 \tag{2}$$

and

$$\sum_{n=k}^{\infty} \left[\frac{n}{k}\right] x^{n-k} = (1 - x)^{-1} (1 - x^k)^{-1}, \quad |x| < 1 . \tag{3}$$

These results and some extensions of (1) in a recent paper [2] suggest that there is more than a casual relation between the binomial coefficients and the bracket function. In the present paper this relation is made evident by exhibiting an expansion of the binomial coefficients in terms of the bracket function, and conversely. These expansions give congruences equivalent to (1), and the expansions are a special case of a general inversion theorem. In the course of the analysis we obtain novel results concerning the compositions (ordered partitions) of a natural number into relatively prime summands. Expansions involving unordered partitions are also developed.

*Research supported by National Science Foundation Grant GP-482.

The compositions (Zergliederungen) of n into positive summands are given as the solution of the Diophantine equation

$$a_1 + a_2 + \ldots + a_k = n, \quad (a_i \geq 1) \tag{4}$$

whereas the partitions (Zerfällungen) of n into positive summands are given by the same equation together with the restriction that

$$1 \leq a_1 \leq a_2 \leq \ldots \leq a_k .$$

Thus the compositions of 4 into positive summands in all are:

4; 3+1; 1+3; 2+2; 2+1+1; 1+2+1; 1+1+2; 1+1+1+1 .

The partitions are: 4; 1+3; 2+2; 1+1+2; 1+1+1+1 .

Catalan [3], [4], [6, Vol. 2, 114, 126] proved in 1838 that the equation

$$a_1 + a_2 + \ldots + a_k = n, \quad (a_i \geq 0) \tag{5}$$

has $\binom{n+k-1}{k-1}$ solutions. He then observed in 1868 that equation (4) has $\binom{n-1}{k-1}$ solutions. In fact this follows by adding 1 to each summand in (5). A direct proof of the enumeration is not difficult. Indeed (Cf. Bachmann [1, Vol. 2, 105-7]; MacMahon [8, Vol. 1, 150-1]; Riordan [10, 124]) if $C_k(n)$ be the number of compositions of n into k positive summands, then

$$(x + x^2 + x^3 + \ldots)^k = \sum_{n=k}^{\infty} C_k(n)\, x^n$$

$$= \left(\frac{x}{1-x}\right)^k = \sum_{n=k}^{\infty} \binom{n-1}{k-1} x^n ,$$

from which the result is evident. P. Paoli [6, Vol. 2, 107] anticipated Catalan in 1780.

We may state this basic result in the enumerative form

$$C_k(n) = \binom{n-1}{k-1} = \sum_{\substack{a_1 + \ldots + a_k = n \\ a_i \geq 1}} 1 \tag{6}$$

The simple identity

$$\binom{n}{k} = \sum_{j=k}^{n} \binom{j-1}{k-1}$$

now allow us to infer that

$$(7) \qquad \binom{n}{k} = \sum_{j=k}^{n} C_k(j) = \sum_{j=k}^{n} \sum_{\substack{a_1 + \ldots + a_k = j \\ a_i \geq 1}} 1 .$$

With this expansion we are now in a position to assert

Theorem 1.

$$(8) \qquad \binom{n}{k} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] \sum_{\substack{a_1 + \ldots + a_k = j \\ (a_1, \ldots, a_k) = 1}} 1 .$$

Proof. The expansion is evident from (7). When we restrict the solutions of the equation $a_1 + a_2 + \ldots + a_k = j$ to those which are relatively prime, it is evident that we may restore the equality by counting how many multiples of j there are, less than or equal to n, and this is precisely the meaning of $[n/j]$.

A simple example will illustrate. On the one hand, by (7)

$$\binom{10}{3} = \sum_{j=3}^{10} \sum_{\substack{a_1+a_2+a_3 = j \\ a_i \geq 1}} 1 = 1+3+6+10+15+21+28+36 = 120 .$$

However, not all the partitions of j are formed by relatively prime integers. These cases are 6 = 2 + 2 + 2; 8 = 2 + 2 + 4 = 2 + 4 + 2 = 4 + 2 +2; 9 = 3 + 3 + 3; 10 = 2 + 2 + 6 = 2 + 6 + 2 = 6 + 2 + 2 = 2 + 4 + 4 = 4 + 2 + 4 = 4 + 4 + 2. Removing the common factors, we could just as well have written such solutions in the forms 3 = 1 + 1 + 1; 4 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1; 3 = 1 + 1 + 1; 5 = 1 + 1 + 3 = 1 + 3 + 1 = 3 + 1

$+ 1 = 1 + 2 + 2 = 2 + 1 + 2 = 2 + 2 + 1$, provided that we regroup and count multiplicities. This gives

$$\binom{10}{3} = 3(1) + 2(3) + 2(6) + (10 - 1) + 15 + (21 - 3) + (28 - 1) + (36 - 6),$$

or

$$\binom{10}{3} = \sum_{j=3}^{10} \left[\frac{10}{j}\right] \sum_{\substack{a_1+a_2+a_3 = j \\ (a_1, a_2, a_3) = 1}} 1$$

We shall obtain expansion (8) by an entirely different approach later in this paper.

For the sake of completeness we wish to show that Theorem 1 is equivalent to the following result due to J. Schröder [11]. Schröder proved the following Theorem 2.

$$(9) \qquad \binom{n}{k} = \sum_{\substack{(a_1, a_2, \ldots, a_k) = 1 \\ 1 \le a_i \le n-k+1}} \left[\frac{n}{a_1 + a_2 + \ldots + a_k}\right].$$

As far as the writer has been able to determine, this is one of the very few expansions in the literature of the sort under discussion. Schröder proved the formula by an enumeration in k-dimensional space and an induction from k to $k+1$. As for the equivalence of (9) and (8), we have

$$\sum_{\substack{(a_1, \ldots, a_k) = 1 \\ 1 \le a_i \le n-k+1}} \left[\frac{n}{a_1 + \ldots + a_k}\right] = \sum_{1 \le j \le n-k+1} \left[\frac{n}{k+j-1}\right] \sum_{\substack{a_1 + \ldots + a_k = k+j-1 \\ (a_1, \ldots, a_k) = 1}} 1$$

$$= \sum_{1 \le j-k+1 \le n-k+1} \left[\frac{n}{j}\right] \sum_{\substack{a_1 + \ldots + a_k = j \\ (a_1, \ldots, a_k) = 1}} 1 ,$$

which is our relation (8) and the steps are reversible.

In view of Schröder's approach, it is of interest to make some remarks here about lattice points. By a lattice point in k-space is meant a point $(a_1, a_2, \ldots, a_k)$ where the coordinates a_i are integers. If we view space from the origin $(0, \ldots, 0)$ and assume that the presence of a point may block our view of points further out along the same ray, then we may speak of visible lattice points. In order for a point to be a visible lattice point it is necessary and sufficient that $(a_1, a_2, \ldots, a_k) = 1$. Thus we may state the theorem of Schröder in the form of

Theorem 3. Let $V_j(k)$ = the number of visible lattice points in k-space, seen from the origin, and lying on the hyperplane $a_1 + a_2 + \ldots + a_k = j + k - 1$. Then

$$\binom{n}{k} = \sum_{j=1}^{n-k+1} V_j(k) \left[\frac{n}{k+j-1} \right] . \tag{10}$$

Thus, in 2-space,

$$\binom{n}{2} = \sum_{j=1}^{n-1} V_j(2) \left[\frac{n}{j+1} \right] ,$$

where $V_j(2)$ is the number of visible lattice points lying entirely within the first quadrant and on the line $x + y = j + 1$. The successive values of $V_j(2)$ $(j = 1, 2, \ldots)$ here are 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, ... and we always have in this case $V_j(2) \leq j$, since the line segment in question has just this many lattice points in all.

In general we evidently have the estimate

$$V_j(k) \leq \binom{j+k-2}{k-1} . \tag{11}$$

As other examples of Theorem 1 we have

$$\binom{n}{3} = \left[\frac{n}{3}\right] + 3\left[\frac{n}{4}\right] + 6\left[\frac{n}{5}\right] + 9\left[\frac{n}{6}\right] + 15\left[\frac{n}{7}\right] + 18\left[\frac{n}{8}\right] + \ldots$$

$$\binom{n}{4} = \left[\frac{n}{4}\right] + 4\left[\frac{n}{5}\right] + 10\left[\frac{n}{6}\right] + 20\left[\frac{n}{7}\right] + 34\left[\frac{n}{8}\right] + 56\left[\frac{n}{9}\right] + \ldots$$

Since the equation $a_1 + a_2 + \ldots + a_k = k$, $(a_i \geq 1)$, has the sole solution $1 + 1 + \ldots + 1 = k$, we have from Theorem 1 the (equivalent) Corollary 1.

(12) $$\binom{n}{k} - \left[\frac{n}{k}\right] = \sum_{j=k+1}^{n} \left[\frac{n}{j}\right] R_k(j)$$

where the number-theoretic function $R_k(j)$ is defined by

(13) $$R_k(j) = \sum_{\substack{a_1 + \ldots + a_k = j \\ (a_1, \ldots, a_k) = 1}} 1$$

and is the number of compositions of j into k relatively prime positive summands.

In order to relate our expansion to congruence (1) we shall now study the arithmetic nature of the function $R_k(j)$.

First of all, it is easy to use (2), (8), and (3) in order to develop a generating function for $R_k(j)$. Indeed we have

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n=k}^{\infty} \binom{n}{k} x^n = \sum_{n=k}^{\infty} x^n \sum_{j=k}^{n} \left[\frac{n}{j}\right] R_k(j)$$

$$= \sum_{j=k}^{\infty} R_k(j) \sum_{n=j}^{\infty} \left[\frac{n}{j}\right] x^n$$

$$= \sum_{j=k}^{\infty} R_k(j) \frac{x^j}{(1-x)(1-x^j)} ,$$

and the lower summation index may be changed to $j = 1$ since $R_k(j) = 0$ if $j < k$. Thus we have established

Theorem 4. The number-theoretic function $R_k(j)$ is the coefficient in the Lambert series

(14) $$\sum_{j=1}^{\infty} R_k(j) \frac{x^j}{1-x^j} = \frac{x^k}{(1-x)^k} ,$$

or, equivalently,

$$(15)\qquad \sum_{j=1}^{\infty} R_k(j) \frac{x^j}{1 - x^j} = \sum_{j=k}^{\infty} C_k(j)\, x^j .$$

It may be of interest to compare this result with the Lambert series for the Euler totient function (Cf. Knopp [7, 466-7]):

$$(16)\qquad \sum_{j=1}^{\infty} \phi(j) \frac{x^j}{1 - x^j} = \frac{x}{(1 - x)^2} .$$

Now [7, 466-7] it is known that the Lambert series

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} A_n x^n$$

is equivalent to the relation

$$A_n = \sum_{d \mid n} a_d ,$$

and so we have from (15) that

$$(17)\qquad C_k(n) = \sum_{d \mid n} R_k(d) .$$

We invert this expansion by the Möbius inversion theorem and so find

Theorem 5. The number of compositions (ordered partitions) of the integer n into k relatively prime positive summands is given by

$$(18)\qquad R_k(n) = \sum_{d \mid n} C_k(d)\, \mu(n/d) = \sum_{d \mid n} \binom{d-1}{k-1} \mu(n/d) .$$

Therefore we also have Theorem 1 in the equivalent form:

Theorem 6.

$$(19)\qquad \binom{n}{k} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] \sum_{d \mid j} \binom{d-1}{k-1} \mu(j/d) .$$

We have presented what seems a natural way to arrive at relation

(19), but we now give a very short derivation on the basis of a famous formula of E. Meissel. First of all we note the general lemma

(20)
$$\sum_{j \le x} \sum_{d \mid j} f(d, j) = \sum_{d \le x} \sum_{\substack{j \le x \\ d \mid j}} f(d, j) = \sum_{d \le x} \sum_{m \le x/d} f(d, md)$$

valid for any number-theoretic function $f(d, j)$.

Meissel (1850 [6, Vol. 1, 441] proved that for all real $x \ge 1$

(21)
$$\sum_{m \le x} \left[\frac{x}{m}\right] \mu(m) = 1 \quad .$$

Thus we have

$$\sum_{1 \le j \le x} \left[\frac{x}{j}\right] \sum_{d \mid j} \binom{d-1}{k-1} \mu(j/d) = \sum_{d \le x} \binom{d-1}{k-1} \sum_{m \le x/d} \left[\frac{x/d}{m}\right] \mu(m) = \sum_{d \le x} \binom{d-1}{k-1} = \binom{[x]}{k} \quad ,$$

and this gives us (more generally than Theorem 6)

<u>Theorem 7.</u> <u>For any real</u> $x \ge 1$, <u>and natural numbers</u> $k \ge 1$,

(22)
$$\binom{[x]}{k} = \sum_{1 \le j \le x} \left[\frac{x}{j}\right] \sum_{d \mid j} \binom{d-1}{k-1} \mu(j/d) \quad .$$

The arithmetical nature of $R_k(n)$ is of interest and in view of (12) the congruence (1) is evidently equivalent to

<u>Theorem 8.</u> <u>The congruence</u>

(23)
$$R_k(n) \equiv 0 \pmod{k}$$

<u>is true for all natural numbers $n \ge k + 1$ if and only if k is prime.</u>

Our proof will depend on some elementary results about the binomial coefficients and the Möbius function.

Now

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases} \qquad (24)$$

Therefore, if p is any prime which divides each divisor d of n, then

$$\sum_{p|d,\, d|n} \mu(n/d) = \sum_{pd'|pm} \mu(pm/pd') = \sum_{d'|m} \mu(m/d') = \begin{cases} 1, & m = 1, \\ 0, & m > 1, \end{cases}$$

or therefore

$$\sum_{p|d,\, d|n} \mu(n/d) = \begin{cases} 1, & n = p \\ 0, & n > p \end{cases} \qquad (25)$$

Now it is familiar that

$$\binom{d-1}{p-1} \equiv \begin{cases} 0 \pmod p, & p \nmid d, \\ 1 \pmod p, & p \mid d, \end{cases} \qquad (26)$$

and so we have

$$\sum_{d|n} \binom{d-1}{p-1} \mu(n/d) \equiv \begin{cases} 0 \pmod p, & \text{for } p \nmid d,\ d|n, \text{ i.e. } p \nmid n, \\ \sum_{d|n} \mu(n/d), & \text{for } p|d,\ d|n, \text{ i.e. } p|n. \end{cases}$$

Thus in any case ($p|n$ or $p \nmid n$) we have by this and (25) that

$$\sum_{d|n} \binom{d-1}{p-1} \mu(n/d) \equiv 0 \pmod p \qquad (27)$$

for all integers $n \geq p + 1$ if p is a prime.

As for the converse, suppose that $R_k(n) \equiv 0 \pmod k$ for all $n \geq k + 1$. Then, in virtue of (17) we should have

$$C_k(n) \equiv \begin{cases} 1 \pmod{k}, & n \geq k+1, \text{ if } k \mid n, \\ 0 \pmod{k}, & n \geq k+1, \text{ if } k \nmid n, \end{cases}$$

or, equivalently,

$$\binom{n-1}{k-1} \equiv \left[\frac{n}{k}\right] - \left[\frac{n-1}{k}\right] \pmod{k} \quad \text{for all } n \geq k+1 \quad .$$

Summing both sides from $n = k + 1$ to $n = m$ we should then have

$$\binom{m}{k} - 1 \equiv \left[\frac{m}{k}\right] - 1 \pmod{k} \quad \text{for all } m \geq k+1 \quad ,$$

or that is

$$\binom{m}{k} \equiv \left[\frac{m}{k}\right] \pmod{k} \quad \text{for all } m \geq k+1 \quad .$$

But this can happen only when k = prime as we know from our original congruence (1) for which a separate proof is known.

As a matter of fact then, congruence (23) is a simple consequence of (1).

Table of Values of $R_k(n)$

	1	2	3	4	5	6	7	8	9	10	11	12	13.....	n
1	1	0	0	0	0	0	0	0	0	0	0	0	0	
2		1	2	2	4	2	6	4	6	4	10	4	12	
3			1	3	6	9	15	18	27	30	45	42	66	
4				1	4	10	20	34	56	80	120	154	220	
5					1	5	15	35	70	125	210	325	495	
6						1	6	21	56	126	252	461	792	
7							1	7	28	84	210	462	924	
8								1	8	36	120	330	792	
9									1	9	45	165	495	
10										1	10	55	220	
11											1	11	66	
12												1	12	
13													1	
⋮														
k														

The numbers $R_k(n)$ form an interesting modification of the familiar Pascal array. We have from (18) the modified binomial theorem relation

$$\sum_{k=1}^{n} R_k(n)\, x^{k-1} = \sum_{d\mid n} \mu(n/d)(x+1)^{d-1} \quad . \tag{28}$$

In particular, when $x = 1$ this sum represents the total number of compositions of n into relatively prime summands. These values, 1, 1, 3, 6, 15, 27, 63, 120, 252, 495, 1023, 2010, 4095, ... afford a check of the table.

We note a few special values of $R_k(n)$:

$$R_k(p^s) = \binom{p^s - 1}{k-1} - \binom{p^{s-1} - 1}{k-1}, \quad s \geq 1,\ p = \text{prime}, \tag{29}$$

$$R_k(pq) = \binom{pq-1}{k-1} - \binom{p-1}{k-1} - \binom{q-1}{k-1} + \binom{0}{k-1}, \quad p, q \text{ primes}, \tag{30}$$

$$R_k(p^2 q) = \binom{p^2 q - 1}{k-1} - \binom{pq-1}{k-1} - \binom{p^2 - 1}{k-1} + \binom{p-1}{k-1} \quad , \tag{31}$$

with similar formulas for other cases. The expansion always contains as even number of binomial coefficients when $n \geq 2$ since $R_1(n) = 0$ for $n \geq 2$.

It is of interest to translate (18) into terms of Dirichlet series. It is easily shown that the formal relation involves Riemann's Zeta function and is

$$\sum_{n=1}^{\infty} C_k(n)\, n^{-s} = \zeta(s) \sum_{n=1}^{\infty} R_k(n)\, n^{-s} \quad , \tag{32}$$

and this also follows from (17).

Having found the expansion (19) of a binomial coefficient in terms of the bracket function, it is natural to look for an inverse expansion.

Put

$$\left[\frac{n}{k}\right] = \sum_{j=k}^{n} \binom{n}{j} A_k(j) .$$

Then

$$\sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \left[\frac{j}{k}\right] = \sum_{r=k}^{n} A_k(r) \sum_{j=r}^{n} (-1)^{n-j} \binom{n}{j} \binom{j}{r}$$

$$= A_k(n) ,$$

since the inner summation is merely a well-known Kronecker delta.

Thus an expansion inverse to (19) is given by

Theorem 9.

(33) $$\left[\frac{n}{k}\right] = \sum_{j=k}^{n} \binom{n}{j} \sum_{d=k}^{j} (-1)^{j-d} \binom{j}{d} \left[\frac{d}{k}\right] .$$

Since $A_k(k) = 1$ we have an analogy to (12)

Corollary 2.

(34) $$\left[\frac{n}{k}\right] - \binom{n}{k} = \sum_{j=k+1}^{n} \binom{n}{j} A_k(j) ,$$

where

(35) $$A_k(j) = \sum_{d=k}^{j} (-1)^{j-d} \binom{j}{d} \left[\frac{d}{k}\right] .$$

For $A_k(j)$ we next develop an expansion inverse to (14). Indeed, we have from (35), (2), and (3)

$$\sum_{j=k}^{\infty} A_k(j) \left(\frac{x}{1-x}\right)^j = \sum_{d=k}^{\infty} (-1)^d \left[\frac{d}{k}\right] \sum_{j=d}^{\infty} \binom{j}{d} \left(\frac{x}{x-1}\right)^j$$

$$= (1-x) \sum_{d=k}^{\infty} \left[\frac{d}{k}\right] x^d = \frac{x^k}{1-x^k} .$$

It is evident from (35) that $A_k(j) = 0$ for $j < k$, so we have

Theorem 10. The expansion inverse to (14) is

$$\sum_{j=1}^{\infty} A_k(j) \left(\frac{x}{1-x}\right)^j = \frac{x^k}{1-x^k} \quad . \tag{36}$$

Now it is evident that (14) and (36) imply a pair of orthogonal relations involving the functions $R_k(j)$ and $A_k(j)$. By a routine calculation we find upon substitution of the one expansion into the other that we have

Theorem 11. The numbers $R_k(j)$ and $A_k(j)$ satisfy the orthogonality relations

$$\sum_{j=k}^{n} R_k(j) A_j(n) = \delta_k^n \tag{37}$$

and

$$\sum_{j=k}^{n} A_k(j) R_j(n) = \delta_k^n \quad . \tag{38}$$

Thus we have also established a general inversion theorem, of which (19) and (33) are special cases. We have

Theorem 12. For any two sequences $f(n, k)$, $g(n, k)$

$$f(n, k) = \sum_{j=k}^{n} g(n, j) R_k(j) \tag{39}$$

if and only if

$$g(n, k) = \sum_{j=k}^{n} f(n, j) A_k(j) \quad , \tag{40}$$

where $R_k(j)$ is given by (18) and $A_k(j)$ by (35).

An alternative form of (35) is easily gotten by way of the recurrence

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1} \quad .$$

Indeed we find that

$$A_k(n) = \sum_{j=k}^{n} (-1)^{n-j} \binom{n-1}{j-1} \left\{ \left[\frac{j}{k}\right] - \left[\frac{j-1}{k}\right] \right\} ,$$

but $[j/k] - [(j-1)/k] = 1$ or 0 accordingly as $k \mid j$ or $k \nmid j$, whence Theorem 13.

$$(41) \qquad A_k(n) = \sum_{\substack{k \le j \le n \\ k \mid j}} (-1)^{n-j} \binom{n-1}{j-1} = \sum_{1 \le m \le n/k} (-1)^{n-mk} \binom{n-1}{mk-1}$$

Table of Values of $A_k(n)$

k \ n	1	2	3	4	5	6	7	8	9	10	11	12	13..n
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2		1	-2	4	-8	16	-32	64	-128	256	-512	1024	-2048
3			1	-3	6	-9	9	0	-27	81	-162	243	-243
4				1	-4	10	-20	36	-64	120	-240	496	-952
5					1	-5	15	-35	70	-125	200	-255	275
6						1	-6	21	-56	126	-252	463	-804
7							1	-7	28	-84	210	-462	924
8								1	-8	36	-120	330	-792
9									1	-9	45	-165	495
10										1	-10	55	-220
11											1	-11	66
12												1	-12
13													1
⋮ k													

The numbers $A_k(n)$ also form an interesting modification of the Pascal array, and the companion to (28) is

$$(42) \qquad \sum_{k=1}^{n} A_k(n)\, x^{k-1} = \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} \sum_{k=1}^{j} \left[\frac{j}{k}\right] x^{k-1} .$$

When $x = 1$ we recall that

$$\sum_{k=1}^{j} \left[\frac{j}{k}\right] = \sum_{k=1}^{j} \tau(k) , \quad \text{where } \tau(k) = \sum_{d \mid k} 1 ,$$

and so we have

$$\sum_{k=1}^{n} A_k(n) = \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} \sum_{k=1}^{j} \tau(k)$$

$$= \sum_{k=1}^{n} \tau(k) \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j}$$

$$= \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} \tau(k)$$

$$= \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} \tau(k+1) .$$

This result is easily inverted, and we may state these formulas as

<u>Theorem 14.</u> <u>For all integers</u> $n \geq 0$, <u>and</u> $\tau(k)$ = <u>number of divisors of</u> k,

$$(43) \qquad \sum_{j=1}^{n+1} A_j(n+1) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \tau(k+1) = \Delta_{x,1}^{n} \tau(x) \Big|_{x=1}$$

<u>and inversely</u>

$$(44) \qquad \tau(n+1) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=1}^{k+1} A_j(k+1) .$$

The first few values of the sum (43) are 1, 1, -1, 2, -5, 13, -33, 80, -184, 402, -840, ... For example, we have the following difference table:

$$\begin{array}{ccccccccccccc}
1 & & 2 & & 2 & & 3 & & 2 & & 4 & \ldots & \tau(n) \\
& 1 & & 0 & & 1 & & -1 & & 2 & & & \\
& & -1 & & 1 & & -2 & & 3 & & & & \\
& & & 2 & & -3 & & 5 & & & & & \\
& & & & -5 & & 8 & & & & & & \\
& & & & & 13 & & & & & & &
\end{array}$$

The arithmetical nature of $A_k(n)$ is of interest. In view of (34) and (1) we evidently have a result analogous to (23). In fact we have

Theorem 15. The congruence

$$A_k(n) \equiv 0 \pmod{k} \tag{45}$$

is true for all natural numbers $n \geq k + 1$ if and only if k is prime.

Indeed this congruence follows easily from (1) since we have

$$\begin{aligned}
A_k(n) &= \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \left[\frac{j}{k}\right] \\
&\equiv \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \binom{j}{k} \pmod{k} \text{ for all } n \geq k \text{ if and only if } k = \text{prime}, \\
&= \delta_k^n \equiv 0 \pmod{k} \text{ for all } n \geq k+1.
\end{aligned}$$

We should like next to return to relation (28) and give another congruence involving $R_k(n)$. It is known [6, Vol. 1, 84-86] that

$$\sum_{d|n} \mu(d)\, a^{n/d} \equiv 0 \pmod{n} \tag{46}$$

for all integers $a \geq 1$. In fact Gegenbauer showed that

$$\sum_{d|n} f(d)\, a^{n/d} \equiv 0 \pmod{n} \text{ whenever } \sum_{d|n} f(d) \equiv 0 \pmod{n}.$$

Gauss proved (46) when a = prime. Thus we have from (28) that

$$a \sum_{k=1}^{n} R_k(n)\, (a-1)^{k-1} \equiv 0 \pmod{n}, \quad (a \geq 1,\ n \geq 1) \tag{47}$$

and in particular this holds for $a = 2$. Thus the numbers 2, 2, 6, 12, 30, 54, 126, 240, 504, 990, 2046, 4020, 8190, ... are, respectively, divisible by 1, 2, 3, 4, 5, ... thereby affording a check of the column sums in the table of values of $R_k(n)$ given previously.

It should be remarked that any formula, such as (18), which gives the number of compositions of n into k relatively prime positive summands also solves the problem of counting how many compositions are possible when the summands have greatest common divisor g; for clearly if $R_k(n, g)$ is this number, then

$$(48) \quad R_k(n, g) = \sum_{\substack{a_1 + \dots + a_k = n \\ (a_1, \dots, a_k) = g}} 1 = \sum_{\substack{b_1 + \dots + b_k = n/g \\ (b_1, \dots, b_k) = 1}} 1 = \begin{cases} 0, & g \nmid n, \\ R_k(n/g), & g \mid n. \end{cases}$$

Thus far we have restricted out attention to <u>compositions</u>. It may therefore be of some interest to consider the possibility of expansion of a binomial coefficient in terms of bracket functions and <u>partitions</u>. Let

$$(49) \qquad p(n, k) = \sum_{\substack{1 \le b_1 \le b_2 \le \dots \le b_k \le n \\ b_1 + b_2 + \dots + b_k = n}} 1$$

so that $p(n, k)$ is the number of partitions of n into k positive summands. Consider a typical partition $n = b_1 + \dots + b_k$. If 1 occurs a_1 times, 2 occurs a_2 times, etc., then it is well known (e.g. Cf. [1, Vol. 2, 102]) that we may restate (49) in the form

$$(50) \qquad p(n, k) = \sum_{\substack{a_1 + 2a_2 + 3a_3 + \dots + na_n = n \\ a_1 + a_2 + \dots + a_n = k, \ a_i \ge 0}} 1 .$$

We recall that if we form an arrangement of k marks $(c_1 c_2 c_1 c_4 c_3 \dots)$, where c_1 occurs a_1 times, c_2 occurs a_2 times, etc., with $k = a_1 + \dots + a_n$, $a_i \geq 0$, then the total number of distinct such arrangements (permutations) which may be formed is enumerated by the expression

$$\frac{k!}{a_1!\,a_2!\dots a_n!}\ .$$

This expression then enumerates the compositions of n into k positive summands corresponding to a given partition $n = b_1 + b_2 + \dots + b_k$. It follows from this that we may change relation (50) into an enumeration of compositions by introducing the above ratio of factorials (instead of just counting 1 for each partition). Thus we evidently have proved

Theorem 16. For all natural numbers n and k

$$(51)\quad \binom{n-1}{k-1} = \sum_{\substack{a_1 + 2a_2 + 3a_3 + \dots + na_n = n \\ a_1 + a_2 + \dots + a_n = k,\ a_i \geq 0}} \frac{k!}{a_1!\,a_2!\dots a_n!}\ .$$

Again we may argue as we did in going from (6) to (7), whence we have established

Theorem 17.

$$(52)\quad \binom{n}{k} = \sum_{j=k}^{n} \sum_{\substack{a_1 + 2a_2 + 3a_3 + \dots + ja_j = j \\ a_1 + a_2 + \dots + a_j = k,\ a_i \geq 0}} \frac{k!}{a_1!\,a_2!\dots a_j!}\ .$$

We may next apply the same argument here which we used to obtain Theorem 1, which is to say that we may restrict our attention to relatively prime summands, but have the same total enumeration of compositions, by introducing the bracket function. We evidently have

Theorem 18.

$$(53)\quad \binom{n}{k} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] \sum_{\substack{a_1 + 2a_2 + 3a_3 + \dots + ja_j = j \\ a_1 + a_2 + \dots + a_j = k \\ (a_1, a_2, \dots, a_j) = 1}} \frac{k!}{a_1!\,a_2!\dots a_j!}$$

It follows that the inner sum gives another way of expressing $R_k(j)$, that is, we conclude that

$$(54)\qquad R_k(n) = \sum_{\substack{a_1 + 2a_2 + 3a_3 + \dots + na_n = n \\ a_1 + a_2 + \dots + a_n = k \\ (a_1, a_2, \dots, a_n) = 1}} \frac{k!}{a_1!\, a_2! \dots a_n!},$$

and of course the arithmetical properties we found for $R_k(n)$ then apply to this summation also. Thus, also, in Theorem 12, our main inversion theorem, we have several ways of expressing the coefficients $R_k(n)$ and $A_k(n)$.

Some further consequences of Theorem 12 and the other expansions in this paper will be presented later.

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LETTER TO THE EDITOR

B.G. BAUMGART
Glencoe, Illinois

Dear Sir:

In the article "On the Periodicity of the Last Digits of the Fibonacci Numbers" Vol. 1 No. 4, it was proved that for $n \gtrsim 3$ the n-th digit (from the right) had a period of $1.5 \cdot 10^n$ thus accounting for the observation made at the University of Alaska on an IBM 1620; that the last Fibonacci digit cycles every 60 numbers; the second to last digit, every 300 numbers; the third, every 1500; the fourth, 15000; the fifth, 150000.

I,too, have observed the periodicity of the last Fibonacci digits on an IBM 709 at Northwestern University (before discovering the Fibonacci Quarterly). However, I also considered the so called:

Tribonacci Series

1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201, 2209, 4063, 7473...

and found that its last digit repeats every 31 numbers, its second to last digit repeats every 620 numbers and its third to last digit repeats every 6200 numbers;

Tetranacci Series

1, 1, 1, 1, 4, 7, 13, 25, 49, 94, 181, 349, 673, 1297, 2500, 4819, 9289...

and found that the last digit repeats every 1560 numbers as does the second to the last digit. That is the period of the last and the second to the last is the same. The period of the third to last digit is 7800 and I believe the period of the fourth to last digit is also 7800 but I can not say for sure with my present results (I got all my data from one program which truncated at the fourth digit, at the time I was only thinking about the very last digit. However, it will be easy to find out and I shall do so when I get a chance. Actually, this sort of problem is a programmer's dream, because one may lose the most significant part of his calculations with impunity.)

Pentanacci Series

1, 1, 1, 1, 1, 5, 9, 17, 33, 65, 129, 253, 497, 977, 1921, 3777, 7425, 14597...

(Continued on page 302.)

FOURTH POWER FIBONACCI IDENTITIES FROM PASCAL'S TRIANGLE

VERNER E. HOGGATT, JR. and MARJORIE BICKNELL
San Jose State College, San Jose, California

In this paper, matrix methods are used to derive some new fourth power Fibonacci identities. We let S be the 5×5 matrix which contains the first five rows of Pascal's triangle beneath and on its secondary diagonal; that is,

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

The right column elements of $S^n = (s_{ij})$ are given by

$$s_{i5} = \binom{4}{i-1} F_n^{5-i} F_{n+1}^{i-1}, \quad i = 1, 2, \ldots, 5.$$

Proof is by induction. Obviously, S has this form. Since

$$S^{n+1} = (t_{ij}) = SS^n ,$$

by definition of matrix multiplication,

$$t_{15} = F_{n+1}^4 ,$$

$$t_{25} = 4F_{n+1}^4 + 4F_n F_{n+1}^3 = 4F_{n+1}^3 F_{n+2} ,$$

$$t_{35} = 6F_{n+1}^4 + 12F_n F_{n+1}^3 + 6F_n^2 F_{n+1}^2 = 6F_{n+1}^2 (F_{n+1}^2 + 2F_n F_{n+1} + F_n^2)$$

$$= 6F_{n+1}^2 F_{n+2}^2 ,$$

$$t_{45} = 4F_{n+1}^4 + 12F_n F_{n+1}^3 + 12F_n^2 F_{n+1}^2 + 4F_n^3 F_{n+1}$$

$$= 4F_{n+1}(F_{n+1}^3 + 3F_{n+1}^2 F_n + 3F_{n+1} F_n^2 + F_n^3) = 4F_{n+1} F_{n+2}^3 ,$$

$$t_{55} = (F_n + F_{n+1})^4 = F_{n+2}^4 .$$

Since only the recursion relation of the Fibonacci sequence was used above, we have almost immediately a matrix identity for generalized Fibonacci numbers. Let u_n be the nth member of the generalized Fibonacci sequence defined by $u_1 = a$, $u_2 = b$, and $u_{n+1} = u_n + u_{n-1}$. Let $U = (a_{ij})$ be the column matrix defined by

$$a_{i1} = \binom{4}{i-1} u_1^{5-i} u_2^{i-1}, \quad i = 1, 2, \ldots, 5 .$$

By our earlier proof, we can write

$$S^n U = S^n \begin{bmatrix} u_1^4 \\ 4u_1^3 u_2 \\ 6u_1^2 u_2^2 \\ 4u_1 u_2^3 \\ u_2^4 \end{bmatrix} = \begin{bmatrix} u_{n+1}^4 \\ 4u_{n+1}^3 u_{n+2} \\ 6u_{n+1}^2 u_{n+2}^2 \\ 4u_{n+1} u_{n+2}^3 \\ u_{n+2}^4 \end{bmatrix} = U_{n+1} .$$

By the Cayley-Hamilton Theorem, S must satisfy the matrix equation

$$\text{(1)} \qquad S^n(S^5 - 5S^4 - 15S^3 + 15S^2 + 5S - I) = 0 .$$

Consideration of Equation (1) leads us to the matrix equation

$$\text{(1')} \qquad U_{n+5} - 5U_{n+4} - 15U_{n+3} + 15U_{n+2} + 5U_{n+1} - U_n = 0 ,$$

where U_n is defined as the matrix $S^{n-1}U$. Since the elements in the first rows of the matrices of Equation (1') must also satisfy the recursion relation of (1'), we have the identity

$$u_{n+5}^4 - u_n^4 = 5(u_{n+4}^4 + 3u_{n+3}^4 - 3u_{n+2}^4 - u_{n+1}^4) .$$

Equation (1) can be rewritten as

$$(S - I)^5 = 25S^2(S - I) .$$

It can easily be shown by induction that

$$(S - I)^{4n+1} = 25^n S^{2n}(S - I) .$$

We plan to investigate

$$(2) \qquad (S - I)^{4n+1} = 25^n S^{2n}(S - I) ,$$

$$(3) \qquad (S - I)^{4n+2} = 25^n S^{2n}(S - I)^2 ,$$

$$(4) \qquad (S - I)^{4n+3} = 25^n S^{2n}(S - I)^3 ,$$

$$(5) \qquad (S - I)^{4n+4} = 25^n S^{2n}(S - I)^4 ,$$

From Equation (2),

$$S^j(S - I)^{4n+1} = \sum_{i=0}^{4n+1} (-1)^i \binom{4n+1}{i} S^{i+j} = 25^n S^{2n+j}(S - I) .$$

Thus, equating elements in the upper right corner of these matrices,

$$(2') \qquad \sum_{i=0}^{4n+1} (-1)^i \binom{4n+1}{i} F_{i+j}^4 = 25^n(F_{2n+j+1}^4 - F_{2n+j}^4) = A_j .$$

Similarly, from Equations (3), (4), and (5), we obtain

$$(3') \qquad \sum_{i=0}^{4n+2} (-1)^i \binom{4n+2}{i} F_{i+j}^4 = 25^n(F_{2n+j+2}^4 - 2F_{2n+j+1}^4 + F_{2n+j}^4)$$

$$= A_{j+1} - A_j ,$$

$$(4') \qquad \sum_{i=0}^{4n+3} (-1)^i \binom{4n+3}{i} F_{i+j}^4 = 25^n(F_{2n+j+3}^4 - 3F_{2n+j+2}^4 + 3F_{2n+j+1}^4 - F_{2n+j}^4)$$

$$= A_{j+2} - 2A_{j+1} + A_j ,$$

$$(5') \qquad \sum_{i=0}^{4n+4} (-1)^i \binom{4n+4}{i} F_{i+j}^4 = 25^n(F_{2n+j+4}^4 - 4F_{2n+j+3}^4 + 6F_{2n+j+2}^4 - 4F_{2n+j+1}^4 + F_{2n+j}^4)$$

$$= A_{j+3} - 3A_{j+2} + 3A_{j+1} - A_j .$$

But, also from (2),

$$\sum_{i=0}^{4n+5} (-1)^i \binom{4n+5}{i} F_{i+j}^4 = 25^{n+1}(F_{2n+j+3}^4 - F_{2n+j+2}^4) = 25\, A_{j+2} ,$$

so that we have the recursion relation

$$A_{j+4} - 4A_{j+3} + 6A_{j+2} - 4A_{j+1} + A_j = 25\, A_{j+2} . \tag{6}$$

By use of well-known Fibonacci identities, we can rewrite Equations (2') and (4') respectively to yield the following:

$$\sum_{i=0}^{4n+1} (-1)^i \binom{4n+1}{i} F_{i+j}^4 = 25^n F_{2n+j-1}F_{2n+j+2}F_{2(2n+j)+1} ,$$

$$\sum_{i=0}^{4n+3} (-1)^i \binom{4n+3}{i} F_{i+j}^4 = 25^n L_{2n+j}L_{2n+j+3}F_{2(2n+j)+3} .$$

Equating elements in the first row of the column matrices formed by multiplying Equations (2) and (4) on the right by the matrix U, and taking $u_1 = 1$, $u_2 = 3$, we can rewrite Equations (2') and (4') to yield the identities

$$\sum_{i=0}^{4n+1} (-1)^i \binom{4n+1}{i} L_{i+j}^4 = 25^n(5\,L_{2n+j-1}L_{2n+j+2}F_{2(2n+j)+1})$$

and

$$\sum_{i=0}^{4n+3} (-1)^i \binom{4n+3}{i} L_{i+j}^4 = 25^n(5\,F_{2n+j}F_{2n+j+3}F_{2(2n+j)+3})$$

for fourth powers of members of the Lucas sequence $\{L_n\}$.

Returning to the recursion relation of Equation (6), we define

$$G(j) = F_{n+j+4}^4 - 4F_{n+j+3}^4 - 19F_{n+j+2}^4 - 4F_{n+j+1}^4 + F_{n+j}^4 .$$

By (6),

$$25^n(G(j+1) - G(j)) = A_{j+4} - 4A_{j+3} - 19A_{j+2} - 4A_{j+1} + A_j = 0 \quad .$$

Thus, $G(j+1) - G(j) = 0$, so that $G(j)$ is a constant. Taking $n = j = 0$, $G(j) = -6$, leading to an identity given by Zeitlin in [1]. If we redefine $G(j)$ by replacing the Fibonacci numbers by the corresponding Lucas numbers, we find that $G(j) = -150$. Further, if we replace members of the Fibonacci sequence by the corresponding generalized Fibonacci number, we obtain $G(j) = -6D^2$, where D is the characteristic of the sequence,

$$D = u_2^2 - u_1^2 - u_1u_2 \quad .$$

(See [2] and [3] for properties of the characteristic of Fibonacci-type sequences.)

Finally, we derive another property of the characteristic of a sequence. It is well-known that $F_{2n+3} = 3F_{2n+1} - F_{2n-1}$ and that $F_{2n+1} = F_{n+1}^2 + F_n^2$. Define

$$G(n) = F_{n+1}^2 - 3F_n^2 + F_{n-1}^2 \quad .$$

Then,

$$G(n+1) + G(n) = F_{2n+3} - 3F_{2n+1} + F_{2n-1} = 0 \quad ,$$

so that $G(n) = (-1)^n 2$. That is,

$$(-1)^n 2 = F_{n+1}^2 - 3F_n^2 + F_{n-1}^2 \quad .$$

For generalized Fibonacci numbers, it can be shown by induction that

$$R^nU = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}^n \begin{bmatrix} u_1^2 \\ 2u_1u_2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} u_{n+1}^2 \\ 2u_{n+1}u_{n+2} \\ u_{n+2}^2 \end{bmatrix} \quad .$$

The 3X3 matrix R given above has been discussed in an earlier article [4]. From the characteristic equation of R, we obtain

$$u_{n+3}^2 - 2u_{n+2}^2 - 2u_{n+1}^2 + u_n^2 = 0 .$$

Rewriting and defining H(n),

$$H(n) = u_{n+2}^2 - 3u_{n+1}^2 + u_n^2 = -u_{n+3}^2 - u_{n+1}^2 + 3u_{n+2}^2 = - H(n+1) .$$

Thus,

$$H(n) = (-1)^n(u_2^2 - 3u_1^2 + u_0^2) = (-1)^n 2(u_2^2 - u_1^2 - u_1 u_2) = (-1)^{n+1} 2D ,$$

where D is the characteristic of the sequence. That is,

$$(-1)^n 2D = u_{n+1}^2 - 3u_n^2 + u_{n-1}^2 .$$

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AN ANALYTIC PROOF OF THE FORMULA FOR F_n

PETER HAGIS, JR.
Temple University, Philadelphia, Pennsylvania

The Fibonacci sequence is defined recursively by the relationship $F_{n+1} = F_n + F_{n-1}$ for $n \geqq 1$, while $F_0 = F_1 = 1$. The two most common procedures for expressing F_n as an explicit function of n are, rather naturally, "finite" in nature. The first of these methods employs the principle of finite induction, while the second involves the solution of a simple finite difference equation. In the present paper I wish to make an analytic attack on the problem employing in particular the theory of residues. The use of such powerful weapons to solve such a simple problem may seem rather absurd, but I am hopeful that the paper may serve as an elementary example of the analytic techniques which have been employed so successfully in attacking very deep and difficult questions in the theory of numbers.

As is well known, the generating function of the Fibonacci numbers is given by

$$f(z) = 1/(1-z-z^2) = \sum_{n=0}^{\infty} F_n z^n .$$

If we consider z to be a complex variable then f(z) is an analytic function whose only singularities are simple poles at the points

$$r = (-1 + \sqrt{5})/2 \text{ and } s = (-1 - \sqrt{5})/2 .$$

r and s, of course, are the roots of the equation $z^2 + z - 1 = 0$. By Cauchy's integral theorem we have

$$F_n = f^{(n)}(0)/n! = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z^{n+1}}$$

where C is the circle $|z| = 1/2$. If Γ is any circle with center at the origin and radius greater than $|s|$ then by Cauchy's residue theorem

$$(1) \qquad F_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)dz}{z^{n+1}} - (R_1 + R_2)$$

where R_1 and R_2 are the residues of $f(z)/z^{n+1}$ at the poles r and s respectively.

Now

$$R_1 = \lim_{z \to r} (z-r)f(z)/z^{n+1} = 1/\left((s-r)r^{n+1}\right),$$

and

$$R_2 = \lim_{z \to s} (z-s)f(z)/z^{n+1} = -1/\left((s-r)s^{n+1}\right).$$

Since $rs = -1$ and $r-s = \sqrt{5}$ we have after simplification

$$(2) \qquad -(R_1+R_2) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

If Γ is the circle $|z| = S$ then, since on Γ $|f(z)| \leqq \frac{1}{S^2-S-1}$, we have

$$(3) \qquad \left| \frac{1}{2\pi i}\int_\Gamma \frac{f(z)dz}{z^{n+1}} \right| \leqq \frac{2\pi S}{2\pi S^{n+1}(S^2-S-1)} = \frac{1}{S^n(S^2-S-1)}.$$

Since S may be taken arbitrarily large we conclude from (1), (2), and (3) that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

Editorial Note: Since $rs = -1$, then

$$r^{-(n+1)} = (-s)^{n+1}$$

and

$$s^{-(n+1)} = (-r)^{n+1},$$

where

$$-s = \frac{1+\sqrt{5}}{2} \text{ and } -r = \frac{1-\sqrt{5}}{2}.$$

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CONTINUED FRACTIONS OF FIBONACCI AND LUCAS RATIOS

BROTHER U. ALFRED
St. Mary's College, California

The purpose of this article is to lay the groundwork for continued fraction representations of Fibonacci and Lucas ratios. We assume the general theory of such fractions to be known and refer the unfamiliar or rusty reader to the very readable work of C. D. Olds [1]. This paper will deal with ratios in which the Fibonacci and Lucas numbers enter linearly since such results are the simplest and most fundamental, being necessary for more advanced developments.

1. THE RATIO F_n/F_{n-a}

Two cases may be distinguished depending on whether a is odd or even.

Case I. a = 2k-1

$$F_n/F_{n-2k+1} = L_{2k-1} + F_{n-4k+2}/F_{n-2k+1}$$

This devolves from the relation:

$$F_n - F_{n-4k+2} = L_{2k-1}F_{n-2k+1}$$

The next partial quotient results from the reciprocal of the fraction F_{n-4k+2}/F_{n-2k+1} and hence is again L_{2k-1}. Thus for odd a, all the partial quotients are L_{2k-1}, the termination depending on the value of n modulo 2k-1.

Example. F_{54}/F_{47}. There will be six partial quotients $L_7(29)$ after which there will be a remainder F_5/F_{12}. This latter gives partial quotients 28, 1, 4. Thus

$$F_{54}/F_{47} = (29_6, 28, 1, 4),$$

where the subscript 6 adjacent to 29 indicates the number of times 29 appears as a partial quotient.

Case 2. a = 2k

It can be shown that

$$F_n/F_{n-2k} = L_{2k} - 1 + \frac{F_{n-2k} - F_{n-4k}}{F_{n-2k}}$$

Then

$$\frac{F_{n-2k}}{F_{n-2k} - F_{n-4k}} = 1 + \frac{F_{n-4k}}{F_{n-2k} - F_{n-4k}}$$

Next

$$\frac{F_{n-2k} - F_{n-4k}}{F_{n-4k}} = L_{2k} - 2 + \frac{F_{n-4k} - F_{n-6k}}{F_{n-4k}}$$

Thus, there is a repeating pattern. The first partial quotient is $L_{2k}-1$; this is followed by $(1, L_{2k}-2)$ as a repeated pattern, the remainder after r such partial quotient pairs being

$$\frac{F_{n-2(r+1)k} - F_{n-2(r+2)k}}{F_{n-2(r+1)k}}$$

Example. F_{40}/F_{32} has a first partial quotient of $L_8-1 = 46$ followed by three sets (1, 45) and a remainder

$$\frac{F_8 - F_0}{F_8} = 1$$

Thus

$$F_{40}/F_{32} = [46, (1, 45)_3, 1]$$

which could also be represented $[46, (1, 45)_2, 1, 46]$.

2. THE RATIO L_n/F_{n-a}

Case 1. a odd

$$L_n/F_{n-a} = 5F_a - 1 + \frac{F_{n-a} - L_{n-2a}}{F_{n-a}}$$

where the relation $5F_aF_{n-a} = L_n + L_{n-2a}$ has been used in arriving at this result.

Then

$$\frac{F_{n-a}}{F_{n-a} - L_{n-2a}} = 1 + \frac{L_{n-2a}}{F_{n-a} - L_{n-2a}}$$

Next

$$\frac{F_{n-a} - L_{n-2a}}{L_{n-2a}} = F_a - 2 + \frac{L_{n-2a} - F_{n-3a}}{L_{n-2a}}$$

where the relation $F_a L_{n-2a} = F_{n-a} + F_{n-3a}$ has been employed.

Then

$$\frac{L_{n-2a}}{L_{n-2a} - F_{n-3a}} = 1 + \frac{F_{n-3a}}{L_{n-2a} - F_{n-3a}}$$

Finally

$$\frac{L_{n-2a} - F_{n-3a}}{F_{n-3a}} = 5F_a - 2 + \frac{F_{n-3a} - L_{n-4a}}{F_{n-3a}}$$

The form of the remainder is the same as that of the first remainder so that a cycle has been completed. In summary, the first term is $5F_a - 1$; the cycle that is repeated is 1, $F_a - 2$, 1, $5F_a - 2$; the remainder after r cycles is:

$$\frac{F_{n-(2r+1)a} - L_{n-(2r+2)a}}{F_{n-(2r+1)a}}$$

Example.

$$L_{86}/F_{79} = [64, (1, 11, 1, 63)_5, 1, 10, 3]$$

The verification of this development is shown below.

	0	1
	1	0
64	64	1
1	65	1
11	779	12
1	844	13
63	53951	831
1	54795	844
11	6 56696	10115
1	7 11491	10959
63	454 80629	7 00532
1	461 92120	7 11491
11	5535 93949	85 26933
1	5997 86069	92 38424
63	3 83401 16296	5905 47645
1	3 89399 02365	5997 86069
11	46 66790 42311	71881 94404
1	50 56189 44676	77879 80473
63	3232 06725 56899	49 78309 64203
1	3282 62915 01575	50 56189 44676
11	39340 98790 74224	605 96393 55639
1	42623 61705 75799	656 52583 00315
63	27 24628 86253 49561	41967 09122 75484
1	27 67252 47959 25360	42623 61705 75799
10	303 97153 65846 03161	4 68203 26180 33474
3	939 58713 45497 34843	14 47233 40246 76221
	L_{86}	F_{79}

Case 2. a even

$$L_n/F_{n-a} = 5F_a + \frac{L_{n-2a}}{F_{n-a}}$$

where the relation $5F_aF_{n-a} = L_n - L_{n-2a}$ has been used in the transformation. Then

$$\frac{F_{n-a}}{L_{n-2a}} = F_a + \frac{F_{n-3a}}{L_{n-2a}}$$

by virtue of the relation $F_aL_{n-2a} = F_{n-a} - F_{n-3a}$. Thus the pattern is

$$(5F_a, F_a)_r$$

with a remainder after r periods of

$$\frac{F_{n-(2r+1)a}}{L_{n-2ra}} .$$

Example. $L_{79}/F_{71} = (5F_8, F_8)_4$ with a remainder of F_7/L_{15}.
Thus

$$L_{79}/F_{71} = \left[(105, 21)_4, 104, 1, 12\right]$$

3. THE RATIO F_n/L_{n-a}

The algebra is quite similar to that in the case of L_n/F_{n-a} so that only the final results will be given. If a is even, the partial quotients are given by

$$(F_a, 5F_a)_r$$

with a remainder of

$$\frac{L_{n-(2r+1)a}}{F_{n-2ra}} .$$

If a is odd, there is a first partial quotient of $F_a - 1$ followed by cycles

$$(1, 5F_a - 2, 1, F_a - 2)_r$$

with a remainder of

$$\frac{L_{n-(2r+1)a} - F_{n-(2r+2)a}}{L_{n-(2r+1)a}}$$

4. THE RATIO L_n/L_{n-a}

Case I. a even

$$L_n/L_{n-2k} = L_{2k} - 1 + \frac{L_{n-2k} - L_{n-4k}}{L_{n-2k}}$$

the relation $L_n - L_{2k}L_{n-2k} = -L_{n-4k}$ being used in the transformation.

Then

$$\frac{L_{n-2k}}{L_{n-2k} - L_{n-4k}} = 1 + \frac{L_{n-4k}}{L_{n-2k} - L_{n-4k}}$$

and

$$\frac{L_{n-2k} - L_{n-4k}}{L_{n-4k}} = L_{2k} - 2 + \frac{L_{n-4k} - L_{n-6k}}{L_{n-4k}} .$$

Hence the pattern is: $L_{2k}-1,\ (1,\ L_{2k}-2)_r$ with a remainder

$$\frac{L_{n-2(r+1)k} - L_{n-2(r+2)k}}{L_{n-2(r+1)k}}$$

Case 2. a odd

$$\frac{L_n}{L_{n-2k+1}} = L_{2k-1} + \frac{L_{n-4k+2}}{L_{n-2k+1}} .$$

Thus the process is a repeating one, the remainder after r partial quotients being

$$\frac{L_{n-(r+1)(2k-1)}}{L_{n-r(2k-1)}}$$

5. GENERAL FIBONACCI SEQUENCE

Let the sequence be taken in the standard form [2] in which

$$f_1 = a, \quad f_2 = b, \quad a < b/2$$

Then

$$f_n = F_{n-1}b + F_{n-2}a$$

so that

$$\frac{f_n}{f_{n-k}} = \frac{F_{n-1}b + F_{n-2}a}{F_{n-1-k}b + F_{n-2-k}a}$$

If k is odd,

$$F_n/F_{n-k} = L_k + F_{n-2k}/F_{n-k}$$

so that

$$\frac{f_n}{f_{n-k}} = L_k + \frac{(F_{n-1} - L_k F_{n-1-k})b + (F_{n-2} - F_{n-2-k}L_k)a}{b\,F_{n-1-k} + a\,F_{n-2-k}}$$

$$= L_k + \frac{f_{n-2k}}{f_{n-k}} .$$

Hence, there is a series of partial quotients $(L_k)_r$ with a remainder

$$\frac{f_{n-(r+1)k}}{f_{n-rk}}$$

Example. Using the series (1, 4),

$$f_{62}/f_{55} = (L_7)_7 \text{ with a remainder } f_6/f_{13} = 23/665$$

Thus

$$f_{62}/f_{55} = \left[(29)_7,\ 28,\ 1, 10\right]$$

If k is even,

$$f_n/f_{n-k} = L_k - 1 + \frac{f_{n-k} - f_{n-2k}}{f_{n-k}}$$

Then

$$\frac{f_{n-k}}{f_{n-k} - f_{n-2k}} = 1 + \frac{f_{n-2k}}{f_{n-k} - f_{n-2k}}$$

$$\frac{f_{n-k} - f_{n-2k}}{f_{n-2k}} = L_k - 2 + \frac{f_{n-2k} - f_{n-3k}}{f_{n-2k}}$$

so that the pattern is

$$L_k - 1,\ (1,\ L_k - 2)_r$$

with a remainder

$$\frac{f_{n-(r+1)k} - f_{n-(r+2)k}}{f_{n-(r+1)k}}$$

Example. f_{93}/f_{83} in the (1, 4) series.

$$f_{93}/f_{83} = \left[122,\ (1, 121)_7\right]$$

with a remainder

$$\frac{f_{13} - f_3}{f_{13}}$$

the latter yielding partial quotients 1, 132. Thus

$$f_{93}/f_{83} = \left[122, (1, 121)_7, 1, 132\right]$$

The verification of this expansion is shown below.

	0	1
	1	0
122	122	1
1	123	1
121	15005	122
1	15128	123
121	18 45493	15005
1	18 60621	15128
121	2269 80634	18 45493
1	2288 41255	18 60621
121	2 79167 72489	2269 80634
1	2 81456 13744	2288 41255
121	343 35360 35513	2 79167 72489
1	346 16816 49257	2 81456 13744
121	42229 70155 95610	343 35360 35513
1	42575 86972 44867	346 16816 49257
121	51 93909 93822 24517	42229 70155 95610
1	52 36485 80794 69384	42575 86972 44867
132	696410036 58721 83205	56 62244 50519 18054

Since f_{93} and f_{83} both have a factor of 5, these final quantities differ from them by this factor.

CONCLUSION

The continued fraction developments of the Fibonacci and Lucas ratios featured in this article are not only of interest in themselves by their mathematical patterns. They provide a ready means of recognizing Fibonacci and Lucas ratios that arise in attempting to formulate laws for the continued fraction developments of non-linear relations. This wider field offers many a challenge to the searcher after additional relations characterizing the Fibonacci sequences.

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A GENERALIZATION OF FIBONACCI NUMBERS

V .C. HARRIS and CAROLYN C. STYLES
San Diego State College and San Diego Mesa College,
San Diego, California

1. INTRODUCTION

Presented here is a generalization of Fibonacci numbers which is intimately connected with the arithmetic triangle. It at once goes beyond and falls short of other generalizations. In section 2 the numbers are defined and denoted by $u(n; p, q)$ where p is a non-negative integer and q is a positive integer. The characteristic equation is shown to be

$$x^p(x - 1)^q - 1 = 0. \tag{1.1}$$

The numbers are represented in the usual manner in terms of powers of roots of the equation and certain initial conditions. In section 3 certain sums and properties involving sums are developed and in section 4 there is made a beginning in the study of divisibility properties.

The generalization made here may be compared with characteristic equations obtained in other generalizations:

by Dickinson [2], $x^c - x^a - 1 = 0$ (a, c integers)

by Miles [4], $x^k - x^{k-1} - \ldots - x - 1 = 0$ (k integral, ≥ 2)

by Raab [5], $x^{r+1} - ax^r - b = 0$ (a, b real; r integral, ≥ 1)

by Feinberg [8], $x^{nu+1} - \sum_{i=0}^{n} x^{ui} = 0$, various positive integral values of u, n.

Generalizations by Basin [1] and Horadam [3] involve altering only the initial conditions of the Fibonacci sequence.

The numbers studied here are special cases of sums defined in Netto [6] and Dickinson [2] and their definition and relation to the arithmetic triangle appear in Hochster [7].

2. THE NUMBERS u(n; p, q)

Let p and q be integers with $p \geqq 0$ and $q > 0$. Then by definition the n-th generalized Fibonacci number of step p, q is

$$(2.1) \qquad u(n;\, p,\, q) = \sum_{i=0}^{\left[\frac{n}{p+q}\right]} \binom{n - i\,p}{i\,q}, \quad n \geq 1, \qquad u(0;\, p,\, q) = 1$$

Here $[x]$ denotes the greatest integer $\leq x$. In particular,

$$u(n-1;\, 1,\, 1) = f_n \text{ (the n-th Fibonacci number)}, \; n = 1,\, 2,\, \ldots$$

$$u(n;\, 0,\, 1) = 2^n$$

When the definition is related to the arithmetic triangle one sees that $u(n;\, p,\, q)$ is the sum of the term in the first column and the n-th row (counting the top row as the zero-th row) and the terms obtained starting from this term by taking steps $p,\, q$ -- that is, p units up and q units to the right.

It follows that

$$u(0;\, p,\, q) = u(1;\, p,\, q) = \ldots = u(p+q-1;\, p,\, q) = 1, \quad u(p+q;\, p,\, q) = 2$$

If ∇ is the backward difference operator, so that

$$\nabla f(x) = f(x) - f(x-1),$$

then

$$(2.2) \qquad \nabla^q u(n;\, p,\, q) = u(n-p-q;\, p,\, q), \qquad n \geq p+q\,.$$

From properties of binomial coefficients and

$$\nabla^q u(n;\, p,\, q) = \nabla^{q-1} \nabla u(n;\, p,\, q)$$

it follows that

$$\nabla^q u(n;\, p,\, q) = \sum_{i=0}^{\left[\frac{n-p-q}{p+q}\right]} \binom{n - p - q - i\,p}{i\,q}$$

$$= u(n - p - q;\, p,\, q) \qquad , \; n \geq p+q\,.$$

This proves (2.2). In terms of forward differences this is

$$\nabla^q u(n-q;\ p,\ q) = u(n-p-q;\ p,\ q) \quad,\quad n \geq p+q \ .$$

The characteristic equation and initial conditions consequently are

$$x^p(x-1)^q - 1 = 0 \tag{2.3}$$

$$u(n;\ p,\ q) = 1, \qquad n = 0,\ 1,\ \ldots,\ p+q-1.$$

Let

$$u(n;\ p,\ q) = \sum_{i=1}^{p+q} c_i\, x_i^{n+1}$$

where x_i, $i = 1, 2, \ldots, (p+q)$ are the roots of (2.3).
The derivative of

$$f(x) = x^p\,(x-1)^q - 1 \quad \text{is} \quad f'(x) = px^{p-1}(x-1)^q + q\,x^p\,(x-1)^{q-1}$$
$$= x^{p-1}(x-1)^{q-1}((p+q)\,x-p) \ .$$

Since no root of $f'(x)$ is a root of $f(x)$, it follows that $f(x)$ has no multiple root. Hence the determinant of the coefficients of

$$\sum_{i=1}^{p+q} c_i\, x_i^{n+1} = u(n;\ p,\ q) = 1,\ n = 0,\ \ldots,\ p+q-1$$

is different from zero. The system can be solved by Cramer's rule with Vandermondians (as in several of the references). It results that

$$c_i = 1/((p+q)x_i - p)$$

and

$$u(n;\ p,\ q) = \sum_{i=1}^{p+q} \frac{x_i^{n+1}}{(p+q)\,x_i - p}\ ,\quad n = 0,\ 1,\ 2,\ \ldots \tag{2.4}$$

There is a positive real root $x_1 > 1$. This follows from $f(1) < 0$ and $f(2) \geqq 0$. Since $f'(x) \neq 0$ for $x > 1$ there is no other real root > 1. Also $|x_1|$ exceeds the absolute value of each other root. For if $x_2 \neq x_1$ is a root and $|x_2| \geq x_1$ then

$$|x_2^p(x_2-1)^q| = |x_2|^p\,|x_2-1|^q > |x_1|^p\,|x_1-1|^q > 1$$

so that (2.2) cannot be satisfied, a contradiction. From this it follows

$$\lim_{n\to\infty} \frac{u(n+1;\,p,q)}{u(n;\,p,q)} = x_1 \tag{2.5}$$

To show this, merely note

$$\lim_{n\to\infty} \frac{u(n+1;\,p,q)}{u(n;\,p,q)} = \lim_{n\to\infty} \frac{u(n+1;\,p,q)/x_1^{n+2}}{u(n;\,p,q)/x_1^{n+2}} = x_1 \ .$$

We remark that if we choose initial conditions $u(0;\,p,q) = u(1;\,p,q) = \ldots = u(p+q-2;\,p,q) = 1$, $u(p+q-1\,;\,p,q) = p+q+1$, then we have a sequence $(w(n;\,p,q))$, where

$$w(n;\,p,\,q) = \sum_{i=1}^{p+q} x_i^{n+1} \quad ,\quad n = 0,\,1,\,2,\,\ldots$$

Moreover, a convenient form for expressing $u(n,\,p,q)$ arises from writing the difference equation as

$$u(n;\,p,\,q) = \binom{q}{1} u(n-1;\,p,\,q) - \binom{q}{2} u(n-2;\,p,\,q) + - \ldots \tag{2.6}$$

$$+ (-1)^{q-1}\, u(n-q;\,p,q) + u(n-p-q;\,p,\,q),\ n \geq p+q \ .$$

3. SUMS

Theorem 3.1. The relation

$$\sum_{i=0}^{n} u(i;\,p,\,q) = \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} u(n+p+q-i;\,p,\,q) - \delta_{1q} \tag{3.1}$$

holds, where δ_{1q} is Kroneeker's δ and $\binom{q-1}{i} = 1$ in the case

$$q = 1,\quad i = 0\ .$$

If (3.1) holds for n, for $q \geq 2$, then

$$\sum_{i=0}^{n+1} u(i;\, p,\, q) = u(n+1;\, p,\, q) + \sum_{i=0}^{n} u(i;\, p,\, q)$$

$$= \sum_{i=0}^{q} (-1)^i \binom{q}{i} u(n+1+p+q-i;\, p,\, q)$$

$$+ \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} u(n+p+q-i;\, p,\, q) - \delta_{1q}$$

$$= \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} u(n+1+p+q-i;\, p,\, q) - \delta_{1q}$$

Hence (3.1) holds for $n + 1$. When $n = 0$, with $q \geq 2$, then (3.1) becomes

$$\sum_{i=0}^{0} u(i;\, p,\, q) = \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} u(p+q-i;\, p,\, q) - \delta_{1q}$$

$$= u(p+q;\, p,\, q) + \sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} = 1 = u(0;\, p,\, q)$$

To complete the proof, we consider $q = 1$. Then

$$u(i;\, p,\, 1) = u(p+1+i;\, p,\, 1) - u(p+i;\, p,\, 1)$$

Hence

$$\sum_{i=0}^{n} u(i;\, p,\, 1) = u(n+p+1;\, p,\, 1) - u(p;\, p,\, 1)$$

$$= u(n+p+1;\, p,\, 1) - \delta_{11} .$$

Theorem 3.2.

$$(3.2)\quad \sum_{i=0}^{m} (-1)^{m-i}\, u(i;p,q) = \frac{1}{1-(-1)^{p+q}2^q}\left[\sum_{k=0}^{q-1}\sum_{j=0}^{k} (-1)^k \binom{q}{j} u(m+p+q-k;p,q) + (-1)^{m+p+q}\, 2^q \sum_{i=m+1}^{m+p} (-1)^i\, u(i;p,q) + (-1)^{m-1}\, 2^{q-1} + (-1)^{m+p+q-1}\, 2^q \epsilon\right],$$

where $\epsilon = 0$, $p+q$ even, and $\epsilon = 1$, $p+q$ odd.

Proof. Writing

$$(-1)^j\, u(m-j;\, p,q) = (-1)^{m-j}\, u(m+p+q-j;\, p,q) + (-1)^{m-j-1}\binom{q}{1} u(m+p+q-j-1;\, p,q) + \cdots + (-1)^{m+q}\binom{q}{q} u(m+p-j;\, p,q)$$

and summing for $j = 0, 1, \ldots, m$ gives for the sum S,

$$S = \sum_{k=0}^{q-1}\sum_{j=0}^{k} (-1)^k \binom{q}{j} u(m+p+q-k;\, p,q) + (-1)^q 2^q \sum_{r=0}^{m-q} (-1)^r\, u(m+p-r;\, p,q) + (-1)^{m-1}\, 2^{q-1}$$

$$= \sum_{k=0}^{q-1}\sum_{j=0}^{k} (-1)^k \binom{q}{j} u(m+p+q-k;\, p,q) + (-1)^q\, 2^q \sum_{r=0}^{m+p} (-1)^r\, u(m+p-r;\, p,q) + (-1)^{m-1}\, 2^{q-1} + (-1)^{q-1}\, 2^q \sum_{r=m-q+1}^{m+p} (-1)^r\, u(m+p-r;\, p,q)$$

$$= \sum_{k=0}^{q-1}\sum_{j=0}^{k} (-1)^k \binom{q}{j} u(m+p+q-k;\, p,q) + (-1)^{p+q}\, 2^q \sum_{i=0}^{m+p} (-1)^{m-i} u(i;\, p,q) + (-1)^{m-1}\, 2^{q-1} + (-1)^{m+p+q-1}\, 2^q \sum_{i=0}^{p+q-1} (-1)^i\, u(i;\, p,q)$$

Solving for S, and noting

$$\sum_{i=0}^{p+q-1} (-1)^i u(i; p, q) = \begin{Bmatrix} 0 & p+q & \text{even,} \\ 1 & p+q & \text{odd} \end{Bmatrix} = \epsilon ,$$

we get the result (3.2).

From (3.1) and (3.2) we can obtain expressions yielding

$$\sum_{i=0}^{n} u(2i; p, q) \quad \text{and} \quad \sum_{i=0}^{n} u(2i+1; p, q) .$$

In the simpler case where $q = 1$, we find

$$(3.3) \quad \sum_{i=0}^{n} u(2i+1; p, 1) = \frac{1}{2} (u(2n+p+2; p, 1) - 1 + \sum_{i=0}^{\frac{2n-p-\eta}{2}} u(2i+\eta; p, 1)$$

and

$$(3.4) \quad \sum_{i=0}^{n} u(2i; p, 1) = \frac{1}{2} \left[u(2n+p+2; p, 1) - 1 - \sum_{i=0}^{\frac{2n-p-\eta}{2}} u(2i+\eta; p, 1)\right]$$

where $\eta = 0$ when p is even and $\eta = 1$ when p is odd. In this case it is simpler to start with

$$u(2i+1; p, 1) = u(2i; p, 1) + u(2i-p; p, 1) , \quad 2i \geq p$$

$$= u(2i; p, 1) , \quad 0 \leq 2i < p$$

and sum. We obtain in this way

$$(3.5) \quad \sum_{i=0}^{n} u(2i+1; p, 1) = \sum_{i=0}^{n} u(2i; p, 1) + \sum_{i=0}^{\frac{2n-p-\eta}{2}} u(2i+\eta; p, 1) .$$

Since we also can write

$$\sum_{i=0}^{2n+1} u(i;\, p, 1)$$

as

$$(3.6)\qquad \sum_{i=0}^{n} u(2i+1;\, p, 1) + \sum_{i=0}^{n} u(2i;\, p, 1) = u(2n+p+2;\, p, 1) - 1$$

by (3.1), the results (3.3) and (3.4) follow by addition and subtraction and solving for the sum.

For $p = 1$ these results reduce to the well-known relations of Fibonacci numbers:

$$(3.1')\qquad \sum_{i=1}^{n} f_i = f_{n+2} - 1$$

$$(3.2')\qquad \sum_{i=1}^{n} (-1)^{n-i} f_i = f_{n-1} + (-1)^{n-1}$$

$$(3.3')\qquad \sum_{i=1}^{n} f_{2i} = f_{2n+1} - 1$$

$$(3.4')\qquad \sum_{i=1}^{n} f_{2i-1} = f_{2n} \quad .$$

Theorem 3.3. Let $q = 1$ and define $u(i;\, p, 1) = 0$ for i a negative integer. Then

$$(3.7)\quad u(n+m;\, p, 1) = u(n;\, p, 1)u(m;\, p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\, p, 1)u(m-p+i;\, p, 1) \quad ,$$

where n, m are any positive integers or zero. To prove this we note first that this is true for n any positive integer or zero and $m = 0$. For n any positive integer or zero and $0 < m = k \leq p$ we have

$$u(n;\, p, 1)u(k;\, p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\, p, 1)u(k-p+i;\, p, 1)$$

$$= u(n;\, p, 1) + \sum_{i=p-k}^{p-1} u(n-1-i;\, p, 1)$$

$$= u(n;\, p, 1) + \sum_{j=n-p}^{n+k-p-1} u(j;\, p, 1)$$

$$= u(n;\, p, 1) + \sum_{j=0}^{n+k-p-1} u(j;\, p, 1) - \sum_{j=0}^{n-p-1} u(j;\, p, 1)$$

$$= u(n;\, p, 1) + u(n+k;\, p, 1) - u(n;\, p, 1)$$

$$= u(n+k;\, p, 1)$$

where the sums have been evaluated using (3.1). Hence (3.7) is true for n any positive integer or zero and $m = 0, 1, \ldots, p$. For $m = p+1$ we get

$$u(n;\, p, 1)u(p+1;\, p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\, p, 1)u(p+1-p+i;\, p, 1)$$

$$= 2\, u(n;\, p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\, p, 1)$$

$$= u(n;\, p, 1) + \sum_{j=n-p}^{n} u(j;\, p, 1)$$

$$= u(n+p+1;\, p, 1) \quad .$$

Assume now, finally, that (3.7) is true for n any positive integer or zero and $m = 0, 1, \ldots, p, \ldots, k$ where $k \geq p+1$. Then

$$u(n+k-p;\ p, 1) = u(n;\ p, 1)u(k-p;\ p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\ p, 1)u(k-2p+i;\ p, 1)$$

$$u(n+k;\ p, 1) \quad = u(n;\ p, 1)u(k;\ p, 1) + \sum_{i=0}^{p-1} u(n-1-i;\ p, 1)u(k-p+i;\ p, 1)$$

Hence

$$\begin{aligned} u(n+k+1;\ p, 1) &= u(n+k;\ p, 1) + u(n+k-p;\ p, 1) \\ &= u(n;\ p, 1)\ [u(k;\ p, 1) + u(k-p;\ p, 1)] \\ &\quad + \sum_{i=0}^{p-1} u(n-1-i;\ p, 1)\ [u(k-p+i;\ p, 1) + u(k-2p;p, 1)] \\ &= u(n;p, 1)u(k+1;p, 1) + \sum_{i=0}^{p-1} u(n-1-i;p, 1) \cdot \\ &\qquad \cdot u(k+1-p+i;p, 1) \end{aligned}$$

But this is (3.7) with $m = k+1$ and the theorem is proved.

For $m = n$, equation (3.7) becomes

$$(3.8)\quad u(2n;p, 1) = u^2(n;p, 1)+u^2(n-\tfrac{p+1}{2};p, 1)+2\sum_{i=1}^{\frac{p-1}{2}} u(n-i;p, 1)u(n-(p+1)+i;p, 1), \quad p \text{ odd}$$

and

$$(3.9)\quad u(2n;p, 1) = u^2(n;p, 1) + 2\sum_{i=1}^{\frac{p}{2}} u(n-i;p, 1)u(n-(p+1)+i;p, 1), \quad p \text{ even.}$$

For $m = n+1$, equation (3.7) becomes

$$3.10)\quad u(2n+1;p, 1) = u^2(n;p, 1) + 2\sum_{i=0}^{\frac{p-1}{2}} u(n-i;p, 1)u(n-p+i;p, 1), \quad p \text{ odd}$$

and

$$(3.11)\qquad u(2n+1;p,1) = u^2(n;p,1) + u^2(n-\tfrac{p}{2};\,p,1) + 2\sum_{i=0}^{\frac{p}{2}-1} u(n-i;p,1)u(n-p+i;p,1),\quad p \text{ even}$$

When $p = 1$ equations (3.7), (3.8) and (3.10) reduce to the known relationships

$$(3.7')\qquad f_{n+m+1} = f_{n+1}f_{m+1} + f_n f_m$$

$$(3.8')\qquad f_{2n+1} = f_{n+1}^2 + f_n^2$$

$$(3.10')\qquad f_{2n} = f_n^2 + 2f_n f_{n-1}$$

4. DIVISIBILITY PROPERTIES

Theorem 4.1. Any $p + q$ consecutive terms are relatively prime.

The terms $u(0; p,q), \ldots, u(p + q - 1; p, q)$ are all unity and so relatively prime. Any $p + q$ consecutive terms containing one of these will have greatest common divisor 1. Assume $(u(n; p,q), u(n + 1; p,q), \ldots, u(n + p + q - 1; p, q)) = d$, where $n > p + q - 1$. Then because of (2.2) it follows

$$d \,|\, (u(n - 1; p,q),\ u(n; p,q),\ \ldots,\ u(n + p + q - 2; p, q)).$$

Successive applications will show

$$d \,|\, (u(p + q - 1; p,q),\ u(p + q; p, q),\ \ldots,\ u(2p + 2q - 2; p, q)) .$$

This contains $u(p + q - 1; p, q)$ so that $d = 1$ and the theorem follows.

Theorem 4.2. The least non-negative residues modulo any positive integer m of $\{u(n; p, q)\}$ are periodic with period P not exceeding m^{p+q}. There is no preperiod. Each period begins with $p + q$ terms all unity.

There are m possible least non-negative residues modulo m for each $u(n; p,q)$ and m^{p+q} possible arrangements of residues in $p + q$ consecutive terms. Since by (2.2) the residue of $u(n; p,q)$

depends upon the residues of the preceding $p + q$ terms, after m^{p+q} terms at most the residues must repeat with a period P. Suppose $u(n + p; p, q)$ is the first term such that the residues repeat and assume $n > 0$. Then

$$u(n + P + j; p, q) \equiv u(n + j; p, q) \pmod{m}, \; j = 0, 1, \ldots, p + q \;.$$

In view of the recursion formula, this shows

$$u(n - 1 + P; p, q) \equiv u(n - 1; p, q) \pmod{m} \;,$$

a contradiction to the assumption $u(n + P; p, q)$ is the first term such that the residues repeat. Thus $n = 0$ and there is no preperiod. Hence each period begins with $p + q$ terms each unity.

As an example, we have residues (mod 7) for $u(n; 2, 1)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
r	1	1	1	2	3	4	6	2	6	5	0	6	4	4	3	0	4	0	0	4	4	4

n	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
r	1	5	2	3	1	3	6	0	3	2	2	5	0	2	0	0	2	2	2	4	6	1

n	44	45	46	47	48	49	50	51	52	53	54	55	56			
r	5	4	5	3	0	5	1	1	6	0	1	0	0	1	1	1

Here $P = 57$.

Theorem 4.3 Any prime divides infintely many $u(n; p, q)$. If the period of the residues (mod m) is P, then m divides each of

$$u(P - 1 + Pk; p, q), \; u(P - 2 + Pk; p, q), \; \ldots, \; u(P - p + Pk; p, q), \quad k = 0, 1, 2, \ldots$$

Since the residues are periodic it is sufficient, to establish the first part of the theorem, to show that any prime divides one $u(n; p, q)$. Let m be any given prime or multiple of any given prime. Then with P the period,

$$u(P; p, q) \equiv u(P + 1; p, q) \equiv \ldots \equiv u(P + p + q - 1; p, q) \equiv 1 \pmod{m}.$$

From the recursion formula,

$$u(P-1;\, p, q) \equiv \sum_{i=0}^{q} (-1)^i \binom{q}{i} u(P - 1 + p + q - i;\, p,\, q)$$

$$\equiv \sum_{i=0}^{q} (-1)^i \binom{q}{i}$$

$$\equiv 0 \pmod m$$

Hence $m \mid u(P-1;\, p,\, q)$. Similarly for $u(P-2;\, p,\, q), \ldots, u(P-p;\, p, q)$.

In the previous example, we note $7 \mid u(56;\, 2,\, 1)$, and $7 \mid u(55;\, 2,\, 1)$. Of course, 7 also divides other terms, as the table indicates.

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EXPANSIONS OF   IN TERMS OF AN INFINITE CONTINUED FRACTION WITH PREDICTABLE TERMS

N.A. DRAIM
Ventura, California

(1) $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots \text{ etc.}$$

Then, $$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \ldots \text{ etc.}$$

Whence, by identity of successive convergents, 4, 8/3, 52/15, 304/105, etc., in the above series, and in the following expansion, we have:

$$\pi = 4 - \cfrac{4}{3 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \text{etc.}}}}}}$$

i.e., $$\pi = \left\{ 4, \; -\frac{4}{3}, \; \frac{9}{2}, \; \frac{25}{2}, \; \frac{49}{2}, \; \text{etc.} \right\} .$$

(2) $$\frac{\pi}{4} - \frac{1}{2} = \sum_{1}^{\infty} (-1)^{n-1} \frac{1}{4n^2 - 1} , \quad \text{Jolley}$$

$$= \sum_{1}^{\infty} (-1)^{n} \frac{1}{(2n-1)(2n+1)}$$

$$\therefore \pi = 4\left\{ \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \text{etc.} \right\}$$

$$= \frac{10}{3} - \frac{4}{15} \quad \frac{4}{35} \quad \frac{4}{63} + \text{etc.}$$

whence, by identity of successive convergents, 10/3, 46/15, 334/105, 2946/945, etc., in the above series and in the following expansion, we have

$$\pi = 3 + \cfrac{1}{3 + \cfrac{12}{1 + \cfrac{16 - 1}{4 + \cfrac{36 - 1}{4 + \cfrac{64 - 1}{4 + \text{etc.}}}}}}$$

i.e., $$\pi = \left\{ 3, \; \frac{1}{3}, \; \frac{12}{1}, \; \frac{16-1}{4}, \; \frac{36-1}{4}, \; \ldots, \frac{(2k)^2 - 1}{4}, \; \ldots \text{ etc.} \right\}$$

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AN APPLICATION OF UNIMODULAR TRANSFORMATIONS

DMITRI THORO
San Jose State College, San Jose, California

1. INTRODUCTION

The purpose of this paper is to investigate the Diophantine equation

$$(1) \qquad f(x, y) = x^2 - xy - y^2 = A .$$

In particular, we will prove the following [1]

Theorem. Equation (1) has a solution in relatively prime integers x and y if and only if

(i) $A = 5^e A' \neq 0$, where $e = 0$ or 1 and

(ii) if p is a prime factor of A', then $p \equiv 1$ or $-1 \pmod{10}$.

An application to Fibonacci numbers may be found in [2].

2. TECHNIQUES

Our primary tool will be unimodular transformations

$$\begin{cases} x = \alpha X + \beta Y \\ y = \gamma X + \delta Y \end{cases}$$

with determinant $\alpha\delta - \beta\gamma = \pm 1$.

If we define the product of two transformations in the customary manner, it is a straightforward procedure to verify that the set of all unimodular transformations forms a non-abelian group. We shall make tacit use of this fact.

For convenience, let us designate the binary quadratic form

$$ax^2 + bxy + cy^2 \text{ by } [a, b, c] .$$

Note that the discriminant $b^2 - 4ac$ is invariant under a unimodular transformation (cf. analytic geometry: rotation of axes).

First we observe that

$$\text{(iii)} \quad \text{if } (\alpha, \gamma) = 1, \text{ then } f(\alpha, \gamma) \neq 0$$

since $(\alpha, \gamma) = 1$ implies α and γ are both odd or of opposite parity, hence $f(\alpha, \gamma)$ is odd;

$$\text{(iv)} \quad f(1, 0) = 1;$$

$$\text{(v)} \quad \text{if } f(\alpha, \gamma) = A, \text{ then } f(\gamma, -\alpha) = -A.$$

Thus in the following discussion we may, whenever it is convenient, assume $A > 2$.

3. THE PROOF: PART I

Suppose the Diophantine equation (1) has a solution in relatively prime integers α and γ : $f(\alpha, \gamma) = A$, $(\alpha, \gamma) = 1$. Since the g.c.d. of any two integers α and γ (not both zero) may be expressed as a linear combination of α and γ, there exist integers β and δ such that $\alpha\delta - \beta\gamma = 1$.

Applying the unimodular transformation whose coefficient matrix is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

to $f(x, y) \equiv [1, -1, -1]$ yields a new binary quadratic form $[A, B, C]$. But the discriminants are invariant under this transformation; thus $B^2 - 4AC = 5$.

Putting it another way, $f(\alpha, \gamma) = A$, where $(\alpha, \gamma) = 1$ implies the congruence

$$(2) \qquad x^2 \equiv 5 \pmod{4A}$$

is solvable. However, this congruence has a solution if and only if conditions (i) and (ii) are satisfied. For any x, $x^2 \equiv 0, 1,$ or $4 \pmod 8$. Therefore (2) has no solution if A is even. If $A = 25A'$, $x^2 \equiv 5 \pmod{100}$, whence $x = 5t$, which leads to the contradiction $5t^2 \equiv 1 \pmod 5$.

To complete this discussion, the reader should use the quadratic reciprocity theorem (first proved by Gauss at the age of 18). In particular, we note that $x^2 \equiv 5 \pmod{p}$ has a solution if and only if $x^2 \equiv p \pmod{5}$ has a solution.

4. THE PROOF: PART II

To establish the <u>sufficiency</u> of conditions (i) and (ii), we will show that there exist unimodular transformations $T_1, T_2, \ldots, T_k$, H, and L such that

$$[A_1, B_1, C_1] \xrightarrow{T_1} [A_2, B_2, C_2] \xrightarrow{T_2} [A_3, B_3, C_3] \xrightarrow{T_3} \cdots$$

$$\xrightarrow{T_k} [A_{k+1}, B_{k+1}, C_{k+1}] \xrightarrow{H} [A_{k+2}, B_{k+2}, C_{k+2}] \xrightarrow{L} [1, -1, -1]$$

where $A_1 = A$ (cf. (1)), $B_1 = B$ (a solution of the congruence (2)), $A_{k+1} = \pm 1$ is the <u>first</u> A_i numerically equal to unity, $|B_{k+2}| = 1$, and the C_i are determined by the invariance of the discriminant.

If T is the product of these transformations,

$$[A, B_1, C_1] \xrightarrow{T} [1, -1, -1] \quad \text{or} \quad [1, -1, -1] \xrightarrow{T^{-1}} [A, B_1, C_1]$$

$$\equiv F(x, y).$$

Thus if the coefficient matrix of T^{-1} is

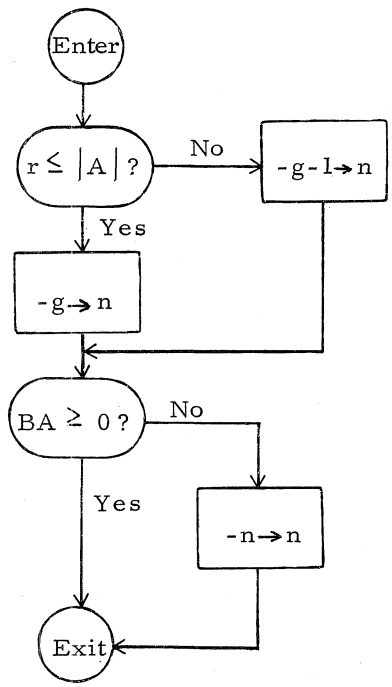
$$\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \quad ,$$

$F(1, 0) = A$ implies $f(t_1, t_3) = A$. I.e., the desired solution of (1) is simply $x = t_1$, $y = t_3$. Moreover, since T^{-1} is unimodular, $t_1t_4 - t_3t_2 = \pm 1$ forces $(t_1, t_3) = 1$.

<u>A Useful Lemma</u>. Given any two integers B and $A \neq 0$, there exists an integer n such that

$$|B + 2nA| \leq |A| \quad .$$

Proof. If we define $g = [|B|/2|A|]$ and $r = |B| - 2|A|g$, then the following flow chart exhibits n. (As usual "$s \to t$" means "replace s by t".)



We may now define the (matrices of the) required transformations. Let

$$T_i = \begin{pmatrix} n_i & 1 \\ -1 & 0 \end{pmatrix} ,$$

where n_i satisfies the inequality

$$|-B_i + 2n_iA| \leq |A_i| .$$

Then it turns out that $B_{i+1} = - B_i + 2\, n_iA_i$, $A_{i+1} = (B_{i+1}^2 - 5)/4A_i$. As previously mentioned, $A_1 = A$ (given) and $B_1 = B$ (a solution of (2)). Note that B must be odd; hence all the B_i are odd. Similarly, all the A_i are odd.

Choose

$$H = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

so that h satisfies

$$|B_{k+1} + 2h\,A_{k+1}| \le |A_{k+1}| \ .$$

Then $B_{k+2} = B_{k+1} + 2h\,A_{k+1}$. Note that $B_{k+2} \neq 0$ (since B_{k+1} is odd); but $A_{k+1} = \pm 1$ (by definition), hence $|B_{k+2}| = 1$.

The reader may quickly establish the inequality

$$|A_{i+1}| < |A_i|/4, \quad i = 1, 2, \ldots, k \ .$$

Since the A_i can be shown to be odd, this establishes the existence of A_{k+1}.

Finally, L is chosen to be

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

according as the penultimate form is

$$[1, -1, -1], \quad [1, 1, -1], \quad [-1, 1, 1], \quad \text{or} \quad [-1, -1, 1],$$

respectively.

Thus we have established the existence of the transformations $T_1, T_2, \ldots, T_{k+1}$, H, L and hence the desired solution of (1).

5. REMARKS

We have, however, more than an existence proof. The procedures developed in Part II of the proof constitute an efficient algorithm. The algorithm was programmed in FORTRAN successfully. For $|A| \le 4^k$, no more than k+2 unimodular transformations are required to obtain a solution.

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GENERALIZED BINOMIAL COEFFICIENTS

ROSEANNA F. TORRETTO and J. ALLEN FUCHS*
University of Santa Clara, Santa Clara, California

We consider the general second order recurrence relation (r.r.)

$$y_{n+2} = gy_{n+1} - hy_n, \quad h \neq 0 \quad . \tag{1}$$

Let a and b be the roots of the auxiliary polynomial $f(x) = x^2 - gx + h$ of (1). Using the notation of the classic paper [1] of E. Lucas, we let U_n and V_n be the solutions of (1) defined by $U_n = (a^n - b^n)/(a - b)$ if $a \neq b$ and $U_n = na^{n-1}$ if $a = b$ and by $V_n = a^n + b^n$.

In [3], D. Jarden defined generalized binomial coefficients by

$$\begin{bmatrix} m \\ j \end{bmatrix} = \frac{U_m U_{m-1} \cdots U_{m-j+1}}{U_1 U_2 \cdots U_j}, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1 \quad . \tag{2}$$

(We have changed Jarden's notation $\binom{m}{j}_U$ to $\begin{bmatrix} m \\ j \end{bmatrix}$.) If $g = 2$ and $h = 1$ then $U_n = n$ and $\begin{bmatrix} m \\ j \end{bmatrix}$ is the ordinary binomial coefficient $\binom{m}{j}$.

Jarden showed that the product z_n of the n-th terms of $k - 1$ sequences satisfying (1) satisfies the k-th order r.r.

$$\sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} h^{j(j-1)/2} z_{n+k-j} = 0. \tag{3}$$

The definition (2) of $\begin{bmatrix} m \\ j \end{bmatrix}$ for all j and m with $0 \leq j \leq m$ obviously requires that $U_n \neq 0$ for $n > 0$ since otherwise (2) may involve division by zero. We call the r.r. (1) ordinary if $U_n \neq 0$ for all $n > 0$ and exceptional if $U_n = 0$ for some $n > 0$. In (7) and (8) below we give an alternate definition of $\begin{bmatrix} m \\ i \end{bmatrix}$ which is valid in all cases. In [2], D. H. Lehmer considered the exceptional r.r.'s (1) for which $g = \sqrt{f}$ and for which f and h are relatively prime. Lehmer's paper is concerned with divisibility properties of the sequences U_n and V_n.

It follows from $h \neq 0$ that $a \neq 0$ and $b \neq 0$. It is then clear from the definition of U_n that (1) is exceptional if and only if $a \neq b$ and $a^p = b^p$ for some positive integer p. If (1) is exceptional, $a \neq b$ and so every solution of (1) is of the form $y_n = c_1 a^n + c_2 b^n$. Then

*This work was supported by the Undergraduate Research Participation Program of the National Science Foundation through G-21681. The authors express their gratitude to NSF and to Dr. A. P. Hillman, Dr. D. G. Mead, Mr. R. M. Grassl, and Mr. J. A. Erbacher for much valuable assistance.

$y_{n+p} = c_1a^{n+p} + c_2b^{n+p} = a^p(c_1a^n + c_2b^n) = a^py_n$ for all n. Conversely, one easily sees that $y_{n+p} = a^py_n$ for all n and all solutions y_n of (1) implies that (1) is exceptional.

We show below that the following four conditions are equivalent to each other and hence to (1) being ordinary:

(a) Either $a = b$ or $a^n \neq b^n$ for all $n > 0$.

(b) Any solution y_n of (1) with two different terms equal to zero is identically zero.

(c) For all $k \geq 2$ the r. r. (3) is the lowest order r. r. satisfied by all term by term products of $k - 1$ sequences satisfying (1).

(d) Every solution of (3) is of the form

$$(4) \quad z_n = c_1U_n^{k-1} + c_2U_n^{k-2}U_{n+1} + c_3U_n^{k-3}U_{n+1}^2 + \ldots + c_kU_{n+1}^{k-1} ,$$

i.e., the sequences $U_n^{k-j}U_{n+1}^{j-1}$ for $j = 1, \ldots, k$ form a basis for the vector space of all solutions of (3).

We shall also establish some identities involving the $\begin{bmatrix} m \\ j \end{bmatrix}$, one of which is the addition formula:

$$(5) \quad \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} h^{(j+1)j/2} U_{a_1+k-j} U_{a_2+k-j} \cdots U_{a_k+k-j} y_{n+k-j} = U_1 \cdots U_k y_{n+a_1+\ldots+a_k+\left[k(k+1)/2\right]},$$

for y_n and U_n satisfying (1) and n and the a's any integers.

If $a \neq b$, every solution of (1) is of the form $y_n = c_1a^n + c_2b^n$ and the term-by-term product of $k - 1$ sequences satisfying (1) is given by

$$(6) \quad z_n = c_1(a^{k-1})^n + c_2(a^{k-2}b)^n + c_3(a^{k-3}b^2)^n + \ldots + c_k(b^{k-1})^n .$$

We therefore let

$$(7) \qquad f_k(x) = (x - a^{k-1})(x - a^{k-2}b) \ldots (x - b^{k-1})$$

and define $\begin{bmatrix} k \\ j \end{bmatrix}$ so that

(8) $$f_k(x) = \sum_{j=0}^{k} (-1)^j \left[{k \atop j} \right] h^{j(j-1)/2} x^{k-j} .$$

The $\left[{k \atop j} \right]$ defined by (8) is a generalization of the $\left\{ {k \atop j} \right\}$ of L. Carlitz [4] defined by

$$(1 - t)(1 - qt) \ldots (1 - q^{k-1}t) = \sum_{j=0}^{k} (-1)^j q^{j(j-1)/2} \left\{ {k \atop j} \right\} t^j .$$

See especially formulas (6.3) through (6.16) of [4].)

Then $f_k(x)$ is the auxiliary polynomial for the r.r. (3). The lowest order r.r. satisfied by the z_n of (6) is (3) if and only if the numbers a^{k-1}, $a^{k-2}b$, ..., b^{k-1} are distinct. Since $a \neq 0$ and $b \neq 0$, this is equivalent to $a^j \neq b^j$ for $j = 1, \ldots, k-1$. Hence condition (c) is equivalent to (a) for $a \neq b$.

If $a = b$, every solution of (1) is given by $y_n = (c_1 + c_2 n)a^n$, the term-by-term product of k-1 sequences satisfying (1) is of the form

(9) $$z_n = (c_1 + c_2 n + \ldots + c_k n^{k-1})(a^{k-1})^n ,$$

and (3) is the lowest order r.r. satisfied by all the z_n of form (9). Thus (c) and (a) are equivalent in this case too. It is also easily seen that $h = a^2$ and $\left[{m \atop j} \right] = \binom{m}{j} a^{j(m-j)}$ when $a = b$.

Lemma.

A solution y_n of (1) that is not identically zero has $y_n = 0$ for two different values of n if and only if $a \neq b$ and there is a positive integer p such that $a^p = b^p$.

Proof.

First let $a = b$. Then $y_n = (c_1 + c_2 n)a^n$. If $y_u = 0 = y_v$ with $u \neq v$, then $(c_1 + c_2 u)a^u = 0 = (c_1 + c_2 v)a^v$. Since $a \neq 0$, it follows that $c_1 + c_2 u = 0 = c_1 + c_2 v$, $c_2(u - v) = 0$, and so $c_2 = 0$. Then $c_1 = 0$ and $y_n = 0$ for all n.

Now let $a \neq b$. Then $y_n = c_1 a^n + c_2 b^n$. If $y_u = 0 = y_v$ with $u > v$, $c_1 a^u + c_2 b^u = 0 = c_1 a^v + c_2 b^v$, and there exists a non-trivial solution for the c's if and only if the determinant $a^u b^v - b^u a^v = 0$. This is equivalent to $a^{u-v} = b^{u-v}$.

This shows that (a) and (b) are equivalent.

Corollary.

If v_n and w_n are solutions of (1) and $v_n = w_n$ for two values of n, then $v_n = w_n$ for all n.

This follows from the lemma and the fact that $v_n - w_n$ is also a solution of (1).

We next consider condition (d). First let (1) be ordinary. Let z_n be the term-by-term product of $k - 1$ solutions of (1). If we can find constants $c_1, \ldots, c_k$ such that (4) holds for $n = 1, 2, \ldots, k$ then the r.r. (3), which is satisfied by the sequences $U_n^{k-j} U_{n+1}^{j-1}$ and z_n, will make (4) hold for all n. Such c's can be found if the k by k determinant D with $d_{ij} = U_i^{k-j} U_{i+1}^{j-1}$ is not zero. Since (1) is ordinary, each of $U_1, U_2, \ldots, U_k$ is not zero and we can factor U_i^{k-1} out of the elements of the i-th row of D thus obtaining the Vandermonde determinant E with $e_{ij} = (U_{i+1}/U_i)^{j-1}$. Then E, and hence D, is not zero if and only if the ratios U_{i+1}/U_i are distinct. It is easily seen that $U_{s+1}/U_s = U_{t+1}/U_t$ if and only if $a^{s-t} = b^{s-t}$. This shows that (a) implies (d).

If (1) is exceptional, $a^p = b^p$ for some $p > 0$ and so $U_{n+p+1}/U_{n+p} = U_{n+1}/U_n$. Then for $k > p$, the determinant D is zero since it has proportional rows. It follows that one of the sequences $U_n^{k-j} U_{n+1}^{j-1}$ is a linear combination of the others, first for $1 \leq n \leq k$ and then, using (3), for all n. This implies that there is a solution of (3) not of the form (4) and so (d) implies (a).

We now go back to (7) and note that $ab = h$. Therefore we can write

$$f_{k+2}(x) = \left[(x-a^{k+1})(x-b^{k+1})\right] \left[(x-a^k b) \ldots (x-ab^k)\right]$$

$$f_{k+2}(x) = \left[x^2 - (a^{k+1} + b^{k+1}) + h^{k+1}\right] \left[(x-a^{k-1}h)(x-a^{k-2}bh) \ldots (x-b^{k-1}h)\right]$$

$$f_{k+2}(x) = h^k (x^2 - V_{k+1} x + h^{k+1}) \, f_k(x/h) \, , \tag{10}$$

where V_n is the general Lucas sequence $a^n + b^n$. Formula (10) implies the following:

$$\text{(11)} \qquad \begin{bmatrix} k \\ j \end{bmatrix} h^{k-j} + \begin{bmatrix} k \\ j+1 \end{bmatrix} V_{k+1} + \begin{bmatrix} k \\ j+2 \end{bmatrix} h^{j+2} = \begin{bmatrix} k+2 \\ j+2 \end{bmatrix} ,$$

$$\text{(12)} \qquad f_{2m} = \prod_{j=1}^{m} (x^2 - V_{2j-1} h^{m-j} + h^{2m-1}) ,$$

$$\text{(13)} \qquad f_{2m+1} = (x - h^m) \prod_{j=1}^{m} (x^2 - V_{2j} h^{m-j} + h^{2m}) .$$

We next prove identity (5) when (1) is ordinary by induction on k. When $k = 1$, (5) becomes

$$\text{(14)} \qquad U_{a+1} y_{n+1} - hU_a y_n = y_{n+a+1} .$$

We consider n to be a constant and let a be the running index. Then both sides of (14) satisfy (1) and they are equal to one another for $a = 0$ and $a = -1$ since $U_{-1} = -1/h$, $U_0 = 0$, and $U_1 = 1$. Hence (14) holds for all a (and all n) by the Corollary.

Now we assume that (5) holds for $k = m-1$ and show that this implies (5) for $k = m$. We consider $a_1, \ldots, a_{m-1}$ and n to be constants and let a_m be the running index. Both sides of (5) satisfy (1). When $a_m = 0$, (5) becomes U_m times the identity for $k = m-1$ with each a_j replaced by $1 + a_j$. When $a_m = -m$, (5) reduces to U_m times the identity for $k = m-1$ using the easily established fact that $U_{-n} = -U_n h^{-n}$. Hence (5) is true for two values of a_m and thus true for all values by the Corollary.

We now turn to identity (5) in the exceptional case. From symmetric function theory and the definitions (7) and (8), it follows that for fixed h the $\begin{bmatrix} m \\ j \end{bmatrix}$ are polynomials in g. For fixed values of y_0 and y_1 and h, the two sides of (5) are then continuous functions of g. Thus (5) for complex numbers g_0 and h_0 that make (1) exceptional can be established by having g approach g_0 (while h is fixed at h_0) through values for which (1) is ordinary. A sufficient condition for (1) to be ordinary is that $|a| \neq |b|$. Any point (g_0, h_0) is a limit of points (g, h_0) satisfying this sufficient condition for (1) to be ordinary.

A purely algebraic proof of identity (5) in the exceptional case can also be given.

Finally we consider the $\left[{m \atop j}\right]$ when (1) is exceptional and g and h are both real. Since $a^p = b^p$ for some $p > 0$, $|a| = |b|$. Since $a \neq b$ this means that $a = -b$, $g = 0$, and $h = -a^2$ if a and b are real. In this case

$$f_{2m}(x) = (x^2 + h^{2m-1})^m, \quad f_{2m+1}(x) = (x^2 - h^{2m})^m(x - (-h)^m) \quad ,$$

and it can then be shown that

$$\left[{2m \atop 2j}\right] = h^{2j(m-j)}\binom{m}{j}, \left[{2m \atop 2j-1}\right] = 0 ,$$

$$\left[{2m+1 \atop 2j}\right] = (-1)^j h^{j(2m-2j+1)}\binom{m}{j}, \left[{2m+1 \atop 2j+1}\right] = (-1)^{j+m} h^{(m-j)(2j+1)}\binom{m}{j} .$$

If a and b are complex, we can let $a = \rho e^{i\theta}$ and $b = \rho e^{-i\theta}$ with $h = \rho^2$ and $\rho > 0$. Then $a^p = b^p$ implies that $p\theta = -p\theta + 2m\pi$ and hence θ is a rational multiple $m\pi/p$ of π. Let $m/p = c/d$ with c and d relatively prime and $d > 0$. Then a/ρ and b/ρ are d-th roots of 1 if c is even and d-th roots of -1 if c is odd. The roots $a^{k-j}b^{j-1}$ of $f_k(x)$ are now of the form $\rho^{k-1}e^{(k-1-2j)\theta i}$. If $k > d$, these roots repeat in blocks of d as j varies from 1 to k. Let $k = qd + r$ with q and r integers and $0 \leq r < d$. Then

$$(15) \quad f_k(x) = (-1)^{cqr}\rho^{qdr} f_r([-1\ ]^{cq}x/\rho^{qd})\left[x^d - (-1)^{c(k-1)}\rho^{(k-1)d}\right]^q \quad .$$

Now let $j = q'd + r'$ with q' and r' integers and $0 \leq r' < d$. It then follows from (15) that

$$\left[{k \atop j}\right] = (-1)^e h^f \binom{q}{q'} \left[{r \atop r'}\right]$$

where $e = q'(d + cr + cqd + c + 1) + cqr'$ and

$$2f = d^2\left[qq' - (q')^2\right] + d(qr' + q'r - 2q'r') \quad .$$

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(Continued from page 260.)

the last digit repeats on a period of 781, the second to last digit has a period of 3900, and the

Hexanacci Series

1, 1, 1, 1, 1, 1, 6, 11, 21, 41, 81, 161, 321, 636, 1261, 2501, 4961, 9841...

the last digit as can easily be seen above repeats on a period of 7, the sequence being:

6111111611111161111116111111 6...

the second to last digit however has the somewhat larger period of 7280.

Finally, for sometime, I have wanted to apply these observations on the periodicity of the last digits to some other Fibonacci problems. So far, I have only the somewhat lame observation that the Prime-Fibonacci-Number Density (that is the ratio between the number of Fibonacci numbers which are prime below a given number n and that number n) is less than

$$4/15 \int_2^x dx/\ln x \ .$$

This observation follows from the theorem that if a Fibonacci number is prime, then its subscript is prime. Thus if all Fibonacci numbers with prime subscripts were prime the density would be Euler's famous expression

$$\pi(n) = \int_2^x dx/\ln x \ .$$

However, a good number of Fibonacci Numbers are <u>not</u> prime but do have prime subscripts, some of these numbers can now be excluded from the prime-density considerations because every prime greater than 3 must end in a 1, 3, 7, or 9 and can be expressed as $6x\pm1$. Now consider the sequence of the last digit of the Fibonacci series:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	5	8	3	1	4	5	9	4	3	7	0	7	7	4	1	5
*						*				*		*				*		*	

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
6	1	7	8	5	3	8	1	9	0	9	9	8	7	5	2	7	9	6	5
		*						*		*						*			

41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
1	6	7	3	0	3	3	6	9	5	4	9	3	2	5	7	2	9	1	0
*		*				*		*				*						*	

(Continued on page 313.)

ADVANCED PROBLEMS AND SOLUTIONS

Edited by VERNER E. HOGGATT, JR.
San Jose State College, San Jose, California

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-46 *Proposed by F.D. Parker, SUNY at Buffalo, Buffalo, New York*

Prove

$$D_n = |a_{ij}| = (-1)^n K ,$$

where $a_{ij} = F^4_{n+i+j-2}$ $(i, j = 1, 2, 3, 4, 5)$ and find the value of K.

H-47 *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Show that

$$\sum_{n=0}^{\infty} \binom{n+k-1}{n} L_n x^n = \frac{\psi_k(x)}{(1-x-x^2)^k} ,$$

where

$$\psi_k(x) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} L_r x^r .$$

H-48 *Proposed by J.A.H. Hunter, Toronto, Ontario, Canada*

Solve the non-homogeneous difference equation

$$C_{n+2} = C_{n+1} + C_n + m^n ,$$

where C_1 and C_2 are arbitrary and m is a fixed positive integer.

H-49 *Proposed by C.R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that, for $n > 0$,

$$2^n F_{n+1} = \sum_{m=0}^{n} \frac{5^{[m/2]} n^{(m)}}{m!}$$

where $[x]$ denotes the integral part of x, and $x^{(n)} = x(x-1)\ldots(x-n+1)$.

H-50 *Proposed by Ralph Greenberg, Philadelphia, Pa. and H. Winthrop, University of South Florida, Tampa, Florida*

Show

$$\sum_{n_1+n_2+n_3+\ldots+n_i = n} \Pi\, n_i = F_{2n},$$

where the sum is taken over all partitions of n into positive integers and the order of distinct summands is considered.

H-51 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, California and L. Carlitz, Duke University, Durham, N.C.*

Show that if

$$\text{(i)} \qquad \frac{xt}{1-(2-x)t+(1-x-x^2)t^2} = \sum_{k=1}^{\infty} Q_k(x)t^k$$

and

$$\text{(ii)} \qquad \sum_{n=0}^{\infty} \binom{n+k-1}{n} F_n x^n = \frac{\phi_k(x)}{(1-x-x^2)^k}$$

that

$$\phi_k(x) = \sum_{r=0}^{k} (-1)^{r+1} \binom{k}{r} F_r x^r = Q_k(x)$$

See also H-47.

LOG OF THE GOLDEN MEAN

H-29 *Proposed by Brother U. Alfred, St. Mary's College, California*

Find the value of a satisfying the relation

$$n^n + (n + a)^n = (n + 2a)^n$$

in the limit as n approaches infinity.

Solution by George Ledin, Jr., San Francisco, Calif.

Since $$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x,$$ then dividing

$$(n+a)^n + n^n = (n+2a)^n$$

through by $n^n \neq 0$ yields

$$(1+\frac{a}{n})^n + 1 = (1+\frac{2a}{n})^n,$$

which upon passing to the limit on n, gives the equation

$$e^a + 1 = e^{2a}$$

whose positive solution is $a = \ln \frac{1+\sqrt{5}}{2} = \ln\phi$, the log of the Golden Mean.

Also solved by R. Weinshenk, Sunnyvale, California, J.L. Brown, Jr., State College, Pa., Raymond Whitney, Lock Haven, Pa., Zvi Dresner, *and the proposer.*

MORE DIOPHANTUS AND FIBONACCI

H-30 *Proposed by J.A.H. Hunter, Toronto, Ontario, Canada*

Find all non-zero integral solutions to the two Diophantine equations,

(a) $$X^2 + XY + X - Y^2 = 0$$

(b) $$X^2 - XY - X - Y^2 = 0 .$$

Report by the proposer

All solutions of $X^2 + XY + X - Y^2 = 0$ are

$$X = F_{2n}^2$$

$$Y = F_{2n}F_{2n+1}$$

All solutions of $X^2 - XY - X - Y^2 = 0$ are

$$X = F^2_{2n+1}$$

$$Y = F_{2n}F_{2n+1}$$

All solutions of $X^2 + XY - X - Y^2 = 0$ are

$$X = F^2_{2n+1}$$

$$Y = F_{2n+1}F_{2n+2}$$

All solutions to $X^2 - XY + X - Y^2 = 0$ are

$$X = F^2_{2n+2}$$

$$Y = F_{2n+1}F_{2n+2}.$$

The "only if" portion of the report was incomplete. The Editor awaits further comments from our readers.

UNIMODULAR BILINEAR TRANSFORMATIONS

H-31 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Prove the following:

Theorem: Let a, b, c, d be integers satisfying $a > 0$, $d > 0$ and $ad - bc = 1$, and let the roots of $\lambda^2 - \lambda - 1 = 0$ be the fixed points of

$$W = \frac{az + b}{cz + d}.$$

Then it is necessary and sufficient for all integral $n \neq 0$, that $a = F_{2n+1}$, $b = c = F_{2n}$, and $d = F_{2n-1}$, where F_n is the n^{th} Fibonacci number. ($F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all integral n.)

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Since the equation $\lambda^2 - \lambda - 1 = 0$ has two distinct roots $\lambda_1 = \frac{1 + \sqrt{5}}{2}$ and $\lambda_2 = \frac{1 - \sqrt{5}}{2}$, we note that $c \neq 0$. From the fixed point conditions,

$$\lambda_1 = \frac{a\lambda_1 + b}{c\lambda_1 + d} \quad \text{and} \quad \lambda_2 = \frac{a\lambda_2 + b}{c\lambda_2 + d},$$

it is simple to derive the following necessary conditions:

$$a = c + d$$

$$b = c .$$

Conversely, if $b \neq 0$, $b = c$ and $a = c+d$, then the transformation becomes

$$w = \frac{az + b}{bz + (a-b)} ,$$

and the equation for the fixed points of this transformation is

$$z^2 - z - 1 = 0 ,$$

so that λ_1 and λ_2 are the fixed points.

We have thus shown that the bilinear transformation $w = \frac{az+b}{cz+d}$ has fixed points $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$ <u>if and only if</u> $c \neq 0$, $b = c$ and $a = c+d$.

Substituting these latter conditions into the condition $ad - bc = 1$, we obtain the following diophantine equation relating c and d:

$$(*) \qquad c^2 - d^2 - cd + 1 = 0 .$$

Let (c, d) be an <u>arbitrary</u> pair of <u>positive</u> integers which satisfy (*). Then, it is clear that $c \geq d > 0$. It is easily verified that $(c-d, 2d-c)$ is also an integer solution pair for (*) with first term ≥ 0 and second term > 0. [If $2d-c < 0$, then $0 < d < \frac{c}{2}$ and $c^2 - d^2 - cd + 1 > c^2 - \frac{c^2}{4} - \frac{c^2}{2} + 1 = \frac{c^2}{4} + 1 > 1$, contradicting the fact that $c^2 - d^2 - cd + 1 = 0$.] If the first term $c-d$ is actually > 0, then we may form another solution $(2c-3d, 5d-3c)$ in the same manner and the new solution will again have a non-negative first term and positive second term. After n such iterations (assuming positive first terms), we arrive at the solution $(F_{2n-1}c - F_{2n}d, F_{2n+1}d - F_{2n}c)$. Now, consider the first terms of the solution pairs thus generated. For any n such that the first term is positive, we may construct an $n+1^{\underline{st}}$ solution which either has a positive first term or has a first term of zero. Also note the first term of each successive solution is <u>smaller</u> than the first term of the preceding solution. It is clear that our construction process must lead, in a finite number of steps, to a solution with first term 0, namely the solution (0, 1). For it not, we could produce by the foregoing process an arbitrarily large number of solution pairs (c_d, d_n) in positive integers with $c > c_1 > c_2 > c_3 \ldots$ and $0 < d_n \le c_n$ for each n. This infinite descent is obviously impossible; hence, there exists an integer $k > 0$ such that the solution pair $(F_{2k-1}c - F_{2k}d, F_{2k+1}d - F_{2k}c)$ is identically the pair (0, 1). We have, therefore,

$$F_{2k-1}c - F_{2k}d = 0$$

$$F_{2k+1}d - F_{2k}c = 1 ,$$

from which $c = F_{2k}$, $d = F_{2k-1}$ and $a = c+d = F_{2k+1}$. This shows the necessity of the condition that the coefficients are Fibonacci numbers of a certain form; the sufficiency follows directly using the identity $F_{2k-1}F_{2k+1} - F_{2k}^2 = 1$. This proves the stated theorem and also shows that $c = F_{2k}$ and $d = F_{2k-1}$ for $k = 1, 2, 3, \ldots$ constitute all possible solutions in positive integers of the diophantine equation $c^2 - d^2 - cd + 1 = 0$.

The reader is directed to an application of the result of H-31 in S. L. Basin's "The Appearance of Fibonacci Numbers and the Q-Matrix in Electrical Network Theory" Mathematics Magazine Volume 3b No. 2 March 1962, pp. 84-97 (see specifically Theorem 1, page 94). This theorem was first proved in an unpublished paper "The Many Facets of the Fibonacci Numbers" by V. E. Hoggatt, Jr., and Charles H. King. Also solved by Zvi Dresner.

NO FIBONACCI TRIANGLES

H-32 *Proposed by R.L. Graham, Bell Telephone Laboratories, Murray Hill, N.J.*

Prove the following:

Given a positive integer n, if there exist m line segments L_i having lengths a_i, $1 \le a_i \le n$, for all $1 \le i \le m$, such that no three L_i can be used to form a non-degenerate triangle then $F_m \le n$, where F_m is the m^{th} Fibonacci number.

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

By hypothesis, $a_1 \ge 1 = F_1$ and $a_2 \ge 1 = F_2$. Since L_1, L_2 and L_3 do not form a non-degenerate triangle, we must have (assuming the L_i have been reordered, if necessary, so that $a_1 \le a_2 \le a_3 \le \ldots \le a_m$)

$$a_3 > a_1 + a_2 \ge F_1 + F_2 = F_3 \quad .$$

Similarly, L_2, L_3 and L_4 do not form a non-degenerate triangle so that

$$a_4 > a_3 + a_2 > F_3 + F_2 = F_4 \quad .$$

Proceeding inductively in this fashion, we conclude $a_m > F_m$, and the desired result follows (actually with strict inequality) from $n \ge a_m$.

Also solved by the proposer and Zvi Dresner.

LUCAS PRIMALITY

H-33 *Proposed by Malcolm Tallman, Brooklyn, N.Y.*

If a Lucas number is a prime number and its subscript is composite, then the subscript must be of the form 2^m, $m \ge 2$.

Solution by John L. Brown, Jr., Pennsylvania State University, State College, Pa.

Assume L_n is prime and has a composite subscript n. Then $n = (2r-1)\cdot 2^m$ for some $m \ge 0$ and some $r \ge 1$. It is well-known (see e.g. equation (6) of "A Note on Fibonacci Numbers" by L. Carlitz, this Quarterly, Vol. 2, No. 1, p. 15) that $L_k | L_{(2r-1)k}$ if $r > 1$ and hence

$$L_{2^m} | L_{(2r-1)2^m} \quad \text{if} \quad r > 1 \quad .$$

Since $L_{(2r-1)2^m}$ is prime by hypothesis, we conclude $r = 1$. (The alternative $m = 0$ would force n to be a prime contrary to hypothesis). Thus $n = 2^m$ and m must be ≥ 2 in order for n to be composite.

Also solved by the proposer and Zvi Dresner.

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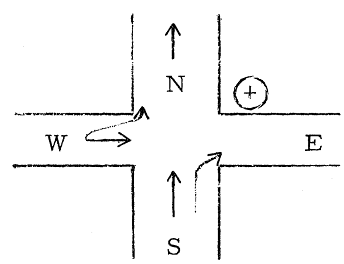
THE PROBLEM OF THE LITTLE OLD LADY TRYING TO CROSS THE BUSY STREET or FIBONACCI GAINED AND FIBONACCI RELOST

RICHARD BRIAN
San Jose State College, San Jose, California

It is no surprise to readers of this journal or to Fibonacci enthusiasts in general to find the numbers of the Fibonacci sequence popping up in the most peculiar places. This is an essay concerning an unusual situation in which these numbers appear in an interval of transience but are then overpowered by a linear function.



Consider the problem of an old lady standing on the northeast corner of the intersection of two one-way streets (one running north and the other running east) during rush hour. The traffic from the south may go east and north when its light is green but the traffic from the west may also go east and north when its light is green, hence a rather timid old lady might do well to bring a bag lunch if she anticipates such a situation.

Having viewed such a situation one evening I wondered if there might be some traffic pattern which would always allow the old lady to cross safely to any corner at any time.

Let us consider a network of one-way streets which alternate directions for both east and west and similarly north and south. If one is allowed to make a turn only at every other intersection, then one must always turn in the same direction. It is possible then to construct a traffic pattern in which one is allowed only to make turns to the right (see Figure 1).

A little study of this traffic pattern will show that one can drive to any location although it may require a trip around an extra block or two. But what of the old lady? Consider the corner letter AA. If she is standing on the northeast corner and wishes to cross to the west, she need only wait till the light stops the northbound traffic for the eastbound traffic cannot turn north. If she wishes to cross to the south,

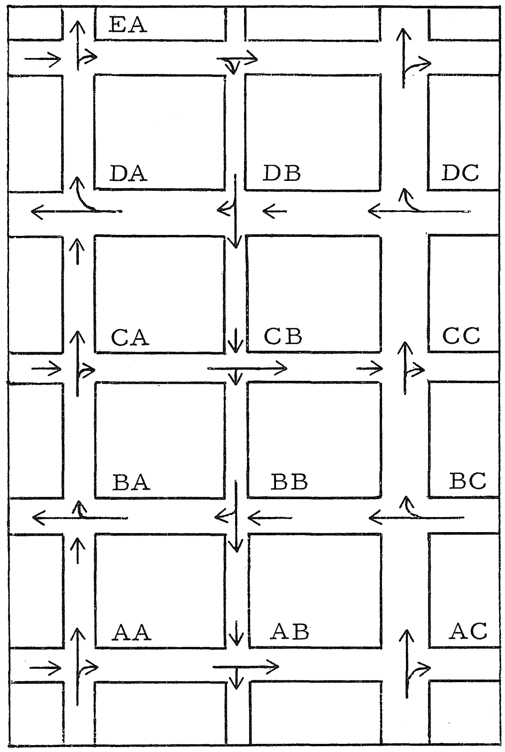


Figure 1

she cannot go directly but she can cross to the west, then to the south and finally to the east, still achieving her goal in complete safety.

Having solved the old lady's problem in every respect except convincing the traffic commission of the virtue of this scheme, I turned to other questions suggested by this same traffic scheme. Suppose one begins to drive north from corner (AA). How many blocks are accessible if one drives n (n = 1, 2, 3, 4, . . .) blocks? When one reaches corner (BA), going north, one must continue north since no turn is allowed northbound traffic here. When one reaches corner (CA) one may either turn to the east or proceed north and so on. Let us call f(n) the number of blocks which one <u>adds</u> to the total number of accessible blocks when driving on the nth block from corner (AA) then:

n	1	2	3	4	5	-	--	--	--
f(n)	1	1	2	3	5	-	--	--	-

Lo and behold f(n) appears to be the Fibonacci sequence. But there is a difficulty. One of the available paths after 5 blocks brings us back to corner (BA) travelling west. A turn to the north is allowed here but the block thus gained is one which we have already counted. Hence for $n = 6$ we have 7 new elements rather than 8 which is the next element of the Fibonacci sequence (F_n). This problem continues to plague us and if we count all the elements every time they occur, we do indeed get a Fibonacci sequence. However, if we do not count the duplications, our block acquisition sequence proceeds thus:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	--	--
f(n)	1	1	2	3	5	7	11	16	22	30	38	46	..	--	--
F_n	1	1	2	3	5	8	13	21	34	55	89	--	--	--	--

Now the question comes as to how fast this alteration takes place. Perhaps we notice that each of the last four entries differ by eight.

With this in mind consider the situation where one has a traffic pattern such that starting at corner (AA) one is allowed to go in any of the four directions and at the next corner any of the three remaining directions and so on. In this situation one acquires new elements at the rate $g(n) = 8n-4$.

It turns out that after the 9th step the acquisition of the new elements in the previous traffic pattern take on a linear form $f(n) = 8n-50$ $(n \geq 9)$.

In summary:

	L. O. L.	Traffic Pattern	Standard	Traffic Pattern
n	$f(n)$	$\sum_{i=1}^{n} f(i)$	$g(n)$	$\sum_{i=1}^{n} g(i)$
1	1	1	4	4
2	1	2	12	16
3	2	4	20	36
4	3	7	28	64
5	5	12	36	100
6	7	19	44	144
7	11	30	52	196
8	16	46		
9	22	68		
	$8n-50$ $(n \geq 9)$	$4n^2-46n+158$ $(n \geq 9)$	$8n-4$	$4n^2$

In this situation, then, the Fibonacci sequence appears only as a transient effect but such effects are, I think, relatively infrequent in purely abstract mathematical models.

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(Continued from page 302.)

Thus every time that this sequence repeats there are only a possible 16 Fibonacci Numbers (the starred ones) out of 60 which both end in 1, 3, 7, or 9 and can be expressed as $6x \pm 1$ and which just may be prime. Therefore we have established 16/60 or rather 4/15 of Euler's expression as an upper bound of the Fibonacci Prime Density.

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NO WONDER NO SOLUTION

H-26 (Corrected) *Proposed by L. Carlitz, Duke University, Durham, N.C.*

Let $R_k = (b_{rs})$, where $b_{rs} = \binom{r-1}{k+1-s}$ $(r, s = 1, 2, \ldots, k+1)$ then show

$$R_k^n = \left(\sum_{j=1}^{s} \binom{r-1}{j-1} \binom{k+1-r}{s-j} F_{n-1}^{k+1-r-s+j} F_n^{r+s-2j} F_{n+1}^{j-1} \right).$$

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APPLICATION OF FIBONACCI NUMBERS TO SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

BEN L. SWENSEN
Wentworth Military Academy, Lexington, Missouri

A casual glance at a system of linear equations such as

$$2584x + 4181y = 20$$

$$4181x + 6765y = 21$$

might lead one to think that it is unlikely that the solution could consist of integers. However, a closer look by regular readers of this journal will reveal that the coefficients of x and y in both equations are Fibonacci numbers and that if the general notation of Fibonacci numbers in which F_n denotes the nth term of the sequence 1, 1, 2, 3, ..., is used, then the equations above turn out to be the special case n = 20 of the general form of the system of equations:

(1) $$(F_{n-2})x + (F_{n-1})y = n \ ,$$

(2) $$(F_{n-1})x + (F_n)y = n + 1.$$

The solution to such a system of equations is

(3) $$\frac{n(F_n) - (n+1)(F_{n-1})}{(F_n)(F_{n-2}) - (F_{n-1})(F_{n-1})} = x, \quad \text{and}$$

(4) $$\frac{n(F_{n-1}) - (n+1)(F_{n-2})}{(F_{n-1})(F_{n-1}) - (F_{n-2})(F_n)} = y \ .$$

The denominators of the fractions in equations (3) and (4) have the interesting property

(5) $$(F_n)(F_{n-2}) - (F_{n-1})(F_{n-1}) = \begin{cases} +1 & \text{if } n \text{ is odd,} \\ -1 & \text{if } n \text{ is even} \end{cases} .$$

(6) $$(F_{n-1})(F_{n-1}) - (F_{n-2})(F_n) = \begin{cases} -1 & \text{if } n \text{ is odd,} \\ +1 & \text{if } n \text{ is even} \end{cases} .$$

It may be noted that statements (5) and (6) are equivalent.

Equations (5) and (6) permit one to write equations (3) and (4) in more convenient form:

(7a) $x = n(F_n) - (n+1)(F_{n-1})$ if n is odd,

(7b) $x = (n+1)(F_{n-1}) - n(F_n)$ if n is even.

(8a) $y = (n+1)(F_{n-2}) - n(F_{n-1})$ if n is odd,

(8b) $y = n(F_{n-1}) - (n+1)(F_{n-2})$ if n is even.

Since the numbers n, $n+1$, F_n, F_{n-1}, F_{n-2} are integers, and since the set of integers is closed under the operations of addition, subtraction, and multiplication, it follows that all solutions to the system of linear equations represented in equations (1) and (2) are integers.

In the accompanying table, the symbols ξ_n and ξ_{n+1} denote equations of the form (1) and (2) respectively and the solution to such a system is symbolized as $\xi_n \cap \xi_{n+1}$. It may be noted that any equation* ξ_n is the sum of the two equations* above it in the table, provided that $n \geq 3$. It may also be noted that the coefficients of both x and y occur in the well known Fibonacci sequence.

Equations (5) and (6) were derived intuitively by the writer who suggests that readers attempt formal proofs of them. The suggestion is also made one might consider investigating the sequences of numbers which constitute the solutions to such systems.

$$X_{2n+1} = (2n+1)\ F_{2n+1} - (2n+2)\ F_{2n}$$

$$= (2n+1)\ F_{2n-1} - F_{2n}$$

$$X_{2n+2} = (2n+1)\ F_{2n-1} - 2nF_{2n} \qquad \text{etc.}$$

$$= (2n+1)\ F_{2n-1} - 2n\ (F_{2n-1} + F_{2n-2})$$

$$= F_{2n-1} - 2n\ F_{2n-2}$$

*Left sides of these equations only.

n	ξ_n	$\xi_n \cap \xi_{n+1}$
1	$x = 1$	(1, 2)
2	$y = 2$	(1, 2)
3	$x + y = 3$	(2, 1)
4	$x + 2y = 4$	(-2, 3)
5	$2x + 3y = 5$	(7, -3)
6	$3x + 5y = 6$	(-13, 9)
7	$5x + 8y = 7$	(27, -16)
8	$8x + 13y = 8$	(-51, 32)
9	$13x + 21y = 9$	(96, -56)
10	$21x + 34y = 10$	(-176, 109)
11	$34x + 55y = 11$	(319, -197)
12	$55x + 89y = 12$	(-571, 353)
13	$89x + 144y = 13$	(1013, -626)
14	$144x + 233y = 14$	(-1783, 1102)
15	$233x + 377y = 15$	(3118, -1927)
16	$377x + 610y = 16$	(-5422, 3351)
17	$610x + 987y = 17$	(9383, -5799)
18	$987x + 1597y = 18$	(-16169, 9993)
19	$1597x + 2584y = 19$	(27759, -17156)
20	$2584x + 4181y = 20$	(-47499, 29356)
21	$4181x + 6765y = 21$	....
..	....	....
..	....	....
..	....	....

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A FIBONACCI-PRIME NUMBER RELATION

B.B. SHARPE
State University of New York at Buffalo

Fibonacci numbers may be related to prime numbers as follows:

Conjecture.

1. $F_i + F_j$ will be a prime number for at least one value of i, provided $i + j$ is a prime number.

2. $F_i - F_j$ will be a prime number for at least one value of i, provided $i + j$ is prime and greater than 3 ($i > j$). An initial verification:

i+j	F_i+F_j	F_i-F_j
2	$F_1+F_1 = 2$	
3	$F_2+F_1 = 2$	
5	$F_3+F_2 = 3$	$F_4-F_1 = 2$
7	$F_4+F_3 = 5$	$F_6-F_1 = 7$
11	$F_6+F_5 = 13$	$F_6-F_5 = 3$
13	$F_9+F_4 = 37$	$F_7-F_6 = 5$
17	$F_{11}+F_6 = 97$	$F_9-F_8 = 13$
19	$F_{10}+F_9 = 89$	$F_{12}-F_7 = 131$
23	$F_{12}+F_{11} = 233$	$F_{18}-F_5 = 2579$
29	$F_{17}+F_{12} = 1741$	$F_{17}-F_{12} = 1453$
31	$F_{16}+F_{15} = 1597$	$F_{18}-F_{13} = 2351$
37	$F_{24}+F_{13} = 46601$	$F_{28}-F_9 = 317877$
41	$F_{30}+F_{11} = 832129$	$F_{24}-F_{17} = 44771$
43	$F_{24}+F_{19} = 50549$	$F_{24}-F_{19} = 42187$
47	$F_{27}+F_{20} = 203183$	$F_{27}-F_{20} = 189653$
53	$F_{29}+F_{24} = 560597$	

No further verification is possible using Lehmer's Factor Table to 10, 000, 000.

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EXPLORING GEOMETRIC-ALGEBRAIC FIBONACCI PATTERNS

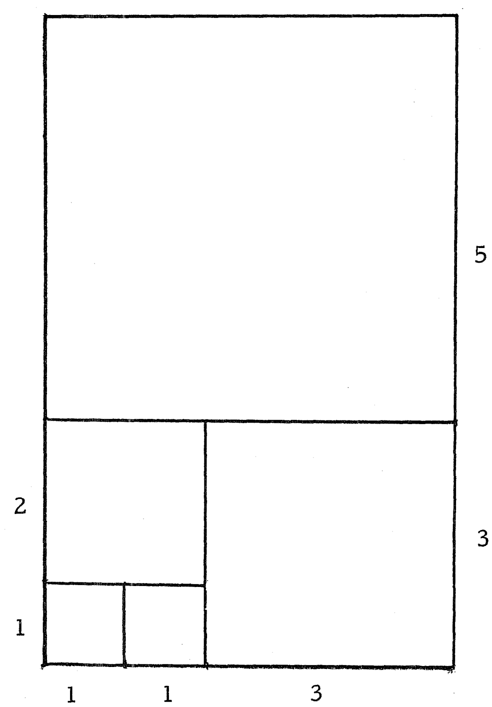
BROTHER U. ALFRED
St. Mary's College, California

Probably one of the early experiences of most Fibonacci enthusiasts is becoming acquainted with the following geometric figure. Two squares of side 1 are placed next to each other horizontally. On top of them is constructed a square of side 2. Next to the resulting figure is located a square of side 3. On top of this a square of side 5 is adjoined. And so on. The demonstrator points out triumphantly that he has added together

$$F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 \quad .$$

And what is the result? F_5F_6. This quickly leads to an intuitive result:

$$\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$$



Now I suppose many of us have asked ourselves the question: Where is example number two? Why don't we have more of this?

The problem may be stated as follows. We wish to find a geometric pattern involving Fibonacci numbers so that there will be a correlation between the geometry and some algebraic formula. In working along these lines, for example, a square F_n on each side can be filled with Fibonacci squares, the idea being to absorb the area with Fibonacci squares of as large a side as possible. For this purpose, a square of side F_{n-1} can be located in one corner; three squares of side F_{n-2} can then be placed in the remaining corners. This leaves as a balance two rectangles of dimensions F_{n-2} by F_{n-3}. Hence we have a formula

$$F_n^2 = F_{n-1}^2 + 3F_{n-2}^2 + 2 \sum_{k=1}^{n-3} F_k^2$$

which correlates with a geometric figure.

Here are additional suggestions though not all have been tried and hence there is no guarantee that results will be forthcoming.

(1) Covering Fibonacci rectangles, such as F_nF_{n-2}, F_nF_{n-3}, etc. by Fibonacci squares.

(2) Filling Lucas squares or rectangles with Fibonacci squares.

(3) Reversing the operation and using Lucas squares instead of Fibonacci squares.

(4) Finding space analogues using Fibonacci cubes.

Discoveries pertaining to this type of problem will be published in the April 1965 issue of the Fibonacci Quarterly. It would be advisable to have this material with the Editor by March 1st.

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FIBONACCI SUMMATION ECONOMICS PART I

ALBERT J. FAULCONBRIDGE
Chicago, Illinois

Summation series numbers appear in the differences in years between stock price cycle maxima and minima as follows: 1909 to 1920-21, 13 years; 1920-21 to 1942, 21; 1907 to 1915-16, 8; 1907 to 1928-29, 21; 1928 to 1962, 34; 1907 to 1941-42, 34; 1907 to 1962, 55; 1895-96 to 1929, 34; 1898-99 to 1932, 34; 1877 to 1932, 55; 1928 to 1949, 21; 1932 to 1937, 5; 1937 to 1942, 5; 1946 to 1949, 3, or more exactly 34 months; 1941-42 to 1962, 21; 1949 to 1962, 13; 1937-38 to 1946, 8; 1836 to 1857, 21; 1840 to 1929, 89; 1949 to 1957, 8; 1921 to 1928-29, 8; 1929 to 1932, 3 years. In addition to time, there appears to be a tendency of the movement to coincide quantitatively from troughs to peaks to the extent of the ratio between two successive Fibonacci numbers, 1.618. For example the 1957 Dow Jones low of 416 carried to the high in January 1962 of 735, a move of 319 points. The succeeding drop to 530 from 730 was 200 points. 319 is 1.618 of 200. Similarly the number of points from maxima to minima in smaller cycles have borne the same relationship, such as the decline from 1937 to 1938 was 61.8% of the advance from 1932-37, and the 1921-26 advance was 61.8% of the 1926 to the so called "orthodox" top of 1928. For a large number of similar apparent coincidences, refer to reference [1], [6]. The large quantity of apparent coincidence that is obvious to even cursory examination attracted little attention and the literature connected with it is very meager.

That Fibonacci summation series principles might be operational in economics is not difficult to imagine since it can be reasoned that it is possible for an economic state at a given time to be a function of those things which immediately preceded it, and they in turn a result of that which immediately preceded them.

The first author to derive principle from the elaborate set of coincidences was Elliot [2] between 1937 and 1947. In his original work, half of which exists in only three known copies, he describes pattern in the heretofore unintelligible movements of Dow Jones stock prices.

Primary rising trends are divided into five segments, three ascending and two descending, whereas the falling correction of that primary trend is divided into three sections, two descending, one rising. Closer examination of the component segments show that they in turn are composed of smaller subdivisions so that one of the original rising segments was itself divided into five waves, three ascending, two descending. When a falling correction of the primary rise was taking place, three components were divided in turn into five sections for those descending and three segments for the one rising. The entire movement of stock prices since their recording he divided into cycles within cycles labelled Grand Super Cycle, Super Cycle, Cycle, Primary Cycle, and Intermediate, Minor, Minute, Minuette, and Sub-Minuette, all of which conform to the five-three design already described. Elliot was able to detect these patterns because he introduced the concept of orthodox and correction tops. He described 1928 as an orthodox top and 1929 as a correction, or "blow-off" top. These corrections are of three basic types, appear to alternate in occurrence, and appear at predictable positions in a cycle.

This description of events was novel but not speculative since his work was easily checked by many and found to be correct, yet his theory that political events were a reflection of the economic cycles and not a cause of them was universally rejected. It was only after the predictions had remained valid as the world passed through depression, World War II, postwar boom, The Korean War, and final boom that the principle of a cyclical economic cause of political feeling began to be accepted.

In the discussion of corrections Elliot described movements that were triangular in form that conformed to Fibonacci sequences in both time and amplitude. For example, the movement between the high of 1928-29 to the low of 1942 was 13 years which was divided into Fibonacci segments of 1929-32, three years, 1932-37, five years, and 1937-42, five years. Each wave of the triangle was 62% of its predecessor in amplitude. When the concept of corrections is introduced the conformity becomes even more precise; for example, the bull run from 1921 to the orthodox top in 1928 lasted 89 months whereas the bull run to the extension top of September 1929 lasted 8 years, the difference between which was the precise time length of the extension. The run from the extension top of 1929 to the bottom of the run in 1932 lasted 34 months. In his work he detected movements possessing similar summation properties in such things as new insurance writings, temperative, gold prices, epidemics, commodities, and volume of securities traded.

A number of anomalies are encountered when attempts at prediction on the basis of Elliot's original description of Fibonacci wave theory are made. Although summation time periods turn up with frequent occurrence it is difficult to predict which one will turn up at a given time. In addition, there is as yet no indication whether these time periods will produce bottoms to tops, bottoms to bottoms, tops to tops, or tops to bottoms. Investigation of the reasons for these apparent limitations could lead to information that is not already understood.

Despite the limited publicity of Elliot's work, its importance came to the attention of one who was later to become the world authority on bank credit analysis, A. Hamilton Bolton, of Montreal, who in 1960 published a review and critical appraisal of Elliot's work. Bolton, aided by the events of the intervening years, expanded the work as far as logic could carry it, and there the matter rested like astronomy awaiting Brahé and Kepler. It did not have long to wait, for simultaneously Edward Dewey of the Foundation for the Study of Cycles at the University of Pittsburgh was gradually assembling his monumental work.

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4. Elliot, R. N. The Wave Principle, Financial World Magazine Vol. 71, Nos. 14 to 21; Vol. 71, Nos. 24 and 26; Vol. 72, No. 5
5. Bolton, A. Hamilton The Elliot Wave Principle - A Critical Appraisal, Montreal, Canada, 1960

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. Any problem believed to be new in the area of recurrent sequences and any new approaches to existing problems will be welcomed. The proposer should submit each problem with solution in legible form, preferably typed in double spacing with name and address of the proposer as a heading.

Solutions to problems listed below should be submitted on separate signed sheets within two months of publication.

B-27 ***Proposed by D.C. Cross, Exeter, England***

Corrected and restated from Vol. 1, No. 4: The Chebyshev Polynomials $P_n(x)$ are defined by $P_n(x) = \cos(n\text{Arccos } x)$. Letting $\phi = \text{Arccos } x$, we have

$$\cos\phi = x = P_1(x),$$

$$\cos(2\phi) = 2\cos^2\phi - 1 = 2x^2 - 1 = P_2(x),$$

$$\cos(3\phi) = 4\cos^3\phi - 3\cos\phi = 4x^3 - 3x = P_3(x),$$

$$\cos(4\phi) = 8\cos^4\phi - 8\cos^2\phi + 1 = 8x^4 - 8x^2 + 1 = P_4(x), \text{ etc.}$$

It is well known that

$$P_{n+2}(x) = 2xP_{n+1}(x) - P_n(x) \quad .$$

Show that

$$P_n(x) = \sum_{j=0}^{m} B_{jn} x^{n-2j}$$

where

$$m = [n/2] \quad ,$$

the greatest integer not exceeding $n/2$, and

(1) $B_{on} = 2^{n-1}$

(2) $B_{j+1,n+1} = 2B_{j+1,n} - B_{j,n-1}$

(3) If $S_n = |B_{on}| + |B_{1n}| + \ldots + |B_{mn}|$, then $S_{n+2} = 2S_{n+1} + S_n$.

B-52 *Proposed by Verner E. Hoggatt, Jr., San Jose State Collete, San Jose, Calif.*

Show that $F_{n-2}F_{n+2} - F_n^2 = (-1)^{n+1}$, where F_n is the n-th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

B-53 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Show that

$$(2n - 1)F_1^2 + (2n - 2)F_2^2 + \ldots + F_{2n-1}^2 = F_{2n}^2 \quad .$$

B-54 *Proposed by C.A. Church, Jr., Duke University, Durham, N. Carolina*

Show that the n-th order determinant

$$f(n) = \begin{vmatrix} a_1 & 1 & 0 & 0 & & 0 & 0 \\ -1 & a_2 & 1 & 0 & & 0 & 0 \\ 0 & -1 & a_3 & 1 & & 0 & 0 \\ 0 & 0 & -1 & a_4 & \ldots & 0 & 0 \\ \ldots \\ \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & a_n \end{vmatrix}$$

satisfies the recurrence $f(n) = a_n f(n-1) + f(n-2)$ for $n > 2$.

B-55 *From a proposal by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Show that $x^n - xF_n - F_{n-1} = 0$ has no solution greater than a, where $a = (1 + \sqrt{5})/2$, F_n is the n-th Fibonacci number, and $n > 1$.

B-56 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

Let F_n be the n-th Fibonacci number. Let $x_0 \geq 0$ and define $x_1, x_2, \ldots$ by $x_{k+1} = f(x_k)$ where

$$f(x) = \sqrt[n]{F_{n-1} + xF_n} .$$

For $n > 1$, prove that the limit of x_k as k goes to infinity exists and find the limit. (See B-43 and B-54.)

B-57 *Proposed by G.L. Alexanderson, University of Santa Clara, Santa Clara, Calif.*

Let F_n and L_n be the n-th Fibonacci and n-th Lucas number respectively. Prove that

$$(F_{4n}/n)^n > L_2 L_6 L_{10} \cdots L_{4n-2}$$

for all integers $n > 2$.

SOLUTIONS

RECURSIVE POLYNOMIAL SEQUENCES

B-26 *Proposed by S.L. Basin, Sylvania Electronic Systems, Mt. View, Calif.*

Corrected statement: Given polynomials $b_n(x)$ and $B_n(x)$ defined by

$$b_0(x) = 1, \quad B_0(x) = 1$$

$$(1) \quad b_n(x) = xB_{n-1}(x) + b_{n-1}(x) \qquad (n > 0)$$

$$(2) \quad B_n(x) = (x+1)B_{n-1}(x) + b_{n-1}(x) \qquad (n > 0)$$

show that $b_n(x) = P_{2n}(x)$ and $B_n(x) = P_{2n+1}(x)$ where

$$P_m(x) = \sum_{j=0}^{[m/2]} \binom{m-j}{j} x^{[m/2] - j} ,$$

$[m/2]$ being the greatest integer not exceeding $m/2$.

Solution by Lucile Morton, Santa Clara, California

We see that both $b_n(x)$ and $B_n(x)$ satisfy

$$u_{n+2}(x) = (x+2)u_{n+1}(x) - u_n(x) \quad (n > 0)\ , \tag{3}$$

as follows: Subtracting corresponding sides of (1) from those of (2), we have $B_n(x) - b_n(x) = B_{n-1}(x)$. Then $b_n(x) = B_n(x) - B_{n-1}(x)$ and it follows from (2) that

$$B_n(x) = (x+1)B_{n-1}(x) + B_{n-1}(x) - B_{n-2}(x) = (x+2)B_{n-1}(x) - B_{n-2}(x).$$

Hence $B_n(x)$ satisfies (3). Then so does $B_{n-1}(x)$ and the difference $B_n(x) - B_{n-1}(x) = b_n(x)$.

A lengthy but not difficult induction confirms that $P_{2n}(x)$ and $P_{2n+1}(x)$ both satisfy (3). Since they have the same initial values as $b_n(x)$ and $B_n(x)$ respectively, this establishes the desired result.

Also solved by the proposer.

ARITHMETIC PROGRESSIONS

B-38 *Proposed by Roseanna Torretto, University of Santa Clara, Santa Clara, California*

Characterize simply all the sequences c_n satisfying

$$c_{n+2} = 2c_{n+1} - c_n\ .$$

Solution by J.L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania

From

$$c_{n+2} - c_{n+1} = c_{n+1} - c_n\ ,$$

it is clear that the differences between successive terms must be a constant independent of n. Letting c_1 and c_2 be two arbitrary specified initial values, we obtain

$$c_n = c_2 + (n-2)(c_2 - c_1)\ .$$

Also solved by George Ledin, Jr., University of California, Berkeley, Calif; Douglas Lind, Falls Church, Virginia; Raymond Whitney, Pennsylvania State University, Hazleton, Pennsylvania; J.A.H. Hunter, Toronto, Ontario, Canada; Dermott A. Breault, Sylvania–A.R.L., Waltham, Mass.; and the proposer.

BOUNDS FOR FIBONACCI NUMBERS

B-39 *Proposed by John Allen Fuchs, University of Santa Clara, Santa Clara, California*

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$. Prove that

$$F_{n+2} < 2^n \text{ for } n \geq 3 .$$

Solution by Brian Scott, Ripon, Wisconsin

The solution is by induction on n. $F_{3+2} = F_5 = 5 < 8 = 2^3$ and $F_{4+2} = F_6 = 8 < 16 = 2^4$. Assume as the induction hypothesis that $F_{(n-2)+2} < 2^{n-2}$ and $F_{(n-1)+2} < 2^{n-1}$. Then $F_{n+2} = F_{(n-1)+2} + F_{(n-2)+2} < 2^{n-1} + 2^{n-2} = 2^{n-2}(2+1) < 2^{n-2} \cdot 2^2 = 2^n$. Therefore $F_{n+2} < 2^n$ for all $n \geq 3$.

Also solved by Gladwin E. Bartel, University of Wisconsin, Madison, Wisconsin; Dermott A. Breault, Sylvania–A.R.L., Waltham, Massachusetts; John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; George Ledin, Jr., University of California, Berkeley, California; Douglas Lind, Falls Church, Virginia; Howard Walton, Yorktown H.S., Arlington, Virginia; John Wessner, Melbourne H.S., Melbourne, Florida; Raymond Whitney, Pennsylvania State University, Hazelton, Pennsylvania; Charles Ziegenfus, Madison College, Harrisonburg, Virginia; and the proposer.

Lind mentioned the related $F_n < (7/4)^n$ on page 7 of Topics in Number Theory by W. J. LeVeque and $a^{n-1} < F_n < a^n$, where

$$a = (1 + \sqrt{5})/2 ,$$

on page 93 of An Introduction to the Theory of Numbers, by Niven and Zuckerman. Ziegenfus mentioned similar problems in LeVeque's Elementary Theory of Numbers.

A SUMMATION FORMULA

B-40 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

If H_n is the n-th term of the generalized Fibonacci sequence, i.e., $H_1 = p$, $H_2 = p+q$, $H_{n+2} = H_{n+1} + H_n$ for $n \geq 1$, show that

$$\sum_{1}^{n} k H_k = (n+1) H_{n+2} - H_{n+4} + 2p + q .$$

Solution by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pensylvania

$H_1 = 2H_3 - H_5 + 2p + q$, so that the assertion is true for $n = 1$. Assume as an induction hypothesis that the result has been proved for all n satisfying $1 \le n \le m$, where $m \ge 1$. We will show the result must necessarily hold for $m + 1$.

$$\begin{aligned}\sum_{1}^{m+1} kH_k &= (m+1)H_{m+1} - \sum_{1}^{m} kH_k \\ &= (m+1)H_{m+1} + (m+1)H_{m+2} - H_{m+4} + 2p + q \\ &= (m+1)H_{m+3} - H_{m+4} + 2p + q \\ &= (m+2)H_{m+3} - (H_{m+4} + H_{m+3}) + 2p + q \\ &= (m+2)H_{m+3} - H_{m+5} + 2p + q \;.\end{aligned}$$

Hence, the assertion holds for $n = m+1$ and the proof is completed by the usual inductive argument.

Also solved by Dermott A. Breault, Sylvania–A.R.L., Waltham, Massachusetts; Douglas Lind, Falls Church, Virginia; George Ledin, Jr., University of California, Berkeley, California; Howard Walton, Yorktown H.S., Arlington, Virginia; and the proposer.

AN IMPOSSIBLE CONDITION

B-41 *Proposed by David L. Silverman, Beverley Hills, California*

Do there exist four distinct positive Fibonacci numbers in arithmetic progression?

Solution by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania

No. For, assume $F_i < F_j < F_h < F_k$ are in arithmetic progression, so that $F_j - F_i = d = F_k - F_h$. Then

$$d = F_j - F_i < F_j$$

while

$$d = F_k - F_h \ge F_k - F_{k-1} = F_{k-2} \ge F_j \;,$$

since $k \geq j+2$. This is a contradiction, so that four distinct positive Fibonacci numbers cannot be in arithmetic progression.

Also solved by Brian Scott, Ripon, Wisconsin and the proposer

F_{n+1} IN TERMS OF F_n

B-42 *Proposed by S.L. Basin, Sylvania Electronics Systems, Mountain View, California*

Express the $(n+1)$-st Fibonacci number F_{n+1} as a function of F_n. Also solve the same problem for L_n.

Solution by H.H. Ferns, University of Victoria, Victoria, B.C., Canada

The following three identities are readily proved by applying Binet's formula.

(1) $$2F_{n+1} = F_n + L_n$$

(2) $$L_n^2 - 5F_n^2 = 4(-1)^n$$

(3) $$2L_{n+1} = 5F_n + L_n$$

Eliminating L_n from (1) and (2) gives

$$F_{n+1} = \frac{F_n + \sqrt{5F_n^2 + 4(-1)^n}}{2},$$

Eliminating F_n from (2) and (3) gives

$$L_{n+1} = \frac{L_n + \sqrt{5}\sqrt{L_n^2 - 4(-1)^n}}{2},$$

Also solved by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; Douglas Lind, Falls Church, Virginia; and the proposer

ITERATION FOR THE GOLDEN MEAN

B-43 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

(1) Let $x_0 \geq 0$ and define a sequence x_k by $x_{k+1} = f(x_k)$ for $k \geq 0$, where $f(x) = \sqrt{1+x}$. Find the limit of x_k as $k \to \infty$.

(2) Solve the same problem for $f(x) = \sqrt[3]{1 + 2x}$.

(3) Solve the same problem for $f(x) = \sqrt[4]{2 + 3x}$.

(4) Generalize.

Solution by John L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania

(1) If

$$\lim_{k \to \infty} x_k$$

exists, call it x. Then we have, since f(x) is continuous,

$$\lim_{k \to \infty} x_{k+1} = x = \lim_{k \to \infty} \sqrt{1 + x_k} = \sqrt{1 + x}, \text{ or } x^2 = 1 + x \quad ,$$

yielding

$$x = \frac{1 + \sqrt{5}}{2}$$

as the unique positive solution for the limit.

(2) The same process yields

$$x^3 = 1 + 2x, \text{ or } x^3 - 2x - 1 = (x+1)(x^2-x-1) = 0 \quad .$$

Again, there is only one positive solution, namely $x = \dfrac{1 + \sqrt{5}}{2}$.

(3) Similarly, the equation $x^4 = 2 + 3x$ or $x^4 - 3x - 2 = 0$ clearly has only one positive root since the quantity $x^4 - 3x - 2$ is negative for $0 \le x \le 1$ and is monotonic increasing for $x > 1$. This unique positive root, which is easily verified to be $\dfrac{1 + \sqrt{5}}{2}$, is the required limit.

(4) If f(x) is continuous and is such that

$$\lim_{k \to \infty} x_k = L$$

exists when $x_{k+1} = f(x_k)$, then the limit L is a solution of the equation $x = f(x)$. If, further, f(x) is positive for all x, then x must be non-negative. Note: The solution above does not prove that $x^n = F_{n-1} + F_n x$ has no solution with $x > (1 + \sqrt{5})/2$. This is left to the reader as B-55 below.

Also solved by George Ledin, Jr., San Francisco, Calif.; Douglas Lind, Falls Church, Virginia; and the proposer

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