Topological Galois Theory

(solvability and non solvability of equations in finite terms)

How to solve explicitly given algebraic or differential equation?

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cardano-Tartaglia formula

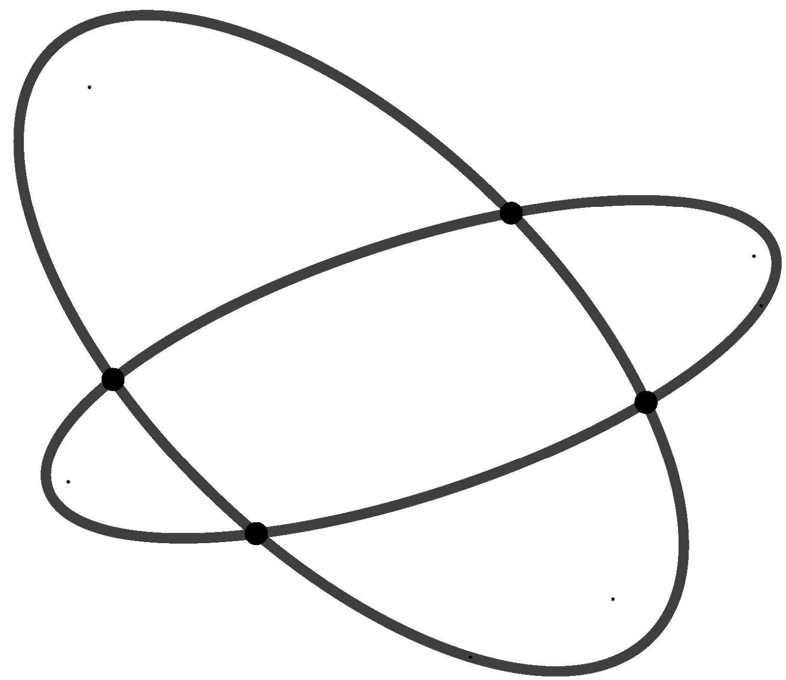
$$x^3 + px + q = 0$$

$$\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Niccolò Fontana Tartaglia (1499/1500 – 13 December 1557, Italian mathematician) &
Gerolamo Cardano (24 September 1501 – 21 September 1576) was an Italian polymath)

A1

Intersection of two quadrics



One can find by radicals intersection points of 2 quadrics.
It provides solution of any equation of degree four.

Slide 5

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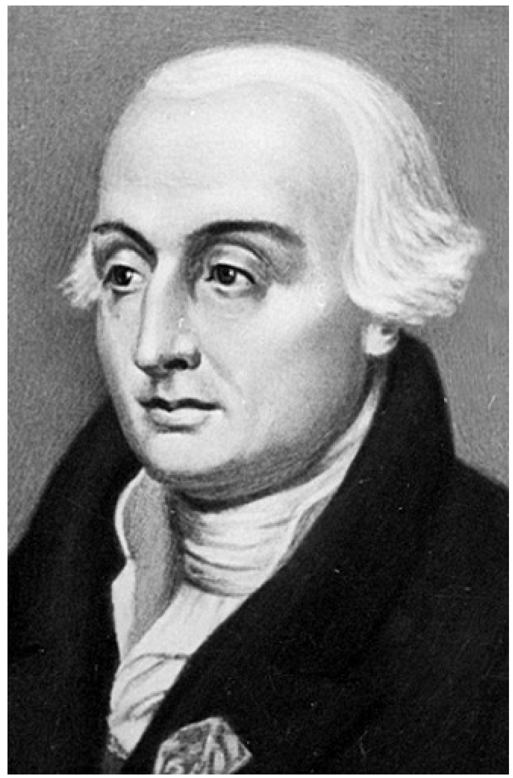
Joseph-Louis Lagrange (25 January 1736 – 10 April 1813, Italian mathematician)

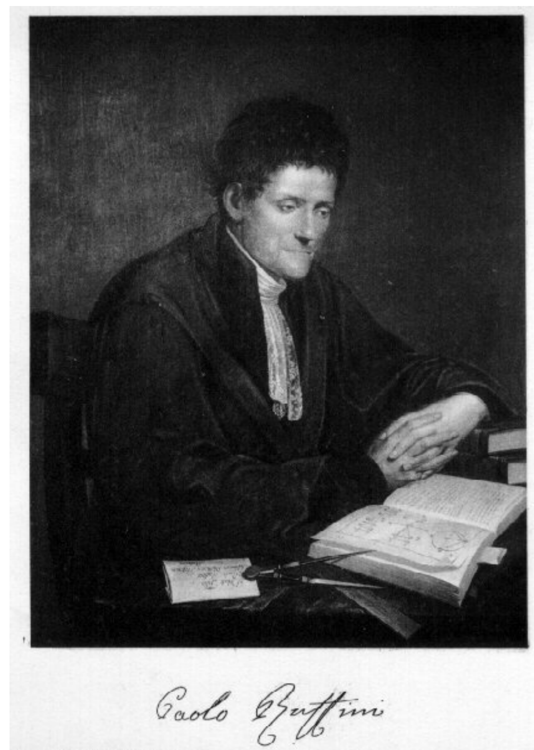


Abel–Ruffini theorem

The Abel Ruffini theorem states that there is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients.

The theorem is named after Paolo Ruffini, who made an incomplete proof in 1799, and Niels Henrik Abel, who provided a complete proof in 1824.

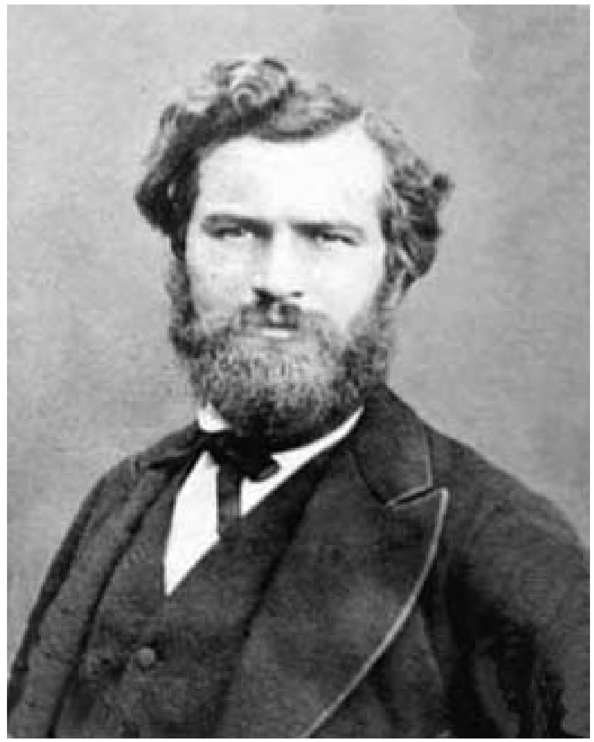
Niels Henrik Abel (5 August 1802 – 6 April 1829, Norwegian mathematician) &
Paolo Ruffini (September 22, 1765 – May 10, 1822, Italian mathematician)



Évariste Galois (25 October 1811 – 31 May 1832, French mathematician)

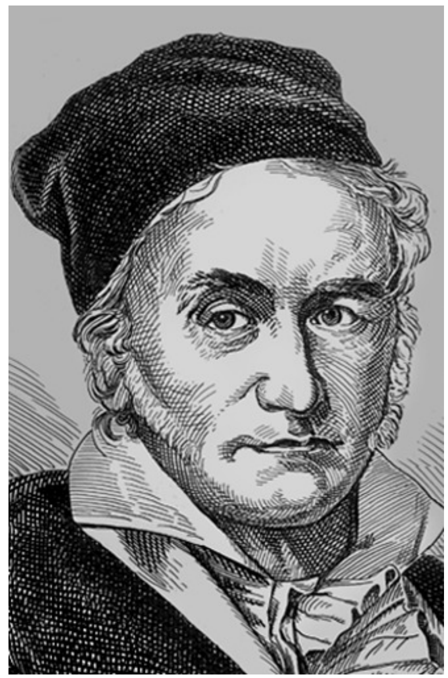
Galois started a very elegant and power Galois theory. It explains completely which algebraic equations can be solved by radicals.

Camille Jordan (5 January 1838 – 22 January 1922, French mathematician)



Jordan found topological meaning of Galois groups for a wide class of algebraic equations.

Carl Friedrich Gauss (30 April 1777 – 23 February 1855, German mathematician)



Gauss proved the constructibility of the regular 17-gon in 1796. Five years later, he formulated a condition for the constructibility of regular polygon.

Joseph Liouville (24 March 1809 – 8 September 1882, French mathematician)

First Liouville's theorem (1833)

The theorem provides conditions for integrability of elementary functions in finite terms. For example it shows that one can not write elementary formulas for the following integrals:

$$\int_a^x \frac{dx}{\sqrt{x(x+1)(x+2)}}$$

$$\int_a^x e^{-x^2}\,dx$$

$$\int_a^x \frac{dx}{\ln x}$$

Second Liouville's theorem (1838)

The theorem provides conditions for solvability by quadratures of second order linear differential equations. For example it shows that one can not solve by quadratures the following equation:

$$x'' + xy = 0$$

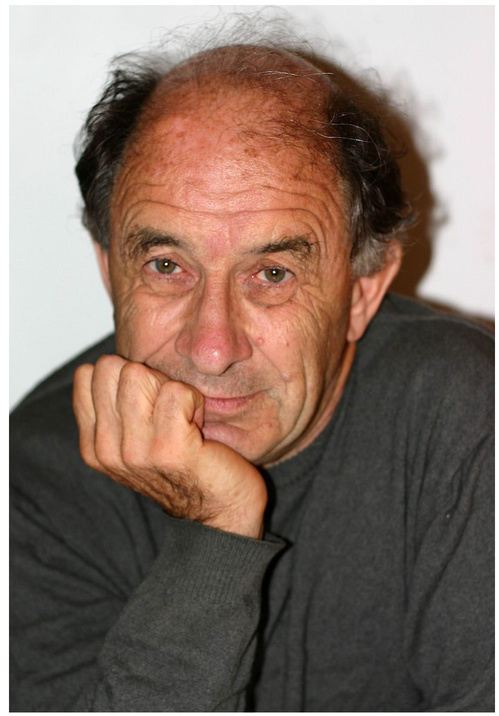
Picard-Vessiot theory (1910)

Picard discovered a deep analogy between algebraic equations and linear differential equations.

Picard-Vessiot theory is a version of Galois theory for such equations.

Émile Picard (24 July 1856 – 11 December 1941, French mathematician) &
Ernest Vessiot (8 March 1865 – 17 October 1952, French mathematician)

Vladimir Igoverich Arnold (12 June 1937 – 3 June 2010, Soviet and Russian mathematician)



Vladimir Igorevich was my beloved teacher. He found a topological proof of Abel-Ruffini theorem.

My first paper on this subject was published half century ago

THE REPRESENTABILITY OF ALGEBROIDAL FUNCTIONS
BY SUPERPOSITIONS OF ANALYTIC FUNCTIONS
AND ALGEBROIDAL FUNCTIONS OF ONE VARIABLE

A. G. Khovanskii

In this note we give a necessary condition for the representability of an algebroidal function in the form of superpositions of analytic functions and algebroidal functions of one variable. In particular, we show that an algebroidal function of two variables, $z(a, b)$, defined by the equation

$$z^5 + az + b = 0,$$

cannot be represented in the form of such superpositions in any neighborhood of the origin.

Definition 1. An analytic germ f_a, defined at a point a of a neighborhood $U \subseteq C^n$, is called algebroidal in this neighborhood if it satisfies some irreducible equation

$$f^n + p_1 f^{n-1} + \ldots + p_n = 0, \quad (1)$$

where p_i, $i = 1, \ldots, n$, are functions which are analytic in the neighborhood U.

Denote the set of all solutions of Eq. (1) in a neighborhood $V \subseteq U$ by $f(V)$.

Definition 2. Two analytic germs f_a and g_b, defined at the points a and b, respectively, of a neighborhood $U \subseteq C^n$, are called equivalent in this neighborhood, $f_a \underset{U}{\sim} g_b$, if the germ g_b can be obtained from the germ f_a by continuation along some curve lying in U.

Denote the value of the germ f_a at the point a by $f(a)$.

Springer Monographs in Mathematics

Askold Khovanskii

Topological Galois Theory

Solvability and Unsolvability of Equations in Finite Terms

Springer

THANK YOU

Lectures 1, September 10
INTRODUCTION

1. UNSOLVABILITY IN FINITE TERMS

A lot of beautiful results on unsolvabilily of equations in finite terms were obtained by

Gauss, Abel, Galois, Liouville, Picard, Vessiot, Ritt, Kolchin, Rosenlicht and by other mathematicians.

What does it mean that an equation can not be solved explicitly?

One can fix a class of functions and say that an equation is solved explicitly if its solution belongs to this class. Different classes of functions correspond to different notions of solvability.

A class of functions can be introduced by specifying:

a list of basic functions and

a list of admissible operations.

Given the two lists, the class of functions is defined as

the set of all functions that can be obtained from the basic functions by repeated application of admissible operations.

2. CLASSICAL CLASSES OF FUNCTIONS

To define a classical class of functions we have to fix its list of basic functions and its list of admissible operations.

A many of them use the **list of basic elementary functions** and the **list of classical operations**.

LIST OF BASIC ELEMENTARY FUNCTIONS

all constants, x (or $x_1, \ldots, x_n$);
exp, ln, $x \to x^\alpha$;

sin, cos, tan;

arcsin, arccos, arctan.

9. CLASSICAL OPERATIONS
LIST OF CLASSICAL OPERATIONS

1) composition: $f, g \in L \Rightarrow f \circ g \in L$;

2) arithmetic operations: $f, g \in L \Rightarrow f \pm g, f \times g, f/g \in L$;

3) differentiation: $f \in L \Rightarrow f' \in L$;

4) integration: $f \in L$ and $y' = f$, i.e. $y = C + \int^{x} f(t)dt \Rightarrow y \in L$;

5) extension by exponent of integral: $f \in L$ and $y' = fy$, i.e. $y = C \exp \int^{x} f(t)dt \Rightarrow y \in L$;

6) algebraic extension: $f_1, \ldots, f_n \in L$ and $y^n + f_1 y^{n-1} + \cdots + f_n = 0 \Rightarrow y \in L$;

7) exponent: $f \in L$ and $y' = f'y$, i.e. $y = C \exp f \Rightarrow y \in L$;

8) logarithm: $f \in L$ and $dy = df/f$, i.e. $y = C + \ln f \Rightarrow y \in L$;

9) meromorphic operation: if $F : \mathbb{C}^n \to \mathbb{C}$ is a meromorphic function, $f_1 \ldots, f_n \in L$, and $y = F(f_1, \ldots, f_n) \Rightarrow y \in L$.

The operations 2) and 7) are meromorphic operations.

10. RADICALS, QUADRATURES, etc.

I. Radicals.

Basic functions: rational functions.

Operations: arithmetic operations and extensions by radicals.

II. Elementary functions.

Basic functions: basic elementary functions.

Operations: composition, arithmetic operations, differentiation.

III. Generalized elementary functions.

The same as elementary functions + algebraic extensions.

IV. Quadrature.

Basic functions: basic elementary functions.

Operations: composition, arithmetic operations, differentiation and integration.

IV'. "Liouville's quadratures".

Basic functions: all complex constant.

Operations: the arithmetic operations, integration, extension by the exponent of integral.

V. Generalized quadratures.

The same as quadratures + algebraic extensions.

11. LIOUVILLE'S THEORY

SLOGAN OF LIOUVILLE'S THEORY.

"Sufficiently simple" equations have either "sufficiently simple" solutions or no explicit solutions at all.

Theorem 1 (Liouville)**.** *Class of "Liouville's quadratures" = class of quadratures.*

Theorem 8 shows that the class of quadratures can be constructed wirhout highly non algebraic operation of taking composition of two given functions. Thus Theorem 8 reduces the problem of solvability by quadratures to differential algebra.

The similar result holds for all classical classes of functions.

Theorem 2 (Liouville 1833)**.** *An integral $y(x)$ of an algebraic function is a generalized elementary function if and only if*

$$y(x) = A_0(x) + \sum_{i=1}^{n} \lambda_i \ln A_i(x),$$

where $\lambda_i \in \mathbb{C}$ and A_i are algebraic functions.

Similarly Liouvilee answered on the following question:

Which generalized elementary functions have untyderivative representable by generalized elementary functions?

In 1968 Rosenlicht found very constructive, pure algebraic proof of Liouville's result.

In 2018 I reprove again Liouville's result. My proof is basically geometric. It uses ideas of Galois Theory.

12. SECOND LIOUVILLE'S THEOREM

Liouville proved the following fantastic theorem:

Theorem 3 (Liouville, 1841). *An equation*

$$y'' + py' + qy = 0$$

, where p, q are rational functions, is solvable by generalized quadratures if and only if it has a solution

$$y_1(x) = \exp \int^x a(t)dt,$$

where $a(t)$ is an algebraic function.

In fact as Liouville proved Theorem can be modified for the case when coefficients p, q are , say, representable by genera;ized quadratures , or belong to agiven differential field.

Theorem 4 (Picard-Vessio 1910, M. Rosenlicht 1973, Kh. 2018)**.** *A linear order n differential equation is solvable by generalized quadratures if and only if:*

1) it has a solution $y_1 = \exp \int^x a(t)dt$ *where* $a(t)$ *is an algebraic function, and*

2) if the equation of order $(n-1)$ *obtained from the original equation by the reduction of order is solvable by generalized quadratures.*

To prove Theorem 12 Picard and Vessio developed the differential Galois theory. Rosenlicht used the valuation theory.

My proof is based on the original ideas due to Liouville. and Ritt. It is very elementary. This Spring I found a wide generalization of Liouville thechnoque which is very simple and very powerful (not published yet).

13. PICARD–VESSIOT THEORY

Picard discovered a similarity between linear differential equations and algebraic equations. He initiated the development of a differential of analogue of Galois theory.

Theorem 5 (Picard–Vessiot, 1910)**.** *A linear differential equation is solvable by quadratures if and only if its differential Galois group is solvable. It is solvable by generalized quadratures if and only if the connected component of the identity in its differential Galois group is solvable.*

Picard–Vessiot theory has many applications. For example, for an equation whose coefficients are rational functions with integral coefficients it allows to determine explicitly is the equation solvable by generalized quadratures or not.

14. TOPOLOGICAL GALOIS THEORY

Theorem 6 (C.Jordan). *The Galois group of an algebraic equation over the field of rational functions 9n several complex variables is isomorphic to the monodromy group of the (multivalued) algebraic function defined by the same equation.*

Jordan's theorem implies that the Galois group of an algebraic equation over the field of rational functions in several complex variables has a pure topologycal meaning. One dimensional topological Galois theory deals with functions in one variable There is also a multydimensional version of topological Galois theory but we will not talk about it now.

Corollary 7 (On nonrepresentability of rational functions by radicals). *If the monodromy group of an algebraic function is unsolvable then the function is not representable by radicals.*

CONSTRUCTING TOPOLOGICAL GALOIS THEORY PROGRAM:

I. Find a wide class of functions which is closed under classical operations, such that for all functions from the class the monodromy group is well defined.

II. Use the monodromy group within this class instead of the Galois group.

15. CLASS OF $\mathcal{S}$-FUNCTIONS

A multivalued analitic function of one complex variable is called $\mathcal{S}$-function if the set of its singular points is at most countable.

Theorem 8. *The class of $\mathcal{S}$-functions is closed under*
composition,
arithmetic operations,
differentiation,
integration, meromorthic operations,
solving algebraic equations,
solving linear differential equations.

Corollary 9. *A function representable by generalized quadratures is $\mathcal{S}$-function.*

Thus a function having an uncountable number of singular points can not be expressed by generalized quadratures.

Example. Consider a function

$$f = \ln(\sum_{i=1}^{n} \lambda_i \ln(x - a_i)).$$

If $n \geq 3$, λ_i are generic and $a_i \neq a_j$ if $i \neq j$ then:

1) the monodromygroup of f contains continuum elements,

2) the set of singular points of f is everywhere dense on the complex line.

16. SOLVABLE MONODROMY GROUP

Theorem 10. *The class of $\mathcal{S}$-functions whose monodromy group is solvable is closed under:*

integration,

differentiation

composition

and meromorphic operations
(in particular arithmetic operations).

Corollary 11. *If the monodromy group of a function f is unsolvable, then f can not be represented via meromorphic functions using integration, differentiation, composition and meromorphic operations.*

Theorem 12. *If the monodromy group of an algebraic function is unsolvable then one can not represent it by a formula which involves meromorphic functions and elementary functions and uses integration, composition and meromorphic operations.*

17. FUCHS-TYPE LINEAR DIFFENTIAL EQUATIONS

Theorem 13. *If the monodromy group of a Fuchs-type linear differential equation or of a system of Fuchs-type linear differential equations is solvable then*

this equation or this system of equations is solvable by quadratures.

But if it is unsolvable one can not represent their general solutions by a formula which involves

integration, composition and meromorphic operations and uses meromorphic and elementary functions.

Corollary 14. *Consider a system*

$$y' = \sum \frac{A_i}{x - a_i} y,$$

where y is n-vector and A_i are $n \times n$ matrices with constans entries.

Assume that the matrices A_i have sufficiently small entries. Then

by generalized quadratures if and only if all the matrices A_i are triangular in some basis.

Moreover if such system is not triangular in some basic, one can not write an finete formula for its generic solution which uses arbitrary meromorphic functions.

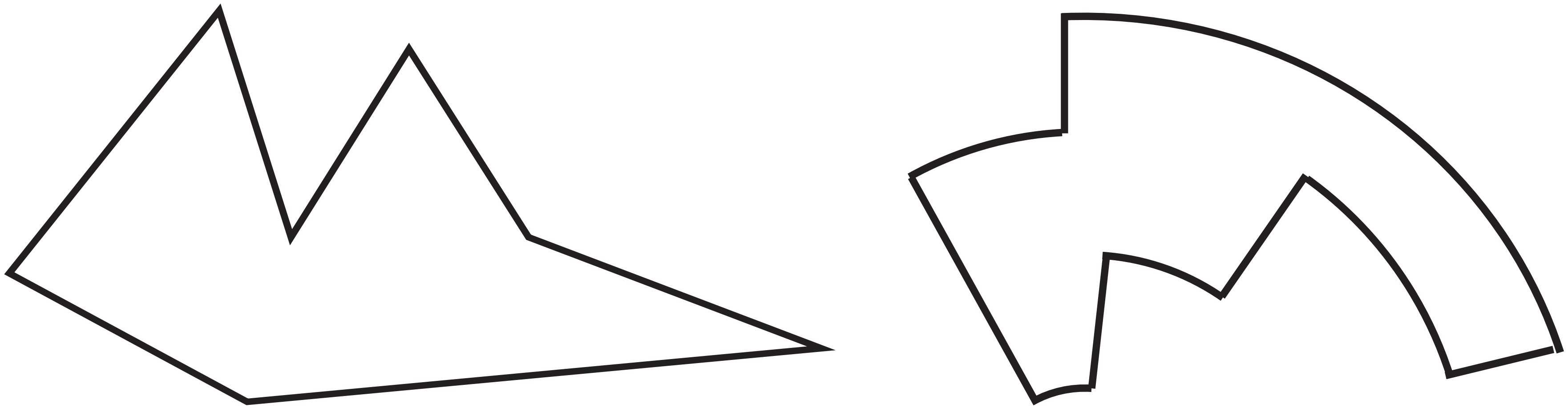
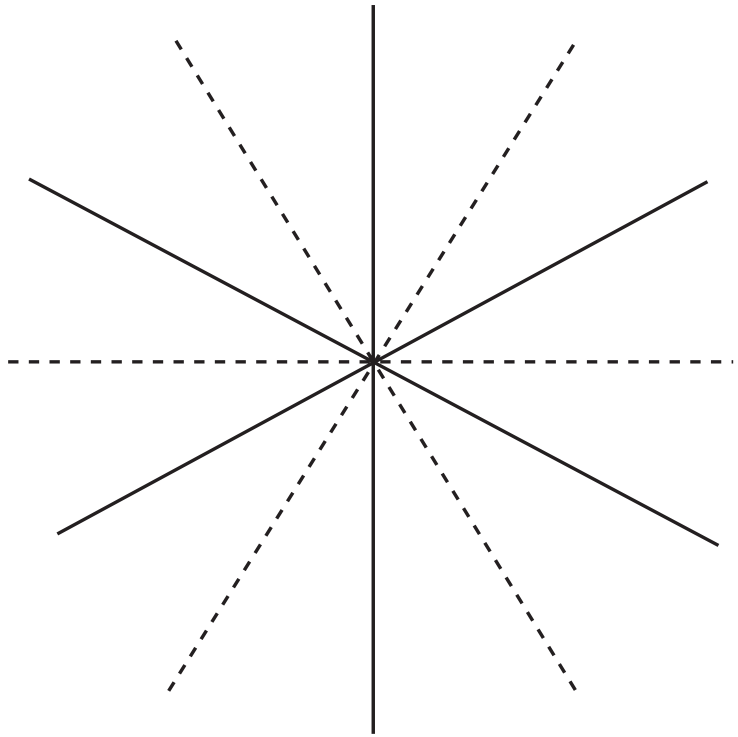
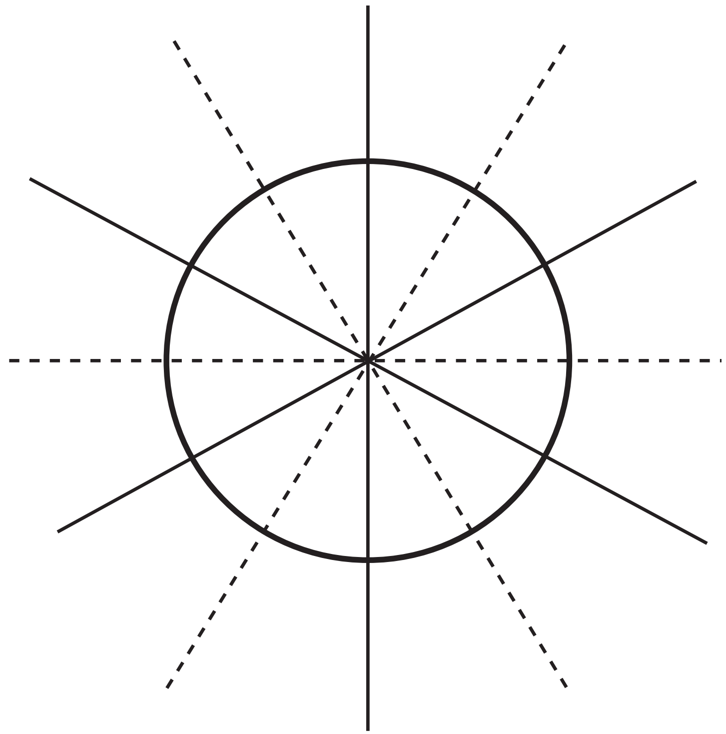


Figure 1: The first and the second cases of integrability

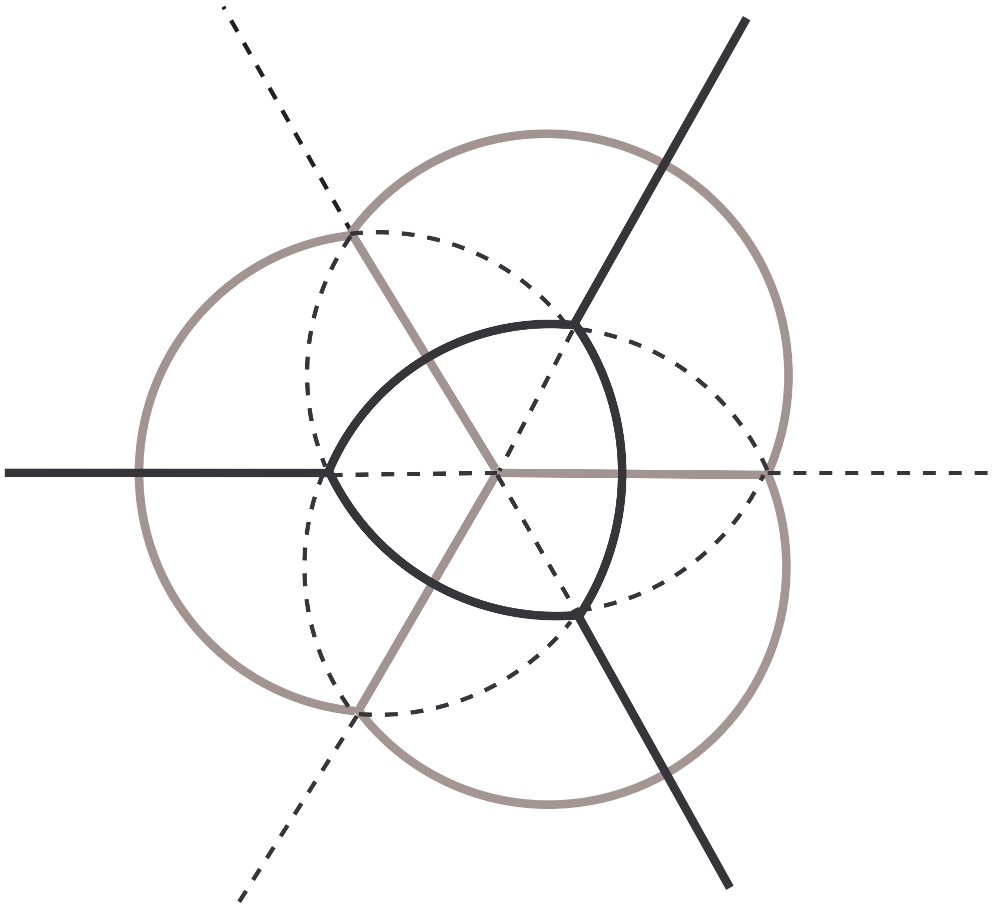
18. MAPPING FROM A BALL TO A CURVED POLYGON

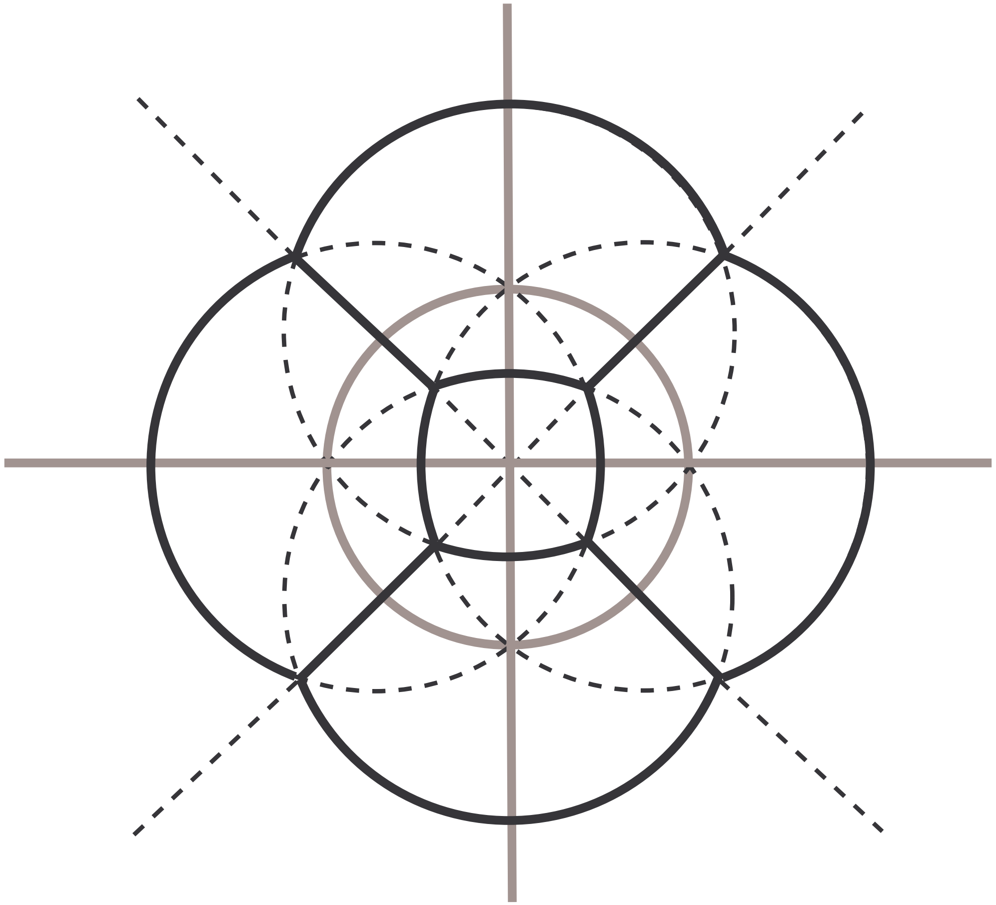
Corollary 15. *Let G be a polygon bounded by arcs of circles on the complex line. Let $f_G : B_1 \to G$ be a Riemann map from a unit ball onto G. One can classify all polygons G such that the function f_G is representable by quadratures.*

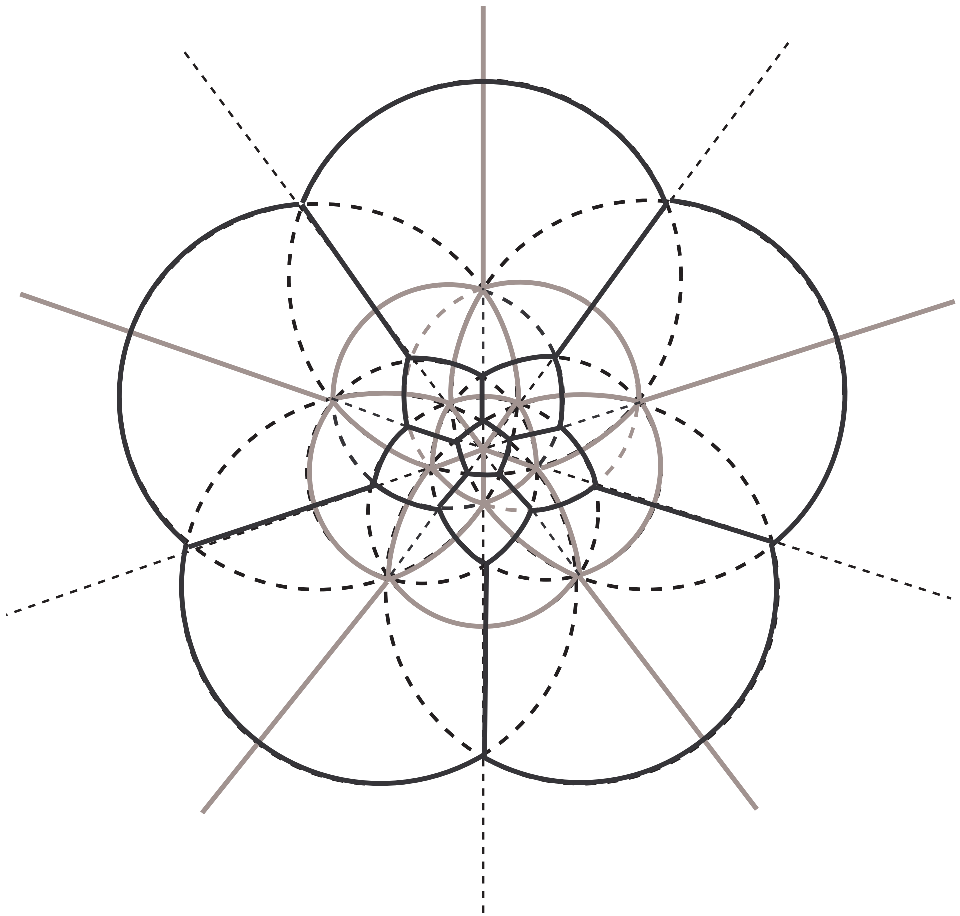




19. THIRD CASE OF INTEGRABILITY







20. POLYNOMIALS INVERTIBLE IN RADICALS

Theorem 16 (Ritt 1922). *A polynomial invertible in radicals if and only if it is a composition of the power polynomials* $z \to z^n$*, Chebyshev polynomials and polynomials of degree* ≤ 4.

Theorem 17 (Yu.Burda, Kh. 2012). *A polynomial invertible in radicals and solutions of equations of degree at most* k *is a composition of power polynomials, Chebyshev polynomials, polynomials of degree at most* k *and, if* $k \leq 14$*, certain exceptional polynomials polynomials (a description of these polynomials is given).*

The proof is based on classification of finite simple groups and results on primitive polynomials obtained by Muller and many other authors.