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ON A RESULT OF BRUNS

BY

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ABSTRACT. Bruns' Theorem states that the classical integrals of the gravitational three-body problem generate all algebraic integrals. We show that the first step in his proof, together with Ziglin's non-integrability criterion for complex systems, can be used to prove the non-existence of energy independent algebraic integrals in certain real analytic systems. We also show that this aspect of Bruns' argument is purely algebraic: We offer a proof based on elementary differential algebraic methods.

Introduction. A recent result of Ziglin gives a computable criterion for establishing the non-meromorphic integrability of complex analytic Hamiltonian systems [10]. Unfortunately, when successfully applied to the complexification of a real analytic system the general conclusion for the real domain is merely that the original system has no meromorphic integral expressible as the quotient of entire functions (e.g., polynomial and rational integrals). In this note we point out, using a classical result of Bruns, that the applicability of Ziglin's method can be extended to cover the non-existence of real *algebraic* integrals when the given real system has a special form.

The Bruns Theorem considered here is only the first step in his proof that the classical integrals of the three-body problem generate all algebraic integrals. Specifically (as discussed in [8, Chapter XIV]), Bruns initially proves that the existence of an algebraic integral for a (weighted) homogeneous vectorfield implies that of a (weighted) homogeneous integral of rational degree (see [9]). In fact this aspect of Bruns' result is purely algebraic, and to emphasize this point we offer a differential algebraic proof, somewhat in the spirit of Rosenlicht's proof of Liouville's Theorem on integration in finite terms [7] (also see [6, p.v.]). By using a standard result from elimination theory our treatment avoids a technicality for which Whittaker [8, p. 358] references Forsyth [3, pp. 332–335], but which Forsyth presents only in a special case. We use nothing from differential algebra beyond what can be found in standard basic algebra references (e.g. [5]).

We use **Q**, **R** and **C** to denote the fields of rational, real and complex numbers, and **Z** to denote the ring of integers.

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1. **Bruns Theorem.** Let M be an analytic manifold with a distinguished vectorfield Y, where "analytic" means real or complex analytic according as the field $\mathbf{F}$ denotes $\mathbf{R}$ or $\mathbf{C}$. All vectorfields, functions and forms on M are assumed analytic, and such an object θ is *Y-homogeneous of degree $r \in \mathbf{F}$* when the Lie derivative satisfies $L_Y(\theta) = r\theta$. We let $\{f_j : M \longrightarrow \mathbf{F}\}_{j=1}^n$, $1 \leqq n \leqq \dim_{\mathbf{F}}(M)$, be a collection of algebraically independent, non-constant, Y-homogeneous functions of degrees r_j, where $0 < r_j = (\nu_j/\rho_j) \in \mathbf{Q}$ and $(\nu_j, \rho_j) = 1$. $A = \mathbf{F}[f_1, \ldots, f_n]$ is the polynomial algebra in these functions.

Set $\rho = \operatorname{lcm}\{\rho_1, \ldots, \rho_n\}$, let $G = \langle \rho^{-1} \rangle$ be the additive subgroup of $\mathbf{Q}$ generated by ρ^{-1}, and for each $0 \leqq j \in \mathbf{Z}$ let A_j denote the subspace of A consisting of Y-homogeneous functions of degree $j\rho^{-1}$. Then $A_0 = \mathbf{F}$, and the derivation L_Y gives A the structure of a graded algebra, i.e. $A = \coprod_{j \geqq 0} A_j$ (direct sum) and $A_j \cdot A_k \subset A_{j+k}$. The only reason for the rationality assumption on the r_j is to achieve this grading. We assume there is a second vectorfield X on M satisfying $L_Y(X) = rX$ for some $r \in G$. It follows that $X(A_j) \subset A_{j+\ell}$ for each j, where $\ell = r\rho$, as $L_Y(X(f)) = [Y, X](f) + X(Y(f)) = ((r\rho + j)\rho^{-1})X(f)$.

Example: Let $M = \mathbf{R}^{2n} = \{(x, y)\} = \{(x_1, \ldots, x_n; y_1, \ldots, y_n)\}$ with symplectic $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. Assume an analytic Hamiltonian $H : M \longrightarrow \mathbf{R}$ is given by $H(x, y) = (1/2)|y|^2 + V(x)$, where the potential V is a homogeneous polynomial in the variables $\{x_1, \ldots, x_n\}$ of integer degree $m > 2$. Set $\delta = 2/(m-2)$ and $Y = \delta \sum_{i=1}^n x_i(\partial/\partial x_i) + (\delta + 1)\sum_{i=1}^n y_i(\partial/\partial y_i)$, where we note that $(\delta + 1) = (1/2)\delta m$. Then $L_Y(x_j) = \delta x_j$ and $L_Y(y_j) = (\delta + 1)y_j$ show that the global coordinate functions $\{x_1, \ldots, x_n; y_1, \ldots, y_n\}$ satisfy the above conditions. Moreover, a straightforward computation shows that the Hamiltonian vectorfield X_H satisfies $L_Y(X_H) = X_H$. Here $G = \langle \rho^{-1} \rangle$, where $\rho = (m-2)$ for m odd and $\rho = (1/2)(m-2)$ for m even. Then X satisfies the required conditions with $\ell = \rho$, and the polynomial algebra $A = \mathbf{F}[x_1, \ldots, x_n; y_1, \ldots, y_n]$ is graded w.r.t. Y as above.

Returning to generalities, let $\mathbf{Q}_A$ denote the quotient field of A. Bruns' Theorem asserts that the existence of a non-empty open set $U \subset M$, and a non-constant function $g : U \longrightarrow \mathbf{F}$, algebraic over $\mathbf{Q}_A$ and satisfying $L_X(g) = 0$, implies the existence of a non-constant quotient $(p/q) \in \mathbf{Q}_A$ of Y-homogeneous functions satisfying $L_X(p/q) = 0$.

As stated in the introduction, Bruns' result is purely algebraic. To give such a formulation allow $\mathbf{F}$ to be an arbitrary field, and replace the polynomial algebra $A = \mathbf{F}[f_1, \ldots, f_n]$ introduced above by an arbitrary finitely generated graded $\mathbf{F}$-algebra $A = \coprod_{j \geqq 0} A_j$ satisfying $A_0 = \mathbf{F}$ and which is a UFD when regarded as a ring. Elements of A_j are called *homogeneous of degree j*. Note that the grading subsumes the role of Y. Assume in addition that $X : A \longrightarrow A$ is an $\mathbf{F}$-derivation satisfying $X|A_j : A_j \longrightarrow A_{j+\ell}$ for some fixed $\ell \geqq 0$. Extend X to a derivation on the quotient field $\mathbf{Q}_A$ in the only possible way (i.e. using the quotient rule), and if g is separable algebraic over $\mathbf{Q}_A$ again use X to denote the unique extension of this derivation to $\mathbf{Q}_A(g)$ (e.g. see [5, pp. 385–386]). Finally, let $\mathcal{D}^*$ denote the dual of the $\mathbf{Q}_A(g)$-vector space $\mathcal{D}$ of $\mathbf{F}$-derivations

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of $\mathbf{Q}_A(g)$ over $\mathbf{F}$ and define $d : \mathbf{Q}_A(g) \longrightarrow \mathcal{D}^*$ by $df(\delta) = \delta(f), f \in \mathbf{Q}_A(g), \delta \in \mathcal{D}$ ([5, p. 387]). In this framework the role of non-constant functions is assumed by elements $f \in \mathbf{Q}_A(g)$ for which the differential df does not vanish, and linear independence of $f_1, \ldots, f_k$ is replaced by the condition $0 \neq df_1 \wedge \ldots \wedge df_k \in \Lambda^k(\mathcal{D}^*)$, $k = 2, 3, \ldots$.

THEOREM (Bruns). *Suppose there is a separable algebraic element g over $\mathbf{Q}_A$ satisfying $dg \neq 0$ and $X(g) = 0$. Then there is a quotient $(u/v) \in \mathbf{Q}_A$ of homogeneous elements u, $v \in A$ which also satisfies $d(u/v) \neq 0$ and $X(u/v) = 0$. Moreover, if $f_1, \ldots, f_n \in \mathbf{Q}_A$ are such that $dg \wedge df_1 \wedge \ldots \wedge df_n \neq 0$, then (u/v) can also be chosen so as to satisfy $d(u/v) \wedge df_1 \wedge \ldots \wedge df_n \neq 0$.*

The proof requires two preliminary lemmas. For the first, observe that each $p \in A$ has a unique representation $p = \sum p_j$ (finite sum) in terms of homogeneous elements $p_j \in A_j$. If t is transcendental over $\mathbf{Q}_A$, this allows us to define an $\mathbf{F}$-algebra embedding $\alpha : A \longrightarrow A[t]$ by sending $p = \sum p_j \in A$ to $\alpha(p) = p(t) = \sum p_j t^j \in A[t]$.

LEMMA 1. *Assume the nonzero elements p, $q \in A$ have no common prime factor. Then t is the only possible common prime factor in $A[t]$ of $\alpha(p)$, $\alpha(q)$.*

PROOF. Assume $\alpha(p) = p(t) = r(t)u(t)$ and $\alpha(q) = q(t) = r(t)v(t)$ factor in $A[t]$. Then $p = r(1)u(1)$ and $u = r(1)v(1)$ imply $r(1) \in \mathbf{F}$ since $\mathbf{F} = A_0$ contains all units. We will show that this implies $r(t) = ct^m$ for some $0 \neq c \in \mathbf{F}$ and $0 \leqq m \in \mathbf{Z}$.

Each $b(t) \in A[t]$ has a unique representation as a finite sum $b(t) = \sum_j (\sum_i b_{ij}) t^j$ with $b_{ij} \in A_i$. Let $\lambda = \lambda(b(t))$ denote the maximum index i such that $b_{ij} \neq 0$ for some j, and set $b_H(t) = \sum_j b_{\lambda j} t^j$. Note that if $a = \sum_{j=0}^n a_j \in A$ with $a_n \neq 0$ and $b(t) = \alpha(a)$, then $b_H(t) = a_n t^n$.

For $p = \sum_{j=0}^n p_j$ with $p_n \neq 0$ we have $p_n t^n = r_H(t) u_H(t)$, hence $r_H(t) = r_\lambda t^m$ for $\lambda = \lambda(r(t))$ and $r_\lambda \in A_\lambda$, $0 \leqq m \in \mathbf{Z}$. But no coefficient in $r(t)$ can cancel r_λ in $r(1)$. Since $r(1) \in \mathbf{F}$, this forces $\lambda = 0$ and $r(t) = r_H(t) = r_0 t^m$ which gives the result. $\square$

The second lemma is a fundamental result of elimination theory. For a proof, see [4, p. 57].

LEMMA 2. *Let $u(t) = \sum_{j=0}^n u_j t^j$ and $v(t) = \sum_{j=0}^m v_j t^j \in \mathbf{Q}_A[t]$ with $u \neq 0 \neq v$ and u_n, v_m not both zero. Then u and v have a common nonconstant factor in $\mathbf{Q}_A[t]$ if and only if there are nonzero polynomials $r_u(t)$ and $r_v(t)$ in $\mathbf{Q}_A[t]$ with $\deg(r_u(t)) < n$ and $\deg(r_v(t)) < m$ such that*

$$u(t) r_v(t) = v(t) r_u(t).$$

PROOF OF BRUNS' THEOREM. (a) There is an irreducible monic polynomial $P(t) = \sum_{i=0}^m g_i t^i \in \mathbf{Q}_A[t]$ such that $P(g) = 0$. Writing df for $df_1 \wedge \ldots \wedge df_n$, we have the identity

$$0 = d(P(g)df) = \sum_{i=0}^{m-1} g^i dg_i \wedge df + P'(g) dg \wedge df.$$

Since $P'(g)dg \wedge df \neq 0$ as $\deg(P(t))$ is minimal, this forces $dg_i \wedge df \neq 0$ for at least one i (in particular, $dg_i \neq 0$).

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Since X is a derivation and $X(g) = 0$, we also have

$$0 = X(P(g)) = \sum_{i=0}^{m-1} X(g_i)g^i,$$

hence $X(g_i) = 0$ for all i. In summary: the hypotheses imply that we can find $g_i \in \mathbf{Q}_A$ such that $X(g_i) = 0$ and $dg_i \wedge df \neq 0$.

(b) Write $g_i = (p/q)$ where $p = \sum p_j$ and $q = \sum q_k \in A$ have no common prime factor. Since

$$0 \neq dg_i \wedge df = d(p/q) \wedge df = q^{-2}(\sum q_k \sum dp_j - \sum p_j \sum dq_k) \wedge df,$$

and since $(q_k dp_j - p_j dq_k) \wedge df = 0$ if $d(p_j/q_k) \wedge df = 0$, we conclude that $d(p_j/q_k) \wedge df \neq 0$ for at least one quotient (p_j/q_k). In particular, $d(p_j/q_k) \neq 0$.

(c) To finish, we show that $X(p_j/q_k) = 0$ for all (p_j/q_k) with $q_k \neq 0$.

By Lemma 1 we can write $\alpha(p)/\alpha(q)$ as $u(t)/v(t)$, where $u(t)$, $v(t) \in A[t]$ have no common prime factor and

$$t^m u(t) = \alpha(p) = \sum p_j t^j, t^m v(t) = \alpha(q) = \sum q_k t^k \tag{1}$$

for some $0 \leqq m \in \mathbf{Z}$. If we set $X(t) = 0$, then from

$$X(A_j) \subset A_{j+\ell}, \ell \geqq 0 \text{ fixed},$$

we see that

$$\alpha \circ X = t^\ell \cdot (X \circ \alpha) \tag{2}$$

on $A[t]$. From $qX(p) = pX(q)$ and the fact that $\alpha : A \longrightarrow A[t]$ is a homomorphism we obtain

$$v(t) \cdot X(u(t)) = u(t) \cdot X(v(t)). \tag{3}$$

If $X(u(t)) \cdot X(v(t)) = 0$, then $X(u(t)) = 0 = X(v(t))$ by (3), hence $X(p_j) = 0 = X(q_k)$ by (1), and we are done. Thus we assume $X(u(t)) \cdot X(v(t)) \neq 0$, hence $\deg(X(w(t))) = \deg(w(t))$ for $w = u$, v, by (3) and Lemma 2. Then

$$X(w(t)) = \lambda_w \cdot w(t) + r_w(t), w = u, v, \tag{4}$$

where $0 \neq \lambda_w \in \mathbf{Q}_A$, and $r_w(t) \in \mathbf{Q}_A[t]$ with $\deg(r_w(t)) < \deg(w(t))$. Substituting into (3) and rearranging gives

$$(\lambda_u - \lambda_v)u(t)v(t) = r_u(t)v(t) - r_v(t)u(t), \tag{5}$$

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whereupon comparison of highest order terms in t yields $\lambda_u = \lambda = \lambda_v$. This reduces (5) to $r_u(t)v(t) = r_v(t)u(t)$, and Lemma 2 then shows that $r_u(t) = 0 = r_v(t)$. But then (4) and (1) imply $X(p_j) = \lambda p_j$ and $X(q_k) = \lambda q_k$, and the result follows. □

2. **An Application to Hamiltonian Systems.** Let H be a weighted homogeneous polynomial Hamiltonian on $\mathbf{R}^{2n}$, such as in the example of the previous section, and let X_H be the associated vector field. Let $H_\mathbf{C}$ and $X_\mathbf{C} = X_{H,\mathbf{C}}$ be the associated analytic extensions to $\mathbf{C}^{2n}$.

THEOREM 2. *If $X_\mathbf{C}$ has no meromorphic integral independent of $H_\mathbf{C}$, then X_H has no algebraic integral independent of H.*

PROOF. Otherwise Bruns' Theorem guarantees a real rational integral for X_H which is independent of H, whereupon analytically extending numerator and denominator to $\mathbf{C}^{2n}$ gives a meromorphic integral for $X_\mathbf{C}$ which is independent of $H_\mathbf{C}$. □

For a specific example in two degrees of freedom consider the *n-saddle Hamiltonian*

$$H = (1/2)(y_1^2 + y_2^2) + (1/n)\mathrm{Real}(x_1 + ix_2)^n, n \geqq 3. \tag{2.1}$$

Ziglin's methods have been applied to the analytic extension of (2.1) to $\mathbf{C}^4$ to show that the resulting system has no meromorphic integrals independent of $H_\mathbf{C}$ [2]; hence by Theorem 2 the real system (2.1) can have no algebraic integrals independent of H.

In fact for *odd* $n \geqq 3$ one can embed a Smale horseshoe mapping into the flow of (2.1), and as a result conclude that the system must be chaotic as well as nonintegrable [1]. Corresponding results for n even are not known.

Using results of Ziglin [11], Theorem 2 can also be applied to the two-degree of freedom system on $\mathbf{R}^4$ with Hamiltonian $H = (1/2)(y_1^2 + y_2^2) + (1/2)x_1^2x_2^2$. This is the Yang-Mills system describing a homogeneous two-component field with gauge group $SU(2)$.

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