TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 1, Pages 275–285
S 0002-9947(01)02848-3
Article electronically published on July 11, 2001

REPRESENTATION TYPE OF HECKE ALGEBRAS OF TYPE A

KARIN ERDMANN AND DANIEL K. NAKANO

ABSTRACT. In this paper we provide a complete classification of the representation type for the blocks for the Hecke algebra of type A, stated in terms of combinatorical data. The computation of the complexity of Young modules is a key component in the proof of this classification result.

1. INTRODUCTION

1.1. Let R be an integral domain and q be a unit in R with $q \neq 1$. Let us regard the symmetric group Σ_r as the set of bijections of $\{1, \ldots, r\}$. The Hecke algebra $\mathcal{H}_q(r) = \mathcal{H}_{R,q}(\Sigma_r)$ is the free R-module with basis $\{T_w : w \in \Sigma_r\}$. The multiplication in the algebra is defined by the rule

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w & \text{otherwise,} \end{cases}$$

where $s = (i, i+1) \in \Sigma_r$ is a basic transposition and $w \in \Sigma_r$. The function $\ell : \Sigma_r \to \mathbb{N}$ is the usual length function. Here we will assume that $R = k$ is a field; in this case the Hecke algebra $\mathcal{H}_q(r)$ is a finite-dimensional self-injective algebra. Let l be the multiplicative order of q in k^*, so that either q is a primitive lth root of unity, or $l = \infty$.

In [DJ2], Dipper and James gave a necessary and sufficient condition for the semisimplicity of $\mathcal{H}_q(r)$. For $q \neq 1$, this is the case if and only if $l > r$. So the case when l is finite (and $l \leq r$) is most interesting here.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition of r. That is, $|\lambda| = \sum_{i=1}^{t} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0$. A partition λ is *l-regular* if there does not exist an i such that $\lambda_{i+1} = \cdots = \lambda_{i+l} > 0$. It is well-known that the simple modules for the Hecke algebra $\mathcal{H}_q(r)$ are naturally indexed by l-regular partitions of r. For each partition λ of r there is a q-Specht module of the Hecke algebra $\mathcal{H}_q(r)$, denoted by S^λ. If λ is l-regular, then S^λ has a unique simple quotient denoted by D^λ.

The blocks of the Hecke algebra can be described via the Nakayama Rule, that is D^λ and D^μ belong to the same l-block if and only if λ and μ have the same l-core. The l-core is the partition whose Young diagram is obtained by removing as many rim l-hooks from Young diagram corresponding to the original partition. For details we refer to [Jam], [DJ2] or [JM]. Equivalently, S^λ and S^μ belong to the same l-block if and only if λ and μ have the same l-core. Let $\mathcal{B}_\lambda$ denote the block of $\mathcal{H}_q(r)$ containing S^λ. Given a partition λ of r, the *weight* $w := w(\lambda)$ of the block

Received by the editors September 24, 1999 and, in revised form, August 18, 2000.
2000 *Mathematics Subject Classification.* Primary 16G60, 20C08.
Research of the second author partially supported by NSF grant DMS-9800960.

$\mathcal{B}_\lambda$ is given by $|\gamma| + lw = r$ where γ is the l-core corresponding to $\mathcal{B}_\lambda$. The block containing the trivial module is called principal block, its core is the partition (t) where $0 \leq t < l-1$ and $t \equiv r \pmod{l}$.

A finite-dimensional algebra A has *finite representation type* if and only if there are only finitely many indecomposable modules in $\mathrm{mod}(A)$. Otherwise, A has *infinite representation type*. Algebras of infinite representation type fall into two mutually exclusive categories—ones which are either *tame* or *wild*. An algebra A has tame representation type if for each dimension d there are finitely many one parameter families of modules which contain all d-dimensional indecomposable modules. In this situation there is a good chance one may be able to classify all indecomposable representations. On the other hand, algebras of wild representation type are those algebras whose representation theory is as complicated as that of the free associative algebra $k\langle x, y\rangle$.

1.2. **Main result.** The main purpose of this paper is to classify the representation type of the blocks of the Hecke algebra $\mathcal{H}_q(r)$. This classification is given in terms of the "weight" of the corresponding partition λ for the block $\mathcal{B}_\lambda$.

Theorem. *Let $\mathcal{B}_\lambda$ be a block for $\mathcal{H}_q(r)$.*

(a) *$\mathcal{B}_\lambda$ is semisimple if and only if $w(\lambda) = 0$;*
(b) *$\mathcal{B}_\lambda$ has finite representation type (and is not semisimple) if and only if $w(\lambda) = 1$;*
(c) *$\mathcal{B}_\lambda$ has tame representation type if and only if $w(\lambda) = 2$ and q is a primitive 2nd root of unity;*
(d) *$\mathcal{B}_\lambda$ has wild representation type if and only if q is a primitive lth root of unity, $w(\lambda) \geq 2$ and $l \geq 3$, or $l = 2$ and $w(\lambda) \geq 3$.*

For blocks of group algebras of finite groups, the representation type may be characterized in terms of the defect groups of the block; see the introduction in [Erd] for a discussion. The defect groups for blocks of symmetric groups are iterated wreath products of cyclic groups of order p, see [JK, 6.2.39]. This gives a characterization of the representation type for blocks of symmetric groups which is analogous to our theorem; but the proof is quite different.

Acknowledgements. The second author would like to acknowledge the Mathematical Institute at Oxford University for their support and hospitality during his visit in June 1997.

2. Complexity

2.1. **Complexity and representation type.** Let A be a finite-dimensional algebra over a field k and M be a finite-dimensional A-module. Let $\Omega^0(M) = M$ and P_0 be the projective cover of $\Omega^0(M)$. Now let $\Omega^1(M)$ be the kernel of the map $P_0 \to \Omega^0(M)$. Recursively, one can now let $\Omega^{j+1}(M)$ be the kernel of map $P_j \to \Omega^j(M)$ where P_j is the projective cover of $\Omega^j(M)$. In this process we have constructed a minimal projective resolution of M in $\mathrm{mod}(A)$

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

For $\mathcal{V} = \{V_t : t \in \mathbb{N}\}$ a sequence of k-vector spaces, let $r(\mathcal{V})$ be the smallest integer $s \geq 0$ such that $\dim_k V_t \leq C \cdot t^{s-1}$ for some constant $C \geq 0$ and all $t \in \mathbb{N}$. If no such integer exists we say that $r(\mathcal{V})$ is infinite. The *complexity* $c_A(M)$ of M in $\mathrm{mod}(A)$ is then equal to $r(\{P_t : t \in \mathbb{N}\})$.

There are some elementary properties about the complexity of modules which are presented below.

(2.1.1) $c_A(M) = r(\{\mathrm{Ext}^t_A(M,S) : t \in \mathbb{N}\})$ where S is the direct sum of all simple A-modules.

(2.1.2) If B is a subalgebra of A such that A is a free (left) B-module, then $c_B(M) \leq c_A(M)$ for $M \in \mathrm{mod}(A)$ and $c_A(U \otimes_B A) \leq c_B(U)$ for $U \in \mathrm{mod}(B)$.

(2.1.3) If N is a direct summand of M in $\mathrm{mod}(A)$, then $c_A(N) \leq c_A(M)$.

The following proposition demonstrates that the notion of complexity is is related to the representation type of the algebra provided that the algebra A is self-injective.

Proposition. *Let A be a finite-dimensional self-injective algebra and $M \in \mathrm{mod}(A)$ be indecomposable.*

(a) *If $c_A(M) \geq 1$, then A is not semisimple.*
(b) *If $c_A(M) \geq 2$, then A has infinite representation type.*
(c) *If $c_A(M) \geq 3$, then A has wild representation type.*

Proof. (a) Observe that if A is self-injective, then $c_A(M) = 0$ occurs precisely when M is a projective A-module. Therefore, A is semisimple if and only if $c_A(M) = 0$ for all $M \in \mathrm{mod}(A)$. (b) The modules $\Omega^\bullet(M)$ are indecomposable and there are infinitely many non-isomorphic ones. Thus A has infinite representation type. (c) Let T be a composition factor of M in $\mathrm{mod}(A)$. Then $c_A(T) \geq c_A(M) \geq 3$. Therefore, by [Ric, Thm. 2] A has wild representation type. □

2.2. **Young modules.** Let λ be a composition of r and let Σ_λ be the corresponding Young subgroup of Σ_r, so $\Sigma_\lambda \cong \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots$. Let

$$x_\lambda = \sum_{w \in \Sigma^\lambda} T_w.$$

For any partition λ, let $\mathcal{H}_q(\lambda)$ be the subalgebra generated by T_w, $w \in \Sigma_\lambda$; this is isomorphic to the tensor product $\bigotimes_i \mathcal{H}_q(\lambda_i)$ of Hecke algebras. It is important to note that $\mathcal{H}_q(r)$ is free as a module over $\mathcal{H}_q(\lambda)$ of rank equal to the index of Σ_λ in Σ_d. The element x_λ above lies in this algebra, and $\langle x_\lambda \rangle$ is a module for $\mathcal{H}_q(\lambda)$ which is isomorphic to the trivial module. Set $M^\lambda := x_\lambda \mathcal{H}_q(r) \cong \langle x_\lambda \rangle \otimes_{\mathcal{H}_q(\lambda)} \mathcal{H}_q(r)$. The module M^λ is often called the q-permutation module.

The module M^λ has a unique submodule isomorphic to the Specht module S^λ, and there is a unique indecomposable direct summand of M^λ containing S^λ; this is the Young module Y^λ. In particular, Y^λ, S^λ belong to the same block $\mathcal{B}_\lambda$. Furthermore, $Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$. Each M^λ is a direct sum of Young modules Y^μ (see [Mar]).

In [DDu] it is proved that indecomposable $\mathcal{H}_q(r)$-modules have "vertices" and "sources", where vertices are certain parabolic subalgebras of the form $\mathcal{H}_q(\mu)$. Moreover it is proved that the vertices of Young modules are $(l-p)$-parabolics given as follows. Fix a partition λ and take the $(l-p)$-adic decomposition of λ as

$$\lambda = \lambda(0) + \sum_{i=0}^{m} \mu(i) l p^i$$

where $\lambda(0)$ is an l-restricted partition and the $\mu(i)$ are p-restricted partitions. This decomposition is unique. Define the partition

$$\rho = (l^a, l^{b_0}, (lp)^{b_1}, \cdots, (lp^m)^{b_m})$$

where $a = |\lambda(0)|$ and $b_i = |\mu(i)|$. Then Y^λ has Young vertex $\mathcal{H}_q(\rho)$, and its source is the trivial module for $\mathcal{H}_q(\rho)$. So we have that Y^λ is a direct summand of M^ρ and the restriction of Y^λ to $\mathcal{H}_q(\rho)$ has k as a direct summand. The following result describes the complexity of Young modules.

Theorem. *Let $\mathcal{H}_q(\rho)$ be a Young vertex of Y^λ. Then $c_{\mathcal{H}_q(r)}(Y^\lambda) = c_{\mathcal{H}(\rho)}(k)$.*

Proof. Recall that $H := \mathcal{H}_q(r)$ is free as a module over $L := \mathcal{H}(\rho)$. Therefore, induction and restriction are exact and projective resolutions of L-modules induce to projective resolutions. By (2.1.2), it follows that $c_L(k) \geq c_H(k \otimes_L H)$. Since Y^λ is a direct summand of $k \otimes_L H$, we also have that $c_H(k \otimes_L \mathcal{H}) \geq c_H(Y^\lambda)$ by (2.1.3), thus $c_L(k) \geq c_H(Y^\lambda)$.

On the other hand, $c_H(Y^\lambda) \geq c_L(Y^\lambda)$ by (2.1.2). By [DDu], k is a direct summand of Y^λ as an L-module, thus $c_L(Y^\lambda) \geq c_L(k)$ and $c_H(Y^\lambda) \geq c_L(k)$. □

2.3. Let A and B be algebras and let F be a functor from $\mathrm{mod}(A)$ to $\mathrm{mod}(B)$. We say that the functor F is bounded if for all $d \geq 1$ there is a constant $c = c(d)$ such that for all $M \in \mathrm{mod}(A)$ of length $\leq d$, the module $F(M)$ has length $\leq c(d)$.

Proposition. *Let A and B be algebras and $F : \mathrm{mod}(A) \to \mathrm{mod}(B)$ be a bounded functor. Moreover, let $G : \mathrm{mod}(B) \to \mathrm{mod}(B)$ be a functor such that M is a direct summand of $GF(M)$ for all $M \in \mathrm{mod}(A)$. If A is wild then so is B.*

Proof. Drozd has proved that a finite-dimensional algebra is either tame or wild and not both ([Dr]). Assume that B is not wild; then by Drozd's theorem B is tame, and we will show that then A is also tame. By [dlP] it suffices to show that A is weakly tame, which is to say that for every $d \geq 1$ there are finitely many $k[T]-A$-bimodules M_i which are finitely generated free as $k[T]$-modules such that every indecomposable A-module of dimension $\leq d$ is a direct summand of some $S \otimes_{k[T]} M_i$ where S is a simple $k[T]$-module. (That is, the weakening consists of allowing direct summands, instead of taking the whole module.)

Let $d \geq 1$, consider A-modules M of dimension $\leq d$. Since B is tame, there are finitely many $k[T] - B$-bimodules M_i such that every B-module X of dimension $\leq c(d)$ is of the form $X \cong S \otimes M_i$. Now take $X = F(M)$, and let $X = F(M) \cong S \otimes M_i$. Then $G(X) = GF(M)$ which is $S \otimes M_i$ restricted to A has M as a direct summand by our hypothesis. Hence, A is weakly tame. □

3. Blocks of the Hecke algebra

3.1. If q is not a root of unity then the Hecke algebra is semisimple and $w(\lambda) = 0$ for all partitions λ. So assume that q is a primitive l-root of unity, and let $\mathcal{B}_\lambda$ be a block of $\mathcal{H}_q(r)$ of weight $w(\lambda)$. It is known that the block $\mathcal{B}_\lambda$ is semisimple if and only if $w(\lambda) = 0$, and the block is of finite type if $w(\lambda) = 1$ [U] (see also §3.2). Therefore, Theorem 1.2 (main result) will be established by proving the following theorem.

Theorem. *Let $\mathcal{B}_\lambda$ be a block of $\mathcal{H}_q(r)$.*

(a) *If either $w(\lambda) \geq 2$ and $l \geq 3$, or if $l = 2$ and $w(\lambda) \geq 3$, then $\mathcal{B}_\lambda$ is wild.*
(b) *If $w(\lambda) = 2$ and $l = 2$, then $\mathcal{B}_\lambda$ is tame.*

The remainder of this section will be devoted to the proof of this theorem. We will proceed by proving the following steps.

(3.1.1) If $w(\lambda) \geq 3$ then $\mathcal{B}_\lambda$ is wild. Also for $l \geq 3$ the principal block of $\mathcal{H}_q(2l)$ is wild.

(3.1.2) If $w(\lambda) = 2$ and $l \geq 3$ then $\mathcal{B}_\lambda$ is wild.

(3.1.3) If $w(\lambda) = 2$ and $l = 2$ then $\mathcal{B}_\lambda$ is tame.

The statement (3.1.2) is verified by investigating "[2 : t]-pairs" of blocks. The definition of these pairs of blocks is explained in detail at the beginning of Section 3.4. For (3.1.3) we determine the basic algebra of $\mathcal{B}_\lambda$ by quiver and relations.

3.2. **Quiver and relations for blocks of weight one.** These were determined in [U]. Also by [Jo], any such block is Morita equivalent to the principal block $\mathcal{A}$ say, of $\mathcal{H}_q(l)$, which can be described as follows.

There are $l-1$ simple modules in the block, $X_j := D^\lambda$ where $\lambda = (l-j, 1^j)$ for $j = 0, 1, 2, \ldots, l-2$. The projective covers for these simple modules have the following structures.

$$
\begin{array}{lcl}
P^{(l)} & P^{(j,1^{l-j})} : \ j = l-1, \ldots, 3 & P^{(2,1^{l-2})} \\
 & & \\
D^{(l)} & D^{(j,1^{l-j})} & D^{(2,1^{l-2})} \\
D^{(l-1,1)} & D^{(j+1,1^{l-j-1})} \quad D^{(j-1,1^{l-j+1})} & D^{(3,1^{l-3})} \\
D^{(l)} & D^{(j,1^{l-j})} & D^{(2,1^{l-2})}
\end{array}
$$

The quiver for the algebra is Figure 1 below with relations $\alpha_j\alpha_{j+1} = 0 = \beta_{j+1}\beta_j$, and $\beta_j\alpha_j = \alpha_{j+1}\beta_{j+1}$ for $j = 1, 2, \ldots, l-3$. The only indecomposable modules of radical length three are the projective modules. Therefore, $\mathcal{A}$ has the same representation type as the algebra $\mathcal{A}/J^2$ ($J = \mathrm{Rad}\,\mathcal{A}$). But, the quiver for $\mathcal{A}/J^2$ which is the same as for $\mathcal{A}$ separates into a union of two quivers of type A_{l-1}. Hence, the principal block of $\mathcal{H}_q(l)$ is of finite representation type, by Gabriel's Theorem.

$$
\begin{tikzcd}
\underset{X_0}{\bullet} \arrow[r, shift left, "\alpha_1"] & \underset{X_1}{\bullet} \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \underset{X_2}{\bullet} \arrow[l, shift left, "\beta_2"] & \cdots & \underset{X_{l-3}}{\bullet} \arrow[r, shift left, "\alpha_{l-2}"] & \underset{X_{l-2}}{\bullet} \arrow[l, shift left, "\beta_{l-2}"]
\end{tikzcd}
$$

FIGURE 1.

3.3. From the previous section the complexity of k as an $\mathcal{H}_q(l)$-module is 1. Namely, $\mathcal{H}_q(l)$ is a self-injective Brauer tree algebra, and k is a non-projective periodic module. This implies that the complexity of k as a module for $L := \mathcal{H}_q(l) \otimes \mathcal{H}_q(l) \otimes \cdots \otimes \mathcal{H}_q(l)$ (m factors) is m, by the Künneth formula. Now suppose that $A = \mathcal{H}_q(\rho)$ contains L as a parabolic subalgebra and that A is a free L-module. Then by (2.1.2), $c_{\mathcal{H}_q(\rho)}(k) \geq m$. We can now verify (3.1.1).

Proposition (A). *Let $\mathcal{B}_\lambda$ be a block of $\mathcal{H}_q(r)$ of weight $w(\lambda) \geq 3$. Then $\mathcal{B}_\lambda$ is wild.*

Proof. Let b be the l-core of $\mathcal{B}_\lambda$, and we may assume that λ is the largest partition in this block, that is

$$\lambda = b + (lw) = (b_1 + lw, b_2, \cdots).$$

Any l-core is l-restricted and hence λ has $(l-p)$-adic expansion

$$\lambda = b + \sum_i (w_i) l p^i$$

where $w = \sum w_i p^i$ is the p-adic expansion of the integer w. Hence, Y^λ has Young vertex $A := \mathcal{H}_q(\rho)$ where ρ has $|b|$ parts equal to 1, w_0 parts equal to l, w_1 parts equal to lp and so on. We see that A contains a subalgebra $L = \mathcal{H}_q(l) \otimes \mathcal{H}_q(l) \otimes \mathcal{H}_q(l)$ so that A is free over L. Consequently, $c_A(k) \geq 3$, and by Theorem 2.2, $c_{\mathcal{H}_q(r)}(Y^\lambda) \geq 3$. Hence, $\mathcal{B}_\lambda$ is wild, by Proposition 2.1. ☐

Proposition (B). *The principal block of $\mathcal{H}_q(2l)$ is wild when $l \geq 3$.*

Proof. It suffices to show that $\mathcal{H}_q(2l)$ is wild, since all non-principal blocks have finite type; these blocks have weight less than or equal to one. For this, it suffices by Proposition 2.3 to show that $L := \mathcal{H}_q(l) \otimes \mathcal{H}_q(l)$ is wild because the induction functor is bounded and it is well-known that for any L-module M, M is a direct summand of $M \otimes_L \mathcal{H}_q(2l)$ restricted to L.

The principal block of $\mathcal{H}_q(l)$ has quiver as Figure 1 with $l-1$ vertices. Hence the quiver of L has "lattice form", with $(l-1)^2$ vertices. If $l > 3$ then this has a proper subquiver which is a circuit, so by [EN, Prop. 2.3(B)] it must be wild. This leaves us with the case $l = 3$. Here we need to look at the projectives in the principal block of L. We see easily that the module category of the connected quiver in [Ri, p.168] with three vertices and three arrows (and no relations) can be embedded into the module category of L. This is wild (see [Ri]) and hence the algebra L is wild as well. In order to obtain this embedding, note that the algebra L has four simple modules, $S_i \otimes S_j$ where S_1, S_2 are the two simple modules for the principal block $\mathcal{H}_q(l)$. The projective cover of the $\mathcal{H}_q(l)$-module S_i is uniserial of length three, with factors S_i, S_j, S_i. It follows that the third Loewy layer of $P(S_1 \otimes S_1)$ is the direct sum of two copies of $S_1 \otimes S_1$ and one copy of $S_2 \otimes S_2$. So label the vertices of the above wild quiver with $S_2 \otimes S_2$ and $S_1 \otimes S_1$ twice (we take the separated quiver of the block). ☐

3.4. For the study of blocks of weight $w = 2$, we will work with the set-up as in [Sc1] and [Jo]. For general w and t they define $[w : t]$-pairs of blocks $\bar{\mathcal{B}}$, $\mathcal{B}$ where $\bar{\mathcal{B}}$ is a block of $\mathcal{H}_q(r-t)$ and $\mathcal{B}$ is a block of $\mathcal{H}_q(r)$, both of weight w. Every block of weight w of some Hecke algebra (over the same field) is related to the principal block of $\mathcal{H}_q(lw)$ by a finite sequence of $[w : t]$-pairs, where t varies. In [Jo] it was proved that for $t \geq w$ the blocks $\mathcal{B}$ and $\bar{\mathcal{B}}$ are Morita equivalent.

For the set-up in [Sc1], [Jo] one needs to use the abacus display of partitions. Let $\mathcal{B}$ be a block of $\mathcal{H}_q(r)$ with l-core $b = (b_1, \cdots, b_s)$, $b_s \neq 0$, and l-weight w. The partition b is a partition of $r - lw$. Display the $(r + lw)$-element β-set of b on an abacus with l runners. If there exists an i, $2 \leq i \leq l$, such that the ith runner contains t more beads than the $(i-1)$th runner, we can swap these two runners to obtain the abacus display of a partition $\bar{b}$ of $r - t - lw$. Let $\bar{\mathcal{B}}$ be the block of $\mathcal{H}_q(r-t)$ which has l-core $\bar{b}$ (and therefore l-weight w). We then say that $\mathcal{B}$ and $\bar{\mathcal{B}}$ form a $[w : t]$-pair.

We assume that $w = 2$ and $t = 1$. Then the functors used in [Sc1] and [Jo] are easy to describe. Take the pair of adjoint functors (G, F) where $G : \mathrm{mod}(\mathcal{B}) \to \mathrm{mod}(\bar{\mathcal{B}})$ and $F : \mathrm{mod}(\bar{\mathcal{B}}) \to \mathrm{mod}(\mathcal{B})$, where G is restriction to $\mathcal{H}_q(r-1)$ followed by taking the block component, and F is (co)induction to $\mathcal{H}_q(r)$ followed by taking the block component. This can be described by taking the $\mathcal{B}-\bar{\mathcal{B}}$-bimodule $W = e\mathcal{H}_q(r)\bar{e}$ where e is the block idempotent for $\mathcal{B}$ and $\bar{e} \in \mathcal{H}_q(r-1)$ is the block idempotent for $\bar{\mathcal{B}}$. Then

$$G \cong (-) \otimes_{\mathcal{B}} W \quad \text{and} \quad F \cong \mathrm{Hom}_{\bar{\mathcal{B}}}(W, -).$$

The results in [Sc2] for symmetric groups generalize to Hecke algebras: the proofs use only the abacus combinatorics and the functor set-up, and these work the same way for Hecke algebras (see [Jo]). The only other ingredient is column removal, but this is also known for Hecke algebras in [Don, 4.2(9), (15)].

From [Sc2] there are three partitions each in $\bar{\mathcal{B}}$, $\mathcal{B}$ respectively which are "exceptional" denoted by $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ and α, β, γ respectively. If $\lambda \neq \alpha$ and λ in the block $\mathcal{B}$ is l-regular, then $G(D^\lambda)$ is simple, and if $\bar{\lambda}$ in the block $\bar{\mathcal{B}}$ with $\bar{\lambda} \neq \bar{\alpha}$ is l-regular, then $F(D^{\bar{\lambda}})$ is simple. The partitions α and $\bar{\alpha}$ are always l-regular. Moreover $G(D^\alpha)$ is not simple but has a simple socle and top isomorphic to $D^{\bar{\alpha}}$ and precisely two composition factors isomorphic to $D^{\bar{\alpha}}$ and $F(D^{\bar{\alpha}})$ is not simple but has a simple socle and top isomorphic to D^α and precisely two composition factors isomorphic to D^α. The projectives $P(\alpha)$ and $P(\bar{\alpha})$ have Specht filtrations with $S^\alpha, S^\beta, S^\gamma$; and with $S^{\bar{\alpha}}$, $S^{\bar{\beta}}$ and $S^{\bar{\gamma}}$ respectively. So $P(\alpha)$ has precisely three composition factors isomorphic to D^α and similarly $P(\bar{\alpha})$ has three composition factors isomorphic to $D^{\bar{\alpha}}$, by reciprocity.

Note that the $\bar{\mathcal{B}}$-module W is a projective generator. Namely it is projective since it is a block component of $e\mathcal{H}_q(r)$. To see that it is a generator, observe that $\mathrm{Hom}_{\bar{\mathcal{B}}}(W, D^{\bar{\lambda}}) \neq 0$ for all l-regular partitions $\bar{\lambda}$ in $\bar{\mathcal{B}}$ (which is just $F(D^{\bar{\lambda}})$).

Lemma. *The module $GF(D^{\bar{\alpha}})$ is isomorphic to $D^{\bar{\alpha}} \oplus P(\bar{\alpha})$.*

Proof. Let $X = GF(D^{\bar{\alpha}})$. By the adjointness we get from the above that $\mathrm{soc}_{\bar{\mathcal{B}}}(X)$ and $\mathrm{top}_{\bar{\mathcal{B}}}(X)$ are both the direct sum of two copies of $D^{\bar{\alpha}}$. Also, in total X has four composition factors isomorphic to $D^{\bar{\alpha}}$.

It suffices to show that $\mathrm{soc}_{\bar{\mathcal{B}}}(X) \not\subseteq \mathrm{rad}_{\bar{\mathcal{B}}}(X)$. If so then the simple $D^{\overline{\alpha}}$ is a direct summand. The remaining part has a simple socle and top and has three composition factors $D^{\bar{\alpha}}$, and the only module with this property is the indecomposable projective.

So suppose that $\mathrm{soc}_{\bar{\mathcal{B}}}(X) \subseteq \mathrm{rad}_{\bar{\mathcal{B}}}(X)$. The projective $P(\bar{\alpha})$ has precisely three composition factors $D^{\bar{\alpha}}$. Therefore there is precisely one module (up to isomorphism) which is not simple or projective which has a simple socle and top isomorphic to $D^{\bar{\alpha}}$, and hence this is $G(D^\alpha)$.

We have an epimorphism $\pi : X \to G(D^\alpha)$. Let $x \in X$ such that $\pi(x)$ generates the image of π. Then $\langle x \rangle$ is a submodule of X which has simple top $D^{\bar{\alpha}}$ and its socle has only composition factors $D^{\bar{\alpha}}$. It cannot have socle of length two because the projective $P(\bar{\alpha})$ does not have such a quotient. Therefore, the socle is simple and it follows by the uniqueness that $\langle x \rangle$ is isomorphic to $G(D^\alpha) \cong \mathrm{Im}(\pi)$. It follows that π is split. Moreover we have an embedding of $G(D^\alpha)$ into the kernel of π and it follows that X is the direct sum of two copies of $G(D^\alpha)$.

We will now get a contradiction by considering stable homomorphisms. For modules Y, Z let $\underline{\mathrm{Hom}}(Y, Z)$ be the quotient of $\mathrm{Hom}(Y, Z)$ factored out by maps which factor through a projective module. First we have $\underline{\mathrm{Hom}}(X, D^{\bar{\alpha}})$ is isomorphic to $\mathrm{Hom}(X, D^{\bar{\alpha}})$ since X has no projective summand and $D^{\bar{\alpha}}$ is simple, and this is 2-dimensional.

On the other hand, the adjointness is compatible with stable homomorphisms, see [DJ1]. So we have

$$\underline{\mathrm{Hom}}(X, D^{\bar{\alpha}}) \cong \underline{\mathrm{Hom}}(F(D^{\bar{\alpha}}), F(D^{\bar{\alpha}})).$$

Clearly $\mathrm{Hom}(F(D^{\bar{\alpha}}), F(D^{\bar{\alpha}}))$ is 2-dimensional. So we will get a contradiction by showing that there are non-zero maps in this space which factor through a projective module.

Let $Z = F(D^{\bar{\alpha}})$; this has a simple socle and top isomorphic to D^{α}, so we have an inclusion $j : Z \to P(\alpha)$ and an epimorphism $\pi : P(\alpha) \to Z$. Since D^{α} occurs precisely once in the heart of $P(\alpha)$ it follows that $\pi \circ j$ cannot be zero and its image must be the socle of Z. So $\pi \circ j$ is a non-zero endomorphism of Z which factors through a projective. □

The following result allows us to relate blocks with weight two to blocks with weight two for smaller Hecke algebras.

Theorem. *Let $(\bar{\mathcal{B}}, \mathcal{B})$ be a $[2:1]$-pair. Then there are (bounded) functors $G : \mathrm{mod}(\mathcal{B}) \to \mathrm{mod}(\bar{\mathcal{B}})$ and $F : \mathrm{mod}(\bar{\mathcal{B}}) \to \mathrm{mod}(\mathcal{B})$ such that for any $\bar{\mathcal{B}}$-module M we have*

$$GF(M) \cong M \oplus P,$$

where P is a projective $\bar{\mathcal{B}}$-module.

Proof. We proceed now by induction on the composition length. Note that by the adjointness we have a natural transformation $\epsilon : GF(-) \to id$ which is "evaluation". Since the $\bar{\mathcal{B}}$-module W is a projective generator, ϵ is onto.

If M is simple then the claim follows by Lemma 3.4. So assume now that M is not simple, then there is a commutative diagram with exact rows, where E is simple in $\bar{B}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & GF(N) & \longrightarrow & GF(M) & \longrightarrow & GF(E) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_N} & & \downarrow{\scriptstyle \epsilon_M} & & \downarrow{\scriptstyle \epsilon_E} & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

By the above remark, ϵ_N and ϵ_E are onto and then by inductive hypothesis, they are split epimorphisms. Now use the Snake Lemma and the fact that projectives are injective, to complete the proof. □

We know that for $l \geq 3$ the principal block of $\mathcal{H}_q(2l)$ is wild by Proposition 3.3(B). From Theorem 3.4 and Proposition 2.3, if $\bar{\mathcal{B}}$ is wild then so is $\mathcal{B}$. Since Morita equivalent blocks have the same representation type, we can conclude by that all blocks of weight $w = 2$ for $l \geq 3$ are wild which completes (3.1.2).

3.5. **Blocks of weight two for $l = 2$.** The goal of this section is to verify statement (3.1.3) that any block for the Hecke algebra of weight two when $l = 2$ is tame. By [Jo] there are only two Morita equivalence classes of such blocks, and they are represented by the principal blocks of $\mathcal{H}_q(4)$ and $\mathcal{H}_q(5)$. Note that here $q = -1$ and hence the trivial module is isomorphic to the alternating module.

Proposition (A). *The Hecke algebra $\mathcal{H}_q(4)$ for $l = 2$ has quiver given in Figure 2 with relations*

$$\beta\epsilon = 0 = \epsilon\alpha, \quad (\alpha\beta)^2 = \epsilon^2.$$

This algebra is tame.

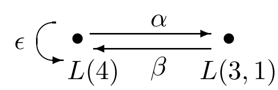

FIGURE 2.

Proof. All partitions of 4 have empty 2-core, so there is only one block. The decomposition matrix for the Hecke algebra $\mathcal{H}_q(4)$ is

$$\begin{pmatrix} [S^{(4)} : D^\mu] \\ [S^{(3,1)} : D^\mu] \\ [S^{(2,2)} : D^\mu] \\ [S^{(2,1^2)} : D^\mu] \\ [S^{(1^4)} : D^\mu] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We consider Young modules; clearly $k = Y^{(4)}$ and from information from the decomposition matrix, $Y^{(3,1)}$ is uniserial of length three, with heart the other simple module which we call E. Moreover, we see that $Y^{(2,2)}$ has Specht quotients $S^{(3,1)}$ and $S^{(2,2)}$, and hence it is uniserial with socle and top E and heart k. Clearly $S^{(1^4)} \cong k$.

So we have all decomposition numbers and the Cartan matrix. The projective $P(E)$ has a quotient $Y^{(2,2)}$ with kernel (necessarily) uniserial of length two and factors E and k. It follows by self-duality that $P(E)$ is uniserial of length five, with "alternating" composition factors.

This shows that $\mathrm{Ext}^1_{\mathcal{H}_q(4)}(E, k)$ is 1-dimensional. Next, observe that $P(k)$ has a quotient $Y^{(3,1)}$ with kernel of length three, and also $Y^{(3,1)}$ as a submodule. Using the fact that $\mathrm{Ext}^1_{\mathcal{H}_q(4)}(E, k)$ is 1-dimensional, it follows that the heart of $P(k)$ is the direct sum of k with a uniserial of length three, actually isomorphic to $Y^{(2,2)}$. Then it is straightforward to determine relations.

The algebra is special biserial and hence is tame. Furthermore, the algebra is of infinite type because the endomorphism ring of $P(k)$ is a Kronecker algebra of infinite type. □

Proposition (B). *The principal block of $\mathcal{H}_q(5)$ for $l = 2$ has quiver given in Figure 3 with relations*

$$\epsilon\alpha = 0 = \beta\epsilon, \quad \alpha\gamma = 0 = \gamma\beta,$$
$$\epsilon^2 = \alpha\beta, \quad \gamma^2 = \beta\alpha.$$

This algebra is tame.

$$\begin{tikzcd} \underset{L(5)}{\bullet} \arrow[loop left, "\epsilon"] \arrow[r, shift left, "\alpha"] & \underset{L(3,2)}{\bullet} \arrow[l, shift left, "\beta"] \arrow[loop right, "\gamma"] \end{tikzcd}$$

FIGURE 3.

Proof. The partitions of five in the principal block are $(5), (3,2), (3,1^2)$ and their conjugates. The decomposition matrix is given by

$$\begin{pmatrix} [S^{(5)} : D^\mu] \\ [S^{(3,2)} : D^\mu] \\ [S^{(3,1^2)} : D^\mu] \\ [S^{(2^2,1)} : D^\mu] \\ [S^{(1^5)} : D^\mu] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From this information we have $Y^{(3,2)} = D^{(3,2)} =: E$ say; and the Young module $Y^{(3,1^2)}$ is indecomposable with Loewy length two, and both socle and top are $k \oplus E$. The Cartan matrix has off-diagonal entries 1 and diagonal entries 3 and this together with self-injectivity and self-duality already implies the structure completely. This algebra is again special biserial of infinite type. Hence, the principal block of $\mathcal{H}_q(5)$ is tame for $l = 2$. $\square$

We remark that both $\mathcal{H}_q(4)$ and $\mathcal{H}_q(5)$ when $l = 2$ are not quite of "dihedral type" as defined in [Erd], since they have tubes of rank two. These algebras are of domestic representation type.

REFERENCES

[DDu] R. Dipper, J. Du, Trivial and alternating source modules for Hecke algebras of type A, *Proc. London Math. Soc* **66**(1993), 479-506. MR **94a:**20022

[DJ1] R. Dipper, G.D. James, Representations of Hecke algebras of general linear groups, *Proc. London Math. Soc.* **52** (1986), 20-52. MR **88b:**20065

[DJ2] R. Dipper, G.D. James, Blocks and idempotents of Hecke algebras of general linear groups, *Proc. London Math. Soc.* **54** (1987), 57-82. MR **88m:**20084

[Don] S. Donkin, *The q-Schur algebra*, London Mathematical Society Lecture Notes Series **253** (1998), Cambridge University Press. CMP 2000:01

[Dr] Yu. A. Drozd, *Tame and wild matrix problems, representations and quadratic forms* Institute of Mathematics, Academy of Sciences, Ukrainian SSR, Kiev 1979. A.M.S Transl. **128** (1986), 31-55.

[Erd] K. Erdmann, *Blocks of tame representation type*, (Lecture Notes in Math. **1428**), Springer-Verlag, New York 1990. MR **91c:**20016

[EN] K. Erdmann, D.K. Nakano, Representation type of q-Schur algebras, Trans. Amer. Math. Soc. **353** (2001), 4729–4756.

[Jam] G.D. James, The decomposition matrices for $GL_n(q)$ for $n \leq 10$, *Proc. London Math. Soc.* **60** (1990), 225–265. MR **91c:**20024

[JK] G.D. James, A. Kerber, *The representation theory of the symmetric group,* Encyclopedia of Mathematics, **16** (1981), Cambridge University Press. MR **83k:**20003

[JM] G.D. James, A. Mathas, A q-analogue of the Jantzen-Schaper theorem, *Proc. London Math. Soc.* **74** (1997), 241-274. MR **97j:**20013

[Jo] T. Jost, Morita equivalence for blocks of Hecke algebras of symmetric groups, *J. Algebra* **194** (1997), 201-223. MR **98h:**20014

[Mar] S. Martin, *Schur Algebras and Representation Theory*, Cambridge Univ. Press, 1993. MR **95f:**20071

[dlP] J.A. de la Peña, *Tame algebras: Some fundamental notions*, Bielefeld preprint 095-010.

[Ric] J. Rickard, The representation type of self-injective algebras, *Bull. London Math. Soc.* **22** (1990), 540-546. MR **92f:**16015

[Ri] C.M. Ringel, *Tame Algebras,* in Representation Theory I, (Lecture Notes in Math. **831**), Springer-Verlag (1980), 137-287.

[Sc1] J. Scopes, Cartan matrices and Morita equivalence for blocks of the symmetric groups, *J. Algebra* **142** (1991), 441-455. MR **92h:**20023

[Sc2] J. Scopes, Symmetric group blocks of defect two, *Quart. J. Math. Oxford* **46** (1995), 201-234. MR **96g:**20016

[U] K. Uno, On representations of non-semisimple specialized Hecke algebras, *J. Algebra* **149** (1992), 287-312. MR **93k:**20044

Mathematical Institute, Oxford University, 24-29 St. Giles, Oxford, OX1 3LB, UK
E-mail address: erdmann@maths.ox.ac.uk

Department of Mathematics, University of Georgia, Athens, Georgia 30602