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EQUIVARIANT BORDISM AND SMITH THEORY. II

BY
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Abstract. This paper analyzes the homomorphism from equivariant bordism to Smith homology for spaces with an action of a finite group G.

1. **Introduction.** Let G be a finite group, and let (X, A, ψ) be a pair with G action. One then has defined the G-equivariant bordism group $\mathfrak{N}_*^G(X, A, \psi)$ and the Smith homology group $H_*^G(X, A, \psi; Z_2)$. These define equivariant homology theories on the category of G pairs and G-equivariant maps, and the object of this paper is to explore the relationship between these theories.

Briefly, being given an equivariant bordism element $f\colon (M, \partial M, \varphi) \to (X, A, \psi)$, the image of the fundamental Smith theory class of $(M, \partial M, \varphi)$ gives a natural transformation

$$\tilde{\mu}\colon \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}_*^G} Z_2 \to H_*^G(X, A, \psi; Z_2).$$

It was shown in [9] that $\tilde{\mu}$ is an isomorphism if $G = Z_2$.

The main results of this paper are

THEOREM 1. *$\tilde{\mu}$ is always epic*

and

THEOREM 2. *$\tilde{\mu}$ is an isomorphism for all G pairs (X, A, ψ) if and only if G is 2-nilpotent and has Sylow 2 subgroup a Z_2 vector space.*

(G is called 2-nilpotent if the elements of odd order in G form a subgroup.)

2. **The representation theorem.** Let G be a finite group, X a simplicial complex, $\psi\colon G \times X \to X$ a simplicial G action and $A \subset X$ a subcomplex invariant under G. It will be assumed that X is "finely" triangulated so that the fixed set of any subgroup H is a subcomplex and the projection $\pi\colon X \to X/H$ is simplicial (E. E. Floyd [4] shows that this may be accomplished by taking the second barycentric subdivision).

Let $C(X) \otimes Z_2$ denote the chains of X with Z_2 coefficients and let $g_\#\colon C(X) \otimes Z_2 \to C(X) \otimes Z_2$ be the chain map induced by $\psi(g,\ )\colon X \to X\colon x \to \psi(g, x)$. One then lets $C^0(X) \subset C(X) \otimes Z_2$ denote the subgroup consisting of chains σ so that

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$g_{\#}\sigma=\sigma$ for all $g\in G$. Since $g_{\#}$ commutes with the boundary, one has an induced boundary $\partial\colon C^0(X)\to C^0(X)$, and if $C^0(X, A)=C^0(X)/C^0(A)$ one has induced a homomorphism ∂ making this a chain complex. The *Smith homology groups* of (X, A, ψ), $H_*^G(X, A, \psi; Z_2)$, are then defined to be the homology groups of the complex $(C^0(X, A), \partial)$.

By using Čech [7], [5] or singular [3] methods to obtain a complex, this may be extended to all topological G pairs.

Being given a compact differentiable manifold M^n with differentiable G action φ, one may triangulate M "finely" so that G acts simplicially. Clearly the fundamental cycle $\mu=\sum\Delta^i$, the sum of all n-simplices, is then an invariant chain, defining a fundamental class $[M, \partial M, \varphi]\in H_n^G(M, \partial M, \varphi; Z_2)$. This lifts the ordinary fundamental class back to Smith theory.

One then has a natural transformation

$$\mu\colon \mathfrak{N}_*^G(X, A, \psi)\to H_*^G(X, A, \psi; Z_2)$$

assigning to the equivariant bordism element $f\colon (M, \partial M, \varphi)\to (X, A, \psi)$ the class $f_*[M, \partial M, \varphi]$.

Letting $\varepsilon\colon \mathfrak{N}_*^G\to Z_2$ be the augmentation to $\mathfrak{N}_0\cong Z_2$ given by ignoring G action and the positive dimensional part, one has $\mu(\alpha\cdot\beta)=\varepsilon(\alpha)\mu(\beta)$ for $\alpha\in\mathfrak{N}_*^G$, $\beta\in\mathfrak{N}_*^G(X, A, \psi)$ as in [9] (*Note*: $H_i^G(M^n, \partial M^n, \varphi; Z_2)=0$ if $i>n$) and thus μ induces a natural transformation

$$\tilde{\mu}\colon \mathfrak{N}_*^G(X, A, \psi)\otimes_{\mathfrak{N}_*^G} Z_2\to H_*^G(X, A, \psi; Z_2).$$

One has the analogue of [9, Proposition 2.1]:

LEMMA 2.1. *If G is a 2 group then $\tilde{\mu}$ is epic.*

Proof. The result is known for $G=\{1\}$ or $G=Z_2$ and so one may induct on the order of G. Let $T=\{1, t\}$ be a central subgroup of G of order 2.

Being given a G complex (X, A, ψ), any element of $C^0(X)$ decomposes uniquely into a sum of invariant chains $\sigma_1+\sigma_2$, where σ_1 is a sum of simplices Δ with $t_{\#}\Delta=\Delta$ and σ_2 is a sum of terms $\Delta+t_{\#}\Delta$ with $t_{\#}\Delta\neq\Delta$, Δ a simplex. This gives a natural decomposition

$$H_*^G(X, A, \psi)\cong H_*^G(X, F_T\cup A, \psi)\oplus H_*^G(F_T, F_T\cap A, \psi)$$

and

$$H_*^G(X, F_T\cup A, \psi)\cong H_*^{G/T}(X/T, A/T\cup F_T, \psi'),$$
$$H_*^G(F_T, F_T\cap A, \psi)\cong H_*^{G/T}(F_T, F_T\cap A, \psi')$$

where F_T is the fixed set of T and ψ' denotes the induced action (see [9, Theorem 2.1]).

Now $\mathfrak{N}_*^{G/T}(F_T, F_T\cap A)$ maps onto $H_*^{G/T}(F_T, F_T\cap A)$ by induction, and if $f\colon (M, \partial M, \varphi')\to (F_T, F_T\cap A, \psi')$ is a G/T bordism element representing α, f may be considered a G bordism element with T acting trivially, to represent α as an element of $H^G_*(X, A, \psi; Z_2)$.

Also $\mathfrak{N}^{G/T}_*(X/T, A/T \cup F_T)$ maps onto $H^{G/T}_*(X/T, A/T \cup F_T)$ by induction, and by excision arguments as in [9, Proposition 2.1], a bordism element $f\colon (M, \partial M, \varphi') \to (X/T, A/T \cup F_T)$ may be lifted to

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{f} & X/T \end{array}$$

with $\tilde{f}$ being G equivariant and representing the class in $H^G_*(X, A, \psi)$ corresponding to $\tilde{\mu}([f]) \in H^{G/T}_*(X/T, A/T \cup F_T)$. □

If $G=(Z_2)^k$ one also has an analog of [9, Proposition 2.2].

LEMMA 2.2. *If $G=(Z_2)^k$, then $\tilde{\mu}$ is an isomorphism.*

Proof. This is known for $k=0, 1$ and hence one may apply induction. Let $t_1, \ldots, t_k$ with $t_i^2=1$, $t_i t_j = t_j t_i$ be generators of G, with $T_i=\{1, t_i\} \subset G$.

Then for any (X, A, ψ) one has an exact sequence of $\mathfrak{N}^G_*$ modules, split as $\mathfrak{N}_*$ modules

$$0 \to \mathfrak{N}^G_*(F_{T_1}, A \cap F_{T_1}) \to \mathfrak{N}^G_*(X, A) \to \mathfrak{N}^G_*(X, F_{T_1} \cup A) \to 0$$

and hence a commutative diagram

$$\begin{array}{ccccccc} Q \longrightarrow & \mathfrak{N}^G_*(F_{T_1}, F_{T_1} \cap A) \otimes Z_2 & \longrightarrow & \mathfrak{N}^G_*(X, A) \otimes Z_2 & \longrightarrow & \mathfrak{N}^G_*(X, F_{T_1} \cup A) \otimes Z_2 & \to 0 \\ & \downarrow{\scriptstyle\tilde{\mu}_0} & & \downarrow{\scriptstyle\tilde{\mu}_1} & & \downarrow{\scriptstyle\tilde{\mu}_2} & \\ 0 \longrightarrow & H^G_*(F_{T_1}, F_{T_1} \cap A) & \longrightarrow & H^G_*(X, A) & \longrightarrow & H^G_*(X, F_{T_1} \cup A) & \longrightarrow 0 \\ & \wr\| & & & & \wr\| & \\ & H^{G/T_1}_*(F_{T_1}, F_{T_1} \cap A) & & & & H^{G/T_1}_*(X/T_1, F_{T_1} \cup A/T_1) & \end{array}$$

with Q a "Tor"-term.

To see that $\tilde{\mu}_2$ is monic, one notes that $(X, F_{T_1} \cup A)$ is relatively free as a T_1 pair, so $\mathfrak{N}^G_*(X, F_{T_1} \cup A) \cong \mathfrak{N}^{G/T_1}_*(X/T_1, F_{T_1} \cup A/T_1)$ by assigning to a T_1 free bordism element $f\colon M \to X$ the induced map $\bar{f}\colon M/T_1 \to X/T_1$. Further, this is a homomorphism of $\mathfrak{N}^{G/T_1}_*$ modules, where $\mathfrak{N}^{G/T_1}_* \to \mathfrak{N}^G_*$ by considering a G/T_1 manifold as a G manifold with trivial T_1 action. One then has a commutative diagram

$$\begin{array}{ccc}
\mathfrak{N}_*^G(X, F_{T_1} \cup A) \otimes_{\mathfrak{N}_*^{G/T_1}} Z_2 & \xrightarrow{\text{epic}} & \mathfrak{N}_*^G(X, F_{T_1} \cup A) \otimes_{\mathfrak{N}_*^G} Z_2 \\
\cong \downarrow & & \downarrow \tilde{\mu}_2 \text{ epic} \\
\mathfrak{N}_*^{G/T_1}(X/T_1, F_{T_1} \cup A/T_1) \otimes_{\mathfrak{N}_*^{G/T_1}} Z_2 & \longrightarrow & H_*^G(X, F_{T_1} \cup A) \\
& \cong \searrow \quad \swarrow \cong & \\
& H_*^{G/T_1}(X/T_1, F_{T_1} \cup A/T_1) &
\end{array}$$

and hence $\tilde{\mu}_2$ is monic.

By an elementary diagram chase, $\tilde{\mu}_1$ will be monic provided $\tilde{\mu}_0$ is monic. Thus, it suffices to prove the lemma for pairs (X, A, ψ) fixed by T_1. A similar analysis may then be applied to each T_i, and hence it suffices to prove the lemma for pairs (X, A, ψ) fixed by each T_i, hence by G.

If (X, A, ψ) is a trivial G space, one has $\mathfrak{N}_*^G(X, A, \psi) \cong \mathfrak{N}_*^G \otimes_{\mathfrak{N}_*} \mathfrak{N}_*(X, A)$, so $\mathfrak{N}_*^G(X, A, \psi) \otimes Z_2$ coincides with $H_*(X, A; Z_2)$ (the $G = \{1\}$ result) and the lemma is valid. □

Now turning to the general case one has:

THEOREM 2.1. *For every G, $\tilde{\mu}$ is always epic.*

Proof. Let G be a finite group and (X, A, ψ) a G pair. Let $S \subset G$ be a Sylow 2 subgroup and $\psi_S: S \times X \to X$ the S action given by restriction to $S \times X$ of ψ.

Considering a G invariant chain of X as being only S invariant defines a homomorphism

$$\theta: H_*^G(X, A, \psi; Z_2) \to H_*^S(X, A, \psi_S; Z_2).$$

Being given an S invariant chain $\sigma \in C(X) \otimes Z_2$ let $t\sigma = \sum g_\# \sigma$ where the sum is taken over a collection of g which represent the cosets G/S. (*Note.* if $g' \in gS$, $g'_\# \sigma = g_\# \sigma$ since σ is S invariant.) Clearly $t\sigma$ is G invariant and this induces a homomorphism

$$t: H_*^S(X, A, \psi_S; Z_2) \to H_*^G(X, A, \psi; Z_2).$$

If σ is G invariant, $g_\# \sigma = \sigma$, so $t\sigma$ is $[G:S]\sigma$ where $[G:S]$ is the index of S in G, and is odd, so $t\sigma = \sigma$. Thus $t\theta = 1$, or θ is monic and t is epic.

Now consider the extension homomorphism

$$e_G^S: \mathfrak{N}_*^S(X, A, \psi_S) \to \mathfrak{N}_*^G(X, A, \psi)$$

defined in [8, §4]. If $f: (M, \partial M, \varphi) \to (X, A, \psi_S)$ is an S equivariant bordism element α, $e_G^S(\alpha)$ is represented by $\bar{f}: (\bar{M}, \partial \bar{M}, \bar{\varphi}) \to (X, A, \psi)$ where $\bar{M} = G \times M/(gs^{-1}, \varphi(s, m)) \sim (g, m)$, $\bar{\varphi}(g', (g, m)) = (g'g, m)$ and $\bar{f}(g, m) = \psi(g, f(m))$. If one considers

$i\colon M\to\bar{M}\colon m\to(1,m)$, $\bar{f}\circ i=f$ and the fundamental cycle of $\bar{M}$ is $\sum g_{\#}(i_{\#}\mu)$ where μ is the fundamental cycle of M. Thus $\bar{f}_*[\bar{M},\partial\bar{M},\bar{\varphi}]=tf_*[M,\partial M,\varphi]$. Thus the diagram

$$\begin{array}{ccc} \mathfrak{N}^S_*(X,A,\psi_S) & \xrightarrow{e^S_G} & \mathfrak{N}^G_*(X,A,\psi) \\ \downarrow{\scriptstyle \mu_S} & & \downarrow{\scriptstyle \mu} \\ H^S_*(X,A,\psi_S;Z_2) & \xrightarrow{t} & H^G_*(X,A,\psi;Z_2) \end{array}$$

commutes, with t and μ_S epic, so μ is epic. Hence also $\tilde{\mu}$ is epic. □

LEMMA 2.3. *Let G be a finite group with Sylow 2 subgroup S and suppose the restriction $\rho^G_S\colon\mathfrak{N}^G_*\to\mathfrak{N}^S_*$ is epic. If (X,A,ψ) is a G pair with*

$$\tilde{\mu}_S\colon \mathfrak{N}^S_*(X,A,\psi_S)\otimes_{\mathfrak{N}^S_*}Z_2\to H^S_*(X,A,\psi_S;Z_2)$$

monic, then

$$\tilde{\mu}_G\colon \mathfrak{N}^G_*(X,A,\psi)\otimes_{\mathfrak{N}^G_*}Z_2\to H^G_*(X,A,\psi;Z_2)$$

is also monic.

Proof. Let $\rho^G_S\colon\mathfrak{N}^G_*(X,A,\psi)\to\mathfrak{N}^S_*(X,A,\psi_S)$ denote the restriction homomorphism which "ignores G equivariance". It is then immediate that the diagram

$$\begin{array}{ccc} \mathfrak{N}^G_*(X,A,\psi)\otimes_{\mathfrak{N}^G_*}Z_2 & \xrightarrow{\tilde{\rho}} & \mathfrak{N}^S_*(X,A,\psi_S)\otimes_{\mathfrak{N}^S_*}Z_2 \\ \downarrow{\scriptstyle \tilde{\mu}_G} & & \downarrow{\scriptstyle \tilde{\mu}_S} \\ H^G_*(X,A,\psi;Z_2) & \xrightarrow{\theta} & H^S_*(X,A,\psi_S;Z_2) \end{array}$$

with $\tilde{\rho}$ induced by ρ^G_S commutes, with θ and $\tilde{\mu}_S$ being monic.

Now consider the extension

$$e^S_G\colon \mathfrak{N}^S_*(X,A,\psi_S)\to\mathfrak{N}^G_*(X,A,\psi).$$

By [2, 6.3] e^S_G is an $\mathfrak{N}^G_*$ module homomorphism; i.e. if $\alpha\in\mathfrak{N}^S_*(X,A,\psi_S)$ and $\beta\in\mathfrak{N}^G_*$, then $e^S_G(\rho^G_S(\beta)\cdot\alpha)=\beta\cdot e^S_G(\alpha)$. In particular, if $\beta'\in\mathfrak{N}^S_*$, there is a $\beta\in\mathfrak{N}^G_*$ with $\rho^G_S(\beta)=\beta'$, so $e^S_G(\beta'\cdot\alpha)=\beta\cdot e^S_G(\alpha)$. Since $\varepsilon(\beta')=\varepsilon(\beta)$, e^S_G induces a homomorphism

$$\tilde{e}\colon \mathfrak{N}^S_*(X,A,\psi_S)\otimes_{\mathfrak{N}^S_*}Z_2\to\mathfrak{N}^G_*(X,A,\psi)\otimes_{\mathfrak{N}^G_*}Z_2.$$

(*Note*. This used the fact that $\rho^G_S\colon\mathfrak{N}^G_*\to\mathfrak{N}^S_*$ is epic. I cannot prove that $\tilde{e}$ is meaningful without this, and in fact Theorem 2 of the Introduction would seem to imply that $\tilde{e}$ cannot always exist.)

Then $\tilde{e}\tilde{\rho}\colon\mathfrak{N}^G_*(X,A,\psi)\otimes_{\mathfrak{N}^G_*}Z_2\to\mathfrak{N}^G_*(X,A,\psi)\otimes_{\mathfrak{N}^G_*}Z_2$ is induced by $e^S_G\circ\rho^G_S$. By [8, Proposition 13.2], $e^S_G\circ\rho^G_S$ is multiplication by the class of $[G/S,\mu]\in\mathfrak{N}^G_0$, so that $\tilde{e}\tilde{\rho}$ is multiplication by $\varepsilon[G/S,\mu]=1$.

Thus $\tilde{\rho}$ is monic, and so $\tilde{\mu}_S\tilde{\rho}$ is monic, which gives $\tilde{\mu}_G$ monic. □

One then has the first half of Theorem 2, given by

THEOREM 2.2. *Let G be 2-nilpotent with Sylow 2 subgroup a Z_2 vector space. Then $\tilde{\mu}$ is an isomorphism for all G pairs.*

Proof. Let S be a Sylow 2 subgroup of G. Then $S=(Z_2)^k$ for some k, so by Lemma 2.2, $\tilde{\mu}_S$ is always monic. Letting $K\subset G$ be the subgroup of elements of odd order, K is normal and $G/K\cong S$, giving a homomorphism $\varphi\colon G\to S$ with $\varphi|S=1$. Thus $\rho_S^G\colon \mathfrak{N}_*^G\to\mathfrak{N}_*^S$ is epic, for if (M,ψ) is an S action, $(M,\psi\circ(\varphi\times 1))$ is a G action restricting to (M,ψ). Thus Lemma 2.3 applies to each G pair and $\tilde{\mu}_G$ is always monic. By Theorem 2.1, $\tilde{\mu}$ is then an isomorphism. □

3. **The isomorphism theorem.** In order to simplify notation, temporarily say that the finite group G has the *isomorphism property* if for all G pairs (X, A, ψ), the natural transformation

$$\tilde{\mu}\colon \mathfrak{N}_*^G(X,A,\psi)\otimes_{\mathfrak{N}_*^G} Z_2\to H_*^G(X,A,\psi;Z_2)$$

is an isomorphism.

Following Bredon [1], one knows that two G equivariant homology theories agree for all spaces if and only if they agree for all of the coset spaces $(G/H,\mu)$, with H a subgroup of G.

Letting $H\subset G$ be a subgroup, consider the pair $(X,A,\psi)=(G/H,\phi,\mu)$.

Clearly $C(G/H)\otimes Z_2$ is the Z_2 vector space with base the points of G/H, and these are permuted by G, so $C^0(G/H)\cong Z_2$ with base the sum of all the points. Thus $H_*^G(G/H,\mu;Z_2)\cong Z_2$.

Now consider $\mathfrak{N}_*^G(G/H,\mu)$. If $f\colon (M,\varphi)\to(G/H,\mu)$ is a G bordism element, then $M_0=f^{-1}(H)$, the inverse image of the coset H, is invariant under H, and hence $(M_0,\varphi|H\times M_0)$ is an H bordism element in $\mathfrak{N}_*^H$. It is immediate that $f\colon (M,\varphi)\to(G/H,\mu)$ is the extension to G of the H equivariant bordism element

$$f|M_0\colon (M_0,\varphi|H\times M_0)\to(G/H,\mu)$$

given by the point map. Thus, this correspondence defines an isomorphism $\mathfrak{N}_*^G(G/H,\mu)\cong\mathfrak{N}_*^H$. If $f\colon (M,\varphi)\to(G/H,\mu)$ and $(N,\psi)\in\mathfrak{N}_*^G$, the product is

$$f\circ\pi_M\colon (M\times N,\varphi\times\psi)\to(G/H,\mu)$$

so that $(f\circ\pi_M)^{-1}(H)=M_0\times N$ with action $(\varphi|H\times M_0)\times(\psi|H\times N)$. Thus identifying $\mathfrak{N}_*^G(G/H,\mu)$ with $\mathfrak{N}_*^H$, $\mathfrak{N}_*^H$ is an $\mathfrak{N}_*^G$ module by $\alpha\cdot\beta=\rho_H^G(\alpha)\cdot\beta$ for $\alpha\in\mathfrak{N}_*^G$, $\beta\in\mathfrak{N}_*^H$ with $\rho_H^G\colon\mathfrak{N}_*^G\to\mathfrak{N}_*^H$ the restriction.

Thus, one has

LEMMA 3.1. *If G has the isomorphism property, then for all $H\subset G$,*

$$\mathfrak{N}_*^H\otimes_{\mathfrak{N}_*^G} Z_2\cong Z_2$$

where $\mathfrak{N}_*^H$ is an $\mathfrak{N}_*^G$ module via the restriction $\rho_H^G\colon\mathfrak{N}_*^G\to\mathfrak{N}_*^H$.

LEMMA 3.2. *If $H \subset G$ is a 2 group then*

$$\mathfrak{N}_*^H \otimes_{\mathfrak{N}_*^G} Z_2 \cong Z_2$$

if and only if $\rho_H^G: \mathfrak{N}_*^G \to \mathfrak{N}_*^H$ is epic.

Proof. Clearly if ρ_H^G is epic, the tensor product rule holds. Now suppose ρ_H^G is not epic. Since H is a 2 group $\mathfrak{N}_0^H \cong Z_2$ generated by a point with trivial action, which comes from a trivial G action, $(\rho_H^G)_0$ is epic (see [8, p. 67]). Suppose

$$(\rho_H^G)_i: \mathfrak{N}_i^G \to \mathfrak{N}_i^H$$

is epic for $i<k$ and is not epic for $i=k$. Let $K=\text{cokernel } (\rho_H^G)_k$ and let

$$\varphi: \mathfrak{N}_*^H \to Z_2 \oplus K$$

be the homomorphism $\varepsilon: \mathfrak{N}_0^H \xrightarrow{\cong} Z_2$ and quotient homomorphism $\varphi: \mathfrak{N}_k^H \to K$ and zero in all other degrees. This is clearly a vector space epimorphism.

If $\alpha \in \mathfrak{N}_*^G$, $\beta \in \mathfrak{N}_*^H$, consider $\varphi(\alpha \cdot \beta) = \varphi(\rho_H^G(\alpha)\beta) = x$. If $\dim \alpha > 0$, $x=0$ unless $\dim \alpha + \dim \beta = k$, when $\dim \beta < k$. But then $\beta = \rho_H^G(\beta')$ for some $\beta' \in \mathfrak{N}_*^G$, so $x = \varphi(\rho_H^G(\alpha\beta'))$ and represents zero in the cokernel. Thus $x = \varepsilon(\alpha)\varphi(\beta)$ if $\dim \alpha > 0$. If $\dim \alpha = 0$, $x = \varphi(\varepsilon(\alpha) \cdot \beta) = \varepsilon(\alpha)\varphi(\beta)$ for $(\rho_H^G)_0$ and ε coincide as maps to Z_2 and φ is Z_2 linear.

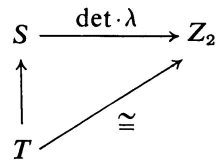
Thus, φ induces an epimorphism $\tilde{\varphi}: \mathfrak{N}_*^H \otimes_{\mathfrak{N}_*^G} Z_2 \to Z_2 \oplus K$, so $\mathfrak{N}_*^H \otimes_{\mathfrak{N}_*^G} Z_2 \not\cong Z_2$. □

Now let G have the isomorphism property, let $S \subset G$ be a Sylow 2 subgroup, and let $T \subset S$ be a central subgroup of order 2, with $T = \{1, t\}$. By the lemmas, ρ_T^G is epic, but ρ_T^G is the composite

$$\mathfrak{N}_*^G \xrightarrow{\rho_S^G} \mathfrak{N}_*^S \xrightarrow{\rho_T^S} \mathfrak{N}_*^T$$

and hence ρ_T^S is epic.

Now let M^3 be the manifold obtained from the 3 disc D^3 by identifying antipodal points of S^2 (i.e. RP (3)) with the T action φ given by the involution $t \cdot x = -x$ on D^3. Since ρ_T^S is epic, there is an S action (N, ψ) cobordant to (M, φ) as T action. The fixed set of T in M is a point ($0 \in D^3$) and RP (2) (image of S^2), and this is cobordant to the fixed set of T in N, $F_T(N)$. In particular, the zero dimensional part $F_T(N)^0$ is an odd number of points. Since T is normal in S, S acts on $F_T(N)$ and hence also on $F_T(N)^0$. Since S is a 2 group, each orbit of S on $F_T(N)^0$ consists of 2^r points, and since $F_T(N)^0$ is odd, there must be a point orbit. Thus, there is a point $p \in F_T(N)^0$ which is fixed by S. Giving N an S invariant Riemannian metric, S acts on the tangent space to N at p orthogonally, giving a homomorphism $\lambda: S \to O_3$ ($\dim N = 3$) and $\lambda(t)$ is multiplication by -1 in R^3. Taking the determinant det: $O_3 \to Z_2$, one has a commutative diagram



so that T splits out of S. Since this is true for all central subgroups of order 2 in S, S must be a Z_2 vector space. (If not, $S=A\times C$, C the central elements of order 2, but if A is nontrivial, its center is nontrivial, giving a central order 2 element of S not in C.) Thus one has

LEMMA 3.3. *If G has the isomorphism property, then the Sylow* 2 *subgroup of G is a Z_2 vector space.*

In order to show that G is 2-nilpotent requires a digression.

Let G be a finite group and $\alpha: G\to G$ an automorphism. One lets $\alpha_*: \mathfrak{N}_*^G \to \mathfrak{N}_*^G$ by $\alpha_*(M, \varphi)=(M, \varphi\circ(\alpha^{-1}\times 1))$, where

$$G\times M \xrightarrow{\alpha^{-1}\times 1} G\times M \xrightarrow{\varphi} M$$

defines a new G action on M. If β is another automorphism of G, $(\alpha\beta)^{-1}\times 1 =(\beta^{-1}\times 1)(\alpha^{-1}\times 1)$ so $(\alpha\beta)_*=\alpha_*\beta_*$, and thus one has a homomorphism

$$*: \mathrm{Aut}\,(G) \to \mathrm{Aut}\,(\mathfrak{N}_*^G): \alpha\to\alpha_*.$$

Now let $g\in G$ and $\alpha: G\to G: h\to ghg^{-1}$ the inner automorphism so that $\alpha^{-1}(h) =g^{-1}hg$. Then if (M, φ) is a G action, $\alpha_*(M,\varphi)=(M,\psi)$ where $\psi(h,m)=\varphi(g^{-1}hg, m)$. Letting $\rho: (M, \varphi)\to(M,\psi): m\to\varphi(g^{-1}, m)$ one has an equivariant diffeomorphism, so $\alpha_*(M, \varphi)=(M, \varphi)$. Thus one has induced a homomorphism

$$*: \mathrm{Aut}\,(G)/\mathrm{Inn}\,(G) \to \mathrm{Aut}\,(\mathfrak{N}_*^G)$$

where Inn (G) is the normal subgroup of inner automorphisms.

Similarly, if $\alpha\in\mathrm{Aut}\,(G)$, α acts on the set of irreducible (real) representations of G, IR (G), by sending $\theta: G\times V\to V$ to $\theta\circ(\alpha^{-1}\times 1): G\times V\to V$. This defines a homomorphism

$$- : \mathrm{Aut}\,(G)\to \mathrm{Perm}\,(\mathrm{IR}\,(G)): \alpha\to\bar\alpha, \quad \text{where } \bar\alpha(V, \theta) = (V, \theta\circ(\alpha^{-1}\times 1)).$$

Notice that Inn (G) acts trivially on IR (G), that $\bar\alpha$ preserves the dimension of the representation, and $\bar\alpha$ sends the trivial representation ($\theta(g, v)=v$ for all (g, v)) to itself.

Now let $\theta: G\times V\to V$ be an irreducible real representation of G. Let M be the manifold obtained from the disc in $V\oplus V$, $D(2V)$, by identifying antipodal points of the sphere, with G action φ given by $\varphi(g, [v_1, v_2]) = [\theta(g, v_1), \theta(g, v_2)]$. Then the fixed set of G in (M, φ) is

$$F_G(M, \varphi) = \{0\} \quad \text{if } \dim V > 1,$$
$$= \{0\} \cup \mathrm{RP}\,(2V) \quad \text{if } \dim V = 1,$$

and G acts in the normal bundle at 0 as two copies of the representation V. Thus letting

$$F_G\colon \mathfrak{N}^G_{2\dim V} \to \mathfrak{N}_0(F'_G(BO_{2\dim V}))$$

be the fixed point homomorphism, $F_G(M, \varphi)$ is given by the inclusion of a point in the component of $F'_G(BO_{2\dim V})$ over which G acts as $2V$. (See [8] for the definition of $F'_G(BO_n)$ and the fixed point homomorphism.)

It is immediate that if (M, φ) is defined by the representation (V, θ), then $\alpha_*(M, \varphi)$ is defined by $\bar{\alpha}(V, \theta)$. Then if $\alpha \in \mathrm{Aut}\,(G)$ with $\alpha_* = 1$, $F_G(M, \varphi) = F_G(\alpha_*(M, \varphi))$ so that (V, θ) and $\bar{\alpha}(V, \theta)$ are equivalent representations, or $\bar{\alpha} = 1$. Thus one has

Lemma 3.4. *If $\alpha \in \mathrm{Aut}\,(G)$ and $\alpha_*\colon \mathfrak{N}^G_* \to \mathfrak{N}^G_*$ is trivial, then $\bar{\alpha}\colon \mathrm{IR}\,(G) \to \mathrm{IR}\,(G)$ is also trivial, or α acts trivially on the irreducible representations of G.*

Now consider a finite group G with the isomorphism property, and let S be a Sylow 2 subgroup of G, so that ρ^G_S is epic (Lemmas 3.1 and 3.2). If N is the normalizer of S in G, $S \subset N \subset G$, then $\rho^G_S = \rho^N_S \rho^G_N$, so $\rho^N_S\colon \mathfrak{N}^N_* \to \mathfrak{N}^S_*$ is epic.

If $n \in N$, $c_n\colon N \to N\colon g \to ngn^{-1}$ is an inner automorphism, so $(c_n)_* \in \mathrm{Aut}\,(\mathfrak{N}^N_*)$ is trivial. Since S is normal in N, $c_n(S) \subset S$ and c_n is an automorphism of S, so $(c_n)_* \in \mathrm{Aut}\,(\mathfrak{N}^S_*)$. Since ρ^N_S is epic, $(c_n)_*$ is trivial in $\mathrm{Aut}\,(\mathfrak{N}^S_*)$, and thus the homomorphism

$$N \to \mathrm{Aut}\,(\mathfrak{N}^S_*)\colon n \to (c_n)_*$$

is trivial, and by Lemma 3.4

$$N \to \mathrm{Perm}\,(\mathrm{IR}\,(S))\colon n \to \bar{c}_n$$

is trivial.

By Lemma 3.3, $S = (Z_2)^k$, and every irreducible representation of S is of the form (R, θ) where $\theta(s, v) = \bar{\theta}(s) \cdot v$, with $\bar{\theta}\colon S \to Z_2 = \{+1, -1\}$. Thus $\mathrm{IR}\,(S) = \mathrm{Hom}\,(S, Z_2)$. If $\alpha \in \mathrm{Aut}\,(S)$, $\bar{\alpha}(R, \theta) = (R, \psi)$ with $\bar{\psi}(s) \cdot x = \bar{\theta}(\alpha^{-1}s) \cdot x$, so $\bar{\alpha} = 1$ implies $\bar{\theta} \circ \alpha^{-1} = \bar{\theta}$ for all $\bar{\theta} \in \mathrm{Hom}\,(S, Z_2)$ and hence $\alpha = 1$. Thus, the homomorphism

$$N \to \mathrm{Aut}\,(S)\colon n \to c_n$$

is trivial, or S is central in its normalizer.

One may now apply the theorem of Burnside [6, Theorem 14.3.1]: If a Sylow subgroup P of G is in the center of its normalizer, then G has a normal subgroup H which has the elements of P as its coset representatives.

Thus one has

THEOREM 3.1. *If G is a finite group with*

$$\tilde{\mu}\colon \mathfrak{N}_*^G(X, A, \psi) \otimes_{\mathfrak{N}_*^G} Z_2 \to H_*^G(X, A, \psi; Z_2)$$

an isomorphism for all G pairs (X, A, ψ), then G is 2-nilpotent and has Sylow 2 *subgroup a Z_2 vector space.*

Combining this with Theorem 2.2 gives Theorem 2 of the Introduction.

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