

MANIFOLD PHENOMENA IN THE THEORY OF POLYHEDRA

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Introduction. When can an isotopy be covered by an ambient isotopy? Let us restrict attention to the compact p.l. category.

Hudson and Zeeman have shown that a locally unknotted isotopy of a manifold in a manifold can be covered by an ambient isotopy of the big manifold. By Zeeman's codimension ≥ 3 unknotting theorem, an isotopy of manifolds is locally unknotted if the codimension is greater than or equal to 3. Hence, any isotopy of a manifold in a manifold of dimension at least 3 higher can be covered. Lickorish has generalized Zeeman's unknotting theorem to the case of a proper embedding of a cone in a ball of dimension at least three higher. From this, Hudson has shown any isotopy of a polyhedron in a manifold can be covered if the polyhedron has codimension at least 3.

As a modest aim we would like a criterion of local unknottedness of a polyhedron in a manifold so that the original Hudson-Zeeman theorem would generalize.

What we actually obtain is more general. We present a characterization of those isotopies of a polyhedron in a polyhedron which can be covered by ambient isotopies. Perhaps surprisingly, this question admits a rather elegant general solution.

Intrinsic Dimension. Our major tool is the theory of intrinsic dimension developed by Armstrong. Given a polyhedral pair (X, X_0) and a point $x \in X_0$, we define the intrinsic dimension of x in (X, X_0) , denoted $d(x; X, X_0)$, to be

$$\max \{t : \text{link pair of } x \text{ in } (X, X_0) \text{ is a } t\text{-fold suspension}\},$$

or equivalently,

$$\max \{t : \text{there is a triangulation of } (X, X_0) \text{ with } x \text{ in the interior of a } t\text{-simplex}\}.$$

Let

$$I^i(X, X_0) = \{x \in X_0 : d(x; X, X_0) \leq i\},$$

called the intrinsic i -skeleton of (X, X_0) . For the absolute notion we associate the polyhedron X with the pair (X, X) , i.e. $d(x; X) = d(x; X, X)$ and $I^i(X) = I^i(X, X)$.

The first fact about intrinsic dimension is that $I^i(X, X_0)$ is a subcomplex of any triangulation of (X, X_0) .

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The second fundamental fact is that $I^i(X, X_0) - I^{i-1}(X, X_0)$ is an i -manifold. In fact, the homogeneity of $I^i(X, X_0) - I^{i-1}(X, X_0)$ is an ambient fact, that is, two points of X_0 lie in the same component of $I^i(X, X_0) - I^{i-1}(X, X_0)$ for some i , iff there is an ambient isotopy of (X, X_0) taking one point to the other.

The most useful form of these homogeneity facts is in the simplicial aspect. Let (K, K_0) be a complex pair with $B = AC \in K_0$ and assume that $A, B \in I^i(K, K_0) - I^{i-1}(K, K_0)$. Then there is an autohomeomorphism of (K, K_0) supported by $St^\circ(A; K)$ and taking B to $x\dot{A}C$ where $x \in \text{Int } B$, i.e. we can ambient isotope the "nice face" A of B into the interior of B .

Using intrinsic dimension there is a definition of local unknottedness which reduces to the usual one in the case of manifold pairs. Call a polyhedral pair (X, X_0) locally unknotted at $x \in X_0$ if $d(x; X, X_0) = d(x; X_0)$. Thus, if (X, X_0) is a manifold pair and $\dim X_0 = n$, then the set of points of X_0 at which the pair is locally knotted is precisely $= I^{n-1}(X, X_0)$. It is easily checked that since S^0 unknotted in any sphere this actually $= I^{n-2}(X, X_0)$. Thus, the intrinsic skeleta of the pair decompose X_0 into submanifolds according to the number of times the link sphere pair desuspends as a pair.

Regular neighborhoods and collapsing. The application of intrinsic dimension from which the other applications flow is to the theory of regular neighborhoods in polyhedra, as developed by Cohen.

First, we define relative collapsing. We say that X collapses to X' relative to X_0 , denoted $(X|X_0) \searrow X'$, if there is a collapse of X to X' , i.e. a sequence $X = X^n, \dots, X^0 = X'$ with $B^i = \text{Cl}(X^{i+1} - X^i)$ a p.l. ball with $B^i \cap X^i = F^i$ a face, for all i , which satisfies the additional condition $B^i \cap X_0 \subset \text{Cl}(\partial B^i - F^i)$.

The definition is chosen so that a relative collapse will preserve the homeomorphism type of the relative regular neighborhood, i.e. we want a regular neighborhood of $X \text{ mod } X_0$ to be a regular neighborhood of $X' \text{ mod } X_0$ in an ambient manifold, and without the additional condition things can go wrong.

Thus, if we let X be a simplex $[abc]$ and $X_0 = [bx]$ where $x \in \text{Int } [ac]$ then in a manifold, a regular neighborhood of $X \text{ mod } X_0$ is two balls joined along the simplex $[bx]$. We can collapse X to $X' = [abx] \cup [xc]$ (not relative to X_0) and a regular neighborhood of $X' \text{ mod } X_0$ is two balls joined along a point.

That relative collapsing does preserve homeomorphism type (and more) of regular neighborhoods in manifolds follows from the more general results in polyhedra.

Now assume that $X < Q_0 < Q$. We call a collapse $(X|X_0) \searrow X'$ homogenous in (Q, Q_0) if $d(x; Q, Q_0)$ is constant for x varying in each $X^{i+1} - X^i$ and weakly homogenous if it is constant for x varying in each $X^{i+1} - (X^i \cup X_0)$. Note that it does not follow that $d(x; Q, Q_0)$ is constant as x varies all over $X - X'$. If there exists such a homogenous (resp. weakly homogenous) collapse, we will write $(X|X_0) \searrow_h X'$ in (Q, Q_0) (resp. $(X|X_0) \searrow_w X'$ in (Q, Q_0)).

The phrase homogenous collapse was introduced by Stallings in the absolute

case ($X_0 = \emptyset$ and $Q = Q_0$), and in that case his definition is equivalent to ours though he does not use the notion of intrinsic dimension.

We note that if $(X|X_0) \xrightarrow{h} X'$ (or \xrightarrow{w}) in Q_0 and (Q, Q_0) is locally unknotted, then $(X|X_0) \xrightarrow{h} X'$ (resp. \xrightarrow{w}) in (Q, Q_0) .

THEOREM. *Let $X_0 < X < Q_0 < Q$ be polyhedra with $(X|X_0) \xrightarrow{h} X'$ in (Q, Q_0) . If (V, V_0) is a regular neighborhood of $X \bmod X_0$ in (Q, Q_0) then (V, V_0) is a regular neighborhood of $X' \bmod X' \cap X_0$ in (Q, Q_0) .*

THEOREM. *Let $X_0 < X < Q_0 < Q$ be polyhedra with $(X|X_0) \xrightarrow{w} X'$ in (Q, Q_0) . If (V, V_0) and (U, U_0) are regular neighborhoods of $X \bmod X_0$ and $X' \bmod X_0$ respectively, in (Q, Q_0) then there is a homeomorphism $f: (V, V_0) \rightarrow (U, U_0)$ which preserves the boundaries and is the identity on X' .*

The idea of the proof is quite simple. We triangulate and use induction to reduce the problem to one simplicial collapse, i.e. $X = X' + B$ where $B = bA$ and $B \cap X' = b\dot{A}$. The condition that the collapse is relative says that $B \cap X_0 < A$. There are two cases, i.e. $B \cap X_0 = A$ or $B \cap X_0 < \partial A$. Then in the second theorem A is a "nice face" of B if $B \cap X_0 < \partial A$ and in the first theorem A is a "nice face" of B regardless.

Using the simplicial method of "parametrization", taken from Cohen, we construct directly regular neighborhoods of both $X' \bmod X_0$ and $X' \cup xb\dot{A} \bmod X_0$ in the second case and both $X' \bmod X_0$ and $X' \cup xb\dot{A} \bmod X_0 \cup x\dot{A}$ in the first case.

Then using the fact that A is a "nice face" of B we find an ambient homeomorphism of (Q, Q_0) which is the identity on X' and maps $X' \cup xb\dot{A}$ to $X' \cup B = X$. This will push the regular neighborhoods out to obtain regular neighborhoods of both $X' \bmod X_0$ and $X \bmod X_0$ in each case.

The remaining possibility is that we are in the first case of the second theorem and A is not a "nice face" of B . In this case the homeomorphism f is just constructed directly using the method of parametrization.

From the above theorems we can then derive several characterizations of a regular neighborhood of $X \bmod X_0$ in Q using collapsing.

Cone unknotting. Given an embedding $f: aX \rightarrow bY$ with $f^{-1}(Y) = X$ (i.e. a proper embedding), call f unknotted if there exists a homeomorphism $\phi: bY \rightarrow bY \text{ rel } Y$, such that ϕf is the cone on $f|X: X \rightarrow Y$. This definition, due to Lickorish, is much stronger than merely assuming that $(bY, f(aX)) \cong c(Y, f(X)) \text{ rel } Y$. In fact, in most applications, f is a self-homeomorphism of a subcone of bY , which we are attempting to cover by a self-homeomorphism of bY . Such a covering homeomorphism will exist iff, regarded as an embedding, the little homeomorphism is unknotted.

The confusion between the two kinds of unknotting arises because for ball pairs they are equivalent. This is because an unknotted ball pair is of the form $(\dot{\Delta} * \bar{\Delta}, \bar{\Delta})$ and if $f: \bar{\Delta} \rightarrow \bar{\Delta}$ is a self-homeomorphism then $\dot{\Delta} * f$ is a covering self-homeomorphism of $\dot{\Delta} * \bar{\Delta}$.

THEOREM. *Let $f: aX \rightarrow bY$ be a proper embedding. Then f is unknotted iff*

1. *For every $x \in X$, $d(f(tx + (1-t)a); bY, f(aX))$ is constant for $0 < t < 1$ (conewise homogeneity), and*
2. *$(bY|Y) \xrightarrow{w} f(aX)$ in bY .*

The second condition is just to insure that bY is homeomorphic to a regular neighborhood of $f(aX) \bmod f(X)$ in bY . Then the result follows from:

THEOREM. *Let $f: aX \rightarrow Q$ be an embedding, which is conewise homogenous. Then if V is a regular neighborhood of $f(aX) \bmod f(X)$ in Q , then there exists a homeomorphism $\phi: V \rightarrow b\dot{V}$ (where $\dot{V} = \text{Bdry } V \cup f(X)$) such $\phi|_{\dot{V}}$ is the identity and ϕf is the cone on $f|_X: X \rightarrow \dot{V}$.*

To get an idea of the proof note that the cone structure gives an obvious ‘‘conewise’’ collapse $(f(aX)|f(X)) \searrow f(a)$. Conewise homogeneity says precisely that this collapse is weakly homogenous in $(Q, f(aX))$. It follows that $(V, f(aX))$ is homeomorphic to a regular neighborhood of $f(a)$ in $(Q, f(aX))$. Thus, the weak ‘‘set-unknottedness’’ condition follows from the theorems of the regular neighborhood section.

The sharper ‘‘map-unknottedness’’ condition of the theorem is proved by going back to the appropriate regular neighborhood theorem and repeating the proof, taking care to preserve the additional structure on the polyhedron $f(aX)$ introduced by the cone structure of aX .

Note that the $d(f(tx + (1-t)a); f(aX))$ is constant for $0 < t < 1$. Thus, if $(Q, f(aX))$, is locally unknotted, conewise homogeneity always holds. Thus, for the first time, the oft-returning ‘‘red herring’’ effect of local unknottedness appears:

In general, the purpose of assuming local unknottedness of a pair (X, X_0) is to transform conditions on the intrinsic dimension in (X, X_0) to conditions on the intrinsic dimension in X_0 , which are usually trivial.

On the other hand, if $(bY, f(aX)) \cong b^*(Y, f(X)) \text{ rel } Y$, then the collapsing condition $(bY|Y) \xrightarrow{w} f(aX)$ in bY holds.

Thus, if $a(Y, X)$ is a locally unknotted cone pair then any self-homeomorphism of $aX \text{ rel } X$ can be covered by a self-homeomorphism of $aY \text{ rel } Y$.

Codimension ≥ 3 . Using sunny collapsing at a crucial stage, we can strengthen the cone unknotting theorem when codimension is ≥ 3 .

THEOREM. *Let $f: aX \rightarrow bY$ be a proper embedding with $\dim f^{-1}I^i(Y) \leq i - 3$ for all i . If $f(a) = b$, and $f^{-1}(bI^i(Y)) = af^{-1}I^i(Y)$ then f is unknotted.*

From this theorem we obtain Lickorish’s theorem and also a criterion for local unknottedness in codimension ≥ 3 .

Covering isotopies. From the cone unknotting theorems we can get our covering isotopy theorems using the Hudson-Zeeman methods.

We define an isotopy $F: X \times I \rightarrow Y \times I$ to be locally collared if for every subinterval J of I , $(Y \times J, F(X \times J)) \subset (Y \times J, F(X \times J))$ is locally collared. Rourke has

proven that an isotopy is locally collared iff there is an ambient isotopy of Y covering the track of F , i.e. there exists $G: Y \times I \rightarrow Y \times I$ such that $G_t F_0(X) = F_t(X)$.

THEOREM. *Let $F: X \times I \rightarrow Y \times I$ be an isotopy. F can be covered by an ambient isotopy of Y iff*

1. F is locally collared, and
2. for each $x \in X$, $d(F(x, t); Y \times I, F(X \times I))$ is constant for $0 < t < 1$.

Call an isotopy $F: X \times I \rightarrow Y \times I$ locally unknotted if it is locally collared and $(Y \times I, F(X \times I))$ is locally unknotted. Since $d(F(x, t); F(X \times I))$ is constant $0 < t < 1$, we have another "red herring" phenomenon:

COROLLARY. *Any locally unknotted isotopy can be covered by an ambient isotopy.*

Now if (X, X_0) is a polyhedral pair and $F: X_0 \times I \rightarrow X_0 \times I$ is an ambient isotopy of X_0 , then regarded as an isotopy in X it is clearly locally collared and hence we have

COROLLARY. *If $F: X_0 \times I \rightarrow X_0 \times I$ is an ambient isotopy then F can be covered by an ambient isotopy of X iff $F^{-1}(I^1(X, X_0) \times I) = I^1(X, X_0) \times I$.*

From this we get a relative covering isotopy theorem.

THEOREM. *Let (X, X_0) be a polyhedral pair and $Y < X$. Assume $F: X_0 \times I \rightarrow X_0 \times I$ is an ambient isotopy of $X_0 \text{ rel } X_0 \cap Y$. If F can be covered by an ambient isotopy of X , then it can be covered by an ambient isotopy of $X \text{ rel } Y$.*

For we can extend the obvious ambient isotopy of $X_0 \cup Y$.

For local unknottedness we get the following:

COROLLARY. *Let (X, X_0) be a locally unknotted pair, then any ambient isotopy of X_0 can be covered by an ambient isotopy of X .*

Finally, from the codimension ≥ 3 cone unknotted theorem, we obtain a corresponding isotopy theorem:

THEOREM. *Let $F: X \times I \rightarrow Y \times I$ be an isotopy with $\dim F_0^{-1}I^1(Y) \leq i - 3$. Then F can be covered by an ambient isotopy of Y iff $F^{-1}(I^1(Y) \times I) = F_0^{-1}(I^1(Y)) \times I$.*

As a corollary of this we have Hudson's result that any isotopy of a polyhedron of codimension ≥ 3 in a manifold can be covered by an ambient isotopy.

Extending triangulations. Armstrong's theorem that any triangulation of a locally unknotted submanifold extends to a triangulation of the manifold generalizes to the following:

THEOREM. *Let (X, X_0) be a polyhedral pair and $(K_0; h)$ a triangulation of X_0 . Then $(K_0; h)$ extends to a triangulation of X iff $h^{-1}I^1(X, X_0)$ is a subcomplex of K_0 for all i .*

Since $I^i(X_0) = I^i(X, X_0)$ iff the pair is locally unknotted, we have another “red herring” effect.

COROLLARY. *Let (X, X_0) be a locally unknotted polyhedral pair. Then any triangulation of X_0 extends to a triangulation of X .*

General position. We call a homotopy $F: X \times I \rightarrow Y$ homogenous if for every $x \in X$ $d(F(x, t); Y)$ is constant for $t \in I$.

Using the isotopy theorems we can prove results on homogenous homotopy of maps:

THEOREM. *Let $f: X \rightarrow Y$ be a map such that $f|_{X_0}$ is nondegenerate (where $X_0 < X$) and $\dim f^{-1}I^i(Y) \leq i$. Then there exists a homotopy of f , homogenous and rel X_0 , to a nondegenerate map.*

THEOREM. *Let $f: X \rightarrow Y$ be a nondegenerate map, then f can be homogeneously homotoped to a map $g: X \rightarrow Y$ such that $\dim S(f) - f^{-1}I^{i-1}(Y) \leq 2 \dim X - i$.*

We also mention some conjectures which relate these results to those of Stallings about $ND(n)$ -spaces.

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I. Notation and background information. We will follow the standard treatments of p.l. topology: Zeeman [Z], Hudson [H] and Stallings [S]. These do not completely overlap so we will refer mostly to [H] and occasionally to [S].

Our polyhedra, polyhedral pairs, polyhedra with (finite) families of subpolyhedra, denoted $X, (X, X_0), (X, \{X_{ij}\})$ will be assumed to have a given family of compatible locally finite triangulations, denoted $(K; h), (K, K_0; h), (K, \{K_{ij}\}, h), (K, \{K_{ij}\}; h)$ for complexes, complex pairs and complexes with a family of subcomplexes. All maps are assumed to be p.l.

For $A \in K$, we will denote by $\text{Lk}(A; K)$, $\text{St}(A; K)$ and $\text{St}^\circ(A; K)$ the link, star and open star of A in K . Note that these are finite, by local finiteness of K . For pairs, we will define $\text{Lk}(A; K, K_0)$, for example, to equal $(\text{Lk}(A; K), \text{Lk}(A; K_0))$ for $A \in K_0$.

Following Armstrong [A₁], for $x \in X$, we will denote by $\text{Lk}(x; X)$ any $\text{Lk}(h^{-1}(x); K)$ where $(K; h)$ is a triangulation of X and $h^{-1}(x)$ is a vertex of K . Similarly, for $x \in X_0$ define $\text{Lk}(x; X, X_0)$ using any triangulation of the pair. These are well defined up to homeomorphism, or homeomorphism of pairs, respectively. Similarly, define $\text{St}(x; X)$ and $\text{St}(x; X, X_0)$.

We recall that joining gives a well-defined functor of pairs of compact polyhedra to compact polyhedra. Denoted $X * Y$. Points of $X * Y$ are denoted $tx + (1-t)y$ with $x \in X, y \in Y$ and $t \in I$. If $f: X \rightarrow X_0$ and $g: Y \rightarrow Y_0$ then $f * g: X * Y \rightarrow X_0 * Y_0$ maps by $f * g(tx + (1-t)y) = tf(x) + (1-t)g(y)$. If X is one point, c , then $c * Y$ or cY is called the cone on Y and if X is two points, i.e. a 0-sphere S^0 , then $S^0 * Y$ or ΣY is called the suspension of Y . The iterated suspension $\Sigma \Sigma^n Y = \Sigma^{n+1} Y = S^n * Y$. We will often use the calculation of the link of a point of the join:

$$\begin{aligned} \text{Lk}(tx + (1-t)y; X * Y) &= X * \text{Lk}(y; Y), & t = 0 \\ &= \Sigma \text{Lk}(x; X) * \text{Lk}(y; Y), & 0 < t < 1 \\ &= \text{Lk}(x; X) * Y, & t = 1. \end{aligned}$$

This has an obvious generalization to $X * (Y, Y_0)$.

Note that any join factor and hence any cone or suspension factor is assumed to be compact.

The product of arbitrary polyhedra, denoted $X \times Y$, is defined and the identity for the link is

$$\text{St}((x, y); X \times Y) = \text{St}(x; X) \times \text{St}(y; Y).$$

This implies that

$$\text{Lk}((x, y); X \times Y) = \text{Lk}(x; X) * \text{Lk}(y; Y)$$

by the Alexander trick which we now describe. Given finite complexes K and L , we can triangulate $cK \times cL$ so that $\text{St}((c, c); cK \times cL) = cK \times cL$ and

$$\text{Lk}((c, c); cK \times cL) = cK \times L \cup K \times cL$$

(where the union is along $K \times L$). Furthermore, $cK \times L \cup K \times cL \cong K * L$ (see [S; §4.3]).

Of particular interest is the case where L is one point so $cL \cong I$. This gives:

$$cK \times I = (c, 1) * ((cK \times 0) \cup (K \times I))$$

with $cK \times 1$ included as the subcomplex $(c, 1) * (K \times 1)$. This construction is also natural on subcomplexes of K .

The Alexander trick proves that a homeomorphism of a cone rel its base is isotopic to the identity rel its base. We define the isotopy (after triangulating) to be the map of $cK \times I$ to itself obtained by coning the given homeomorphism on $cK \times 0$ and the identity of $K \times I$.

As an indispensable tool we will require the general theory of relative regular neighborhoods as developed by Cohen in [C]. Following him, we define (V, V_0) to be a regular neighborhood of $X \text{ mod } X_0$ in (Y, Y_0) , if there is a triangulation $(J, J_0; h)$ of (Y, Y_0) such that $K = h^{-1}(X)$, $K_0 = h^{-1}(X_0)$ and J_0 are full subcomplexes of J , and

$$(V, V_0) = hN(K - K_0; \eta J, \eta J_0)$$

where η is a derived subdivision of J . Define the boundary of the regular neighborhood (\dot{V}, \dot{V}_0) by

$$(\dot{V}, \dot{V}_0) = h\dot{N}(K - K_0; \eta J, \eta J_0).$$

By [C; Lemma 2.13], $\dot{V} = \text{Bdry}_Y V \cup (X_0)_R$ where $(X_0)_R = X_0 \cap (X)_R$ and $(X)_R = \text{Cl}_Y (X - X_0)$.

We will use much of the notation of [C], such as $D(A; K)$ for dual cell, $\dot{D}(A; K)$ for boundary of the dual cell, and so on. We will also depend heavily on the results of [C] particularly the parametrization lemma [C; Lemma 2.14], the uniqueness theorem [C; Theorems 3.1–3.4] and the stellar neighborhood theorem [C; Theorem 6.1].

We will require the theory of collapsing and mention, in particular, the principle of excision.

Let X, X_0 and Y be polyhedra with $Y \cap X \subset X_0$. Then $X \searrow X_0$ iff $X \cup Y \searrow X_0 \cup Y$.

Finally, a little convention, for convenience in some formulae. We distinguish between \emptyset , the empty complex, and $\{0\}$ the complex consisting of the empty simplex 0. Define $\emptyset * X = \emptyset$ and $\{0\} * X = X$, if $X \neq \emptyset$. $\emptyset \times X = \emptyset$ and $\{0\} \times X = \{0\}$, if $X \neq \emptyset$. Also if $A \notin K$, then define $\text{Lk}(A; K) = \emptyset$. We will always explicitly mention this convention whenever it is around.

II. Intrinsic dimension. We give three equivalent definitions of intrinsic dimension:

DEFINITION 1. Let (X, X_0) be a polyhedral pair and $x \in X_0$. Then we define the intrinsic dimension of x in (X, X_0) , denoted $d(x; X, X_0)$ to be

- a. the greatest t , such that $\text{Lk}(x; X, X_0)$ is a t -fold suspension,
- b. the greatest t , such that there exists an embedding $f: \Delta^t \times c(Y, Y_0) \rightarrow (X, X_0)$ such that $f^{-1}(X_0) = \Delta^t \times c Y_0$ and $f(\Delta^t \times c Y)$ is a neighborhood of $x = f(d, c)$ (for d some element of $\text{Int } \Delta^t$),
- c. the greatest t , such that there exists a triangulation $(K, K_0; h)$ of (X, X_0) with $h^{-1}(x)$ in the interior of a t -simplex of K_0 .

Proof of the equivalence.

a \rightarrow b. $\text{Lk}(x; X, X_0) \cong \Sigma^t(Y, Y_0) \rightarrow \text{St}(x; X, X_0) \cong \Delta^t * (Y, Y_0)$ with x corresponding to an interior point of Δ^t . But a regular neighborhood of Δ in $\Delta * (Y, Y_0)$ is homeomorphic to $\Delta^t \times c(Y, Y_0)$.

b \rightarrow a. $\Sigma^t(Y, Y_0) \cong \text{Lk}((d, c); \Delta^t \times c(Y, Y_0)) \cong \text{Lk}(x; X, X_0)$ since $f(d, c) = x$.

c \rightarrow a. If $(K, K_0; h)$ is a triangulation of (X, X_0) with $h^{-1}(x) \in \text{Int } A$, with $A \in K_0$ and $\dim A = t$, then $\text{Lk}(x; X, X_0) \cong \dot{A} * \text{Lk}(A; K, K_0) \cong \Sigma^t \text{Lk}(A; K, K_0)$.

a \rightarrow c. Let $(K, K_0; h)$ be a triangulation of (X, X_0) with $h^{-1}(x)$ a vertex of K_0 . As in the proof a \rightarrow b we come to get a homeomorphism q ,

$$q: \text{St}(h^{-1}(x); K, K_0) \cong \Delta^t * (Y, Y_0)$$

with $q(h^{-1}(x)) = d_0$ an interior point of Δ^t . Let $d_1 \in \text{Int } \Delta^t$ such that $q^{-1}(d_1)$ lies in the interior of a simplex of K_0 of dimension at least t .

Let $g: \Delta^t \cong \Delta^t \text{ rel } \partial\Delta^t$ with $g(d_0) = d_1$. Join with the identity on Y and conjugate with q to obtain

$$k: \text{St}(h^{-1}(x); K, K_0) \rightarrow \text{St}(h^{-1}(x); K, K_0)$$

satisfying $k|_{\text{Lk}(h^{-1}(x); K)} = \text{identity}$ and $k(h^{-1}(x)) = q^{-1}(d_1)$. Extend k by the identity to the rest of K . Let $(K, K_0; hk^{-1})$ triangulate (X, X_0) . $(hk^{-1})^{-1}(x) = q^{-1}(d_1)$ and so lies in the interior of a simplex of K_0 of dimension at least t .

Note. The proof that $a \rightarrow b$ gave a homeomorphism

$$q: \Delta^t * (Y, Y_0) \rightarrow \text{St}(x; X, X_0)$$

taking $\partial\Delta^t * (Y, Y_0)$ to $\text{Lk}(x; X, X_0)$ and hence $q(\Delta^t * Y - \partial\Delta^t * Y)$ was open in X . Also, the resultant embedding $f: \Delta^t \times c(Y, Y_0) \rightarrow (X, X_0)$ satisfied the condition that $f((\text{Int } \Delta^t) \times (cY - Y))$ is open in X .

$d(x; X, X_0)$ is sometimes called the ambient intrinsic dimension to distinguish it from the absolute intrinsic dimension in a polyhedron, rather than a polyhedral pair. We obtain the absolute definitions by identifying a polyhedron X , with the pair (X, X) . Thus conditions on link pairs, triangulation of pairs etc. become conditions on links and triangulations of polyhedra.

DEFINITION 2. For $x \in X$, we define the intrinsic dimension of x in X , denoted $d(x; X)$, by setting $d(x; X) = d(x; X, X)$.

REMARKS. The definitions are obviously p.l. invariants, i.e. if $f: (X, X_0) \rightarrow (Y, Y_0)$ is a homeomorphism, then $d(x; X, X_0) = d(f(x); Y, Y_0)$.

An easy comparison of definitions shows that

$$d(x; X, X_0) \leq \min(d(x; X), d(x; X_0)).$$

DEFINITION 3. For a polyhedral pair (X, X_0) or a polyhedron X , we define the intrinsic i -skeleton of (X, X_0) (resp. X), denoted $I^i(X, X_0)$ (resp. $I^i(X)$) by:

$$I^i(X, X_0) = \{x \in X_0 : d(x; X, X_0) \leq i\}$$

$$I^i(X) = I^i(X, X) = \{x \in X : d(x; X) \leq i\}.$$

The name intrinsic i -skeleton is suggested by Zeeman's alternate definition of the intrinsic i -skeleton as the intersection of all possible i -skeleta. More precisely, part c. of Definition 1 implies

$$I^i(X, X_0) = \bigcap \{h(K_0^i) : (K, K_0; h) \text{ a triangulation of } (X, X_0)\}.$$

That is, the intersection is taken over all triangulations of the polyhedral pair (X, X_0) . This implies that $I^i(X, X_0)$ is a closed subset of X_0 .

THEOREM 4. Let $(K, K_0; h)$ be a triangulation of a polyhedral pair (X, X_0) , then for all i , $h^{-1}(I^i(X, X_0))$ is a subcomplex of K_0 .

Proof. If $h^{-1}(x_0), h^{-1}(x_1) \in \text{Int } A$ for A is a simplex of K_0 , then $\text{Lk}(x_i; X, X_0) \cong \partial A * \text{Lk}(A; K, K_0)$, $i=0, 1$ and hence $d(x_0; X, X_0) = d(x_1; X, X_0)$. So intrinsic dimension is constant on the interior of any simplex of K_0 .

Hence, $h^{-1}(I^i(X, X_0))$ is a union of open simplices of K_0 and as it is closed, it is a subcomplex.

Now we examine the intrinsic skeleta locally.

LEMMA 5. *Let $f: \Delta^i * (Y, Y_0) \rightarrow (X, X_0)$ be an embedding with $f^{-1}(X_0) = \Delta^i * Y_0$ and $f(\Delta^i * Y - \partial\Delta^i * Y)$ open in X , then for $d_0, d_1 \in \text{Int } \Delta^i$, $0 < t_0, t_1 < 1$ and $y \in Y_0$, we have*

- a. $d(f(d_0); X, X_0) = d(f(d_1); X, X_0) \geq i$.
- b. $d(f(t_0d_0 + (1-t_0)y); X, X_0) = d(f(t_1d_1 + (1-t_1)y); X, X_0) > i$.

Proof. a. $\text{Lk}(f(d_p); X, X_0) \cong \text{Lk}(d_p; \Delta^i * (Y, Y_0)) \cong \partial\Delta^i * (Y, Y_0)$, $p = 0, 1$.

- b. $\text{Lk}(f(t_p d_p + (1-t_p)y); X, X_0) \cong \text{Lk}(t_p d_p + (1-t_p)y; \Delta^i * (Y, Y_0))$
 $\cong \Sigma \partial\Delta^i * \text{Lk}(y; Y, Y_0)$, $p = 0, 1$.

THEOREM 6. $I^i(X, X_0) - I^{i-1}(X, X_0)$ is an i -manifold.

Proof. If $d(x; X, X_0) = i$, then there exists an embedding $f: \Delta^i * (Y, Y_0) \rightarrow (X, X_0)$ such that $f^{-1}(X_0) = \Delta^i * Y_0$, $f(\Delta^i * Y - \partial\Delta^i * Y)$ is open in X and $f^{-1}(x) \in \text{Int } \Delta^i$. Lemma 5 implies that

$$f(\Delta^i * Y_0 - \partial\Delta^i * Y_0) \cap I^i(X, X_0) = f(\text{Int } \Delta^i)$$

and

$$f(\text{Int } \Delta^i) \cap I^{i-1}(X, X_0) = \emptyset.$$

Hence, $f(\text{Int } \Delta^i)$ is a neighborhood of x in $I^i(X, X_0) - I^{i-1}(X, X_0)$.

We now demonstrate the relationship between homogeneity of the open intrinsic i -skeleton as an i -manifold and as a subset of X_0 .

DEFINITION 7. For a pair of points x, y in $X_0 < X$, the following are equivalent.

- a. $d(x; X, X_0) = d(y; X, X_0) = i$ and x and y lie in the same component of $I^i(X, X_0) - I^{i-1}(X, X_0)$.

- b. There exists an ambient isotopy $H: (X, X_0) \times I \rightarrow (X, X_0) \times I$ such that $H_0 = \text{id}_x$ and $H(x, 1) = (y, 1)$.

If these conditions hold we will write $x \sim y$ in (X, X_0) .

Proof of the equivalence. $b \rightarrow a$. Let $d(x; X, X_0) = i$, then $t \rightsquigarrow H(x, t)$ is a path in $I^i(X, X_0) - I^{i-1}(X, X_0)$ joining x to y .

$a \rightarrow b$. For $x \in I^i(X, X_0) - I^{i-1}(X, X_0)$, let $f: \Delta^i * (Y, Y_0) \rightarrow (X, X_0)$ be an embedding with $f^{-1}(X_0) = \Delta^i * Y_0$, $f(\Delta^i * Y - \partial\Delta^i * Y)$ open in X and $f^{-1}(x) \in \text{Int } \Delta^i$.

If $d \in \text{Int } \Delta^i$, let $k: \Delta^i \rightarrow \Delta^i$ be a homeomorphism rel $\partial\Delta^i$ with $k(f^{-1}(x)) = d$. $k * 1_Y$ is a self-homeomorphism of $\Delta^i * (Y, Y_0)$, rel $\partial\Delta^i * Y$ and extending k . It is isotopic to the identity on $\Delta^i * (Y, Y_0)$ rel $\partial\Delta^i * Y$ by the Alexander trick.

Hence, if $\bar{k}: (X, X_0) \cong (X, X_0)$ is defined to be $f(k * 1_Y)f^{-1}$ on $f(\Delta^i * Y)$ and the identity elsewhere then \bar{k} is ambient isotopic to the identity and $\bar{k}(x) = f(d)$.

Thus, X ambient isotopes to any other point of $f(\text{Int } \Delta^i)$, which is a neighborhood of X in $I^i(X, X_0) - I^{i-1}(X, X_0)$ as we saw in the proof of Theorem 6.

It follows that the set of points to which x will ambient isotope is open and closed in $I^i(X, X_0) - I^{i-1}(X, X_0)$ and since y is in the same component of $I^i(X, X_0) - I^{i-1}(X, X_0)$ as x , x ambient isotopes to y .

REMARK. It is clear that if $x \sim y$ in (X, X_0) then $\text{Lk}(x; X, X_0) \cong \text{Lk}(y; X, X_0)$.

The embedding of Lemma 5 is usually obtained, as in the proof that $a \rightarrow b$ in Definition 1, by coning an isomorphism of the link of a point with a suspension. To get a stronger hold on the simplicial situation, we must examine the intrinsic properties of suspensions.

LEMMA 8. *Let $(X, X_0) = S^{n-1} * (Y, Y_0)$ and assume that (X, X_0) is not an $(n+1)$ -fold suspension, then*

$$S^{n-1} = I^{n-1}(X, X_0) = \{x \in X_0 : \Sigma \text{Lk}(x; X, X_0) \cong (X, X_0)\}.$$

Proof. Clearly, we have

$$S^{n-1} \subset \{x \in X_0 : \Sigma \text{Lk}(x; X, X_0) \cong (X, X_0)\} \subset I^{n-1}(X, X_0)$$

where the second inequality follows from the assumption that (X, X_0) is not an $(n+1)$ -fold suspension.

But $I^{n-1}(X, X_0) \subset S^{n-1}$, for if $x \in S^{n-1} * Y_0 - S^{n-1}$, $\text{Lk}(x; X, X_0)$ is an n -fold suspension and hence $x \notin I^{n-1}(X, X_0)$.

As a corollary, we have the following result of Armstrong and Morton, [A₁] and [M].

LEMMA 9. *If $\Sigma^r(X, X_0) \cong \Sigma^n(Y, Y_0)$, $r \leq n$, then $(X, X_0) \cong \Sigma^{n-r}(Y, Y_0)$.*

Proof. First we assume that n is maximal, i.e. that $\Sigma^n(Y, Y_0)$ is not an $(n+1)$ -fold suspension and prove the result in this special case by induction on r .

For $r=0$, the result is clear.

Now assume we are given a homeomorphism

$$h: S^{r-1} * (X, X_0) \rightarrow S^{n-1} * (Y, Y_0).$$

Let $x \in S^{r-1}$, then

$$\Sigma^{r-1}(X, X_0) \cong \text{Lk}(x; S^{r-1} * (X, X_0)) \cong \text{Lk}(h(x); S^{n-1} * (Y, Y_0)).$$

By Lemma 8, $h(x) \in S^{n-1}$ and hence

$$\text{Lk}(h(x); S^{n-1} * (Y, Y_0)) \cong \Sigma^{n-1}(Y, Y_0).$$

Note that since $\Sigma^n(Y, Y_0)$ was not an $(n+1)$ -fold suspension, $\Sigma^{n-1}(Y, Y_0)$ is not an n -fold suspension, and so the inductive hypothesis of the special case applies and $(X, X_0) \cong \Sigma^{n-r}(Y, Y_0)$.

Now if n is not maximal, $\Sigma^n(Y, Y_0) \cong \Sigma^m(Z, Z_0)$ where m is maximal. By two applications of the special case result,

$$\Sigma^{n-r}(Y, Y_0) \cong \Sigma^{n-r}(\Sigma^{m-n}(Z, Z_0)) \cong \Sigma^{m-r}(Z, Z_0) \cong (X, X_0).$$

COROLLARY 10. $\Sigma^n(X, X_0) \cong \Sigma^n(Y, Y_0)$ implies $(X, X_0) \cong (Y, Y_0)$.

DEFINITION 11. For a suspension pair (X, X_0) (resp. a suspension X), we define the intrinsic sphere $S(X, X_0)$ (resp. $S(X)$), to be

$$\begin{aligned} S(X, X_0) &= \{x \in X_0 : \Sigma \text{Lk}(x; X, X_0) \cong (X, X_0)\} \\ S(X) &= S(X, X) = \{x \in X : \Sigma \text{Lk}(x; X) \cong X\}, \end{aligned}$$

and if a compact pair (X, X_0) is not a suspension we define $S(X, X_0) = \{0\}$.

LEMMA 12. For any compact pair (X, X_0) there is a homeomorphism $(X, X_0) \cong S(X, X_0) * (Y, Y_0) \text{ rel } S(X, X_0)$, with (Y, Y_0) unique up to homeomorphism and not a suspension.

Proof. If (X, X_0) is not a suspension then let $(Y, Y_0) = (X, X_0)$. Otherwise let (X, X_0) be an n -fold suspension but not an $(n+1)$ -fold suspension. Then there exists a homeomorphism

$$q: (X, X_0) \rightarrow S^{n-1} * (Y, Y_0)$$

and (Y, Y_0) is not a suspension. By Lemma 8, $q(S(X, X_0)) = S^{n-1}$ so

$$q^{-1}|_{S^{n-1}} : S^{n-1} \cong S(X, X_0),$$

and joining with the identity on Y and preceding by q gives

$$(X, X_0) \rightarrow S^{n-1} * (Y, Y_0) \rightarrow S(X, X_0) * (Y, Y_0)$$

rel $S(X, X_0)$. Uniqueness of (Y, Y_0) comes from Corollary 10.

REMARKS. If $f: S^m * (Y, Y_0) \rightarrow (X, X_0)$ is a homeomorphism then

$$f(S^m) \subset S(X, X_0).$$

Hence, (X, X_0) is a $(\dim S(X, X_0) + 1)$ fold suspension and not a $(\dim S(X, X_0) + 2)$ fold suspension.

Thus, if $x \in X_0$, we have the equation

$$d(x; X, X_0) = \dim S(\text{Lk}(x; X, X_0)) + 1.$$

$S(X, X_0)$ is a subcomplex of any triangulation of (X, X_0) . For Lemma 8 implies that if (X, X_0) is a suspension, $S(X, X_0)$ is precisely the first nonempty intrinsic skeleton.

We now examine the important phenomena of the points interior to a face of a simplex having the same intrinsic dimension as the points interior to the simplex:

DEFINITION 13. Let $B = AC$ be a simplex of K_0 in the complex pair (K, K_0) , then the following conditions on A and B are equivalent:

- $d(x; K, K_0) = d(y; K, K_0)$ $x \in \text{Int } A, y \in \text{Int } B$.
- $\text{Lk}(x; K, K_0) \cong \text{Lk}(y; K, K_0)$ $x \in \text{Int } A, y \in \text{Int } B$.
- $\text{Lk}(A; K, K_0) \cong \Sigma \dot{C} * \text{Lk}(B; K, K_0)$.

If these conditions hold we will say that A is a nice face of B in (K, K_0) .

Proof of the equivalence. $a \rightarrow b$. Let $d(y; K, K_0) = i$, then $d(x; K, K_0) = d(y; K, K_0) \rightarrow x \sim y$ in (K, K_0) since $\text{Int } A$ lies in the closure of $\text{Int } B$. Hence, the links are isomorphic by the remark following Definition 7.

$b \rightarrow a$. Clear.

$b \rightleftharpoons c$. $\text{Lk}(x; K, K_0) \cong \dot{A} \text{Lk}(A; K, K_0)$ and $\text{Lk}(y; K, K_0) \cong \dot{B} \text{Lk}(B; K, K_0)$. But,

$$\dot{B} = (A\dot{C} + \dot{A}C) \cong \Sigma\dot{A}\dot{C} = A\Sigma\dot{C}$$

So $b \rightleftharpoons c$ by Corollary 10.

It is the explicit condition c , which relates the links of the two simplices that will be most useful. Let us examine the simplicial situation more closely.

Since $\text{Lk}(B) = \text{Lk}(C; \text{Lk}(A))$, it follows that $C * \text{Lk}(B; K, K_0)$ is actually a subcomplex pair of $\text{Lk}(A; K, K_0)$, and in fact, we have the usual stellar decomposition

$$\text{Lk}(A; K, K_0) = C * \text{Lk}(B; K, K_0) + (P, P_0)$$

where $(P, P_0) \cap C * \text{Lk}(B; K, K_0) = \dot{C} * \text{Lk}(B; K, K_0)$.

Thus, we let $(Q, Q_0) = \dot{C} * \text{Lk}(B; K, K_0)$ and identify $C * \text{Lk}(B; K, K_0)$ with $c(Q, Q_0)$, then we have $c(Q, Q_0) \subset \text{Lk}(A; K, K_0)$ with $cQ - Q_0$ open in $\text{Lk}(A; K, K_0)$. Now condition c , is precisely the statement that $\text{Lk}(A; K, K_0) \cong \Sigma(Q, Q_0)$.

So the question arises: When can we find a homeomorphism $\text{Lk}(A; K, K_0) \cong \Sigma(Q, Q_0)$, which takes $c(Q, Q_0)$ identically onto the "upper half" of $\Sigma(Q, Q_0)$? We will show that we can always "straighten" the homeomorphism on C , and in a first derived mod B , we can straighten the homeomorphism on all of cQ , as required.

First, if A is a nice face of B , then

$$c(Q, Q_0) \subset \text{Lk}(A) \cong \Sigma(Q, Q_0).$$

If x is any point of $cS(Q, Q_0) - S(Q, Q_0)$, $\text{Lk}(x; c(Q, Q_0)) \cong \text{Lk}(x; \text{Lk}(A))$ and the former is easily seen to be homeomorphic to (Q, Q_0) .

Hence, $cS(Q, Q_0) \subset S(\text{Lk}(A))$ and is in fact a face. That is,

$$C * S(\text{Lk}(B; K, K_0)) \text{ is a face of } S(\text{Lk}(A; K, K_0)).$$

Consider the composite homeomorphism

$$\text{Lk}(A) \xrightarrow{\phi} (v_0 + v_1)\dot{C} \text{Lk}(B) \xrightarrow{\psi} (v_0 + v_1)\dot{C}S(\text{Lk}(B)) * R$$

where ψ is the homeomorphism of Lemma 12, and R is some compact pair, which is a join complement of $S(\text{Lk}(B))$ in $\text{Lk}(B)$. ψ can be chosen to be the identity on $(v_0 + v_1)\dot{C}$.

Let c be an interior point of C . We would like to alter ϕ to get $\bar{\phi}$ which on C sends \dot{C} identically to \dot{C} and c to v_0 and joins.

We know that $\psi\phi(C * S(\text{Lk}(B)))$ is a fact of $(v_0 + v_1)\dot{C}S(\text{Lk}(B))$. So we can extend the map

$$\psi\phi(C * S(\text{Lk}(B))) \xrightarrow{\phi^{-1}\psi^{-1}} c\dot{C} * S(\text{Lk}(B)) \xrightarrow{1} v_0\dot{C} * S(\text{Lk}(B)),$$

to an auto-homeomorphism of $(v_0 + v_1)\dot{C} * S(\text{Lk}(B))$, called ρ . Define

$$\bar{\phi} = \psi^{-1}(\rho * 1)\psi\phi.$$

This proves the first part of the following:

LEMMA 14. *Let (K, K_0) be a complex pair with $B=AC$ a simplex of K_0 , and $c \in \text{Int } C$. Assume A is a nice face of B in (K, K_0) , then*

a. *There exists a homeomorphism*

$$\phi: \text{Lk}(A; K, K_0) \rightarrow (c+x)\dot{C} \text{Lk}(B; K, K_0)$$

with $\phi|_C$ mapping identically onto $c\dot{C}$.

b. *If $\eta_B K$ is a first derived of $K \bmod B$, then there exists a homeomorphism:*

$$\phi: \text{Lk}(A; \eta_B K, \eta_B K_0) \rightarrow (c+x)\dot{C} \text{Lk}(B; \eta_B K, \eta_B K_0)$$

with $\phi|_C * \text{Lk}(B; \eta_B K, \eta_B K_0)$ the identity.

Proof. a is proved above, we now prove b.

A simplex of $\text{Lk}(A; \eta_B K)$ is of the form $D_0\eta(AD_0D_1) \cdots \eta(AD_0D_n)$ with $D_0 < C$ and $D_1 < \cdots < D_n$. Mapping this simplex to $D_0\eta(D_0D_1) \cdots \eta(D_0D_n)$ gives a simplicial homeomorphism

$$\text{Lk}(A; \eta_B K, \eta_B K_0) \rightarrow \eta_C \text{Lk}(A; K, K_0).$$

This homeomorphism takes $C \text{Lk}(B; \eta_B(K, K_0)) = C \text{Lk}(C; \text{Lk}(A; \eta_B(K, K_0)))$ to $C \text{Lk}(C; \eta_C \text{Lk}(A; K, K_0)) = \text{St}(C; \eta_C \text{Lk}(A; K, K_0))$. This is a regular neighborhood of $C \bmod \dot{C}$ in $\text{Lk}(A; K, K_0)$.

Hence, $C \text{Lk}(B; \eta_B(K, K_0))$ is a regular neighborhood of $C \bmod \dot{C}$ in

$$\text{Lk}(A; \eta_B(K, K_0)).$$

Using part a, define a homeomorphism

$$\phi: \text{Lk}(A; \eta_B(K, K_0)) \rightarrow (c+x)\dot{C} \text{Lk}(B; \eta_B(K, K_0))$$

rel C . By uniqueness of regular neighborhoods there exists a homeomorphism:

$$k: (c+x)\dot{C} \text{Lk}(B) \rightarrow (c+x)\dot{C} \text{Lk}(B)$$

such that $k|_{c\dot{C}} = \text{identity}$ and $k\phi(C * \text{Lk}(B)) = c\dot{C} \text{Lk}(B)$. In particular,

$$k\phi(\dot{C} * \text{Lk}(B)) = \dot{C} \text{Lk}(B).$$

Finally, define

$$\bar{k}: (c+x)\dot{C} \text{Lk}(B) \rightarrow (c+x)\dot{C} \text{Lk}(B)$$

to be $(k\phi)^{-1}$ on $c\dot{C} \text{Lk}(B)$ and extended by coning with $x \rightarrow x$ over $x\dot{C} \text{Lk}(B)$.

The required homeomorphism is $\bar{k}k\phi$.

From the first part of the above we get a useful "local" version of our homogeneity results.

COROLLARY 15. *Let (K, K_0) be a complex pair with A a nice face of $B=AC$ in (K, K_0) . Let $b \in \text{Int } B$. There exists a homeomorphism*

$$f: \text{St}(A; K, K_0) \rightarrow \text{St}(A; K, K_0) \text{ rel } \dot{A} \text{ Lk}(A, K)$$

and such that $f(B) = b\dot{A}C$.

Proof. Let ϕ be a homeomorphism

$$\text{Lk}(A) \rightarrow (c+x)\dot{C} \text{ Lk}(B)$$

rel $c\dot{C}$. Join with the identity on A to extend ϕ to

$$\phi: \text{St}(A) \rightarrow (c+x)A\dot{C} \text{ Lk}(B).$$

Let $q: (c+x)A\dot{C} \rightarrow (c+x)A\dot{C}$ be a homeomorphism rel $(c+x)A\dot{C} = \partial(c+x)A\dot{C}$ and mapping $cA\dot{C}$ to $cb\dot{A}C$.

Define $f = \phi^{-1}(q * 1)\phi$.

REMARK. By the Alexander trick f is isotopic to the identity rel $\dot{A} \text{ Lk}(A; K)$.

We now relate the intrinsic skeleton and the links of simplices.

LEMMA 16. *Let (K, K_0) be a complex pair $B=AC \in K_0$, $x \in \text{Int } B$ and $y \in \text{Int } C$, then*

- a. $d(x; K, K_0) = \dim S(\text{Lk}(B; K, K_0)) + \dim B + 1$.
- b. $d(x; K, K_0) = d(y; \text{Lk}(A; K, K_0)) + \dim A + 1$.

Proof. a. $S(\text{Lk}(x; K, K_0)) = \dot{B} * S(\text{Lk}(B; K, K_0))$. So

$$d(x; K, K_0) = \dim S(\text{Lk}(x; K, K_0)) + 1 = \dim S(\text{Lk}(B; K, K_0)) + \dim B + 1.$$

b. By two applications of a, using $\text{Lk}(C; \text{Lk}(A)) = \text{Lk}(B)$:

$$\begin{aligned} d(x; K, K_0) &= \dim S(\text{Lk}(B)) + \dim B + 1, \\ d(y; \text{Lk}(A)) &= \dim S(\text{Lk}(C; \text{Lk}(A))) + \dim C + 1, \\ d(x; K, K_0) &= d(y; \text{Lk}(A)) + \dim B - \dim C. \end{aligned}$$

Finally, $\dim B - \dim C = \dim A + 1$.

COROLLARY 17. *Let (K, K_0) be a complex pair and let $A \in K_0$ and*

$$C_0, C_1 \in \text{Lk}(A; K_0).$$

Then

- a. A is a nice face of AC_0 in (K, K_0) iff $C_0 \in S(\text{Lk}(A; K, K_0))$.
- b. AC_0 is a nice face of AC_1 in (K, K_0) iff C_0 is a nice face of C_1 in $\text{Lk}(A; K, K_0)$.

Proof. a. Let $x \in \text{Int } A$, $y \in \text{Int } AC_0$ and $z \in \text{Int } C_0$. By Lemma 16a. $d(x; K, K_0) = \dim S(\text{Lk}(A; K, K_0)) + \dim A + 1$. By Lemma 16b.

$$d(y; K, K_0) = d(z; \text{Lk}(A; K, K_0)) + \dim A + 1.$$

A is a nice face of AC_0 iff $d(x; K, K_0) = d(y; K, K_0)$ iff $d(z; \text{Lk}(A; K, K_0)) = \dim S(\text{Lk}(A; K, K_0))$ iff (by Lemma 8) $z \in S(\text{Lk}(A; K, K_0))$.

b. From Lemma 16a, since $\text{Lk}(AC_i; K, K_0) = \text{Lk}(C_i; \text{Lk}(A; K, K_0))$ for $i = 0, 1$. Hence, a fortiori, $\dim S(\text{Lk}(AC_i)) = \dim S(\text{Lk}(C_i; \text{Lk}(A)))$.

COROLLARY 18. *Let (K, K_0) be a complex pair and A a j simplex of K_0 , then for $j \leq i$*

$$\text{Lk}(A; I^i(K, K_0)) = I^{i-j-1}(\text{Lk}(A; K, K_0)).$$

Proof. This is just a restatement of Lemma 16b.

We should note what Lemma 16 and this corollary do and do not say. If $C \in \text{Lk}(A; K_0)$ and $x \in \text{Int } AC$, $y \in \text{Int } C$, then Lemma 16b relates $d(x; K, K_0)$ and $d(y; \text{Lk}(A; K, K_0))$. We would like to in turn conclude something about $d(y; K, K_0)$. Since C is a face of AC , $d(y; K, K_0) \leq d(x; K, K_0)$ and hence we can conclude

$$d(y; K, K_0) \leq d(y; \text{Lk}(A; K, K_0)) + \dim A + 1.$$

Furthermore, it is easy to construct examples to show that equality need not hold. For equality to hold it is clearly necessary and sufficient that $d(y; K, K_0) = d(x; K, K_0)$, i.e. that C be a nice face of AC in (K, K_0) . Since this comes up in another context we mention the following special case when equality does hold:

LEMMA 19. *Let (K, K_0) be a complex pair and $B = AC \in K_0$. For*

$$y \in \text{Lk}(A; \eta_B K_0) - C, \\ d(y; K, K_0) = d(y; \text{Lk}(A; \eta_B K, \eta_B K_0)) + \dim A + 1.$$

Proof. $y \in \text{Int } D_0 \eta(AD_0 D_1) \cdots \eta(AD_0 D_n)$ with $D_0 < C$, $D_1 < \cdots < D_n \neq 0$, $D_i \in K_0$, $i = 0, \dots, n$.

$$\text{Int } D_0 \eta(AD_0 D_1) \cdots \eta(AD_0 D_n) \cup \text{Int } AD_0 \eta(AD_0 D_1) \cdots \eta(AD_0 D_n) \subset \text{Int } AD_0 D_n.$$

Hence, $D_0 \eta(AD_0 D_1) \cdots \eta(AD_0 D_n)$ is a nice face of $AD_0 \eta(AD_0 D_1) \cdots \eta(AD_0 D_n)$ in $\eta_B(K, K_0)$ and equality holds by the above remarks.

LEMMA 20. *Let (X, X_0) be a compact polyhedral pair, then for $i \geq 0$,*

$$\Sigma I^i(X, X_0) = I^{i+1}(\Sigma(X, X_0)).$$

Proof. Since $\Sigma S(X, X_0) = S(\Sigma(X, X_0))$, the result is true for the minimal non-empty intrinsic skeleton, if (X, X_0) is a suspension. If (X, X_0) is not a suspension, then $I^0(\Sigma(X, X_0)) = \text{suspension vertices}$. In either case, this takes care of the suspension vertices.

If $y = tv + (1-t)x$, with v a suspension vertex, $0 \leq t < 1$ and $x \in X_0$,

$$d(y; \Sigma(X, X_0)) = d(x; X, X_0) + 1.$$

REMARK. As we saw in the proof, if (X, X_0) is not a suspension then $I^0(\Sigma(X, X_0)) = \text{suspension vertices} = \Sigma\{0\}$, so if (X, X_0) is not a suspension, we interpret $I^{-1}(X, X_0) = \{0\}$, which fits with the corresponding convention of Definition 11.

We obtain the following amusing corollary,

COROLLARY 21. *Let X be a polyhedron, then $I^i(X, I^i(X))=I^j(X)$ for $j \leq i$.*

Proof. Let $x \in I^i(X)$, then by Corollary 18,

$$\text{Lk}(x; X, I^i(X)) = (\text{Lk}(x; X), I^{i-1}(\text{Lk}(x; X))).$$

If $d(x; X)=j$, then $\text{Lk}(x; X) \cong \Sigma^j Y$, with Y not a suspension, then by Lemma 20 (and the remark which follows for the case $i=j$),

$$\text{Lk}(x; X, I^i(X)) \cong (\Sigma^j Y, \Sigma^j I^{i-j-1}(Y)) \cong \Sigma^j(Y, I^{i-j-1}(Y)).$$

Since Y does not desuspend, the pair does not and hence $d(x; X, I^i(X))=j$.

We now relate the absolute intrinsic skeleton with the ambient notion:

DEFINITION 22. Let (X, X_0) be a polyhedral pair, with $x \in X_0$, then (X, X_0) is called locally unknotted at x if $d(x; X, X_0)=d(x; X_0)$. If (X, X_0) is locally unknotted at all points of X_0 , we call (X, X_0) a locally unknotted pair.

Local unknottedness translates absolute conditions on X_0 into ambient conditions on the pair, and can be defined in terms of any of the notions we have introduced:

THEOREM 23. *For a complex pair (K, K_0) , the following are equivalent:*

- a. (K, K_0) is locally unknotted.
- b. For all $A \in K_0$, $S(\text{Lk}(A; K, K_0))=S(\text{Lk}(A; K_0))$.
- c. A nice face of B in K_0 implies that A is a nice face of B in (K, K_0) .
- d. $x \sim y$ in K_0 implies $x \sim y$ in (K, K_0) .
- e. $I^i(K, K_0)=I^i(K_0)$ for all i .

Proof. $a \rightarrow e \rightarrow d \rightarrow c$ are clear. To complete the proof, note that $c \rightarrow b$ by Corollary 17a applied to (K, K_0) and K_0 . $b \rightarrow a$ by Lemma 16a applied to (K, K_0) and K_0 .

We relate these notions to the usual definition of locally unknottedness for manifolds, by the lemma.

LEMMA 24. a. *A sphere pair (S^n, S^p) is unknotted iff $S(S^n, S^p)=S^p$.*

b. *A ball pair (B^n, B^p) is unknotted iff $S(B^n, B^p)=\partial B^p$.*

Proof. In each case unknottedness clearly implies the intrinsic sphere condition and conversely $(S^{n-p}, \{0\})$ and (B^{n-p+1}, p) (with $p \in \text{Int } B^{n-p+1}$) are unknotted and hence so are any suspensions of them.

COROLLARY 25. *Let (X, X_0) be a proper manifold pair, i.e. $\partial X \cap X_0 = \partial X_0$, then (X, X_0) is locally unknotted iff it is locally unknotted in the usual sense, i.e. $x \in X_0$ implies $\text{Lk}(x; X, X_0)$ is an unknotted ball or sphere pair.*

Proof. From Lemma 24, and the equivalence of locally unknottedness with $S(\text{Lk}(x; X, X_0)) \cong S(\text{Lk}(x; X_0))$.

REMARK. Note that $(X, I^i(X))$ need not be locally unknotted. Theorem 23 and Corollary 21 imply that this is equivalent to $I^j(I^i(X))=I^j(X)$ for all $j \leq i$. It is easy to see that this need not be true, e.g. the wedge of two different dimensional spheres.

III. **Homogeneous collapsing and regular neighborhoods.** We begin by defining a general form of relative collapsing.

DEFINITION 1. Let X be a polyhedron, with $X', X_0 < X$ and let $X'_0 = X' \cap X_0$, then X collapses to X' relative to X_0 , or more simply (X, X_0) collapses to (X', X'_0) , denoted $(X|X_0) \searrow X'$ if there exists a sequence of elementary collapses

$$X = X^n \searrow X^{n-1} \searrow \dots \searrow X^0 = X'$$

with $\text{Cl}(X^{i+1} - X^i) = B^i$ and $B^i \cap X^i = F^i$, a ball and face, as usual, satisfying the condition that $B^i \cap X_0 < \text{Cl}(\partial B^i - F^i)$.

In such a case, we will call the sequence $(X^n, X^{n-1}, \dots, X^0)$ a collapse of X to X' rel X_0 , or of (X, X_0) to (X', X'_0) .

There is a corresponding simplicial definition:

DEFINITION 2. Let K be a complex, with $K', K_0 < K$ and let $K'_0 = K_0 \cap K'$, then K collapses simplicially to K' relative to K_0 , or (K, K_0) collapses simplicially to (K', K'_0) if there exists a simplicial collapse $(K^n, K^{n-1}, \dots, K^0)$ of K to K' which is relative to K_0 in the sense of Definition 1, i.e. $K^{i+1} = K^i + B^i + A^i$ and $K_0 \cap B^i < A^i$. We denote this by $(K|K_0) \searrow^s K'$.

REMARKS. In the simplicial case, there are two possibilities, which represent the two kinds of things that can occur: Either $K_0 \cap B^i = A^i$, or $K_0 \cap B^i < \partial A^i$. The first corresponds to a punch out from K_0 and the second is in a strong sense, "away from K_0 ".

Any subdivision of a collapse of X to X' relative to X_0 or of a simplicial collapse of K to K' relative to K_0 is clearly still a collapse relative to X_0 , or K_0 respectively. Also, any collapse of X to X' rel X_0 is a collapse relative to any subpolyhedron of X_0 .

If $X_0 < X'$, then for all i in a relative collapse, $B_i \cap X_0 < \partial F^i$ and the definition reduces to Cohen's definition of relative collapsing. In this case, of course, $X'_0 = X_0$.

We now relate the geometrical and simplicial notions using the following elegant proof due to Marshall Cohen.

LEMMA 3. Let $\Delta^n = a\Delta^{n-1} = ad\Delta^{n-1}$ (with $d \in \text{Int } \Delta^{n-1}$) and let $r: \Delta^n \rightarrow a\Delta^{n-1}$ be the simplicial retraction sending d to a . Let $\alpha\Delta^n$ be any subdivision of Δ^n such that $r: \alpha\Delta^n \rightarrow \alpha a\hat{\Delta}^{n-1}$ is simplicial. Then $(\alpha\Delta^n | \alpha\Delta^{n-1}) \searrow^s \alpha a\hat{\Delta}^{n-1}$.

Proof. Let A be a simplex of $\alpha a\hat{\Delta}^{n-1} - \alpha\hat{\Delta}^{n-1}$. $r^{-1}(A)$ is a convex cell of dimension $\dim A + 1$, with A as a face and $r^{-1}(A) \cap \Delta^{n-1}$ as a face. The simplices of $r^{-1}(A) - (A \cup r^{-1}(\hat{A}))$ are mapped by r onto A . They are ordered by the line segment $r^{-1}(\hat{A})$ which joins \hat{A} to $r^{-1}(\hat{A}) \cap \text{Int } \Delta^{n-1}$. We remove $\dim A$ and $\dim A + 1$ simplices of $r^{-1}(A)$ in pairs moving down $r^{-1}(\hat{A})$ from $r^{-1}(\hat{A}) \cap \text{Int } \Delta^{n-1}$. This gives a collapse $r^{-1}(A) \searrow^s r^{-1}(\hat{A}) \cup A$. Furthermore, this collapse is

relative to Δ^{n-1} , or more precisely, $(r^{-1}(A) \cap \alpha\Delta^{n-1})$. For if B and C with $B=bC$ are a pair of simplices removed during this collapse then $rB=rC=A$ and so there is a vertex c of C such that $rb=rc$ is a vertex of A , and hence $b \notin \dot{\Delta}^{n-1}$ and since it lies between c and rb on the segment $r^{-1}rb$, $b \notin \text{Int } \Delta^{n-1}$ so $B \cap \alpha\Delta^{n-1} < C$.

Now order the simplices of $\alpha\dot{\Delta}^{n-1} - \alpha\dot{\Delta}^{n-1}$ in decreasing order of dimension and in order, for each such simplex A , perform the above simplicial collapse $r^{-1}(A) \searrow^s A \cup r^{-1}(A)$, giving a collapse $\alpha\Delta^n \searrow^s \alpha\dot{\Delta}^{n-1}$, which is clearly relative to $\alpha\Delta^{n-1}$.

LEMMA 4. *Let $(X|X_0) \searrow X'$ and let $(K; h)$ be a triangulation of X . Then there exists a subdivision αK such that $h^{-1}(X_0)$ and $h^{-1}(X')$ are subcomplexes of αK and $(\alpha K|h^{-1}(X_0)) \searrow^s h^{-1}(X')$.*

Proof. Let $X = X^n, \dots, X^0 = X'$ be a collapse of X to X' rel X_0 , and let $B^i = h^{-1}(\text{Cl}(X^i - X^{i-1}))$ and F^i the face of B^i complementary to $B^i \cap h^{-1}(X^{i-1})$. So $B^i \cap h^{-1}(X_0) \subset F^i$.

Let $q^i: (\Delta_i, \dot{\Delta}_i) \rightarrow (B^i, F^i)$ be a homeomorphism with $\Delta_i = a_i \dot{\Delta}_i$. If $r^i: \Delta_i \rightarrow a_i \dot{\Delta}_i$ is the retraction of Lemma 3, then define $s^i: h^{-1}(X^i) \rightarrow h^{-1}(X^{i-1})$ to be $q^i r^i (q^i)^{-1}$ on B^i and the identity on $h^{-1}(X^{i-1})$.

First, let βK be a subdivision of K so that $h^{-1}(X_0)$ and $h^{-1}(X^i) \ i=0, \dots, n$ are subcomplexes of βK . Then subdivide to make the following diagram simplicial:

$$\begin{array}{ccccccc}
 \beta K = h^{-1}(X^n) & \xrightarrow{s^n} & h^{-1}(X^{n-1}) & \longrightarrow \dots \longrightarrow & h^{-1}(X^1) & \xrightarrow{s^1} & h^{-1}(X^0) \\
 \uparrow q^n & & \uparrow q^{n-1} & & \uparrow q^1 & & \\
 \Delta_n & & \Delta_{n-1} & & \Delta_1 & &
 \end{array}$$

If αK is the subdivision of βK obtained, then Lemma 3 implies that

$$(\alpha K|h^{-1}(X_0)) \searrow^s h^{-1}(X').$$

REMARK. Notice that the simplicial collapse obtained is a subdivision of the original geometric collapse.

We now relate relative collapsing with intrinsic skeleta.

DEFINITION 5. Assume that $X', X_0 < X < Q_0 < Q$, let $X = X^n, X^{n-1}, \dots, X^0 = X'$ be a collapse of X to X' rel X_0 . Call this collapse homogenous (resp. weakly homogenous) with respect to (Q, Q_0) , if for each $i=0, \dots, n-1$, $d(x; Q, Q_0)$ is constant as x varies over $X^{i+1} - X^i$ (resp. as x varies over $X^{i+1} - (X^i \cup X_0)$). If there exists a collapse of X to X' rel X_0 which is homogenous (resp. weakly homogenous) we will write $(X|X_0) \searrow_h X'$ with respect to (Q, Q_0) (resp. $(X|X_0) \searrow_w X'$ with respect to (Q, Q_0)).

Note that $(X|X_0) \searrow_h X'$ with resp. to (Q, Q_0) does not imply that $d(x; Q, Q_0)$ is constant as x varies all over $X - X'$. The intrinsic dimension of the cells which are removed may vary; $d(x; Q, Q_0)$ is constant only on each cell.

As usual we will write $(X|X_0)_h \searrow X'$ in Q or $(X|X_0)_w \searrow X'$ in Q for $(X|X_0)_h \searrow X'$ in (Q, Q) or $(X|X_0)_w \searrow X'$ in (Q, Q) .

Note that if $X_0 < X'$, then $(X|X_0)_h \searrow X'$ is equivalent to $(X|X_0)_w \searrow X'$.

Assume that $(X|X_0)_h \searrow X'$ in (Q, Q_0) and let $(K, K_0; h)$ be any triangulation of (Q, Q_0) . Subdividing K so that $h^{-1}(X)$ is a subcomplex, and then applying Lemma 4, and the remark that follows it, and finally extending the resulting subdivision of $h^{-1}(X)$ to a subdivision of K , we obtain a subdivision $(\alpha K, \alpha K_0; h)$ satisfying:

a. $h^{-1}(X)$, $h^{-1}(X_0)$ and $h^{-1}(X')$ are subcomplexes of αK_0 .

b. $h^{-1}(X) \searrow^s h^{-1}(X')$ by a sequence of collapses of simplices $B^i = b^i A^i$ across faces A^i , satisfying:

1. $B^i \cap h^{-1}(X_0) < A^i$.

2. A^i is a nice face of B^i in $(\alpha K, \alpha K_0)$.

Similarly, if $(X|X_0)_w \searrow X'$ in (Q, Q_0) and $(K, K_0; h)$ is a triangulation of (Q, Q_0) , then there exists a subdivision $(\alpha K, \alpha K_0; h)$ of (Q, Q_0) satisfying:

a. $h^{-1}(X)$, $h^{-1}(X_0)$ and $h^{-1}(X')$ are subcomplexes of αK_0 .

b. $h^{-1}(X) \searrow^s h^{-1}(X')$ by a sequence of collapses of simplices $B^i = b^i A^i$ across faces A^i , satisfying:

1. $B^i \cap h^{-1}(X_0) < A^i$.

2. If $B^i \cap h^{-1}(X_0) < A^i$ then A^i is a nice face of B^i in $(\alpha K, \alpha K_0)$.

In these simplicial situations, condition b1 says that the collapse is relative to $h^{-1}(X_0)$ and b2 says that the collapse is homogenous, or weakly homogenous.

Now after all these definitions, the collapsing theorems:

THEOREM 6. *Assume $X_0 < X < Q_0 < Q$, and let (V, V_0) be a regular neighborhood pair of $X \bmod X_0$ in (Q, Q_0) . Then if $(X|X_0)_h \searrow X'$ with respect to (Q, Q_0) , then (V, V_0) is a regular neighborhood pair of $X' \bmod X' \cap X_0$ in (Q, Q_0) .*

Proof. We assume that everything is triangulated as above to obtain the situation: a complex pair (K, K_0) with K_0 full in K and subcomplexes L, L', L_0 of K_0 with L_0 full in K_0 , and a simplicial collapse: $L = L^n, L^{n-1}, \dots, L^0 = L'$ with $L^{i+1} = L^i + A^i + B^i$ with A^i a nice face of B^i and $L_0 \cap B^i < A^i$. If we prove that every (K, K_0) neighborhood of $L^{i+1} \bmod L^{i+1} \cap L_0$ is one of $L^i \bmod L^i \cap L_0$, we are clearly finished by induction. So it suffices to prove the result across one simplicial collapse.

Let $L = L' + A + B$ with $B = bA$, $L' \cap B = bA$, $L_0 \cap B < A$ and A a nice face of B in (K, K_0) , where L_0 is a full subcomplex of K and we can assume L is a full subcomplex of K by deriving mod L , if it were not. We show that every (K, K_0) neighborhood of $L \bmod L_0$ is also a (K, K_0) neighborhood of $L' \bmod L_0 \cap L'$. By the uniqueness theorem for regular neighborhoods, it suffices to exhibit any regular neighborhood of $L \bmod L_0$ which is also a regular neighborhood of $L' \bmod L_0 \cap L'$.

There are two cases, according to whether $B \cap L_0 < A$ or $B \cap L_0 = A$.

(1) $B \cap L_0 < A$. Then $B \cap L_0$ is a proper face of A , by the fullness of L_0 .

Let $(H, H_0) = \text{Lk}(B; K, K_0)$. Let $\alpha K = \sigma_{ba}\sigma_A K$ where $\sigma_A K$ is stellar subdivision of K at A introducing a vertex $a \in \text{Int } A$, and $\sigma_{ba}\sigma_A K$ is stellar subdivision of $\sigma_A K$ at ba introducing a vertex $x \in \text{Int } ba$.

$$\alpha(B * H) = \sigma_{ba}\sigma_A bA * H = \sigma_{ba}ba\dot{A} * H = x(a+b)\dot{A} * H = (\alpha B) * H.$$

So

$$\alpha(B * (H, H_0)) = \alpha \text{St}(B; K, K_0) = (\alpha B) * (H, H_0).$$

Define $L_1 = L' + xb\dot{A}$. This is a full subcomplex of αK .

ASSERTION. There exists a derived $\eta\alpha K$ such that $N(L_1 - L_0; \eta\alpha(K, K_0))$ is a regular neighborhood of $L' \text{ mod } L_0$ in (K, K_0) .

Note that as a first derived neighborhood $N(L_1 - L_0; \eta\alpha(K, K_0))$ is by definition, a regular neighborhood of $L_1 \text{ mod } L_0$.

Assuming the above assertion, we are almost done, for by Corollary II.15 there exists a homeomorphism $f: (K, K_0) \rightarrow (K, K_0)$ supported by $\text{St}^\circ(A; K)$ such that $f(B) = xb\dot{A}$. Note that f is the identity on L' and L_0 and $f(L) = L_1$. Hence,

$$f^{-1}N(L_1 - L_0; \eta\alpha(K, K_0))$$

is a regular neighborhood of both $L \text{ mod } L_0$ and $L' \text{ mod } L_0$ in (K, K_0) .

To prove the assertion, we parametrize $\sigma_A \text{St}(B; K)$. As above, $\sigma_A(B * H) = (\sigma_A B) * H = ba\dot{A} * H$. Now let $\Delta = [e_0, e_1, e_2, e_3, e_4]$ be the standard four simplex, and define a simplicial map $p: \sigma_A B * H \rightarrow \Delta$ as the join of the following:

$$\begin{aligned} p(\dot{A} \cap L_0) &= e_0, & p(b) &= e_3, \\ p(C(\dot{A} \cap L_0; \dot{A})) &= e_1, & p(H) &= e_4, \\ p(a) &= e_2. \end{aligned}$$

Note that the inverse of each vertex of Δ is a full subcomplex of $\sigma_A B * H$.

Furthermore, if we subdivide $\Delta: \sigma_{e_2e_3} \Delta$ at the point $e_5 = p(x)$, then $p: \alpha B * H \rightarrow \sigma_{e_2e_3} \Delta$ is simplicial, for $\alpha B * H = x(a+b)\dot{A} * H$ and $\sigma_{e_2e_3} \Delta = e_5(e_2 + e_3)e_0e_1e_4$.

We will now define two derived subdivisions, one of Δ and one of $\sigma_{e_2e_3} \Delta$, called $\eta\Delta$ and $\bar{\eta}\sigma_{e_2e_3} \Delta$ respectively, satisfying:

- (1) $\eta|e_0e_1e_2e_4 + e_0e_1e_3e_4 = \bar{\eta}|e_0e_1e_2e_4 + e_0e_1e_3e_4$.
- (2) $|N(e_0e_1e_3 - e_0; \eta\Delta)| = |N(e_0e_1e_3e_5 - e_0; \bar{\eta}\sigma_{e_2e_3} \Delta)|$.

Let $\bar{\eta}$ be any derived subdivision of $\sigma_{e_2e_3} \Delta$ and then pick $\eta\Delta$ to agree with $\bar{\eta}$ on $e_0e_1e_2e_4 + e_0e_1e_3e_4$ and for C any simplex (including 0) of $e_0e_1e_4$, let $b(e_2e_3C)$ for $\eta\Delta$ be defined to equal $\bar{b}(e_2e_5C)$ of $\bar{\eta}\Delta$. (1) is clear and (2) is easily proved by looking at the complements of the two sets in Δ and seeing that they are made up of the same simplices.

Now, note that $\sigma_A K$ and αK agree outside of $\text{St}^\circ(B; K)$. Define derived subdivisions $\eta\sigma_A K$ and $\bar{\eta}\alpha K$ so that

- (a) $\eta\sigma_A|K - \text{St}^\circ(B; K) = \bar{\eta}\alpha|K - \text{St}^\circ(B; K)$,
- (b) $p: \eta\sigma_A B * H \rightarrow \eta\Delta$ and $p: \bar{\eta}\alpha B * H \rightarrow \bar{\eta}\sigma_{e_2e_3} \Delta$ are simplicial.

Here (1) above is used to make (a) possible.

Now we claim that

$$|N(L' - L_0; \eta\sigma_A(K, K_0))| = |N(L_1 - L_0; \bar{\eta}\alpha(K, K_0))|,$$

which would prove the assertion since L' is full in $\sigma_A K$ and L_1 is full in αK .

To prove the above equality note that the sets are clearly equal outside of $\text{St}^\circ(B; K)$. So it suffices to prove the equality of their intersection with $\text{St}(B; K)$, i.e. to prove

$$|N(b\dot{A} - \dot{A} \cap L_0; \eta\sigma_A B * (H, H_0))| = |N(xb\dot{A} - \dot{A} \cap L_0; \bar{\eta}\alpha B * (H, H_0))|.$$

But, by the parametrization Lemma [C; Lemma 2.14], these sets are respectively $(p, p|B * H_0)^{-1}|N(e_0 e_1 e_3 - e_0; \eta\Delta)|$ and $(p, p|B * H_0)^{-1}|N(e_0 e_1 e_3 e_5 - e_0; \bar{\eta}\sigma_{e_2 e_3} \Delta)|$ and are consequently equal by condition (2) above on $\eta\Delta$ and $\bar{\eta}\sigma_{e_2 e_3} \Delta$.

This completes the proof of the first case.

(2) $B \cap L_0 = A$. Let $L'_0 = L_0 \cap L' = L_0 - A$. Again we assume that L is full in K . Define $\alpha K = \sigma_{ab}\sigma_A K$ and $L_1 = L' + xb\dot{A}$ as in the first case.

We reduce the proof of case (2), precisely as we did case (1) to a proof of the following:

ASSERTION. There exists a derived $\eta\alpha K$ such that $N(L_1 - (L_0 + x\dot{A}); \eta\alpha(K, K_0))$ is a regular neighborhood of $L' \bmod L'_0$ in (K, K_0) .

Completing the proof of case (2) from the assertion uses the same homeomorphism f , and noting this time that $f(L_0) = L'_0 + x\dot{A}$.

To prove the assertion, we parametrize again, this time to $\Delta = [e_0 e_1 e_2 e_3]$ the standard three simplex. Define $p: \sigma_A B * H \rightarrow \Delta$ by joining

$$\begin{aligned} p(\dot{A}) &= e_0, & p(b) &= e_2, \\ p(a) &= e_1, & p(H) &= e_3. \end{aligned}$$

Let $p(x) = e_4$ and let $\sigma_{e_1 e_2} \Delta$ introduce the vertex e_4 , so that $p: \alpha B * H \rightarrow \sigma_{e_1 e_2} \Delta$ is simplicial.

We now define derived subdivisions $\eta\Delta$ and $\bar{\eta}\sigma_{e_1 e_2} \Delta$ to satisfy:

- (1) $\eta|e_0 e_1 e_3 + e_0 e_2 e_3 = \bar{\eta}|e_0 e_1 e_3 + e_0 e_2 e_3$.
- (2) $|N(e_0 e_2 - e_0; \eta\Delta)| = |N(e_0 e_2 e_4 - e_0 e_4; \bar{\eta}\sigma_{e_1 e_2} \Delta)|$.

Let $\bar{\eta}$ be any derived of $\sigma_{e_1 e_2} \Delta$ and then pick $\eta\Delta$ to agree with $\bar{\eta}$ on $e_0 e_1 e_3 + e_0 e_2 e_3$ and let $b(e_1 e_2) = e_4$, $b(e_1 e_2 e_0) = \bar{b}(e_0 e_4)$ of $\bar{\eta}\sigma_{e_1 e_2} \Delta$ and $b(e_1 e_2 e_3)$ and $b(e_0 e_1 e_2 e_3) = \bar{b}(e_2 e_4 e_3)$ and $\bar{b}(e_0 e_2 e_4 e_3)$ of $\bar{\eta}\sigma_{e_1 e_2} \Delta$ respectively.

Define $\eta\sigma_A K$ and $\bar{\eta}\alpha K$ as in case (1) and we prove, precisely, as in case (1) the equality

$$|N(L' - L'_0; \eta\sigma_A(K, K_0))| = |N(L_1 - (L'_0 + x\dot{A}); \bar{\eta}\alpha(K, K_0))|.$$

This completes the proof of case (2) and hence of Theorem 6.

COROLLARY 7. Let $X_0 < X' < X < Q$, and V be a regular neighborhood of $X' \bmod X_0$ in V , with V a topological neighborhood of $X - X_0$. Then, if $(X|X_0) \xrightarrow{h} X'$, then V is a regular neighborhood of X and X_0 in Q .

Proof. Let U be a regular neighborhood of $X \text{ mod } X_0$ with $U - X_0 \subset \text{Int } V$ e.g. triangulate so that everything in the statement is a subcomplex with X and X_0 full and then let $U = N(X - X_0; \eta Q) \subset \text{Int } V$ since V is a topological neighborhood of $X - X_0$ and a subcomplex.

By Theorem 6, U is a regular neighborhood of $X' \text{ mod } X_0$ and hence by [C; Proposition 7.3] V is a regular neighborhood of $U \text{ mod } X_0$ and hence by [C; Proposition 7.4] V is a regular neighborhood of $X \text{ mod } X_0$.

Now for the theorem associated with weakly homogenous collapsing:

THEOREM 8. Assume $X_0 < X < Q_0 < Q$, and let (V, V_0) be a regular neighborhood of $X \text{ mod } X_0$ in (Q, Q_0) . Then if $(X|X_0) \xrightarrow{w} X'$ with respect to (Q, Q_0) , there exists an embedding $f: (V, V_0) \rightarrow (V, V_0)$ with $f|X' = \text{identity}$ and such that if $(U, U_0) = f(V, V_0)$, then (U, U_0) is a regular neighborhood of $X' \cup X_0 \text{ mod } X_0$ in (Q, Q_0) and $(\dot{U}, \dot{U}_0) = f(\dot{V}, \dot{V}_0)$.

Proof. We assume that everything is triangulated as in the remarks preceding Theorem 6, and so obtain the situation: a complex pair (K, K_0) with K_0 full in K and subcomplexes L, L_0, L' of K_0 with L_0 full in K_0 , and a simplicial collapse $L = L^n, \dots, L^0 = L'$, with $L^{i+1} = L^i + A^i + B^i$ with $L_0 \cap B^i < A^i$ and A^i a nice face of B^i if $L_0 \cap B^i < A^i$. If we prove that every (K, K_0) neighborhood of $L^{i+1} \cup L_0 \text{ mod } L_0$ embeds in itself as required with image a regular neighborhood of $L^i \cup L_0 \text{ mod } L_0$, then the result follows by induction, by composing the embeddings. So again it suffices to prove the result across one simplicial collapse.

Let $L = L' + A + B$ with $B = bA, L' \cap B = bA, L_0 \cap B < A$ and if $L_0 \cap B < A$, then A is a nice face of B . L_0 is a full subcomplex of K and we can again assume that L is also full. As in Theorem 6, there are two cases:

(1) $L_0 \cap B < A$ and A is a nice face of B . Then Theorem 6 itself applies and any regular neighborhood of $L \text{ mod } L_0$ is a regular neighborhood of $L' \text{ mod } L' \cap L_0 = L_0$ so we can let $f = \text{identity}$, in this case.

(2) $L_0 \cap B = A$. Note that as in Theorem 6, uniqueness of regular neighborhoods implies that we need only prove the existence of such an f for one regular neighborhood of $L \cup L_0 \text{ mod } L_0$, since any other can be mapped to it by a homeomorphism of $(K, K_0) \text{ rel } L \cup L_0$.

As in the proof of Theorem 6, let $(H, H_0) = \text{Lk}(B; K, K_0)$, $\alpha K = \sigma_{ab} \sigma_A K$ and define $p: \sigma_A B * H \rightarrow \Delta = [e_0 e_1 e_2 e_3]$, by joining

$$\begin{aligned} p(A) &= e_0, & p(b) &= e_2, \\ p(a) &= e_1, & p(H) &= e_3. \end{aligned}$$

Again, let $\sigma_{e_1 e_2} \Delta$ introduce the vertex $e_4 = p(x)$. Let $e_5 \in \text{Int } e_2 e_3$. There exist deriveds $\eta \Delta$ and $\bar{\eta} \Delta$ satisfying

- (1) $|\eta| e_0 \partial(e_1 e_2 e_3) = |\bar{\eta}| e_0 \partial(e_1 e_2 e_3)$.
- (2) $|N(e_0 e_1 e_2 - e_0 e_1; \eta \Delta)| = |e_0 e_1 e_2 e_5|$ and $|\dot{N}(e_0 e_1 e_2 - e_0 e_1; \eta \Delta)| = |e_0 e_1 e_5|$.
- (3) $|N(e_0 e_2 - e_0; \bar{\eta} \Delta)| = |e_0 e_4 e_2 e_5|$ and $|\dot{N}(e_0 e_2 - e_0; \bar{\eta} \Delta)| = |e_0 e_4 e_5|$.

Now define derived $\eta\sigma_A K$ and $\bar{\eta}\sigma_A K$ to agree outside $\text{St}^\circ(B; K)$ and so that $p: \eta\sigma_A B * H \rightarrow \eta\Delta$ and $p: \bar{\eta}\sigma_A B * H \rightarrow \bar{\eta}\Delta$ are simplicial.

We will define a homeomorphism

$$f: |N(L \cup L_0 - L_0; \eta\sigma_A(K, K_0))| \rightarrow |N(L' \cup L_0 - L_0; \bar{\eta}\sigma_A(K, K_0))|$$

taking \dot{N} to \dot{N} and the identity on L' , which will prove the result.

To define f , note first that

$$N(L \cup L_0 - L_0; \eta\sigma_A K) - \text{St}^\circ(B; K) - A = N(L' \cup L_0 - L_0; \bar{\eta}\sigma_A K) - \text{St}^\circ(B; K),$$

and

$$\dot{N}(L \cup L_0 - L_0; \eta\sigma_A K) - \text{St}^\circ(B; K) - A = \dot{N}(L' \cup L_0 - L_0; \bar{\eta}\sigma_A K) - \text{St}^\circ(B; K).$$

So define f to be the identity on these sets. To complete the proof we must define f on the intersection with $\text{St}(B, K)$, i.e. we must extend f to a homeomorphism

$$N(B - A; \eta\sigma_A B * (H, H_0)) \rightarrow N((b + a)\dot{A} - A; \bar{\eta}\sigma_A B * (H, H_0))$$

taking boundary to boundary and the identity on the intersection with $b\dot{A}H$ (to extend the previous definition of f).

By the parametrization lemma, these sets are respectively,

$$(p, p|B * H_0)^{-1}|e_0e_1e_2e_5| \quad \text{and} \quad (p, p|B * H_0)^{-1}|e_0e_4e_2e_5|,$$

and the respective boundaries are the inverses under $(p, p|B * H_0)$ of $|e_0e_1e_5|$ and $|e_0e_4e_5|$.

Define $f: \sigma_A B * H \rightarrow \alpha B * H$ by $f(a) = x$ joined with the identity on $b\dot{A} * H$. We will prove that $f|N(B - A; \eta\sigma_A B * (H, H_0))$ is the required homeomorphism. To do this, it suffices to show that

$$fp^{-1}|e_0e_1e_2e_5| = p^{-1}|e_0e_4e_2e_5| \quad \text{and} \quad fp^{-1}|e_0e_1e_5| = p^{-1}|e_0e_4e_5|.$$

Define $f_\Delta: \Delta \rightarrow \sigma_{e_1e_2}\Delta$ by $f_\Delta(e_1) = e_4$ joined with the identity on $e_0e_2e_3$. Clearly, $pf = f_\Delta p$. Since $f_\Delta^{-1}(|e_0e_4e_2e_5|, |e_0e_4e_5|) = (|e_0e_1e_2e_5|, |e_0e_1e_5|)$ and since $f(\sigma_A B * H)$ contains $p^{-1}|e_0e_4e_2e_5|$, a little diagram chase proves that

$$fp^{-1}(|e_0e_1e_2e_5|, |e_0e_1e_5|) = p^{-1}(|e_0e_4e_2e_5|, |e_0e_4e_5|).$$

Thus, f defines a homeomorphism of a regular neighborhood of $L \cup L_0 \bmod L_0$ onto a regular neighborhood of $L' \cup L_0 \bmod L_0$ within it. Proving case (2) and the theorem.

COROLLARY 9. *Let (X, X_0) be a polyhedral pair contained in a polyhedron Q . Let V be a regular neighborhood of $X \bmod X_0$ in Q . If $(X|X_0) \rightsquigarrow X'$ in Q , then there are regular neighborhoods U and U' of $X' \bmod X_0 \cap X'$ in Q contained in V and there is an isotopy H of V into itself such that $H|V \times 0 + U' \times I = \text{identity}$ and such that $H_1(V) = U$.*

Proof. By Theorem 8, V is homeomorphic rel X' to a regular neighborhood of $X' \bmod X_0 \cap X' = X'_0$. Hence, $V \cap X'_0 = (X'_0)_R$ and V supports a stellar triangulation as required by the hypothesis of part a of the stellar neighborhood theorem [C; Theorem 6.1(a)]. The conclusion is that of that theorem.

Before showing the various resolutions that these theorems give to the recognition problem for regular neighborhoods, we present the following equivalent to homogenous collapsing:

LEMMA 10. Assume $X_0 < X < Q_0 < Q$ with $X - X_0$ finite dimensional, then $(X|X_0) \xrightarrow{h} X'$ iff for all $0 \leq i \leq \dim X - X_0$:

$$(X \cap I^i(Q, Q_0) | X_0 \cap I^i(Q, Q_0)) \xrightarrow{h} (X' \cap I^i(Q, Q_0)) \cup (X \cap I^{i-1}(Q, Q_0)).$$

Proof. Let $I^i = I^i(Q, Q_0)$. The collapsing through intrinsic skeleta clearly implies homogenous collapsing since we can go from $X \xrightarrow{h} X' \cup (X \cap I^i) \xrightarrow{h} X'$ by induction using the hypothesis for the inductive step.

To go the other way, we simply note that if we have a homogenous collapsing of X to X' , we can rearrange the collapses in descending order of intrinsic dimension, i.e. If $X^{i-1} + B^{i-1} = X^i$ and $X^{i+1} = X^i + B^{i+1}$ and $d(x; Q, Q_0)$ is less for $x \in X^{i+1} - X^i$ than for $X^i - X^{i-1}$, then $B^{i+1} \cap X^i < X^{i-1}$ so we can first remove B^{i-1} from X^{i+1} and then remove B^{i+1} .

THEOREM 11. Let $X_0 < X < Q$, with $\text{Cl}(X - X_0) = X_R$ compact, and let $V < Q$ be the closure of an open neighborhood of $X - X_0$, i.e. $V = \text{Cl Int } V$, with $X - X_0 \subset \text{Int } V$. Assume $V \cap X_0 = \text{Cl}(X - X_0) \cap X_0 = (X_0)_R$, and let $\dot{V} = (\text{Bdry } V) \cup (X_0)_R$. Then the following are equivalent:

- a. V is a regular neighborhood of $X \bmod X_0$ in Q .
- b. $(V|\dot{V}) \xrightarrow{h} (X)_R$.
- c. $(V|\dot{V}) \xrightarrow{w} (X)_R$ and $\dot{V} - X_0$ is locally collared in $\text{Cl}(Q - V) = Q - \text{Int } V$.
- d. $(V|(X_0)_R) \xrightarrow{h} (X)_R$ and $\dot{V} - X_0$ is locally bicollared in Q .
- e. $\text{Cl}(V \cap I^i(Q) - X_0)$ is a regular neighborhood of

$$(X \cap I^i(Q)) \cup (V \cap I^{i-1}(Q)) \bmod (X_0 \cap I^i(Q)) \cup (\dot{V} \cap I^{i-1}(Q))$$

in $I^i(Q)$ for all i , and

$$(\text{Bdry}_{I^i(Q)} \text{Cl}(V \cap I^i(Q) - X_0)) \cup (X_0 \cap I^i(Q)) = (\dot{V} \cup X_0) \cap I^i(Q).$$

REMARK. All of the collapsing conditions b-d have equivalent formulations including all of X_0 . The equivalence comes from excision, and the assumption that $V \cap X_0 = (X_0)_R$. The corresponding conditions are

- b'. $(V \cup X_0 | \dot{V} \cup X_0) \xrightarrow{h} X$.
- c'. $(V \cup X_0 | \dot{V} \cup X_0) \xrightarrow{w} X + \text{local collaring on the outside}$.
- d'. $(V \cup X_0 | X_0) \xrightarrow{h} X + \text{local bicollaring}$.

Proof. $a \rightarrow b$. Directly from the definition of regular neighborhoods, we have full subcomplexes L, L_0 of K , with $L - L_0$ finite and we wish to show that

$$(N(L - L_0; \eta K) | \dot{N}(L - L_0; \eta K)) \xrightarrow{h} (L)_R$$

in K . Note that by local finiteness $N(L - L_0; \eta K)$ is a finite complex.

Every simplex of $N(L - L_0; K) - L$ is of the form ABC with $A \in (L_0)_R, B \in C(L_0, L)$ and $C \in \dot{N}((L)_R; \dot{N}(L - L_0; K))$ [C; Proposition 2.9], with $B \neq 0 \neq C$. Order these simplices in descending order of dimension.

$$(N(L - L_0; \eta K), \dot{N}(L - L_0; \eta K)) \cap ABC = (N(AB - A; \eta ABC), \dot{N}(AB - A; \eta ABC))$$

from the parametrization lemma. By [C; Lemma 2.16] this is a ball and face, with complementary face

$$N(AB - A, \eta \partial ABC) = N(AB - A; \eta(B * \partial AC)) = \bigcup_i (N(L - L_0; \eta K) \cap A_i BC_i)$$

for $A_i BC_i \subseteq \partial ABC$. Thus, removing these pairs in decreasing order of dimension provides a collapse $N(L - L_0; \eta K) \searrow (L)_R$ which is clearly rel $\dot{N}(L - L_0; \eta K)$. Since the points removed in the collapse corresponding to ABC all lie in the interior of ABC , the collapse is homogenous in K .

$a \rightarrow c$ and d . Follows from $a \rightarrow b$ and [C; Theorem 5.2] which says that $\dot{V} - X_0$ is locally collared in V and in $Q - \text{Int } V$.

$a \rightarrow e$. Triangulate the situation to obtain $L_0 < L$ full subcomplexes of K and $V = N(L - L_0; \eta K)$. Recall that the intrinsic skeleta I^i are subcomplexes of K . Let $V^i = \text{Cl}(N(L - L_0; \eta K) \cap I^i - L_0)$. By [C; Lemma 2.13], $V^i = N(L \cap I^i - L_0 \cap I^i; \eta I^i)$ for all i . Hence, (V^i, V^{i-1}) is a regular neighborhood pair of $L \cap I^i \text{ mod } L_0 \cap I^i$ in (I^i, I^{i-1}) . This implies by [C; Proposition 7.10] that V^i is a regular neighborhood, in I^i , of

$$(L \cap I^i) \cup V^{i-1} \text{ mod } (\text{Bdry}_{I^{i-1}} V^{i+1}) \cup (L_0 \cap I^i).$$

But $(L \cap I^i) \cup V^{i-1} = (L \cap I^i) \cup (V \cap I^{i-1})$. And

$$\begin{aligned} (\text{Bdry}_{I^{i-1}} V^{i-1}) \cup (L_0 \cap I^i) &= \dot{N}(L \cap I^{i-1} - L_0 \cap I^{i-1}; \eta I^{i-1}) \cup (L_0 \cap I^i) \\ &= (\dot{N}(L - L_0; \eta K) \cap I^{i-1}) \cup (L_0 \cap I^i) \\ &= (\dot{V} \cap I^{i-1}) \cup (L_0 \cap I^i). \end{aligned}$$

For in each of these equalities the only points where the varying terms change lie in $L_0 \cap I^{i-1}$.

$b \rightarrow a$. V is obviously a regular neighborhood of $V \cup X_0 \text{ mod } X_0 \cup \dot{V}$ ($V \cap X_0 = (X_0)_R$ and $V = \text{Cl}(\text{Int } V)$). So a follows from Theorem 6.

$c \rightarrow a$. As above, using Theorem 8, we have (V, \dot{V}) is homeomorphic rel X to a regular neighborhood of $X \text{ mod } X_0$ and its boundary. Hence, $\dot{V} - X_0$ is locally bicollared and the stellar neighborhood theorem [C; Theorem 6.1b] implies that V is a regular neighborhood of $X \text{ mod } X_0$.

d → a. Triangulate and subdivide so that X, X_0 and V are full subcomplexes of Q . The local bicollaring assumption implies that $\dot{V} - X_0$ is locally collared in V and $Q - \text{Int } V$ ($Q - X_0 = (Q - (V \cup X_0)) \cup (\dot{V} - X_0) \cup (\text{Int } V - X_0)$). Using this we can produce a homeomorphism $f: Q \rightarrow Q \text{ rel } X$ with $fV = N(V - X_0; \eta Q)$, see for example the proof of [C; Theorem 6.1]. Theorem 6 implies that $N(V - X_0; \eta Q)$ is a regular neighborhood of $X \text{ mod } X_0$ in Q and hence so is V .

e → b. Let $V^i = \text{Cl}(V \cap I^i - X_0)$. We have assumed the identity

$$(\text{Bdry}_{I^i} V^i) \cup (X_0 \cap I^i) = (\dot{V} \cup X_0) \cap I^i.$$

Then, since $a \rightarrow b'$, the hypothesis of e gives

$$\begin{aligned} (V^i \cup (X_0 \cap I^i) \cup (\dot{V} \cap I^{i-1})) &| (\text{Bdry}_{I^i} V^i) \cup (X_0 \cap I^i) \cup (\dot{V} \cap I^{i-1}) \\ &\searrow (X \cap I^i) \cup (V \cap I^{i-1}). \end{aligned}$$

Apply the assumed identity, this becomes

$$((V \cup X_0) \cap I^i | (\dot{V} \cup X_0) \cap I^i) \searrow (X \cap I^i) \cup (V \cap I^{i-1}).$$

So Lemma 10 implies that $(V \cup X_0 | \dot{V} \cup X_0) \searrow X$.

REMARKS. The identity at the end of e is equivalent to the assumption $(V - \dot{V}) \cap I^i(Q) = \text{Int } V^i - X_0$. Without this assumption $e \nrightarrow a$. Let Q be the upper half plane, $V = [(-2, 0), (-1, 1), (0, 0)] + [(0, 0), (1, 1), (2, 0)]$ and let $X = \{(-1, 0)\}$ and $X_0 = \emptyset$, then the conditions of e except for the identity are satisfied and yet V is not a regular neighborhood of X .

There is an interesting example which shows that the local collaring conditions of c and d are necessary:

Let $\Delta_1 = [(0, 1), (0, -1), (1, 0)]$ and $\Delta_2 = [(0, 0), (1/2, 1/2), (1/2, -1/2)]$ in the plane R^2 . Let $V = \Delta_1 \times [0, 1] + \Delta_2 \times [-1, 0]$ in the half space of $R^3 = \{(x, y, z) : x \geq 0\} = Q$. Let $X = (0, 0, 1/2)$ and $X_0 = \emptyset$. One can check that $(V | \dot{V}) \searrow X$ and $V \searrow X$ (by two different collapsings), but V is not a regular neighborhood of X . In fact, $V \cap \partial Q$ is not a manifold.

We remark that the equivalence of a and d in the case $X_0 = \emptyset$ is due to Stallings, by a rather different approach, see [S; Chapter VI].

If instead of condition e, we assume that V^i is a regular neighborhood of $(X \cap I^i) \cup (V \cap I^{i-1}) = (X \cap I^i) \cup V^{i-1} \text{ mod } (X_0 \cap I^i) \cup (\text{Bdry}_{I^{i-1}} V^{i-1})$, then the identity follows by downward induction since, $x \in (\dot{V} \cap I^i) - (\text{Bdry}_{I^i} V^i \cup X_0)$ implies $x \in \text{Int } V^i$ and hence by hypothesis $x \in \text{Int } V^{i+1}$, contradicting inductive hypothesis for $i+1$. Note that identity is clear for $i = \dim V$.

In the unlikely event that the reader is interested we mention a generalization of Theorems 6 and 8.

In our definition $(X | X_0) \searrow X'$ in (Q, Q_0) we assumed $X < Q_0$. To discard this assumption we can extend the previous definitions of homogenous and weakly homogenous collapsing using the old definitions as follows: Define $(X | X_0) \searrow X'$ in (Q, Q_0) to mean $(X | X_0) \searrow X' \cup (X \cap Q_0)$ in Q and $(X \cap Q_0 | X_0 \cap Q_0)$

$\xrightarrow{h} X' \cap Q_0$ in (Q, Q_0) (old definition). Similarly, we can extend the definition of $(X|X_0) \xrightarrow{w} X'$ in (Q, Q_0) . Theorems 6 and 8 remain true as stated, deleting the assumption that $X, X_0 < Q_0$ and using the new definitions. The proofs are the same but there are four cases, in each theorem, the two of Theorem 6 for collapses in $Q - Q_0$ and the two of Theorem 6 for collapses in Q_0 . Similarly, for Theorem 8.

Finally, we note that our solution of the “recognition problem” for regular neighborhoods, Theorem 11, has the unsatisfactory aspect of requiring $Cl(X - X_0)$ to be compact. To eliminate this hypothesis, presumably one could replace collapsing conditions by the existence of some kind of “homogenously collapsible retraction”, and construct a proof à la [C; Chapter 9] by using the retraction to reduce the problem to collapsing conditions on the dual cells, which are finite, and use Theorem 11 where Cohen uses Whitehead’s shelling process. However, this would seem to require more attention than this outline suggests as the dual cells are cones, not balls in this case. Probably, a proof can be constructed using the results on cones, later in this paper, to generalize a \rightleftarrows d to this context.

IV. Unknotting cone pairs. Given an embedding of a cone into a polyhedron: $f: aX \rightarrow Q$ and V a regular neighborhood of $f(aX) \text{ mod } f(X)$ in Q . We consider the problem of when $f: aX \rightarrow V$ looks like the inclusion of a subcone in a cone, i.e. when does there exist a homeomorphism $h: V \rightarrow a\dot{V}$ such that $hf = \bar{f}$ when \bar{f} is defined as the coning of $f|X$, i.e. $a(f|X): aX \rightarrow a\dot{V}$. This is a much stronger condition than the existence of a homeomorphism of pairs: $(V, f(aX)) \cong (a\dot{V}, f(X))$, for this homeomorphism preceded by f need not equal \bar{f} , but will, in general, be \bar{f} preceded by a homeomorphism $k: aX \rightarrow aX \text{ rel } X$. So this problem has two aspects: Unknotting the pair $(V, f(aX))$ and unknotting the homeomorphism k . The solution of the second is by far the more important and more difficult. In fact, it is an easy exercise to show using Theorem III.8, that $(V, f(aX)) \cong (a\dot{V}, f(X))$ iff there exists a point $p \in f(aX) - f(X)$ such that $(f(aX)|f(X)) \xrightarrow{h} p$ in $(Q, f(aX))$.

The importance of unknotting the embeddings of cones (rather than the set pairs) and the above version of unknotting is due to Lickorish [L] and this whole section is just a generalization of his Proposition 1, and in broad design mimics its proof.

DEFINITION 1. Let $f: aX \rightarrow Q$ be an embedding. Call f conewise homogenous, denoted CH, if for each $x \in X, d(f(ta + (1-t)x); Q, f(aX))$ is constant for $0 < t < 1$.

LEMMA 2. Let $a(X, X_0)$ be a cone pair, then the inclusion $aX_0 \subset aX$ is conewise homogenous.

Proof. If $x \in X_0$, then for any $0 < t < 1$:

$$Lk(ta + (1-t)x; a(X, X_0)) \cong \Sigma Lk(x; X, X_0),$$

which is independent of t .

REMARKS. It follows that conewise homogeneity is surely a necessary condition for unknotting an embedding as described above. The remainder of this section is devoted to showing that it is sufficient.

Note that for each $x \in X$, $d(f(ta + (1-t)x); f(aX)) = d(ta + (1-t)x; aX) = 1 + d(x; X)$ for $0 < t < 1$ and hence if $(Q, f(aX))$ is locally unknotted at every point of $f(aX) - f(X)$, then f is conewise homogenous.

Since every triangulation of a cone has a conical subdivision [Z; Lemma 2.1], we can triangulate and subdivide an embedding $f: aX \rightarrow Q$ to obtain the simplicial situation:

$f: \gamma aK \rightarrow J$ a simplicial embedding with γaK a conical subdivision of aK and $\gamma K = K$. We will call such an embedding $f: \gamma aK \rightarrow J$ a conical simplicial embedding.

We will need some properties of and notation about conical subdivisions which we now describe:

On a cone aX , there is a canonical projection $\pi: aX - a \rightarrow X$ given by

$$\pi(ta + (1-t)x) = x$$

for $t \neq 1$. This map is a "standard mistake" and is not p.l.; however, to say that γaK is a conical subdivision of aK is precisely to say that if A is a simplex of $\gamma aK - a$, then $\pi(A - a)$ is a simplex of γK .

Let γaK be a conical subdivision with $\gamma K = K$. Then for $A \in \gamma aK - a$ we will denote by πA the simplex $\pi(A - a)$ of K . For $A \in \gamma aK - a$, either $\dim A = \dim \pi A$, in which case we call A horizontal, or $\dim A = \dim \pi A + 1$, in which case we call A vertical. If $B \in K$, then $\gamma aB - \gamma a\dot{B}$ consists of precisely those simplices A such that $\pi A = B$, and the simplices of $\text{Int } \gamma aB$ are those A such that $\pi A = B$ and $A \neq B$.

If A is a horizontal simplex of $\gamma aB - \gamma a\dot{B}$, then the component of $aB - A$ which contains a , has precisely one vertical simplex with A as a face. We will call this simplex, the simplex above A . If A is a horizontal simplex of $\text{Int } \gamma aB$, then the component of $aB - A$ which intersects K has precisely one vertical simplex with A as a face. We will call this the simplex below A . Thus, if $A \in \gamma aK - a$ the simplex above A lies in $\text{Int } \gamma a(\pi A)$ and if $A \notin K$ then the simplex below A is also defined and lies in $\text{Int } \gamma a(\pi A)$.

LEMMA 3. *Let $f: \gamma aK \rightarrow J$ be a conical simplicial embedding. Then the following are equivalent:*

- a. f is conewise homogenous.
- b. If $B \in K$, then $d(f(x); J, f(aK))$ is constant for x varying in $\text{Int } aB$.
- c. If $A \in \gamma aK - (a \cup K)$ and B is the simplex above or below A , then fA is a nice face of fB in $(J, f(\gamma aK))$.

Proof. Clearly, $c \rightarrow b \rightarrow a$. We show that $a \rightarrow c$. If $x \in \text{Int } \pi A = \pi B$, then the open cone line $\{ta + (1-t)x; 0 < t < 1\}$ intersects $\text{Int } A$ and $\text{Int } B$. Since f is CH, $d(f(y); J, f(aK))$ is constant for y along this line, so fA is a nice face of fB .

THEOREM 4. *Let $f: aX \rightarrow Q$ be a conewise homogenous embedding. Let V be a regular neighborhood of $f(aX) \bmod f(X)$ in Q . Then there exists a homeomorphism $h: V \rightarrow a\dot{V} \text{ rel } \dot{V}$ such that $hf = \bar{f}$, where \bar{f} is defined to be $a * (f|X)$.*

Proof. We will inductively assume the theorem for $\dim X < n$ and prove it for $\dim X = n$.

Triangulate to obtain $f: \gamma aK \rightarrow J$ a conical simplicial embedding. There is a canonical simplicial collapse $(\gamma aK|K) \searrow a$, obtained by ordering the simplices of K in descending order of dimension $\{D_j\}$ and for each D_j , in order, collapse $\gamma aD \searrow \gamma a\dot{D}$ by removing in pairs a horizontal simplex and the vertical simplex above it. This is clearly relative to K . The image of this collapse under f is a collapse: $f(\gamma aK) = L^k, L^{k-1}, \dots, L^0 = f(a)$, with $L^{i+1} = L^i + A^i + B^i$ with $B^i = b^i A^i$ and $B^i \cap L^i = b^i A^i$ and $f^{-1}A^i$ horizontal in γaK and $f^{-1}B^i$ the vertical simplex above it.

First we assume the following:

ASSERTION. Let $(V^i, V_0^i, \{V_j^i\})$ be a regular neighborhood of $L^i \bmod L^i \cap f(K)$ in $(J, f(aK), \{f(aD_j)\})$. Then there exists a homeomorphism $h_i: (V^i, V_0^i, \{V_j^i\}) \rightarrow (V^0, V_0^0, \{V_j^0\})$, taking boundary to boundary.

All the work and in particular, the inductive application of Theorem 4, is contained in the proof of the assertion.

From the assertion, there exists

$$h_k: (V, f(aK), \{f(aD_j)\}) \rightarrow \text{St}(a; (\eta J, \eta f(\gamma aK), \{\eta f(\gamma aD_j)\}))$$

taking $(\dot{V}, f(K), \{f(D_j)\})$ to the corresponding link family. Thus we can define:

$$g = (a * (h_k| \dot{V})^{-1})h_k: (V, f(aK), \{f(aD_j)\}) \rightarrow a * (\dot{V}, f(K), \{f(D_j)\})$$

rel \dot{V} . Now $gf\bar{f}^{-1}$ is a homeomorphism of $a * f(K)$ with itself rel $f(K)$ and mapping $a * f(D_j)$ to itself for all j . Since \dot{V} can be triangulated with $f(K)$ as a subcomplex (e.g. map \dot{V} to $\dot{N}(f(aK) - f(K); \eta_f J)$ rel $f(K)$ where $\eta_f J$ is a derived of $J \bmod f(\gamma aK)$), $a\dot{V}$ can be triangulated with $af(K)$ as a subcomplex. The homeomorphism of $af(K) \cup \dot{V}$ given by $gf\bar{f}^{-1}$ on $af(K)$ and the identity on \dot{V} maps each simplex of such a triangulation to itself and so extends to a homeomorphism of $a\dot{V}$ by induction up the dimension of the remaining simplices. Call this extension $\bar{g}: a(\dot{V}, f(K)) \rightarrow a(\dot{V}, f(K)) \text{ rel } \dot{V}$. Let

$$h = \bar{g}^{-1}g: V \rightarrow a\dot{V}$$

rel \dot{V} , then $hf = \bar{f}$.

This reduces the proof of the theorem to a proof of the above assertion. Clearly, it suffices to prove the following:

ASSERTION'. There exists a homeomorphism k_i , for each i ,

$$k_i: (V^{i+1}, V_0^{i+1}, \{V_j^{i+1}\}) \rightarrow (V^i, V_0^i, \{V_j^i\})$$

and such that

$$k_i(\dot{V}^{i+1}, \dot{V}_0^{i+1}, \{\dot{V}_j^{i+1}\}) = (\dot{V}^i, \dot{V}_0^i, \{\dot{V}_j^i\}).$$

This assertion is seen to be just a “family” version of Theorem III.8. Thus, the proof will completely parallel that one, but with the difficulty, as will be seen, appearing at an unexpected point.

By the uniqueness theorem for regular neighborhoods, [C; Addendum 3.3] it suffices to construct such a k_i , for any choice of V^{i+1} and any choice of V^i .

Now $L^{i+1} = L^i + A + B$ with $B = bA$, $L^i \cap B = bA$ and $f^{-1}B$ the vertical simplex above the horizontal simplex $f^{-1}A$. As in Theorem III.8, there are two cases: $A \in fK$ or $A \notin fK$.

Case (1) $A \in fK$. This corresponds to case (2) of the proof of Theorem III.8. There it is directly proved that there exist deriveds $\eta\sigma_A J$ and $\bar{\eta}\sigma_A J$ agreeing outside $St^\circ(B, J)$ and a homeomorphism:

$$k: N(L^{i+1} \cup f(K) - f(K); \eta\sigma_A(J, f(\gamma aK))) \rightarrow N(L^i \cup f(K) - f(K); \bar{\eta}\sigma_A(J, f(\gamma aK)))$$

taking boundary pairs to boundary pairs. The naturality of the construction is such that if $D_j \in St^\circ(f^{-1}A; K)$ then

$$k(N(L^{i+1} \cup fK - fK; \eta\sigma_A f(\gamma aD_j))) = N(L^i \cup fK - fK; \bar{\eta}\sigma_A f(\gamma aD_j))$$

and similarly for the boundaries. For in $St^\circ(B; f(\gamma aD_j))$ where the two neighborhoods differ, both are parametrized by the same $p: St(B) \rightarrow \Delta^3$ as before.

Finally, if $D_j \in K - St^\circ(f^{-1}A; K)$, then η and $\bar{\eta}$ are the same on $f(\gamma aD_j)$, $L^{i+1} \cap f(aD_j) = L^i \cap f(aD_j)$ and k on its domain intersect $f(aD_j)$ is the identity.

Thus, if we let $(V^{i+1}, V_0^{i+1}, \{V_j^{i+1}\}) = N(L^{i+1} \cup fK - fK; \eta\sigma_A(J, f(\gamma aK), \{f(\gamma aD_j)\}))$ and $(V^i, V_0^i, \{V_j^i\}) = N(L^i \cup fK - fK; \bar{\eta}\sigma_A(J, f(\gamma aK), \{f(\gamma aD_j)\}))$, then the k constructed as above following the proof of Theorem III.8 is the required k_i .

Case (2) $A \notin fK$. This corresponds to case (1) of Theorem III.8, or more precisely to case (1) of the proof of Theorem III.6. There it is shown directly, that there exist deriveds $\eta\sigma_A J$ and $\bar{\eta}\alpha J$ (where $\alpha J = \sigma_{ba}\sigma_A J$) agreeing outside of $St^\circ(B; J)$ and such that:

$$|N(L^i - L^i \cap fK; \eta\sigma_A(J, f(\gamma aK)))| = |N((L^i + xbA) - L^i \cap fK; \bar{\eta}\alpha(J, f(\gamma aK)))|.$$

As before, naturality of the construction, will imply that for $D_j \in St^\circ(\pi f^{-1}A; K)$,

$$|N(L^i - L^i \cap fK; \eta\sigma_A f(\gamma aD_j))| = |N((L^i + xbA) - L^i \cap fK; \bar{\eta}\alpha f(\gamma aD_j))|.$$

For $D_j \in K - St^\circ(\pi f^{-1}A; K)$ this is clear because $L^i \cap f(aD_j) = (L^i + xbA) \cap f(aD_j)$ and the two subdivisions of $f(\gamma aD_j)$ are the same, as $f(\gamma aD_j) \cap St^\circ(B, J) = \emptyset$ in this case.

Thus, if we let $(V^i, V_0^i, \{V_j^i\}) = N(L^i - L^i \cap fK; \eta\sigma_A(J, f(\gamma aK), \{f(\gamma aD_j)\}))$.

Then, this is also a regular neighborhood of $L^i + xbA \text{ mod } L^i \cap fK$ in $(J, f(aK), \{f(aD_j)\})$.

From this stage to the end of the proof of Theorem III.6 is a simple little application of Corollary II.15 to push xbA out to $B \text{ rel } L^i$. This is, of course, the rub because if we just apply Corollary II.15 as before the resulting homeomorphism

may mess up the cone structure i.e. it may not map the $f(aD_j)$ into themselves. What we need is the following strengthening of Corollary II.15 to this situation:

MAP. There exists a homeomorphism

$$m: (J, f(aK), \{f(aD_j)\}) \rightarrow (J, f(aK), \{f(aD_j)\})$$

such that $m|_{L^i} = \text{identity}$ and $m(B) = x b A$.

Assuming the existence of m , define $(V^{i+1}, V_0^{i+1}, \{V_j^{i+1}\}) = m^{-1}(V^i, V_0^i, \{V_j^i\})$ and let $k^i = m|_{V^{i+1}}$. Since k^i is the restriction of a homeomorphism of the ambient spaces, it takes boundary into boundary.

Instead of proving Map, we prove the following sharpening of Lemma II.14b from which Map follows using the proof of Corollary II.15:

MAP'. Let $\eta_B J$ be a first derived mod B , then there exists a homeomorphism

$$\phi: \text{Lk}(A; \eta_B(J, f(\gamma a K))) \rightarrow (b+x) \text{Lk}(B; \eta_B(J, f(\gamma a K)))$$

such that

1. $\phi|_b * \text{Lk}(B; \eta_B J) = \text{identity}$.
2. For $D \in \text{St}^\circ(\pi f^{-1}A; K)$

$$\phi(\text{Lk}(A; \eta_B f(\gamma a D))) = (b+x) \text{Lk}(B; \eta_B f(\gamma a D)).$$

Condition 2 is precisely what is required to insure that if m is defined using ϕ as in the proof of Corollary II.15, $m(f(aD_j)) = f(aD_j)$ for all $D_j \in K$. We only have to worry about $D \in \text{St}^\circ(\pi f^{-1}A; K)$ because m will be the identity on the other $f(aD_j)$'s.

We prove Map' in a sequence of claims:

CLAIM A. There exists a homeomorphism

$$\phi_0: \text{Lk}(A; \eta_B(J, f(\gamma a K))) \rightarrow (b+x) \text{Lk}(B; \eta_B(J, f(\gamma a K)))$$

with $\phi_0|_b * \text{Lk}(B; \eta_B J) = \text{identity}$.

Proof. By the assumption of conewise homogeneity, Lemma 3 implies that A is a nice face of B in $(J, f(\gamma a K))$. So we can apply Lemma II.14b.

NOTATION. For convenience, we will use A, B and b to denote $f^{-1}A, f^{-1}B$ and $f^{-1}b$, also, and allow context to determine whether it lies in the domain or range. Also, we let $\eta_B \gamma a K$ be the derived mod B (i.e. $f^{-1}B$) of $\gamma a K$, such that $f: \eta_B \gamma a K \rightarrow \eta_B f(\gamma a K)$ is simplicial.

CLAIM B. There exists a homeomorphism

$$\check{\phi}: \text{Lk}(A; \eta_B \gamma a K) \rightarrow (b+x) \text{Lk}(B; \eta_B \gamma a K)$$

such that:

1. $\check{\phi}|_b * \text{Lk}(B; \eta_B \gamma a K) = \text{identity}$.
2. $\check{\phi}(\text{Lk}(A; \eta_B \gamma a D)) = (b+x) \text{Lk}(B; \eta_B \gamma a D)$ for $D \in \text{St}^\circ(\pi A; K)$.

Proof. Let $C = \pi A$. Then since $A \in \text{Int } aC$ and $\dim A = \dim C = \dim aC - 1$, $\text{Lk}(A; \eta_B \gamma aC)$ is two points. One of them is b and call the other x . Then define $\bar{\phi}_{-1}(b+x) = b+x$ and this defines

$$\bar{\phi}_{-1}: \text{Lk}(A; \eta_B \gamma aC) \rightarrow (b+x) \text{Lk}(B; \eta_B \gamma aC).$$

Let $S^i = C * \text{Lk}(C; K)^{(i)}$ and $S^{-1} = C$. We define inductively

$$\bar{\phi}_i: \text{Lk}(A; \eta_B \gamma aS^i) \rightarrow (b+x) \text{Lk}(B; \eta_B \gamma aS^i)$$

with $\bar{\phi}_i$ the identity on $b * \text{Lk}(B; \eta_B \gamma aS^i)$ and satisfying 2 for any $D \in \text{St}^\circ(C; S^i)$, i.e. for any D of dimension $\leq \dim C + i + 1$.

As we have defined $\bar{\phi}_{-1}$ above, we will assume that $\bar{\phi}_{i-1}$ is defined and construct $\bar{\phi}_i$ to extend it.

Let $D \in S^i - S^{i-1}$, i.e. $D = CE$ with $\dim E = i$. We want to define

$$\bar{\phi}_i: \text{Lk}(A; \eta_B \gamma aD) \rightarrow (b+x) \text{Lk}(B; \eta_B \gamma aD)$$

to extend the identity on $b \text{Lk}(B; \eta_B \gamma aD)$ and $\bar{\phi}_{i-1}$ on $\text{Lk}(A; \eta_B \gamma a * C\dot{E})$. Doing this for each $E \in \text{Lk}(C, K)^{(i)} - \text{Lk}(C, K)^{(i-1)}$ defines the required $\bar{\phi}_i$.

Let $j = \dim A = \dim C$, then $\eta_B \gamma aC * (E, \dot{E})$ is an $i+j+2$ ball-and-face pair. A and B are interior j and $j+1$ simplices of $\eta_B \gamma aC\dot{E}$. So $\text{Lk}(A; \eta_B \gamma aC * (E, \dot{E}))$ and $(b+x) \text{Lk}(B; \eta_B \gamma aC(E, \dot{E}))$ are $i+1$ ball-and-boundary pairs.

On the other hand, $b \text{Lk}(B; \eta_B \gamma aCE)$ is an $i+1$ ball contained in each of these and meeting each boundary in $b \text{Lk}(B; \eta_B \gamma aC\dot{E})$, a face.

Hence, by the Alexander-Newman theorem, we can define $\bar{\phi}_i$ on $\text{Lk}(A; \eta_B \gamma aCE)$ to extend $\bar{\phi}_{i-1} + \text{identity on } \text{Lk}(A; \eta_B \gamma aC\dot{E}) + b \text{Lk}(B; \eta_B \gamma aCE)$.

This defines $\bar{\phi}_i$ by induction and we define $\bar{\phi} = \bar{\phi}_i$ for $i = \dim \text{Lk}(C; K)$.

CLAIM C. $\phi_0 f \bar{\phi}^{-1} | x \text{Lk}(B; \eta_B \gamma aK)$ is CH, where

$$\phi_0 f \bar{\phi}^{-1}: x \text{Lk}(B; \eta_B \gamma aK) \rightarrow (b+x) \text{Lk}(B; \eta_B J).$$

Proof. Since the definition of conewise homogeneity is an invariant under homeomorphism of the range, it suffices to show that

$$f \bar{\phi}^{-1}: x \text{Lk}(B; \eta_B \gamma aK) \rightarrow \text{Lk}(A; \eta_B J)$$

is CH. If $y \in x \text{Lk}(B; \eta_B \gamma aK)$ then

$$\begin{aligned} d(f \bar{\phi}^{-1}(y); \text{Lk}(A; \eta_B J), f \bar{\phi}^{-1} x \text{Lk}(B; \eta_B \gamma aK)) &= d(f \bar{\phi}^{-1}(y); \text{Lk}(A; \eta_B(J; \gamma aK))) \\ &= d(f \bar{\phi}^{-1}(y); J, f(aK)) - \dim A - 1 \end{aligned}$$

with the last equality from Lemma II.19.

Thus, it suffices to show that $d(f \bar{\phi}^{-1}(y); J, f(aK))$ is constant as y varies along a cone-line: $\{tx + (1-t)y_1 : 0 < t < 1\}$ with $y_1 \in \text{Lk}(B; \eta_B \gamma aK)$.

There is a unique $D \in \text{St}^\circ(C; K)$ with $y_1 \in \text{Int } aD$. Let $D = CE$ and $y = tx + (1-t)y_1$ with $0 < t < 1$. Then by condition 2 on $\bar{\phi}$, $\bar{\phi}^{-1}(y) \in \text{Lk}(A; \eta_B \gamma aCE) - \text{Lk}(A; \eta_B \gamma aC\dot{E})$. From this it follows that $\bar{\phi}^{-1}(y) \in aD - a\dot{D} = aCE - aC\dot{E} - a\dot{C}E$,

and since A is an interior simplex, $\text{Lk}(A; \eta_B \gamma aK) \cap K$ contains at most b and so $\phi^{-1}(y) \in \text{Int } aD$.

Thus, for all $0 < t < 1$, $\phi^{-1}(tx + (1-t)y_1) \in \text{Int } aD$, so the result follows from Lemma 3.

MAP'. The required ϕ exists.

Proof. $\phi_0 f \phi^{-1}$ on $b * \text{Lk}(B; \eta_B \gamma aK)$ is just $b * f| \text{Lk}(B)$ and hence has the image $b * \text{Lk}(B; \eta_B f(\gamma aK))$. So $\phi_0 f \phi^{-1}$ maps the pair $(x * \text{Lk}(B; \eta_B \gamma aK), \text{Lk}(B; \eta_B \gamma aK))$ homeomorphically onto $(x * \text{Lk}(B; \eta_B f \gamma aK), \text{Lk}(B; \eta_B f \gamma aK))$, and by C , it is CH regarded as an embedding into $(b+x) \text{Lk}(B; \eta_B J)$.

Clearly, $x * \text{Lk}(B; \eta_B J)$ is a regular neighborhood of

$$x \text{Lk}(B; \eta_B f \gamma aK) \text{ mod } \text{Lk}(B; \eta_B f \gamma aK)$$

in $(b+x) \text{Lk}(B; \eta_B J)$, with boundary $\text{Lk}(B; \eta_B J)$.

By an inductive application of Theorem 4, there exists a homeomorphism

$$h: x * \text{Lk}(B; \eta_B J) \rightarrow x * \text{Lk}(B; \eta_B J)$$

rel $\text{Lk}(B; \eta_B J)$ such that, $h \phi_0 f \phi^{-1} = x * (f| \text{Lk}(B; \eta_B \gamma aK))$.

Extend h over $b * \text{Lk}(B; \eta_B J)$ by the identity and call the extension h_1 .

Define $\phi = h_1 \phi_0$.

Clearly, $\phi| b * \text{Lk}(B; \eta_B J)$ is the identity

$$h(x * \text{Lk}(B; \eta_B f \gamma aK)) = x * \text{Lk}(B; \eta_B f \gamma aK)$$

and so $\phi(\text{Lk}(A; \eta_B f \gamma aK)) = (b+x) \text{Lk}(B; \eta_B f \gamma aK)$.

Finally, $\phi \cdot (f| \text{Lk}(A)) = ((b+x) * (f| \text{Lk}(B))) \cdot \phi$ and hence for $D \in \text{St}^\circ(C; K)$:

$$\begin{aligned} \phi(\text{Lk}(A; \eta_B f \gamma aD)) &= \phi f(\text{Lk}(A; \eta_B \gamma aD)) = ((b+x) * f)(b+x) * \text{Lk}(B; \eta_B \gamma aD) \\ &= (b+x) * \text{Lk}(B; \eta_B f \gamma aD). \end{aligned}$$

Thus, ϕ is the required homeomorphism.

This completes the proof of Map' and hence the theorem.

From Theorem 4, we obtain several corollaries about unknotting cone pairs.

Call an embedding $f: aX \rightarrow bY$ a proper embedding if $f^{-1}(Y) = X$.

DEFINITION 5. We call a proper embedding $f: aX \rightarrow bY$ unknotted if there exists a homeomorphism $h: bY \rightarrow bY$ rel Y such that $hf = \bar{f} = \text{cone } f|X$.

COROLLARY 6. Let $f: aX \rightarrow bY$ be a proper embedding, then f is unknotted iff

1. f is conewise homogenous, and
2. $(bY|Y) \xrightarrow{w} f(aX)$ in bY .

Proof. Lemma 2 implies that 1 is necessary and if f is unknotted then the pair $(bY, f(aX)) \cong b * (Y, f(X))$ and the conewise collapse of bY to $bf(X)$ is rel Y and weakly homogenous in bY and induces under the homeomorphism a similar collapse to $f(aX)$.

Since bY is a regular neighborhood of $bY \text{ mod } Y$ in bY , Theorem III.8 implies that there exists a homeomorphism

$$k: (bY, Y) \rightarrow (V, \dot{V})$$

rel $f(aX)$, where V is a regular neighborhood of $f(aX) \text{ mod } f(X)$ in bY .

By Theorem 4, there is a homeomorphism $h_1: V \rightarrow b\dot{V}$ rel \dot{V} such that $h_1 f = \bar{f}$.

Let $h = (b * (k|Y)^{-1})h_1 k$.

COROLLARY 7. *Let $f: aX \rightarrow bY$ be a proper embedding with $(bY, f(aX))$ locally unknotted at every point of $f(aX) - f(X)$. Then f is unknotted iff $(bY|Y) \searrow_w f(aX)$ in bY .*

Proof. Since the local unknottedness condition implies f is CH, this is clear from Corollary 6.

COROLLARY 8. *Let $a(X, X_0)$ be a cone pair and $h: aX_0 \rightarrow aX$ a homeomorphism rel X_0 . Then h can be covered by a homeomorphism $H: aX \rightarrow aX$ rel X , iff regarded as an embedding of $aX_0 \rightarrow aX$, h is conewise homogenous.*

Proof. \bar{h} = inclusion $aX_0 \subset aX$ so $H: aX \rightarrow aX$ rel X extends h iff H^{-1} unknots the embedding $h: aX_0 \rightarrow aX$. Since $(aX|X) \searrow_w aX_0$ in aX , h unknots iff it is conewise homogenous by Corollary 6.

REMARKS. If $f_0 = f_1: aX \rightarrow bY$ are proper unknotted embeddings with $f_0|X = f_1|X$ then f_0 is ambient isotopic to f_1 rel Y . For if $h_0, h_1: bY \rightarrow bY$ rel Y unknot the two embeddings, then $h_0 h_1^{-1}$ is ambient isotopic to the identity rel Y .

Note that if $f: cX_0 \cong cX_0$ rel $X_0 \cup c(X_0 \cap Y)$, where $X_0, Y < X$, is conewise homogenous, regarded as an embedding of cX_0 into cX , then so is $f \cup \text{identity}$: $c(X_0 \cup Y) = cX_0 \cup cY \rightarrow cX_0 \cup cY$, which gives a relative form of Corollary 8.

V. Codimension three unknotting criteria. In this section we apply the sunny collapsing lemma of Stallings-Zeeman-Lickorish-Hudson to obtain much stronger theorems on unknotting cones in what may be called the codimension ≥ 3 situation.

DEFINITION 1. A pair (X, X_0) is called a codimension ≥ 3 pair if it satisfies the condition, for all i

$$\dim X_0 \cap I^i(X) \leq i - 3.$$

This notion is singled out because we will repeatedly use the fact that it is inductive, i.e.

LEMMA 2. *Let (K, K_0) be a codimension ≥ 3 pair of complexes and $A \in K_0$. Then the pair $\text{Lk}(A; K, K_0)$ is a codimension ≥ 3 pair.*

Proof. Let $\dim A = j$. By Corollary II.18

$$\begin{aligned} \text{Lk}(A; K_0) \cap I^i(\text{Lk}(A; K)) &= \text{Lk}(A; K_0) \cap \text{Lk}(A; I^{i+j+1}(K)) \\ &= \text{Lk}(A; K_0 \cap I^{i+j+1}(K)). \\ \dim K_0 \cap I^{i+j+1}(K) &\leq (i+j+1) - 3. \end{aligned}$$

So $\dim \text{Lk}(A; K_0 \cap I^{i+j+1}(K)) \leq (i+j+1-3) - (j+1) = i-3$.

We will also need the following technical remark, which we will dignify as a lemma because we use it repeatedly:

LEMMA 3. *Consider a pair (X, X_0) and an embedding $g: (v_0 + v_1)X_0 \rightarrow (v_0 + v_1)X$ such that*

1. $g|_{v_0 * X_0}$ is the inclusion.
2. $g|_{v_1 * X_0}$ maps $v_1 * X_0$ into $v_1 * X$ and is a proper unknotted embedding of cones.

*Then there exists $G: (v_0 + v_1)X \rightarrow (v_0 + v_1)X$ rel $v_0 * X$ such that $Gg = \text{inclusion}$.*

Proof. If $H: v_1 * X \rightarrow v_1 * X$ unknots $g|_{v_1 * X_0}$ then let $G = H$ on $v_1 * X$ and identify on $v_0 * X$.

Now we state the main theorem of this section:

THEOREM 4. *Let (X, X_0) be a compact codimension ≥ 3 pair and $f: aX_0 \rightarrow aX$ a proper embedding with $f|_{X_0} = \text{inclusion}$. Assume that for all i , $f^{-1}(aI^i(X)) = a * (X_0 \cap I^i(X))$ and $f(a) \in a * S(X)$. Then the embedding is unknotted.*

REMARKS. The fact that $X_0 < X$ is a merely formal restriction for convenience. To apply this theorem to a proper embedding $f: aX_0 \rightarrow aX$, just identify X_0 with fX_0 by f and f with $f \cdot a * (f|_{X_0})^{-1}$.

The assumption $f(a) \in a * S(X)$ can be included in the first assumption if we recall our convention that $I^{-1}(X) = \{0\}$ if X is not a suspension and assume that $a * (X_0 \cap I^i(X)) = a$ if X_0 and $I^i(X)$ are disjoint.

We begin by examining the condition put on the embedding:

LEMMA 5. *Let $f: aX_0 \rightarrow aX$ be a proper embedding. Then the following are equivalent:*

- a. $f^{-1}(a * I^i(X)) = a * (f|_{X_0})^{-1}(I^i(X))$ and $f(a) \in a * S(X)$.
- b. For all $x \in aX_0 - X_0$, $d(\bar{f}(x); aX) = d(f(x); aX)$.
- c. For all $x \in aX_0 - X_0$, $\bar{f}(x) \sim f(x)$ in aX .
- d. $d(a; aX) = d(f(a); aX)$ and if $x = ta + (1-t)x_1$, with $0 < t < 1$ and $x_1 \in X_0$, then $d(f(x); aX) = d(f(x_1); X) + 1$.

Note that if $f|_{X_0}$ is an inclusion, then \bar{f} is an inclusion.

Proof. a \rightarrow c. If $x = a$, this is clear. If $x \in aX_0 - (X_0 \cup a)$ then $x = ta + (1-t)x_1$ for $x_1 \in X_0$ and $0 < t < 1$. Let \bar{xx}_1 be the path along this cone line from x to x_1 . If $x \in a * (f|_{X_0})^{-1}(I^i(X) - I^{i-1}(X))$, then $f(\bar{xx}_1) + \bar{f}(\bar{xx}_1^{-1})$ (addition and inverse of paths) is a path joining $f(x)$ to $\bar{f}(x)$ in $aI^i(X) - aI^{i-1}(X)$ which is an i manifold with boundary $I^i - I^{i-1}$. The path meets the boundary at one point $f(x_1)$ and so can be pushed off the boundary rel the end points by local collaring.

c \rightarrow b. Clear.

b \rightarrow a. $\bar{f}^{-1}(a * I^i) = a * (f|_{X_0})^{-1}(I^i)$ and b implies the left side equals $f^{-1}(a * I^i)$. Explicitly at the vertex we have $d(f(a); X) = d(a; X) = \dim S(X) + 1$. Points of

$aX - (aS(X) \cup X)$ have higher intrinsic dimension and so since $f(a) \notin X$ (f is a proper embedding) $f(a) \in aS(X)$.

$$b \rightleftharpoons d. \quad d(\bar{f}(x); aX) = d(f(x_1); X) + 1.$$

Now we begin the proof of Theorem 4. There are essentially two parts to the proof. There is a collapsing statement proved using the sunny collapsing lemma, and then an inductive argument.

LEMMA 6. *Let (X, X_0) and $f: aX_0 \rightarrow aX$ satisfy the hypothesis of Theorem 4 and in addition, assume that $(X, X_0) \subset (aX, f(aX_0))$ is locally collared. Then $(aX|X) \xrightarrow{w} f(aX_0)$ in aX .*

Proof. The local collaring assumption implies that there is a homeomorphism

$$h: (aX, f(aX_0)) \rightarrow (aX, f(aX_0)) \cup_{(X, X_0)} (X \times I, X_0 \times I)$$

such that $h(x) = (x, 1)$ for $x \in X$ (see Theorem IV.2).

So it suffices to show that

$$(aX \cup X \times I | X \times I) \xrightarrow{w} f(aX_0) \cup X_0 \times I \quad \text{in } aX \cup X \times I.$$

We begin by the obvious cylindrical collapse:

$$(aX \cup X \times I | X \times I) \xrightarrow{w} aX \cup X_0 \times I \quad \text{in } aX \cup X \times I$$

given by $(X \times I | X \times I) \searrow X \times 0 \cup X_0 \times I$ in $X \times I$.

So now, by excision, it suffices to show that $aX \xrightarrow{h} f(aX_0)$ in $aX \cup X \times I$. So we will prove

$$aI^i(X) \searrow f(a * (X_0 \cap I^i(X))) \cup aI^{i-1}(X).$$

Since $aI^i(X) = aX \cap I^{i+1}(aX \cup X \times I)$ and $f(aX_0) \cap aI^i(X) = f(a(X_0 \cap I^i(X)))$ by hypothesis, Lemma III.10 will then imply that $aX \xrightarrow{h} f(aX_0)$ in $aX \cup X \times I$.

To prove $aI^i \searrow f(a(X_0 \cap I^i)) \cup aI^{i-1}$, we will require two sublemmas:

SUBLEMMA A (SUNNY COLLAPSING). *Let $X_0 < X < M$ with M a compact manifold. Assume $M \searrow X_0$, $X \searrow X_0$ and $\dim M - \dim X - X_0 \geq 3$. Then $M \searrow X$.*

Proof. See [H; Theorem 5.2].

SUBLEMMA B (MARSHALL COHEN). *Let $X \searrow X_0$ and V_0 be a regular neighborhood of X_0 in X . Then $X \searrow V_0$ and $X - \text{Int } V_0 \searrow \dot{V}_0$.*

Proof. By uniqueness of regular neighborhoods it suffices to prove it for any V_0 . So triangulate with K_0 full to obtain $K \searrow^s K_0$ and let $V_0 = N(K_0; \eta K)$. It suffices to prove $C(K_0; \eta K) \searrow \dot{N}(K_0; \eta K)$.

If the simplicial collapse $K \searrow^s K_0$ is given by removing pairs of simplices: $(A_n, B_n), \dots, (A_0, B_0)$ with $A_i < B_i$, then a geometrical collapse $C(K_0; \eta K) \searrow \dot{N}(K_0; \eta K)$ is given by removing ball pairs: $(\bar{A}_n, \bar{B}_n), \dots, (\bar{A}_0, \bar{B}_0)$ defined by $\bar{A}_i = C(A_i \cap K_0; \eta A_i)$ and $\bar{B}_i = C(B_i \cap K_0; \eta B_i)$.

To prove $aI^i \searrow f(a(X_0 \cap I^i)) \cup aI^{i-1}$: By collapsing conewise $aI^i \searrow aI^{i-1}$, or in the lowest dimensional case $aS(X) \searrow f(a)$.

Also, by hypothesis $f(a(X_0 \cap I^i)) \cap aI^{i-1} = f(a(X_0 \cap I^{i-1}))$. Hence, by excision, $f(a(X_0 \cap I^i)) \cup aI^{i-1} \searrow aI^{i-1}$, or in the lowest dimensional case, $f(a(X_0 \cap S(X))) \searrow f(a)$. Let $Q = aI^i$ or $aS(X)$, $F = f(a(X_0 \cap I^i)) \cup aI^{i-1}$ or $f(a(X_0 \cap S(X)))$ and $F_0 = aI^{i-1}$, or $f(a)$. Then $Q \searrow F_0$ and $F \searrow F_0$ and we wish to show that $Q \searrow F$. Note that (Q, F_0) is a relative manifold and the codimension ≥ 3 assumption states precisely that $F - F_0$ has codimension at least 3 in $Q - F_0$.

Let (V, V_0) be a regular neighborhood of F_0 in (Q, F) . Let $\bar{Q} = Q - \text{Int } V$ and $\bar{F} = (F - \text{Int } V_0) \cup \dot{V}$ and $\bar{F}_0 = \dot{V}$. By Sublemma B, $\bar{Q} \searrow \bar{F}_0$ and by Sublemma B and excision $\bar{F} \searrow \bar{F}_0$. Now \bar{Q} is a compact manifold and $\bar{F} - \bar{F}_0 \subset F - F_0$ has codimension at least three so sunny collapsing implies that $\bar{Q} \searrow \bar{F}$. So by excision $Q \searrow F \cup V$.

Now by [C; Proposition 7.10], V is a regular neighborhood of $V_0 \text{ mod } \dot{V}_0$ in Q so $V \searrow V_0$ and hence by excision $F \cup V \searrow F$.

Thus, $Q \searrow F \cup V \searrow F$, as required, and we have for all i

$$aI^i \searrow f(a(X_0 \cap I^i)) \cup aI^{i-1}.$$

Proof of Theorem 4. We assume inductively that the theorem is true for $\dim X < n$ and prove it for $\dim X = n$. The proof is in two parts: CH: f is conewise homogenous and LC: $(X, X_0) \subset (aX, f(aX_0))$ is locally collared. Lemma 6 and Corollary IV.6 then imply that f is unknotted.

Triangulate to obtain $f: \gamma aK_0 \rightarrow \beta aK$ a conical simplicial embedding with $\beta K = K$. $f|_{K_0}$ = inclusion of K_0 into K . Let $i: aK_0 \rightarrow aK$ be inclusion $= \bar{f}$. Our assumptions are: (K, K_0) is a codimension ≥ 3 situation and

$$f^{-1}(aI^i(K)) = a(I^i(K) \cap K_0).$$

CH: f is conewise homogenous.

Proof. By Lemma IV.3 it suffices to consider A and $B = bA$ such that $f^{-1}A$ is a horizontal simplex and $f^{-1}B$ is a vertical simplex above or below it, both in $\gamma aK_0 - (a \cup K_0)$, and prove that A is a nice face of B in $(\beta aK, f(\gamma aK_0))$. We shall do this by showing that the link pair of A is a suspension of the link pair of B . This we shall do in a sequence of claims.

NOTATION. Let $\eta_B \gamma aK_0$ and $\eta_B \beta aK$ be deriveds mod $f^{-1}B$ and B respectively such that $f: \eta_B \gamma aK_0 \rightarrow \eta_B \beta aK$ is simplicial.

Also, let $C = \pi f^{-1}A = \pi f^{-1}B \in K_0$.

CLAIM A. There exists a homeomorphism

$$\psi: \text{Lk}(A; \eta_B \beta aK) \rightarrow (b+x) \text{Lk}(B; \eta_B \beta aK) \text{ rel } b * \text{Lk}(B; \eta_B \beta aK).$$

Proof. By Lemma 5, $f(a) \sim i(a) \sim i(b) \sim f(b)$ in aK where a and b are interior points of $f^{-1}A$ and $f^{-1}B$ respectively. Hence, A is a nice face of B in aK and the claim comes from Lemma II.14b.

CLAIM B. There exists a homeomorphism

$$\phi: \text{Lk}(A; \eta_B f(\gamma a K_0)) \rightarrow (b+x) \text{Lk}(B; \eta_B f(\gamma a K_0))$$

rel $b * \text{Lk}(B; \eta_B f(\gamma a K_0))$ and such that if $D \in \text{St}^\circ(C; K_0)$ then

$$\phi(\text{Lk}(A; \eta_B f(\gamma a D))) = (b+x) \text{Lk}(B; \eta_B f(\gamma a D)).$$

Proof. This is essentially Claim B of the proof of Theorem IV.4. If $\bar{\phi}$ is defined as it is there then $\phi = ((b+x) * f)\bar{\phi}f^{-1}$.

CLAIM C. $\psi\phi^{-1}|_x \text{Lk}(B; \eta_B f(\gamma a K_0))$ satisfies the hypotheses of Theorem 4, regarded as embedding

$$x \text{Lk}(B; \eta_B f(\gamma a K_0)) \rightarrow x \text{Lk}(B; \eta_B \beta a K).$$

Proof. First, we show that $\text{Lk}(B; \eta_B \beta a K, \eta_B f(\gamma a K_0))$ is a codimension ≥ 3 pair. As in Lemma 2, we have

$$\text{Lk}(B; \eta_B f(\gamma a K_0)) \cap I^i(\text{Lk}(B; \eta_B \beta a K)) = \text{Lk}(B; \eta_B f(\gamma a K_0) \cap I^{i+j+1}(\eta_B \beta a K)).$$

Since $B \notin K_0$, this equals, in turn

$$\text{Lk}(B; \text{Cl}(\eta_B f(\gamma a K_0) \cap I^{i+j+1}(\eta_B \beta a K) - K)).$$

So it suffices to know that $\dim f(aK_0) \cap I^{i+j+1}(aK) - K \leq (i+j+1) - 3$. But $f(aK_0) \cap I^{i+j+1}(aK) - K = f(aK_0) \cap aI^{i+j}(K) - K$ and by hypothesis on f this is $f(a(K_0 \cap I^{i+j}(K)))$ which has dimension $\leq i+j-2$ as required.

Next, if $y \in x \text{Lk}(B; \eta_B f(\gamma a K_0)) - \text{Lk}(B; \eta_B f(\gamma a K_0))$ we invoke Lemma 5 and prove

$$d(\psi\phi^{-1}(y); x \text{Lk}(B; \eta_B \beta a K)) = d(y; x \text{Lk}(B; \eta_B \beta a K)).$$

These equal, respectively, the intrinsic dimension of $\psi\phi^{-1}(y)$ and y in

$$(b+x) \text{Lk}(B; \eta_B \beta a K).$$

If y does not equal x , then $y = tx + (1-t)y_1$ with $0 < t < 1$ and

$$y_1 \in \text{Lk}(B; \eta_B f(\gamma a K_0)).$$

Clearly $y \sim y_1$ in $(b+x) \text{Lk}(B; \eta_B \beta a K)$.

There exists a unique $D \in \text{St}^\circ(C; K_0)$ with $y_1 \in \text{Int Lk}(B; \eta_B f(\gamma a D))$, and $y_1 \in \text{Int } aD$. Since $y \in x \text{Lk}(B; \eta_B f(\gamma a D)) - x \text{Lk}(B; \eta_B f(\gamma a \dot{D}))$, it follows that $\phi^{-1}(y) \in \text{Int } f(aD)$.

Hence, $d(y_1; aK) = d(\phi^{-1}(y); aK)$ and hence by Lemma II.19,

$$d(y_1; \text{Lk}(A; \eta_B \beta a K)) = d(\phi^{-1}(y); \text{Lk}(A; \eta_B \beta a K)).$$

So

$$d(\psi\phi^{-1}(y); (b+x) \text{Lk}(B; \eta_B \beta a K)) = d(y_1; (b+x) \text{Lk}(B; \eta_B \beta a K)),$$

which completes the proof when $y \neq x$ since $y_1 \sim y$.

As for x , $\phi^{-1}(x) \in \text{Lk}(A; \eta_B f(\gamma a C))$ which consists of b and this point. At any rate A is a nice face of $A\phi^{-1}(x)$ in $\eta_B \beta a K$ since they are both interior to $\eta_B f(\gamma a C)$. Hence, $\phi^{-1}(x) \in S(\text{Lk}(A; \eta_B \beta a K))$ and so

$$\psi\phi^{-1}(x) \in S((b+x) \text{Lk}(B; \eta_B \beta a K)) = (b+x)S(\text{Lk}(B; \eta_B \beta a K)).$$

So $\psi\phi^{-1}(x) \in xS(\text{Lk}(B; \eta_B \beta a K))$.

CLAIM D. $\text{Lk}(A; \eta_B(\beta a K, f(\gamma a K_0))) \cong \Sigma \text{Lk}(B; \eta_B(\beta a K, f(\gamma a K_0)))$.

Proof. Consider the embedding:

$$\psi\phi^{-1}: (b+x) \text{Lk}(B; \eta_B f(\gamma a K_0)) \rightarrow (b+x) \text{Lk}(B; \eta_B \beta a K).$$

By Claim C and an application of the inductive assumption on Theorem 4, this embedding satisfies the requirements of Lemma 3. Thus,

$$\begin{aligned} ((b+x) \text{Lk}(B; \eta_B \beta a K), \psi\phi^{-1}(b+x) \text{Lk}(B; \eta_B f(\gamma a K_0))) \\ \cong (b+x) \text{Lk}(B; \eta_B \beta a K, \eta_B f(\beta a K_0)), \end{aligned}$$

and the first pair is homeomorphic, by ψ to

$$\text{Lk}(A; \eta_B \beta a K, \eta_B f(\gamma a K_0)),$$

which proves Claim D and hence completes the proof of CH.

LC: $(K, K_0) \subset (aK, f(aK_0))$ is locally collared.

Proof. Given a point y of K_0 we have to show that $\text{Lk}(y; aK, f(aK_0)) \cong c \text{Lk}(y; K, K_0)$. Clearly, by going to a subdivision if necessary, we can assume that $y=v$ is a vertex of K_0 .

NOTATION. Let $\eta \gamma a K_0$ and $\eta \beta a K$ be deriveds such that $f: \eta \gamma a K_0 \rightarrow \eta \beta a K$ is simplicial.

CLAIM A. There exists a homeomorphism

$$\psi: \text{Lk}(v; \eta \beta a K) \rightarrow x \text{Lk}(v; \eta K) \text{ rel } \text{Lk}(v; \eta K).$$

Proof. This is because $K \subset aK$ is locally collared.

CLAIM B. There exists a homeomorphism

$$\phi: \text{Lk}(v; \eta f(\gamma a K_0)) \rightarrow x \text{Lk}(v; \eta K_0) \text{ rel } \text{Lk}(v; \eta K_0)$$

and such that if $D \in \text{St}^\circ(v; K_0)$:

$$\phi(\text{Lk}(v; \eta f(\gamma a D))) = x \text{Lk}(v; \eta D).$$

Proof. Pseudo-radial projection gives a homeomorphism

$$\tilde{\phi}: \text{Lk}(v; \eta \gamma a K_0) \rightarrow \text{Lk}(v; aK_0) = a \text{Lk}(v; K_0)$$

such that for $D \in \text{St}^\circ(v; K_0)$:

$$\phi(\text{Lk}(v; \eta \gamma a D)) = \text{Lk}(v; aD).$$

So let $\phi = (x * f | \text{Lk}(v; \eta\gamma aK_0))(x * \phi | \text{Lk}(v; K_0))^{-1} \phi f^{-1}$.

CLAIM C. $\psi\phi^{-1}$ satisfies the hypotheses of Theorem 4.

Proof. $\text{Lk}(v; \eta K, \eta K_0)$ is a codimension ≥ 3 pair by Lemma 2.

Now we check the intrinsic skeleton condition.

First, on the vertex x . $d(\psi\phi^{-1}(x); x \text{Lk}(v; \eta K)) = d(\phi^{-1}(x); \text{Lk}(v; \eta\beta aK)) = d(\phi^{-1}(x); aK) - 1$ by Corollary II.19. Now by Claim B, $\phi^{-1}(x) = \text{Lk}(v; \eta f(\gamma av))$ and so $\phi^{-1}(x) = f(ta + (1-t)v)$ for some $0 < t < 1$. Hence, by Lemma 5d applied to f , $d(\phi^{-1}(x); aK) = d(v; K) + 1$. Note that $d(v; K)$ clearly equals $d(x; x \text{Lk}(v; \eta K))$. So we have

$$d(\psi\phi^{-1}(x); x \text{Lk}(v; \eta K)) = d(x; x \text{Lk}(v; \eta K)).$$

Now let $y = tx + (1-t)y_1$ with $0 < t < 1$ and $y_1 \in \text{Lk}(v; \eta K_0)$. We will show that

$$d(\psi\phi^{-1}(y); x \text{Lk}(v; \eta K)) = d(y_1; \text{Lk}(v; \eta K)) + 1.$$

By Corollary II.19 again, it suffices to show

$$d(\phi^{-1}(y); aK) = d(y_1; K) + 1.$$

However, if $y_1 \in \text{Int Lk}(v; \eta D)$ for $D \in \text{St}^\circ(v; K_0)$ then by Claim B, $\phi^{-1}(y) \in \text{Int } f(aD)$. But by Lemma 5b $d(\phi^{-1}(y); aK) = d(f^{-1}\phi^{-1}(y); aK)$. Since $f^{-1}\phi^{-1}(y) \in \text{Int } aD$, clearly $d(f^{-1}\phi^{-1}(y); aK) = d(y_1; K) + 1$.

Thus, $\psi\phi^{-1}$ satisfies the hypotheses of Theorem 4, by Lemma 5d.

CLAIM D. $\text{Lk}(v; \eta\beta aK, \eta f(\gamma aK_0)) \cong c \text{Lk}(v; \eta K, \eta K_0)$.

Proof. Let $h: x \text{Lk}(v; \eta K) \rightarrow x \text{Lk}(v; \eta K) \text{ rel } \text{Lk}(v; \eta K)$ unknot $\psi\phi^{-1}$, by an inductive application of Theorem 4. Then $h\psi | \text{Lk}(v; \eta f(\gamma aK_0)) = \phi$ and so $h\psi$ is the required homeomorphism of pairs.

This completes the proof of LC and hence completes the induction for Theorem 4.

The most important corollaries of Theorem 4 are local unknottedness criteria in the codimension ≥ 3 situation.

DEFINITION 7. We will say that a complex pair (K, K_0) preserves intrinsic skeleta if it satisfies the following equivalent conditions:

- a. For $x, y \in K_0$, $x \sim y$ in K_0 implies $x \sim y$ in K .
- b. For $A < B$ in K_0 , then A is a nice face of B in K_0 implies A is a nice face of B in K .
- c. For A a simplex of K_0 , $S(\text{Lk}(A; K_0)) \subset S(\text{Lk}(A; K))$.

Proof of equivalence. $a \rightleftharpoons b$ because $x \sim y$ in K_0 implies there exists a sequence A_0, \dots, A_n with $x \in \text{Int } A_0$, $y \in \text{Int } A_n$ and A_i a nice face of A_{i+1} or A_{i+1} a nice face of A_i for each i .

$b \rightleftharpoons c$ by Corollary II.17a.

REMARK. Part a of the above definition is obviously p.l. invariant. So we will say a polyhedral pair (X, X_0) preserves intrinsic skeleta if any triangulation of (X, X_0) does.

THEOREM 8. *Let (X, X_0) be a codimension ≥ 3 pair. If (X, X_0) preserves intrinsic skeleta, then (X, X_0) is locally unknotted.*

Note that preserving intrinsic skeleta is much weaker, in general, than local unknottedness. See Theorem II.23. In particular, if X is a manifold and $X_0 < X - \partial X$, then (X, X_0) clearly preserves intrinsic skeleta.

Proof. Let (K, K_0) triangulate the pair (X, X_0) . To show that (X, X_0) is locally unknotted it suffices, by Theorem II.23, to show that A is a nice face of B in K_0 implies A is a nice face of B in (K, K_0) . Clearly, it suffices to show this where $\dim B = \dim A + 1$, i.e. $B = bA$. So we will show that in this case

$$\text{Lk}(A; K, K_0) \cong \Sigma \text{Lk}(B; K, K_0).$$

Since (K, K_0) preserves intrinsic skeleta, A is a nice face of B in K . Hence, by Lemma II.14b, there exist, for $\eta_B K$ a derived mod B , homeomorphisms

$$\phi: \text{Lk}(A; \eta_B K_0) \rightarrow (b+x) \text{Lk}(B; \eta_B K_0) \text{ rel } b \text{Lk}(B; \eta_B K_0).$$

$$\psi: \text{Lk}(A; \eta_B K) \rightarrow (b+x) \text{Lk}(B; \eta_B K) \text{ rel } b \text{Lk}(B; \eta_B K).$$

By Lemma 3, it suffices to prove that $\psi\phi^{-1}$ is an unknotted embedding. To do this it suffices to show that $\psi\phi^{-1}$ satisfies the hypotheses of Theorem 4, embedding $x \text{Lk}(B; \eta_B K_0)$ in $x \text{Lk}(B; \eta_B K)$.

$\text{Lk}(B; \eta_B(K, K_0))$ is a codimension ≥ 3 pair by Lemma 2.

$\text{Lk}(A; \eta_B(K, K_0))$ preserves intrinsic skeleta by Corollary II.12b. Hence, so does $((b+x) \text{Lk}(B; \eta_B K), \psi\phi^{-1}(b+x) \text{Lk}(B; \eta_B K_0))$.

$\phi^{-1}(x) \in S(\text{Lk}(A; \eta_B K_0)) \subset S(\text{Lk}(A; \eta_B K))$ by part c of Definition 7. Hence,

$$\psi\phi^{-1}(x) \in S((b+x) \text{Lk}(B; \eta_B K)) \cap x \text{Lk}(B; \eta_B K) = xS(\text{Lk}(B; \eta_B K)).$$

Finally, if $y = tx + (1-t)y_1$ with $0 < t < 1$ and $y_1 \in \text{Lk}(B; \eta_B K_0)$. Then, clearly, $y \sim y_1$ in $(b+x) \text{Lk}(B; \eta_B K)$ and in $(b+x) \text{Lk}(B; \eta_B K_0)$. The latter implies that $\psi\phi^{-1}y \sim \psi\phi^{-1}y_1 = y_1$ in $\psi\phi^{-1}(b+x) \text{Lk}(B; \eta_B K_0)$ and since the pair

$$((b+x) \text{Lk}(B; \eta_B K), \psi\phi^{-1}(b+x) \text{Lk}(B; \eta_B K_0))$$

preserves intrinsic skeleta, it follows that $\psi\phi^{-1}y \sim y_1$ in $(b+x) \text{Lk}(B; \eta_B K)$. Hence

$$\begin{aligned} d(y; x \text{Lk}(B; \eta_B K)) &= d(y; (b+x) \text{Lk}(B; \eta_B K)) \\ &= d(\psi\phi^{-1}(y); (b+x) \text{Lk}(B; \eta_B K)) \\ &= d(\psi\phi^{-1}(y); x \text{Lk}(B; \eta_B K)). \end{aligned}$$

Thus, by Lemma 5, $\psi\phi^{-1}$ satisfies the hypotheses of Theorem 4 and the proof is complete.

As immediate corollaries we have the Lickorish unknotting theorems [L]:

COROLLARY 9. a. *Let $f: aX \rightarrow aS$ be a proper embedding of a cone into a ball with $\dim X \leq \dim S - 3$, then f is unknotted.*

b. Let (X, X_0) be a polyhedral pair with X a manifold without boundary and $\dim X_0 \leq \dim X - 3$, then (X, X_0) is locally unknotted.

Proof. a follows from Theorem 4 and b from Theorem 8.

VI. Covering isotopies. The Hudson-Zeeman approach to covering isotopies, which we generalize, is based on a close examination of collaring phenomena. So we begin with a theorem listing equivalents of collaring for pairs.

DEFINITION 1. An inclusion $(X, X_0) < (Y, Y_0)$ of polyhedral pairs is called collared if there exists an embedding $h: (X, X_0) \times I \rightarrow (Y, Y_0)$ with $h^{-1}Y_0 = X_0 \times I$, $h_0 =$ the inclusion $X \rightarrow Y$ and $H(X \times I)$ is a neighborhood of X in Y .

The inclusion is called locally collared if for each point $x \in X_0$ there is a neighborhood N of x in Y such that $(N \cap X, N \cap X_0) < (N, N \cap Y_0)$ is collared and for each point $x \in X - X_0$ there is a neighborhood N of x in Y such that $N \cap X < N$ is collared.

The following theorem is a straightforward generalization to pairs of [C; Proposition 4.2].

Note that the equivalence of a and d was used in the proof of Theorem V.4.

THEOREM 2. For an inclusion of complex pairs: $i: (K, K_0) \subset (J, J_0)$ with $K_0 = J_0 \cap K$, and K full in J , the following are equivalent:

a. There exists a homeomorphism

$$h: (J, J_0) \cup (K, K_0) \times I \rightarrow (J, J_0)$$

(where in the union $x \in K$ is identified with $(x, 0) \in K \times I$) such that $h_1: K \rightarrow J$ is the inclusion.

b. i is collared.

c. i is locally collared.

d. For $x \in K_0$, $\text{Lk}(x; J, J_0) \cong c \text{Lk}(x; K, K_0) \text{ rel } \text{Lk}(x; K)$ and for $x \in K - K_0$, $\text{Lk}(x; J) \cong c \text{Lk}(x; K) \text{ rel } \text{Lk}(x; K)$.

e. For $A \in K_0$, $\text{Lk}(A; J, J_0) \cong c \text{Lk}(A; K, K_0) \text{ rel } \text{Lk}(A; K)$ and for $A \in K - K_0$, $\text{Lk}(A; J) \cong c \text{Lk}(A; K) \text{ rel } \text{Lk}(A; K)$.

f. Let ηJ be a derived of J and for $A \in K$ let $C(A; J) = \dot{N}(D(A; K); D(A; J)) = C(\dot{D}(A; K); \dot{D}(A; J))$ and $\check{C}(A; J) = \dot{N}(\dot{D}(A; K); \dot{D}(A; J))$ and similarly for $A \in K_0$ define $C(A; J_0)$ and $\check{C}(A; J_0)$. By convention, for $A \in K - K_0$, let $C(A; J_0) = \check{C}(A; J_0) = \emptyset$.

Then there exists a homeomorphism

$$g: N(K; \eta(J, J_0)) \rightarrow (K, K_0) \times I \text{ rel } K$$

and such that:

1. $g(D(A; J, J_0)) = D(A; K, K_0) \times I$ for $A \in K$.

2. $g(C(A; J, J_0)) = D(A; K, K_0) \times 1$ for $A \in K$.

Proof. $a \rightarrow b \rightarrow c \rightarrow d$ are clear. $d \rightarrow e$ is a straightforward restatement of Cohen's argument in the proof of [C; Proposition 4.2] for pairs. $f \rightarrow a$ is simply a

matter of pulling the external collar into (J, J_0) by shrinking the internal collar given by f .

The proof that $e \rightarrow f$ is also an easy restatement of the corresponding portion of [C; Proposition 4.2] for pairs, but because its argument is crucial to our later proofs, we will outline it.

The first step is to obtain for each $A \in K$ a homeomorphism

$$\phi_A: C(A; J, J_0) \rightarrow c\dot{C}(A; J, J_0) \text{ rel } \dot{C}(A; J).$$

Using the canonical isomorphism

$$\dot{D}(A; J, J_0) \rightarrow \eta \text{ Lk}(A; J, J_0)$$

which takes, by naturality, $\dot{D}(A; K, K_0) \rightarrow \eta \text{ Lk}(A; K, K_0)$, we get from e a homeomorphism $\dot{D}(A; J, J_0) \cong c\dot{D}(A; K, K_0)$ which takes $N(\dot{D}(A; K); \dot{D}(A; J, J_0))$ to a regular neighborhood of $\dot{D}(A; K, K_0)$ in its cone pair. Uniqueness of regular neighborhoods and their complements gives the homeomorphism ϕ_A , exhibiting $C(A; J, J_0)$ as the cone on $\dot{C}(A; J, J_0)$, the boundary pair.

We decompose $N(K; \eta J, \eta J_0)$ in a “complex” of cone pairs indexed by the simplices of K , namely:

$$N(K; \eta J, \eta J_0) = \bigcup \{D(A; J, J_0) : A \in K\} \cup \bigcup \{C(A; J, J_0) : A \in K\}$$

with $D(A; J, J_0) = \eta(A) * \dot{D}(A; J, J_0)$ and $C(A; J, J_0) \cong c\dot{C}(A; J, J_0)$. Furthermore,

$$\dot{D}(A; J, J_0) = \bigcup \{D(B; J, J_0) = A < B \in K\} \cup C(A; J, J_0)$$

and

$$\dot{C}(A; J, J_0) = \bigcup \{C(B; J, J_0) : A < B \in K\}.$$

Furthermore, distinct cone-pairs intersect in the intersection of the base pairs.

Similarly, we can decompose

$$(K, K_0) \times I = \bigcup \{D(A; K, K_0) \times I : A \in K\} \cup \bigcup \{D(A; K, K_0) \times 1 : A \in K\}.$$

$D(A; K, K_0) \times 1 = (\eta(A), 1) * \dot{D}(A; K, K_0) \times 1$ and by the Alexander trick there is an isomorphism $D(A; K, K_0) \times I \cong (\eta(A), 0) * (\dot{D}(A; K, K_0) \times I \cup D(A; K, K_0) \times 1)$ which is the identity mapping $D(A; K, K_0) \times 0$ to $(\eta(A), 0) * \dot{D}(A; K, K_0) \times 0$.

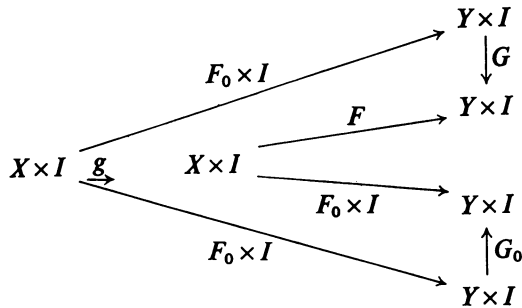
We associate the “complexes” by a map $\gamma: \gamma(D(A; J, J_0)) = D(A; K, K_0) \times I$ and $\gamma(C(A; J, J_0)) = D(A; K, K_0) \times 1$. γ clearly preserves incidences and so we can construct a homeomorphism, g , of polyhedral pairs which is carried by γ and extends the identity mapping $(K, K_0) \rightarrow (K, K_0) \times 0$. Just go by induction up an ordering of the cone pairs of the domain by increasing dimension and at each stage extend g by coning.

We now prove the key lemma, sharpening Theorem 2, and crucial to the covering isotopy theorem.

LEMMA 3. Let $F: X \times I \rightarrow Y \times I$ be an embedding with $F^{-1}(Y \times 0) = X \times 0$ and satisfying:

- a. $(Y \times 0, F(X \times 0)) < (Y \times I, F(X \times I))$ is locally collared.
- b. For every $x \in X$, $d(F(x, t); Y \times I, F(X \times I))$ is constant for $0 < t < 1$.

Then there exists a regular neighborhood $V(X)$ of X in $X \times I$, regular neighborhoods $V(Y)$ and $V_0(Y)$ of Y in $Y \times I$, and collaring homeomorphisms $g: X \times I \rightarrow V(X) \text{ rel } X = X \times 0$, $G: Y \times I \rightarrow V(Y)$ and $G_0: Y \times I \rightarrow V_0(Y) \text{ rel } Y = Y \times 0$ such that the following diagram commutes:



Note. The importance of this lemma is to collar $F(X \times I)$ and $F_0(X) \times I$ in $Y \times I$ and induce the same collaring of $X \times I$.

Proof. By identifying X with $F_0(X)$ by F_0 and identifying $F: X \times I \rightarrow Y \times I$ with $F \cdot (F_0 \times I)^{-1}: F_0(X) \times I \rightarrow Y \times I$, we can assume that $X < Y$ and F_0 is the inclusion. Then by triangulating and subdividing to make the diagram simplicial,

$$Y \times 0 \xleftarrow{\text{proj}} Y \times I \xleftarrow{\text{inc}} X \times I \cup Y \times 0 \xrightarrow{F \cup \text{inc}} Y \times I$$

we obtain the simplicial situation of a complex pair (J, K) and a simplicial map $F: \alpha(K \times I) \rightarrow \beta(J \times I)$ with $\alpha(K \times I)$ a subcomplex of $\alpha(J \times I)$ which is a cylindrical subdivision. Furthermore, we can assume $\alpha(K \times 0) = K \times 0 = K$ and $\alpha(J \times 0) = J \times 0 = \beta(J \times 0) = J$. Our assumptions on F become

- 1. $(J \times 0, K \times 0) < (J \times I, F(K \times I))$ is locally collared.
- 2. For $x \in K$, $d(F(x, t); J \times I, F(K \times I))$ is constant for $0 < t < 1$. This clearly implies, since $\alpha(K \times I)$ is cylindrical with $\alpha(K \times 0) = K \times 0$, that for $A \in K$, $d(F(y); J \times I, F(K \times I))$ is constant for $y \in \text{Int } A \times I$.

Now let $(V(J), V(K)) = N(J \times 0; \eta\beta(J \times I), \eta f(\alpha(K \times I)))$ and $(V_0(J), V_0(K)) = N(J \times 0; \eta\alpha(J \times I), \eta\alpha(K \times I))$ where $\eta\beta J \times I$ and $\eta\alpha J \times I$ are deriveds chosen so that $F: \eta\alpha K \times I \rightarrow \eta\beta J \times I$ is simplicial.

Following Theorem 2f we will construct homeomorphisms $G: (J, K) \times I \rightarrow (V(J), V(K))$ and $G_0: (J, K) \times I \rightarrow (V_0(J), V_0(K)) \text{ rel } J = J \times 0$, and such that $G|_{K \times I} = F(G_0|_{K \times I})$. If we then define $g = G_0|_{K \times I}$ we will be done.

We decompose $(V_0(J), V_0(K))$ into a cone pair complex, as in the proof of Theorem 2, consisting of pairs

$$\{C(A; \alpha J \times I, \alpha K \times I) : A \in J\} \quad \text{and} \quad \{D(A; \alpha J \times I, \alpha K \times I) : A \in J\}.$$

Similarly, decompose $(V(J), V(K))$ into $\{C(A; \beta J \times I, F(\alpha K \times I)) : A \in J\}$ and $\{D(A; \beta J \times I, F(\alpha K \times I)) : A \in J\}$ and $(J, K) \times I$ into $\{D(A; J, K) \times 1 : A \in J\}$ and $\{D(A; J, K) \times I : A \in J\}$.

We first reduce the proof to the following technical assertion:

ASSERTION. For $A \in K$ there exist homeomorphisms

$$\begin{aligned} \phi_A^0: C(A; \alpha J \times I, \alpha K \times I) &\rightarrow c\dot{C}(A; \alpha J \times I, \alpha K \times I) \text{ rel } \dot{C}, \\ \phi_A: C(A; \beta J \times I, F(\alpha K \times I)) &\rightarrow c\dot{C}(A; \beta J \times I, F(\alpha K \times I)) \text{ rel } \dot{C}, \end{aligned}$$

such that $(c * F(\phi_A^0 | C(A; \alpha K \times I))) = (\phi_A | C(A; F(\alpha K \times I)))F$.

Note that since the dual cells are defined in terms of $\eta\beta J \times I$ and $\eta\alpha J \times I$ and $F: \eta\alpha K \times I \rightarrow \eta\beta J \times I$ is simplicial, the restriction of F maps

$$(C(A; \alpha K \times I), \dot{C}(A; \alpha K \times I))$$

to $(C(A; F(\alpha K \times I)), \dot{C}(A; F(\alpha K \times I)))$ by a simplicial isomorphism.

Now for $A \in J - K$ there exist homeomorphisms

$$\begin{aligned} \phi_A^0: C(A; \alpha J \times I) &\rightarrow c\dot{C}(A; \alpha J \times I) \text{ rel } \dot{C}, \\ \phi_A: C(A; \beta J \times I) &\rightarrow c\dot{C}(A; \beta J \times I) \text{ rel } \dot{C}. \end{aligned}$$

Define $G: (J, K) \times I \rightarrow (V(J), V(K))$ and $G_0: (J, K) \times I \rightarrow (V_0(J), V_0(K))$ to be carried respectively by γ and γ_0 , where $\gamma(D(A; J, K) \times I) = D(A; \beta J \times I, F(\alpha K \times I))$ and $\gamma_0(D(A; J, K) \times I) = D(A; \alpha J \times I, \alpha K \times I)$;

$$\gamma(D(A; J, K) \times 1) = C(A; \beta J \times I, F(\alpha K \times I))$$

and

$$\gamma_0(D(A; J, K) \times 1) = C(A; \alpha J \times I, \alpha K \times I).$$

Using ϕ_A and ϕ_A^0 we define the conical extensions over the $C(A)$'s.

Thus, if $G: \dot{D}(A; J, K) \times 1 \rightarrow \dot{C}(A; \beta J \times I, F(\alpha K \times I))$ and $G_0: \dot{D}(A; J, K) \times 1 \rightarrow \dot{C}(A; \alpha J \times I, \alpha K \times I)$ are defined so that $G = FG_0$ on the intersection with $K \times I$ then define G and G_0 on $D(A; J, K) \times 1$ by $\phi_A^{-1}(cG)$ and $\phi_A^0{}^{-1}(cG_0)$ respectively. Since F on $D(A; J, K) \times 1$ is the cone of F on $\dot{D}(A; J, K)$, we have, on $D(A; K) \times 1$,

$$\begin{aligned} FG_0 &= F\phi_A^0{}^{-1}cG_0 = \phi_A^{-1}cFcG_0, \\ \phi_A^{-1}c(FG_0) &= \phi_A^{-1}cG = G. \end{aligned}$$

If $G: \dot{D}(A; J, K) \times I \cup D(A; J, K) \times 1 \rightarrow \dot{D}(A; \beta J \times I, F(\alpha K \times I))$ and

$$G_0: \dot{D}(A; J, K) \times I \cup D(A; J, K) \times 1 \rightarrow \dot{D}(A; \alpha J \times I, \alpha K \times I)$$

are defined so that $G = FG_0$, then the extension by coning using the Alexander trick also satisfies $G = FG_0$ on $K \times I$.

Thus, the proof is reduced to the assertion that the compatible ϕ_A and ϕ_A^0 for $A \in K$ exist.

CLAIM A. There exists a homeomorphism $\psi: \text{Lk}(A; \alpha J \times I) \rightarrow c \text{Lk}(A; J)$ rel $\text{Lk}(A; J)$ such that for $B \in \text{Lk}(A; J)$, $\psi(\text{Lk}(A; \alpha(A * B) \times I)) = cB$.

Proof. As in the proof of Claim B of the proof of Theorem IV.4, we define $S^i = A * \text{Lk}(A; J)^{(i)}$ and define $\psi^i: \text{Lk}(A; \alpha S^i \times I) \rightarrow c \text{Lk}(A; J)^{(i)}$. Starting with $i = -1$, $S^{-1} = A$, $\text{Lk}(A; \alpha S^{-1} \times I) = \text{one point}$ which is mapped to c .

Assume ψ^{i-1} is defined and, for B and i -simplex of $\text{Lk}(A; J)$, define

$$\psi^i: \text{Lk}(A; \alpha(A * B) \times I) \rightarrow cB$$

to extend the identity on B and ψ^{i-1} mapping $\text{Lk}(A; \alpha(A * \dot{B}) \times I) \rightarrow c\dot{B}$. This is just a question of extending a homeomorphism of the boundaries of two balls to the balls.

The induction completed, let $\psi = \psi^i$ for $i = \dim \text{Lk}(A; J)$.

CLAIM B. There exists a homeomorphism $\phi^0: C(A; \alpha J \times I) \rightarrow c\dot{C}(A; \alpha J \times I)$ rel \dot{C} such that, for $D \in \text{St}^0(A; J)$, $\phi^0 C(A; \alpha D \times I) = c\dot{C}(A; \alpha D \times I)$. In particular, $\phi^0 C(A; \alpha K \times I) = c\dot{C}(A; \alpha K \times I)$.

Proof. Using ψ as defined in Claim A, and using the canonical isomorphism $j: \dot{D}(A; \alpha J \times I) \rightarrow \eta \text{Lk}(A; \alpha J \times I)$ we obtain a homeomorphism

$$\dot{D}(A; \alpha J \times I) \rightarrow c\dot{D}(A; J) \text{ rel } \dot{D}(A; J)$$

which takes $N(D(A; J); D(A; \alpha J \times I), \{D(A; \alpha D_j \times I)\})$ to a regular neighborhood of $D(A; J)$ in $(cD(A; J), \{cD(A; D_j)\})$ where $\{D_j\}$ is the set of simplices in $\text{St}^0(A; J)$. Uniqueness of regular neighborhood families and their complements, applied to $\dot{D}(A; J)$ in the cone family, gives the required homeomorphism.

CLAIM C. $F\phi^{0-1}: c\dot{C}(A; \alpha K \times I) \rightarrow \dot{D}(A; \beta J \times I)$ is CH.

Proof. For $y \in C(A; F(\alpha K \times I)) - \dot{C}(A; F(\alpha K \times I))$.

$$\begin{aligned} d(y; \dot{D}(A; \beta J \times I), C(A; F(\alpha K \times I))) &= d(y; \dot{D}(A; \beta J \times I, F(\alpha K \times I))) \\ &= d(y; J \times I, F(K \times I)) - \dim A - 1 \end{aligned}$$

by Lemma II.19. Since

$$F\phi^{0-1}(C(A; \alpha K \times I) - \dot{C}(A; \alpha K \times I)) = C(A; F(\alpha K \times I)) - \dot{C}(A; F(\alpha K \times I)),$$

it follows that for $x \in \dot{C}(A; \alpha K \times I)$ it suffices to prove that

$$d(F\phi^{0-1}(tc + (1-t)x); J \times I, F(K \times I))$$

is constant for $0 < t < 1$.

There is a unique $D \in \text{St}^0(A; K)$ such that $x \in \dot{C}(A; \alpha D \times I) - \dot{C}(A; \alpha \dot{D} \times I)$. By Claim B, $\phi^{0-1}(tc + (1-t)x) \in C(A; \alpha D \times I) - C(A; \alpha \dot{D} \times I)$ for all $0 < t < 1$.

$$C(A; \alpha D \times I) - C(A; \alpha \dot{D} \times I) \subset \text{Int } D \times I.$$

Hence,

$$d(F\phi^{0-1}(tc + (1-t)x); J \times I, F(K \times I))$$

is constant for $0 < t < 1$, by the intrinsic skeleton assumption on F .

Proof of the Assertion. Since $(J \times I, F(K \times I))$ is locally collared, $C(A; \beta J \times I)$ is a regular neighborhood of $C(A; F(\alpha K \times I)) \bmod \dot{C}(A; F(\alpha K \times I))$ in $\dot{D}(A; \beta J \times I)$. For by the proof of Theorem 2 $C(A; \beta J \times I, F(\beta K \times I))$ is homeomorphic to the cone pair on $\dot{C}(A; \beta J \times I, F(\alpha K \times I))$ and is the complement of a regular neighborhood pair in $\dot{D}(A; \beta J \times I)$ and hence is locally collared on the outside. So we can apply Theorem III.11c or the stellar neighborhood theorem to get this regular neighborhood fact.

Theorem IV.4 now implies that there exists a homeomorphism

$$\phi: C(A; \beta J \times I) \rightarrow c\dot{C}(A; \beta J \times I)$$

rel \dot{C} and such that

$$\phi F \phi^{0^{-1}} = \overline{(F \phi^{0^{-1}})} = cF.$$

So let $\phi_A = \phi$ and $\phi_A^0 = \phi^0$.

With Lemma 3 taken care of, the proof of the covering isotopy follows from a sequence of arguments taken directly from Hudson-Zeeman.

LEMMA 4. *Let (X, X_0) be a polyhedral pair and $c: X \times [0, \varepsilon] \rightarrow X \times I$ an embedding with $c^{-1}(X \times 0) = X \times 0$ and $c|_{X_0 \times [0, \varepsilon]}$ level-preserving. Then there exists $0 < \delta < \varepsilon$ and $h: X \times I \rightarrow X \times I$ a homeomorphism, with hc level-preserving on $X \times [0, \delta]$ and h ambient isotopic to the identity rel $X \times I \cup c(X_0 \times [0, \varepsilon])$.*

Proof. Triangulate and subdivide so that $c: \alpha K \times [0, \varepsilon] \rightarrow \beta K \times I$ is simplicial and K_0 is a subcomplex of K_0 covering X_0 .

Let $\varepsilon > \delta > 0$ be chosen so that no vertex of $\alpha K \times [0, \varepsilon]$ or $K \times I$ has level $0 < t \leq \delta$, and such that $c(K \times [0, \delta]) \cap K \times 1 = \emptyset$.

Let $\eta \alpha K \times [0, \varepsilon]$ be a derived with $\eta A \in \text{Int } A \cap K \times \delta$ if $\text{Int } A \cap K \times \delta \neq \emptyset$. Let $\eta_0 \beta K \times I$ be a derived such that $c: \eta \alpha K \times [0, \varepsilon] \rightarrow \eta_0 \beta K \times I$ is simplicial and let $\eta_1 \beta K \times I$ agree with $\eta_0 \beta K \times I$ except for

$$c(N(K \times 0; \alpha K \times [0, \varepsilon])) - c(K_0 \times [0, \varepsilon] \cup K \times 0)$$

and on these simplices, let $\eta_1(A) \in \text{Int } A \cap K \times \delta$. Note that if

$$A \in cN(K_0 \times 0; \alpha(K_0 \times [0, \varepsilon])) - K \times 0, \eta_1(A) = \eta_0(A) \in K \times \delta$$

since c is level preserving on $K_0 \times [0, \varepsilon]$.

Let $h: \eta_0 \beta K \times I \rightarrow \eta_1 \beta K \times I$ be defined by $h(\eta_0 A) = \eta_1 A$.

REMARK. Note that if $X_1 < X$ and then h can be assumed a homeomorphism of pairs $(X \times I, c(X_1 \times [0, \varepsilon]))$.

We now cover an isotopy locally:

LEMMA 5. *Given X, Y compact polyhedra let $F: X \times I \rightarrow Y \times I$ be an isotopy satisfying:*

1. $(Y \times 0, F_0(X) \times 0) \subset (Y \times I, F(X \times I))$ is locally collared.
2. For every $x \in X, d(F(x, t); Y \times I, F(X \times I))$ is constant for $0 < t < 1$.

Then, there exists $\varepsilon > 0$ and a homeomorphism $h: Y \times [0, \varepsilon] \rightarrow Y \times [0, \varepsilon] \text{ rel } Y \times 0$ and level-preserving such that: $h(F_0 \times [0, \varepsilon]) = F|X \times [0, \varepsilon]$.

Proof. Choose compatible collarings g, G_0 and G by Lemma 3. By compactness, we can choose $\delta > 0$ so that $Y \times [0, \delta] \subset G_0(Y \times I)$.

Define $h_0: Y \times [0, \delta] \rightarrow Y \times I$ by $GG_0^{-1}|Y \times [0, \delta]$. $h_0|Y \times 0 = \text{identity}$ and by compatibility, $h_0|F_0(X) \times [0, \delta] = F(F_0 \times [0, \delta])^{-1}$, and, in particular, is level-preserving on $F_0(X) \times [0, \delta]$.

By Lemma 4, there exists $0 < \delta' < \delta$ and $h_1: Y \times I \rightarrow Y \times I \text{ rel } Y \times 0 \cup F(X \times [0, \delta])$ and $h_1 h_0 = h$ is level-preserving on $Y \times [0, \delta']$.

Now $h(Y \times \delta') \subset Y \times \delta'$ but might be proper, we note that $h(Y \times [0, \delta'])$ is a neighborhood of $Y \times 0$ and hence, by compactness, there exists $0 < \varepsilon \leq \delta'$ with $Y \times [0, \varepsilon] \subset h(Y \times [0, \delta'])$. So $h|Y \times [0, \varepsilon]$ being level-preserving, is a homeomorphism.

DEFINITION 6 (FOLLOWING ROURKE [R]). We call an isotopy $F: X \times I \rightarrow Y \times I$ locally collared if for every subinterval J of I , $(Y \times J, F(X \times J)) \subset (Y \times J, F(X \times J))$ is locally collared, or equivalently $(Y \times t, F(X \times t)) \subset (Y \times [t, 1], F(X \times [t, 1]))$ is locally collared for all $0 \leq t < 1$ and $(Y \times t, F(X \times t)) \subset (Y \times [0, t], F(X \times [0, t]))$ is locally collared for all $0 < t \leq 1$.

THEOREM 7. Let $F: X \times I \rightarrow Y \times I$ be an isotopy, with X compact. Then a necessary and sufficient condition that F can be covered by an ambient isotopy of Y is that

1. F be locally collared, and,
2. for every $x \in X$, $d(F(x, t); Y \times I, F(X \times I))$ is constant for $0 < t < 1$.

Proof. Necessity is clear. To prove sufficiency, we lift the compactness argument from Hudson-Zeeman, assuming Y is compact also.

Let $t \in I$, then by Lemma 5, applied in both directions, there exists a subinterval J_t with $t \in \text{Int}_I J_t$ and a homeomorphism

$$h(t): Y \times J_t \rightarrow Y \times J_t$$

with $h(t)$ level-preserving and $h(t)(F_t \times J_t) = F|X \times J_t$.

By compactness of I , there exists a sequence $0 = s_0 < s_1 < \dots < s_n = 1$ such that for each $i = 1, \dots, n$, there exists t_i and $J_i = J_{t_i}$ with $[s_{i-1}, s_i] \subset \text{Int}_I J_i$, let $[0, s_0] = J_0$.

Inductively, we define $H(i): Y \times [0, s_i] \rightarrow Y \times [0, s_i] \text{ rel } Y \times 0$ and level-preserving homeomorphisms, such that $H(i)(F_0 \times [0, s_i]) = F|X \times [0, s_i]$. Let $H(0) = h(0)$.

$$\begin{aligned} H(i) &= H(i-1) \text{ on } Y \times [0, s_{i-1}] \\ &= h(t_i)(h(t_i)_{s_{i-1}} \times [s_{i-1}, s_i])^{-1}(H(i-1)_{s_{i-1}} \times [s_{i-1}, s_i]) \text{ on } Y \times [s_{i-1}, s_i]. \end{aligned}$$

Let $H = H(n): Y \times I \rightarrow Y \times I \text{ rel } Y \times 0$. This is the required ambient isotopy.

In the case that Y is not compact, we let Y^* = a regular neighborhood of $\text{proj}_Y F(X \times I)$ and let $X^* = X \cup \text{boundary of the regular neighborhood}$. Let $F^* = F$ on $X \times I$ and the identity on the boundary. If H^* is the ambient isotopy of Y^* , let H extend H^* by the identity on the rest of $Y \times I$.

We outline a related theorem of Rourke [R]:

DEFINITION 8. If $F: X \times I \rightarrow Y \times I$ is an isotopy and $G: Y \times I \rightarrow Y \times I$ is an ambient isotopy, we shall say that G covers the track of F , if $G(F_0(X)) = F_t(X)$.

THEOREM 9 [R]. *Let $F: X \times I \rightarrow Y \times I$ be an isotopy with X and Y compact. Then a necessary and sufficient condition that the track of F can be covered by an ambient isotopy is that F be locally collared.*

Proof. Necessity is again clear and for sufficiency it suffices to cover the track of $F|X \times [0, \epsilon]$ for some ϵ , as the result then follows by a compactness argument identical with Theorem 7.

By local collaring, there is a homeomorphism

$$G: (Y \times I, F_0(X) \times I) \rightarrow (Y \times I, F(X \times I)) \text{ rel } Y \times 0.$$

By Lemma 4 and the remark thereafter, there exists a homeomorphism

$$h: (Y \times I, F(X \times I)) \rightarrow (Y \times I, F(X \times I)) \text{ rel } Y \times 0$$

and a $1 > \delta > 0$, with $G^* = hG$ level-preserving on $Y \times [0, \delta]$, and so there exists $0 < \epsilon \leq \delta$ such that $G^*(Y \times [0, \epsilon]) = Y \times [0, \epsilon]$ and $G^*(F_0(X) \times [0, \epsilon]) = F(X \times [0, \epsilon])$. Then $G^*|Y \times [0, \epsilon]$ is the required local covering.

REMARK. As in Theorem 7, it is possible to get rid of the assumption that Y is compact.

COROLLARY 10. *Let (X, X_0) be a polyhedral pair with X_0 compact, and let $F: X_0 \times I \rightarrow X_0 \times I$ be an ambient isotopy of X_0 . Then F can be covered by an ambient isotopy of X iff $F(I^i(X, X_0) \times I) = I^i(X, X_0) \times I$.*

Proof. Again necessity is clear. For sufficiency, note that F is obviously locally collared (regarded as an isotopy in X), and in the presence of local collaring, $d(F(x, t); X \times I, F(X_0 \times I)) = d(F(x, t); X \times t, F(X_0 \times t)) + 1$, which is in this case $d(F_t(x); X, X_0) + 1$ which is constant for $0 \leq t \leq 1$ by hypothesis.

COROLLARY 11. *Let X be a compact polyhedron, and $F: I^i(X) \times I \rightarrow I^i(X) \times I$ be an ambient isotopy of $I^i(X)$. Then F can be covered by an ambient isotopy of X iff $F(I^j(X) \times I) = I^j(X) \times I$ for all $j \leq i$.*

Proof. Corollary 10 and Corollary II.21 imply the result.

COROLLARY 12. *Let $X_0, Y < X$ be polyhedra with X_0 compact, and $F: X_0 \times I \rightarrow X_0 \times I$ an ambient isotopy of X_0 rel $X_0 \cap Y$. If F can be covered by an ambient isotopy of X , then there exists an ambient isotopy of X rel Y covering F .*

Proof. Assume X compact. Define $F: (X_0 \cup Y) \times I \rightarrow (X_0 \cup Y) \times I$ to be F on $X_0 \times I$ and the identity on $Y \times I$.

$$d(\bar{F}_t(x); X, X_0 \cup Y) = d(x; X, X_0 \cup Y) \text{ for all } x \in X_0 \cup Y \text{ and } 0 \leq t \leq 1.$$

This is clear for $x \in Y$ and for $x \in X_0 - Y$,

$$d(\bar{F}_i(x); X, X_0 \cup Y) = d(F_i(x); X, X_0) = d(x; X, X_0) = d(x; X, X_0 \cup Y).$$

So \bar{F} can be covered by an ambient isotopy by Corollary 10.

Now we want to introduce local unknottedness conditions instead of the ambient intrinsic dimension conditions.

DEFINITION 13. Call a locally collared isotopy $F: X \times I \rightarrow Y \times I$, with X compact, locally unknotted if it satisfies the following equivalent conditions.

1. $(Y, F_0(X))$ is locally unknotted.
2. $(Y \times I, F(X \times I))$ is locally unknotted at all points $F(x, t)$, $0 < t < 1$.

Proof of the equivalence. Since, by Rourke's theorem, Theorem 9, the track of F can be covered by an ambient isotopy of Y (or a regular neighborhood of $\text{proj}_Y F(X \times I)$ if Y is not compact), it follows that $(Y, F_0(X)) \cong (Y, F_t(X))$ for all t .

Now $d(F(x, t); Y \times I, F(X \times I)) = d(F_t(x); Y, F_t(X)) + 1$ by local collaring and similarly $d(F(x, t); F(X \times I)) = d(F_t(x); F_t(X)) + 1$, for all $0 < t < 1$.

Thus, $(Y, F_0(X))$ is locally unknotted iff $(Y, F_t(X))$ is locally unknotted for all $0 < t < 1$ iff $(Y \times I, F(X \times I))$ is locally unknotted at every point $F(x, t)$ with $x \in X$ and $0 < t < 1$.

REMARK. Note that if $X_1 < X_0 < X$ are polyhedra with (X, X_0) locally unknotted at every point of $X_0 - X_1$ and $F: X_0 \times I \rightarrow X_0 \times I$ is an ambient isotopy of $X_0 \text{ rel } X_1$, then F regarded as an isotopy $X_0 \times I \rightarrow X \times I$ satisfies the hypotheses of Theorem 7. In particular, we have

COROLLARY 14. *Let $F: X \times I \rightarrow Y \times I$ be a locally unknotted isotopy, then F can be covered by an ambient isotopy of Y .*

Proof. $d(F(x, t); F(X \times I))$ is constant for $0 < t < 1$ and hence part 2 of Definition 13 and Theorem 7 imply the result.

COROLLARY 15. *Let $F: X \times I \rightarrow Y \times I$ be an isotopy with X compact and $(Y, F_0(X))$ locally unknotted. Then if the track of F can be covered by an ambient isotopy of Y , F itself can be covered by an ambient isotopy of Y .*

Proof. Theorem 9 and Corollary 14.

Our notion of locally unknotted isotopy reduces to the usual one in the case of manifolds. We leave to the reader the proof:

Let $F: X \times I \rightarrow Y \times I$ be an isotopy with X, Y manifolds and X compact and such that $F^{-1}(\partial Y \times I) = \partial X \times I$. Then the following are equivalent:

1. F is locally unknotted.
2. For every subinterval J of I , $(Y \times J, F(X \times J))$ and $((\partial Y) \times J, F((\partial X) \times J))$ are locally unknotted.
3. For every subinterval J of I , $(Y \times J, F(X \times J))$ and $(Y \times J, F(X \times J))$ are locally unknotted.

To obtain the codimension ≥ 3 isotopy theorem, we use Lickorish's device of converting concordance problems to cone problems by glueing cones on top.

THEOREM 16. *Let $F: X \times I \rightarrow Y \times I$ be an isotopy with X compact and $(Y, F_0(X))$ a codimension ≥ 3 pair. Then F can be covered by an ambient isotopy iff*

$$F^{-1}(I^t(Y) \times I) = F_0^{-1}(I^t(Y)) \times I.$$

Proof. Necessity is clear and for sufficiency, we prove that in such a situation, $(Y \times 0, F(X \times 0)) < (Y \times I, F(X \times I))$ is locally collared and $d(F(x, t); Y \times I, F(X \times I))$ is constant for $0 < t < 1$.

By applying this result to F^s , defined by $F^s(x, t) = F(x, \min(t + s, 1))$ and F^{-s} , defined by $F^{-s}(x, t) = F(x, \max(s - t, 0))$ gives local collaring. Note that by hypotheses $F^{-1}(I^t(Y) \times I) = F_0^{-1}(I^t(Y)) \times I$ so in particular, $F_0^{-1}(I^t(Y)) = F_s^{-1}(I^t(Y))$ and hence, $(Y, F_s(X))$ is a codimension ≥ 3 pair and F^s and F^{-s} satisfy the assumptions of the theorem.

As usual, identify X with $F_0(X)$ and F with $F(F_0 \times I)^{-1}$ and we will assume that $X < Y$ and Y compact. Triangulate (Y, X) and subdivide so that $F: \alpha K \times I \rightarrow \beta J \times I$ is simplicial with $\alpha K \times I$ cylindrical. Let $\alpha K \times 0 = K$ and $\beta J \times 0 = J$. Let $\alpha K \times 1 = K_1$ and $\beta J \times 1 = J_1$. Extend F to

$$\bar{F}: \alpha K \times I \cup cK_1 \rightarrow \beta J \times I \cup cJ_1$$

by coning $F|_{K_1}: K_1 \rightarrow J_1$. We can construct homeomorphisms

$$\phi_\alpha: \alpha K \times I \cup cK_1 \rightarrow cK \text{ rel } K,$$

$$\phi_\beta: \beta J \times I \cup cJ_1 \rightarrow cJ \text{ rel } J,$$

and such that for $A \in K$ (resp. $A \in J$)

$$\phi_\alpha(A \times I \cup c(A \times 1)) = cA \text{ (resp. } \phi_\beta(A \times I \cup c(A \times 1)) = cA).$$

Just order the simplices of K (or J) in increasing order of dimension and construct ϕ_α (or ϕ_β) inductively.

$$\text{Let } G: cK \rightarrow cJ = \phi_\beta \bar{F} \phi_\alpha^{-1}.$$

We wish to apply Theorem V.4 to G . We already know that (J, K) is a codimension ≥ 3 pair.

$$\phi_\beta^{-1}(cI^t(J)) = I^t(J) \times I \cup cI^t(J_1)$$

because $I^t(J)$ is a subcomplex of J . So

$$\begin{aligned} \bar{F}^{-1} \phi_\beta^{-1}(cI^t(J)) &= F^{-1}(I^t(J) \times I) \cup cF_1^{-1}(I^t(J_1)) \\ &= F_0^{-1}(I^t(J)) \times I \cup cF_1^{-1}(I^t(J_1)). \end{aligned}$$

Since $F_1^{-1}(I^t(J_1)) = F_0^{-1}(I^t(J))$, and $F_0^{-1}(I^t(J))$ is a subcomplex of K , it follows that

$$G^{-1}(cI^t(J)) = \phi_\alpha \bar{F}^{-1} \phi_\beta^{-1}(cI^t(J)) = cF_0^{-1}(I^t(J)).$$

Hence, Theorem V.4 applies to G .

We immediately obtain the local collaring result from Theorem V.4. And from Corollary IV.6 we have that G is conewise homogenous.

If $x \in K$, then $x \in \text{Int } A$ for a unique $A \in K$. If $c\bar{x}$ is the cone line segment in cK , then $\text{Int } \phi_\alpha^{-1}c\bar{x} \subset \text{Int } A \times I \cup cA$ by hypothesis on ϕ_α , and $\phi_\alpha^{-1}c\bar{x}$ joins c to x . Since $K \times I$ is a cylindrical triangulation, every simplex of $\text{Int } \alpha A \times I$ separates $\text{Int } A \times I \cup cA$, with x in the closure of the "bottom" component and c in the closure of the "top" component. It follows that $\text{Int } \phi_\alpha^{-1}c\bar{x}$ meets the interior of every simplex of $\alpha A \times I - \partial\alpha A \times I$.

From this, and the fact that G is CH, we get that $d(F(y); J \times I, F(K \times I)) = d(F(y); J \times I \cup cJ_1, \bar{F}(K \times I \cup cK_1))$, for $y \in K \times I - K \times \dot{I}$, is constant for $y \in \text{Int } A \times I$. In particular,

$$d(F(x, t); J \times I, F(K \times I))$$

is constant for $0 < t < 1$.

This proves the case where Y is compact. The extension to the noncompact case is identical to the extension at the end of the proof of Theorem 7.

As a corollary, we have the following unpublished result of Hudson:

COROLLARY 17. *Let $F: X \times I \rightarrow Y \times I$ be an isotopy with Y an n -manifold and X compact such that*

1. $F^{-1}(\partial Y \times I) = F_0^{-1}(\partial Y) \times I$,
2. $n\text{-dim } X \geq 3$,
3. $n\text{-dim } F_0^{-1}(\partial Y) \geq 4$.

Then F can be covered by an ambient isotopy.

VII. Extending triangulations. For a polyhedral pair (X, X_0) if $k_0: K_0 \rightarrow X_0$ is a triangulation of X_0 , then we will say that $k: K \rightarrow X$ extends $(K_0; k_0)$ if the induced map

$$K_0 \xrightarrow{k_0} X_0 \subset X \xleftarrow{k} K$$

is simplicial. In this section we will generalize Armstrong's results [A2] about extending triangulations from manifold pairs to polyhedral pairs.

LEMMA 1. *For a compact polyhedral pair (X, X_0) any triangulation $(K_0; k_0)$ has an iterated derived subdivision which extends to a triangulation of X .*

Proof. If $(K; k)$ is any triangulation of X , then we can subdivide so that $\alpha K_0 \rightarrow X_0 \subset X \leftarrow \beta K$ is simplicial. There is a further subdivision $\gamma \alpha K_0$ which is an iterated derived $\eta^{(r)}K_0$, [Z; Lemma 4] or [H; Corollary 1.6]. We can extend γ over βK to obtain a subdivision $\delta \beta K$ so that $(\delta \beta K; k)$ extends $(\eta^{(r)}K_0; k_0)$.

THEOREM 2. *Let (X, X_0) be a compact polyhedral pair and let $(K_0; k_0)$ be a triangulation of X_0 . $(K_0; k_0)$ extends to a triangulation of X iff $k_0^{-1}I^i(X, X_0)$ is a subcomplex of K_0 for all i .*

Proof. Necessity is clear.

By Lemma 1, there exists some iterated derived subdivision of K_0 which will extend. Since a derived subdivision is obtained by a number of stellar subdivisions,

to prove sufficiency we use induction on the number of stellar subdivisions and reduce the proof to that of the theorem with the additional assumption that $(\sigma K_0; k_0)$ extends to a triangulation of X , where σK_0 is the subdivision obtained by starring K_0 at A .

Let $k: K \rightarrow X$ be a triangulation extending $k_0: \sigma K_0 \rightarrow X_0$. Let $s: \sigma K_0 \rightarrow K$ be the simplicial embedding, $k^{-1}k_0$. We assume that $s\sigma K_0$ is full in K , by deriving $\text{mod } s\sigma K_0$ if necessary.

Let $u = s(\sigma A)$ where σA is the vertex in $\text{Int } A$ introduced by σ , and let v be a vertex of sA . Note that the hypothesis on the triangulation K_0 implies that $d(x; K, sK_0)$ is constant for x in the interior of sA so that u is a nice face of uv in $(K, s\sigma K_0)$.

By deriving $\text{mod } s\sigma K_0$ again if necessary, we can assume that $v \text{ Lk } (uv; K)$ is a regular neighborhood of $v \text{ Lk } (uv; s\sigma K_0) \text{ mod Lk } (uv; s\sigma K_0)$ in $\text{Lk } (u; K)$. Under this assumption, clearly, $v \text{ Lk } (uv; K, s\sigma K_0)$ is a regular neighborhood of

$$v \text{ Lk } (uv; s\sigma K_0) \text{ mod Lk } (uv; s\sigma K_0)$$

in $\text{Lk } (u; K, s\sigma K_0)$. If $C \in \text{Lk } (uv; s\sigma K_0)$ then uC is a nice face of uvC in $(K, s\sigma K_0)$ (since both uC and uvC are images of simplices interior to a simplex of K_0). Hence, by Corollary II.17, C is a nice face of vC in $\text{Lk } (u; K, s\sigma K_0)$. Hence, the collapse $(v \text{ Lk } (uv; s\sigma K_0) | \text{Lk } (uv; s\sigma K_0)) \searrow v$ conewise is homogenous in $\text{Lk } (u, K, s\sigma K_0)$. It follows from Theorem III.6 that $v \text{ Lk } (uv; K, s\sigma K_0)$ is a regular neighborhood of v in $\text{Lk } (u; K, s\sigma K_0)$.

By Lemma II.14a, there exists a homeomorphism

$$\phi: \text{Lk } (u; K, s\sigma K_0) \cong (v+x) \text{ Lk } (uv; K, s\sigma K_0).$$

Since $v \text{ Lk } (uv; K, s\sigma K_0)$ is a regular neighborhood of v in $\text{Lk } (u; K, s\sigma K_0)$, we can assume that ϕ is the identity on $v \text{ Lk } (uv; K)$, using the method of proof of Lemma II.14b.

If P is the pair $\text{Lk } (u; K, s\sigma K_0) - \text{St}^\circ (v; K)$, then

$$\phi(P) = x \text{ Lk } (uv; K, s\sigma K_0) \text{ rel Lk } (uv; K, s\sigma K_0).$$

By coning, this extends to a homeomorphism

$$\text{St } (u; K, s\sigma K_0) \cong u(v+x) \text{ Lk } (uv; K, s\sigma K_0)$$

$\text{rel } uv \text{ Lk } (uv; K) = \text{St } (uv; K)$. Now homeomorph $u(v+x) \cong vx \text{ rel } (v+x)$ and join, to get a homeomorphism

$$\bar{\phi}: \text{St } (u; K, s\sigma K_0) \cong vx \text{ Lk } (uv; K, s\sigma K_0)$$

$\text{rel } v \text{ Lk } (uv; K)$. Note that $\bar{\phi}(P) = \phi(P) = x \text{ Lk } (uv; K, s\sigma K_0)$.

What $\bar{\phi}$ really is, is best illustrated by looking at K_0 .

$$\text{St } (A; K_0) = A \text{ Lk } (A; K_0) = v_0 B \text{ Lk } (A; K)$$

and

$\sigma \text{St}(A; K_0) = u_0 A \text{Lk}(A; K_0) = u_0(v_0 \dot{B} + B) \text{Lk}(A; K) \cong u_0(v_0 + x) \dot{B} \text{Lk}(A; K_0)$
 (where $su_0 = u, sv_0 = v$). Thus,

$$\text{Lk}(u; s\sigma K_0) - \text{St}^\circ(v; s\sigma K_0) = s(B \text{Lk}(A; K)).$$

At this point, we are motivated, having read Armstrong's paper, to introduce the complex

$$K_1 = (K - \text{St}^\circ(u; K)) \cup v * (\text{Lk}(u; K) - \text{St}^\circ(v; K)),$$

where the intersection of the two terms is

$$\text{Lk}(u; K) = (\text{Lk}(u; K) - \text{St}^\circ(v; K)) \cup v * \text{Lk}(uv; K).$$

We will define a homeomorphism $g: K \rightarrow K_1$ such that $(K_1; kg^{-1})$ is a triangulation of X extending $(K_0; k_0)$.

Begin by letting $g|_{K - \text{St}^\circ(u; K)} = \text{identity}$. So we must define g over $\text{St}(u; K)$. First, on $\text{St}(u; s\sigma K_0) = s(v_0 B \text{Lk}(A; K_0))$ define g to be the composite:

$$\text{St}(u; s\sigma K_0) \xrightarrow{s^{-1}} v_0 B \text{Lk}(A; K_0) \xrightarrow{v * s} v(\text{Lk}(u; s\sigma K_0) - \text{St}(v; s\sigma K_0)).$$

Note that this extends the identity on $\text{Lk}(u; s\sigma K_0) = (\text{Lk}(u; s\sigma K_0) - \text{St}^\circ(v; s\sigma K_0)) \cup v * \text{Lk}(uv; s\sigma K_0)$, on the latter term because s is simplicial and hence $= v * s$.

Let $j: vx \text{Lk}(uv; s\sigma K_0) \rightarrow vx \text{Lk}(uv; s\sigma K_0)$ be the embedding $(v * \phi)g(\phi)^{-1}$. This is a homeomorphism $\text{rel } v \text{Lk}(uv; s\sigma K_0) \cup x \text{Lk}(uv; s\sigma K_0)$.

Assuming for a moment that j is conewise homogenous, regarded as an embedding into $vx \text{Lk}(uv; K)$, then j can be covered by a homeomorphism

$$J: vx \text{Lk}(uv; K) \rightarrow vx \text{Lk}(uv; K) \text{ rel } (v + x) \text{Lk}(uv; K)$$

by Corollary IV.8 and the remarks at the end of §IV.

So letting $g = (v * \phi)^{-1} J \phi$ on $\text{St}(u; K)$ extends the previous definitions of g and $(K; kg^{-1})$ extends $(K_0; k_0)$ because $(kg^{-1})^{-1}k_0 = gs$ is simplicial on K_0 .

Thus, it suffices to show that regarded as an embedding $j: vx \text{Lk}(uv; s\sigma K_0) \rightarrow vx \text{Lk}(uv; s\sigma K)$ is conewise homogenous (with respect to either cone structure). It suffices to show that j^{-1} , regarded as an embedding is CH. $j^{-1} = \phi g^{-1} (v * \phi)^{-1}$. If we use the v -cone structure then since ϕ is a homeomorphism and $(v * \phi)^{-1} = v * (\phi^{-1})$ is a cone map, it suffices to show that

$$g^{-1}: v(\text{Lk}(u; s\sigma K_0) - \text{St}^\circ(v; s\sigma K_0)) \rightarrow \text{St}(u; K)$$

is CH and since $(v * s)^{-1} = v * s^{-1}$ is a cone map, that

$$s: v_0 B \text{Lk}(A; K_0) \rightarrow \text{St}(u; K)$$

is CH. This is clear because every open cone-line of $v_0 B \text{Lk}(A; K_0)$ lies in the interior of a simplex of K_0 and by hypothesis $d(x; K, sK_0)$ is constant on $\text{Int } sC$ for $C \in K_0$.

Thus, j is CH and the theorem is proved.

COROLLARY 3. *For a compact polyhedral pair (X, X_0) the following are equivalent:*

- a. (X, X_0) is locally unknotted.
- b. Every ambient isotopy of X_0 extends to an ambient isotopy of X .
- c. Every triangulation of X_0 extends to a triangulation of X .

Proof. $a \rightarrow b$. Corollary VI.

$a \rightarrow c$. $I^i(X, X_0) = I^i(X_0)$ which is a subcomplex of any triangulation of X_0 .

$b \rightarrow a$. b gives immediately the fact that $x \sim y$ in X_0 implies $x \sim y$ in (X, X_0) .

Apply Theorem II.23.

$c \rightarrow a$. If $d(x; X_0) = i$ then X_0 can be triangulated with x in the interior of an i -simplex. By c the same is true of (X, X_0) and hence $d(x; X) = d(x; X_0)$.

Actually, the following more general corollary is true. We leave the verification to the reader:

COROLLARY 3'. *Let $Y < X_0 < X$ be compact polyhedra. The following are equivalent:*

- a. (X, X_0) is locally unknotted at every point of $X_0 - Y$.
- b. Every ambient isotopy of X_0 rel Y , extends to an ambient isotopy of X .
- c. If $(K_0, L; h)$ is a triangulation of (X_0, Y) then if $(L; h|L)$ extends to a triangulation of (X, X_0) , then $(K_0; h)$ extends to a triangulation of X .

Theorem 2 and Corollary 3 answer some questions raised at the end of [A2] and have been stated in some form by Zeeman.

VIII. Homogenous homotopy and general position theorems.

DEFINITION 1. Call a homotopy $F: X \times I \rightarrow Y$ homogenous if for every $x \in X$, $d(F(x, t); Y)$ is constant for $0 \leq t \leq 1$.

Thus, a homogenous homotopy of a map is a homotopy which preserves intrinsic dimension in the image. Equivalently, $F: X \times I \rightarrow Y$ is homogenous if for each i , $F^{-1}(I^i(Y)) = F_0^{-1}(I^i(Y)) \times I$.

We will consider in this section theorems about "improving" a map through homogenous homotopy, i.e. minimizing self-intersection dimensions and making maps nondegenerate.

THEOREM 2. *Let $f: X \rightarrow Y$ be a p.l. map with X compact and $F: X_0 \times I \rightarrow Y$ a homogenous homotopy of $f|X_0$, where $X_0 < X$. Then there exists a homogenous homotopy $G: X \times I \rightarrow Y$ with $G_0 = f$ and $G|X_0 \times I = F$.*

Proof. Let $i: X \rightarrow B^n$ be an embedding of X into $\text{Int } B^n$ with $n - \dim X \geq 3$, then $F \times i: X_0 \times I \rightarrow Y \times B^n \times I$ defined by $F \times i(x, t) = (F(x, t), i(x), t)$ is a homogenous isotopy with $(Y \times B^n, F_0 \times i(X_0))$ a codimension ≥ 3 pair.

Hence, by Theorem VI.16, it extends to an ambient isotopy $H: Y \times B^n \times I \rightarrow Y \times B^n \times I$. Let $G(x, t) = \text{proj}_Y H(f(x), i(x), t)$.

Note that the usual proof of the homotopy extension theorem, by finding a retraction of $X \times I$ to $X_0 \times I \cup X \times 0$ fails because the resulting homotopy fails to be homogenous.

Our first application is to obtain a nondegeneracy theorem, using Theorem 2 to go by induction up the intrinsic skeleta. The inductive step requires a sharpening of the usual nondegeneracy theorem.

LEMMA 3. *Let (X, X_0) and (M, Y) be compact polyhedral pairs and $f: X \rightarrow M$ a p.l. map with $f^{-1}(Y) \subset X_0$, (M, Y) a relative n -manifold, $f|X_0$ nondegenerate, $\dim X \leq n$. Then there exists a homotopy F of f rel X_0 with $F((X - X_0) \times I) \subset M - Y$ and F_1 nondegenerate.*

Proof. Triangulate to obtain: $f: K \rightarrow M^n$, simplicial, with K_0 a full subcomplex of K , $f|K_0$ nondegenerate. L a full subcomplex of M with $f^{-1}L \subset K_0$, and (M, L) a relative n -manifold.

Order the simplices of $M - L$ in decreasing order of dimension A_1, \dots, A_p . Let ηK and ηM be deriveds such that f is simplicial. Define $D(A; f) = f^{-1}D(A; M)$ and $\dot{D}(A; f) = f^{-1}\dot{D}(A; M)$ following Cohen.

$D(A; f)$ is a subcomplex of $f^{-1}\eta A * \dot{D}(A; f)$, with $f|D(A; f)$ the restriction of the join of $f^{-1}\eta A \rightarrow \eta A$ with $f: \dot{D}(A; f) \rightarrow \dot{D}(A; M)$.

We proceed by a double induction.

Define $F(0): K \times I \rightarrow M$ to be $f(\text{proj}_K)$.

Inductively, we define $F(i): K \times I \rightarrow M$ to satisfy:

1. $F(i)_0 = F(i-1)_1$ and $F(i)|(\bigcup_{j < i} D(A_j; f) \cup K_0) \times I$ is the restriction of

$$F(i-1)_1(\text{proj}_K).$$

2. For all j , $F(i)((D(A_j; f) - \dot{D}(A_j; f)) \times I) \subset D(A_j; M) - \dot{D}(A_j; M)$.

3. $F(i)_1|D(A_i; f)$ is nondegenerate.

4. For all $j > i$, $F(i)_1$ is the map $D(A_j; f) \rightarrow D(A_j; M)$ defined by the restriction of the join of $f^{-1}\eta A_j \rightarrow \eta A_j$ and $F(i)_1|\dot{D}(A_j; f): \dot{D}(A_j; f) \rightarrow \dot{D}(A_j; M)$.

Assuming $F(1), \dots, F(p)$ are defined then the composite homotopy $F(1) + \dots + F(p)$ is a homotopy of f (since $(F(1) + \dots + F(p))_0 = F(1)_0 = f$) to a nondegenerate map (namely $F(p)_1$, by 1 and 3) which is rel K_0 (by 1) and maps $(K - K_0) \times I$ into $M - L$ (by 2).

Now assuming $F(i-1)$ is constructed, define $F(i)$ on $(\bigcup_{j < i} D(A_j; f) \cup K_0) \times I \cup K \times 0$ by condition 1. Note that $F(i-1)_1$ is nondegenerate on $\bigcup_{j < i} D(A_j; f) \cup K_0$.

To define $F(i)$ on $D(A_i; f) \times I$ note that $F(i-1)_1$ on $D(A_i; f)$ is the restriction of the join of $f^{-1}\eta A_i \rightarrow \eta A_i$ with the nondegenerate map $F(i-1)_1: \dot{D}(A_i; f) \rightarrow \dot{D}(A_i; M)$. $D(A_i; M)$ is a ball of dimension $n - \dim A_i$ which is $\geq \dim D(A_i; f) = \bigcup \{D(B; K) : fB = A_i\}$, since $\dim K \leq \dim M$, and $\dim B \geq \dim A_i$ for all such B . Hence, by standard general position on Euclidean space arguments, we can find a homotopy $F(i): D(A_i; f) \times I \rightarrow D(A_i; M)$ rel $\dot{D}(A_i; f)$, mapping $(D(A_i; f) - \dot{D}(A_i; f)) \times I$ into $D(A_i; M) - \dot{D}(A_i; M)$ and such that $F(i)_1$ is nondegenerate on $D(A_i; f)$.

To define $F(i)$ on $D(A_j; f) \times I$ for $j > i$ proceed by induction up the sequence A_{i+1}, \dots, A_p . Inductive assumption and condition 4 determine

$$F(i) \times I: \dot{D}(A_j; f) \times I \cup D(A_j; f) \times \dot{I} \rightarrow \dot{D}(A_j; M) \times I \cup D(A_j; M) \times \dot{I}.$$

By following this by a homeomorphism to $\dot{\Delta}$, with $\dim \Delta = n - \dim A_j + 1$, and subdividing to make the map simplicial, we can use linearity and the fact that $D(A_j; f)$ is a subcomplex of $f^{-1}A_j * \dot{D}(A_j; f)$ and $F(i)_0$ and $F(i)_1$ are “cone maps” to extend this to a map taking the rest of $D(A_j; f) \times I$ to $\text{Int } D(A_j; M) \times I$. Projecting to $D(A_j; M)$ extends $F(i)$ over $D(A_j; f) \times I$ to satisfy 2 and 4.

THEOREM 4. *Let $f: X \rightarrow Y$ be a map, nondegenerate on $X_0 < X$, with $\text{Cl}(X - X_0)$ compact and $\dim f^{-1}(I^i(Y)) \leq i$, then f can be homogeneously homotoped to a nondegenerate map, rel X_0 .*

Proof. By replacing X by $(X)_R$, X_0 by $(X_0)_R$ and Y by a regular neighborhood of $f((X)_R)$, if necessary, we can assume X, X_0 and Y are compact. With this assumption we proceed by induction up the intrinsic skeleta of Y . More precisely, let $m_f = \min \{i : X - X_0 \subset f^{-1}(I^i(Y))\}$. Assuming the result for maps f , with $m_f < n$, we prove it for $m_f = n$.

By inductive hypothesis, we can homogeneously homotope $f|_{f^{-1}I^{n-1}(Y)}$ to a nondegenerate map, rel X_0 . By Theorem 2, this extends to a homogenous homotopy rel X_0 of f to a map \tilde{f} nondegenerate on $X_0 \cup f^{-1}(I^{n-1}(Y))$. $(X, X_0 \cup f^{-1}I^{n-1}(Y))$ and $(I^n(Y), I^{n-1}(Y))$ satisfy the hypotheses of Lemma 3, with $\tilde{f}: X \rightarrow I^n(Y)$, since $m_f = n$ and the homotopy of f to \tilde{f} was homogenous. Hence, by Lemma 3, we can homotope rel $X_0 \cup f^{-1}I^{n-1}(Y)$ with $X - X_0 \cup f^{-1}I^{n-1}(Y)$ kept in $I^n(Y) - I^{n-1}(Y)$ and go from \tilde{f} to a nondegenerate map, g . The composite homotopy $f \sim \tilde{f} \sim g$ is the required homotopy of f to a nondegenerate map.

We will say that two subspaces X and Y are in general position in a polyhedron Q , if $\dim X \cap Y - I^{i-1}(Q) \leq \dim X + \dim Y - i$, for all i .

To prove that we can ambient isotope subspaces to general position, we need the analogue of Lemma 3, for general position.

LEMMA 5. *Let (M, Y) be a compact, relative n -manifold with $X_0, X, R_1, \dots, R_q < M$ and $Y \cap X < X_0$. Then there is an ambient isotopy of identity on M to $f: M \rightarrow M$ rel $X_0 \cup Y$ such that $\dim f(X - X_0) \cap R_i \leq \dim X - X_0 + \dim R_i - n$.*

Proof. Triangulate (X, X_0) and order the simplices of $X - X_0$ in increasing order of dimension A_1, \dots, A_p . Proceed inductively with the i th inductive step an ambient isotopy of M rel $X_0 \cup Y \cup A_1 \cup \dots \cup A_{i-1}$ moving the image of A_i under the $i-1$ st step mod its boundary to something in general position with respect to the R_j 's. This is done by restricting the motion to be supported by a regular neighborhood of $f_{i-1}(A_i)$ mod $f_{i-1}(A_i)$. This will define f_i and the composite ambient isotopy to f_p will be the required isotopy.

THEOREM 6. *Let $X, X_0, R_1, \dots, R_q < Q$ with $\text{Cl } X - X_0$ compact. Then there exists an ambient isotopy $\text{rel } X_0$ of identity on Q to $f: Q \rightarrow Q$ with $f(X - X_0)$ in general position with respect to each of the R_i 's.*

Proof. By replacing X by $(X)_R$, Q by a regular neighborhood V of $(X)_R$ in Q , X_0 by $\dot{V} \cup (X_0 \cap V)$ and R_i by $R_i \cap V$, if necessary, we can assume that Q is compact. With this assumption we proceed by induction on

$$m_X = \min \{i : X - X_0 \subset I^i(Q)\}.$$

Assuming the result when $m_X < n$, we prove it for $m_X = n$.

By inductive hypothesis there is an ambient isotopy $\text{rel } X_0$ of Q to $f_1: Q \rightarrow Q$ such that $f_1(X \cap I^{n-1}(Q) - X_0)$ is in general position with respect to the R_i 's.

By Lemma 5, we can ambient isotope $I^n(Q) \text{ rel } (X_0 \cap I^n(Q)) \cup I^{n-1}(Q)$ to $g: I^n(Q) \rightarrow I^n(Q)$ so that $gf_1[X - ((X_0 \cap I^n(Q)) \cup I^{n-1}(Q))]$ is in general position with respect to the $R_i \cap I^n(Q)$.

By Corollaries VI.11 and VI.12, this extends to an ambient isotopy of Q . So the composite of the isotopy to f_1 and this extension give the required ambient isotopy.

Hudson's proof of [H; Lemma 4.7] produces general position for maps, using general position for subspaces. The homotopies he constructs are ambient isotopies on their supports. In a polyhedron, consider a homotopy $F: (X_0 \cup X_1) \times I \rightarrow Y$, which is constant on $X_0 \times I$ and which on $X_1 \times I$ is equal to $G \cdot (F_0 \times I)$ where $G: Y \times I \rightarrow Y$ is an ambient isotopy. Such a homotopy is clearly homogenous. Thus, Hudson's method of proof using Theorem 6, instead of [H; Lemma 4.6] gives the following analogue of [H; Lemma 4.7]:

THEOREM 7. *Let (K, K_0) be a complex pair with $K - K_0$ finite and $f: K \rightarrow Q$ and a map which embeds each simplex of K in the polyhedron Q , and $R_1, \dots, R_q < Q$. Then there is a homogenous homotopy $\text{rel } K_0$ of f to $f': K \rightarrow Q$ such that:*

1. f' embeds each simplex of K .
2. If A_1, \dots, A_r are distinct simplices of K , then

$$\dim \bigcap_{i=1}^r f'(\text{Int } A_i) - I^{m-1}(Q) \leq \sum_{i=1}^r \dim A_i - (r-1)m,$$

$$\dim \bigcap_{i=1}^r f'(\text{Int } A_i) \cap R_j - I^{m-1}(Q) \leq \sum_{i=1}^r \dim A_i + \dim R_j - rm,$$

for all j and m .

In particular, if $f: X \rightarrow Q$ is a nondegenerate map with $\dim S_r(f) - f^{-1}I^{m-1}(Q) \leq r \dim X - (r-1)m$ for all r and m , then we call f a general position map.

COROLLARY 8. *Let $f: X \rightarrow Q$ be a map with $f|_{X_0}$ a general position map $X_0 < X$ (compact), then f can be homogeneously homotoped $\text{rel } X_0$ to a general position map.*

Examining this outline of a theory of general position for polyhedra, we see two major flaws.

First, we do not have simplicial or p.l. approximation theorem, i.e. let $f: X \rightarrow Q$ be a topological map of polyhedra such that for all $m, f^{-1}I^m(Q)$ is a subpolyhedron of X . Can we homogeneously homotope f to a p.l. map?

Secondly, our theory has no ε -approximation theorems. Lemmas 3 and 5 can probably be sharpened to get arbitrarily close approximations. Less obviously, Theorem 2 can be strengthened to extend small homotopies to small homotopies. Here the trick is to construct a retraction $r: X \times I \rightarrow X_0 \times I \cup X \times 0$ such that $F \cdot r$ is a small homotopy and then use instead of $G, G \cdot r'$, where $r': X \times I \rightarrow X \times I \text{ rel } X_0 \times I \cup X \times 0$ is a close approximation to r and has the additional property that $Gr'(x, t) \sim G(x, t)$ in Y . This will introduce ε 's into Theorem 4 and allow arbitrarily close approximation by nondegenerate maps. To introduce ε 's into Theorem 6 and hence Theorem 7 one needs to have them in Theorem VI.11.

Stallings' treatment of general position in [S; Chapter V] is rather different from ours. His results on homotoping maps to general position are stronger than ours (and also more useful) but he has no ambient isotopy theorems and our results are the best possible there. To "localize" Stallings' argument and show its connection with our own we introduce:

DEFINITION 8. Let $x \in X$, then define Stallings' dimension

$$d_s(x; X) = \max \{n : \text{Lk}(x; X) \text{ is an } (n-2)\text{-connected } ND(n-1) \text{ space}\}.$$

From the properties of $ND(n)$ spaces developed in [S; Chapter V], it is clear that $d_s(x; X) \geq d(x; X)$. Furthermore, X is an $ND(n)$ space iff for all $x \in X$,

$$d_s(x; X) \geq n.$$

Letting $S^i(X) = \{x : d_s(x; X) \leq i\}$, we note that $S^i(X) < I^i(X)$ (it is a subpolyhedron because $x \sim y$ in X implies $d_s(x; X) = d_s(y; X)$ and if $d_s(x; X) = n$ then every point of $\text{St}^\circ(x; X)$ has as link an $(n-2)$ connected $ND(n-1)$ space and so has Stallings dimension at least n).

We would conjecture that a map $f: X \rightarrow Q$ can be homotoped to a map such that

$$\dim S_r(f) - f^{-1}S^{m-1}(Q) \leq r \dim X - (r-1)m.$$

A proof along the lines of our proofs in this section would require the analogue of Theorem 2, for homotopies homogeneous with respect to d_s , and relative versions of Stallings' theorems on the order of Lemma 3.

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