VISCOUS FLUIDS, ELASTICITY AND FUNCTION-THEORY. I

BY

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Introduction. In this first paper, we are going to consider the properties of pairs of functions $\{u(x, y), v(x, y)\}$ of two real variables (x, y), which for some fixed k satisfy the system of differential equations

$$(0.1)\qquad \begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = \theta, \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = \omega, \\ (k+1)\dfrac{\partial \theta}{\partial x} + \dfrac{\partial \omega}{\partial y} = 0, \\ (k+1)\dfrac{\partial \theta}{\partial y} - \dfrac{\partial \omega}{\partial x} = 0. \end{cases}$$

k is any real number such that $k+1\neq 0$.

We set $z=x+iy$ and $f(z)=u+iv$ and call $f(z)$ a bi-analytic function of z of type k. The main purpose of the paper is to show that all the elementary properties of analytic functions can be extended to bi-analytic functions. The fundamental reason for this is that the first two equations of (0.1) are merely an inhomogeneous form of the Cauchy-Riemann equations for $u+iv$ while the last two imply that $\{(k+1)\theta-i\omega\}$ is analytic. $\phi(z)=(k+1)\theta-i\omega$ will be called the associated analytic function of $f(z)=u+iv$.

If an elastic body is in a state of plane strain in a plane parallel to the (x, y) plane, then its elastic properties are determined by two functions $u(x, y)$ and $v(x, y)$, the x and y components of the displacement of a particle. The known equations of classical elasticity theory show that $u-iv$ is a bi-analytic functions of type k. There is, then, a geometric interpretation of θ and ω; θ is the "dilatation" and ω the "rotation." What happens in elasticity, therefore, is analogous to the situation in the theory of the two-dimensional flow of a perfect fluid where the velocity components u and v are such that $u-iv$ is analytic.

In problems of plane strain $k=\lambda/\mu+1$, where λ and μ are the usual Lamé's constants. Therefore $k>1/3$. Problems of generalized plane stress give rise to the same equations, only now

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$$k = \frac{2(\lambda/\mu)}{2 + (\lambda/\mu)} + 1 \quad \text{and so} \quad 3 > k > 0.$$

Thus bi-analytic functions of type k with $k>0$ are of interest in mathematical physics. However, up to now it is only functions of type -2 that have attracted mathematical interest. In fact, when $k=-2$, equations (0.1) have the form that, as Haskell has shown, characterize areolar-monogenic functions [1]. The theory of these functions has been fairly well developed, notably by Kriszten [2]. It turns out, however, as a consequence of the theory developed in this paper, that the case $k=-2$ is very special and the extension from $k=-2$ to any k is nontrivial. This is what one might expect, of course, from the fact that it is the cases $k\neq -2$ that occur in physical theories.

We shall also introduce bi-analytic functions of type -1. These are functions $f(z)=u+iv$ such that u and v satisfy the equations

$$\text{(0.2)} \qquad \begin{cases} u_x - v_y = \theta, \\ u_y + v_x = 0, \\ \theta_x - \omega_y = 0, \\ \theta_y + \omega_x = 0. \end{cases}$$

Again $(\theta+i\omega)$ is called the associated analytic function of $f(z)$.

These equations are not a special case of (0.1). However, the formulae we derive for functions of type k ($k\neq -1$) will turn out to be valid for functions of type -1 provided k is set equal to -1, ω to zero and $\omega/(k+1)$ is replaced by $-\omega$.

Equations of the form (0.2) have been introduced by Lauricella in the study of elastic plates [3]. Under the usual assumptions the fundamental problem of the theory of plates reduces to the Dirichlet problem for the biharmonic equation, namely, finding a function $\phi(x, y)$ such that $\Delta(\Delta\phi) = \Delta^2\phi = 0$ in a domain D with ϕ_x and ϕ_y given on the boundary of C of D $[\Delta\equiv\partial^2/\partial x^2+\partial^2/\partial y^2]$. By setting $u=\phi_x$ and $v=-\phi_y$, Lauricella reduces this problem to the problem of determining what we have called a bi-analytic function of type -1 with prescribed values on the boundary. Although Lauricella introduced equations (0.2) he did not notice that they could be made the basis for a generalization of the classical theory of functions of a complex variable. This was done by the present author [4]. However, that paper was mainly concerned with generalizations of (0.2) obtained by putting some nonconstant coefficients into the equations. Consequently, it did not obtain many of the results we shall derive here for bi-analytic functions.

Functions of type infinity can also be defined. These are functions $f(z) = u+iv$ such that

$$(0.3)\qquad \begin{cases} u_x - v_y = 0, \\ u_y + v_x = \omega, \\ \theta_x - \omega_y = 0, \\ \theta_y + \omega_x = 0. \end{cases}$$

Once more $\theta + i\omega$ is called the associated function of $f(z)$.

However, we shall not concern ourselves with such functions since, if $f(z) = u + iv$ is of type infinity with associated function $\phi(z)$ then $if(z) = -v + iu$ is of type -1 with associated function $i\phi(z)$. In this way the theory of these functions is immediately reduced to the theory of functions of type -1.

Bi-analytic functions of type -1 occur not only in the theory of elastic plates but also in the theory of viscous fluids. The equations governing the incompressible two dimensional flow of a viscous fluid with no "body forces" are

$$(0.4)\qquad \begin{cases} \mu\Delta u = p_x, \\ \mu\Delta v = p_y, \\ u_x + v_y = 0 \end{cases}$$

where u and v are the components of the velocity in the x and y directions respectively, μ is the coefficient of viscosity and so is >0 and p is the pressure. Here we have assumed that the velocities are so small that the nonlinear terms in the equations can be neglected. This is the approximation first introduced by Stokes.

We shall show later that equations (0.4) are equivalent to the equations

$$(0.5)\qquad \begin{cases} u_x + v_y = 0, \\ u_y - v_x = \zeta, \\ \dfrac{1}{\mu} p_x - \zeta_y = 0, \\ \dfrac{1}{\mu} p_y + \zeta_x = 0 \end{cases}$$

and so $(u - iv)$ is bi-analytic of type ∞ with associated function $((1/\mu)p + i\zeta)$ where ζ is the vorticity. By our previous result this fact can also be expressed by saying that $(v + iu)$ is bi-analytic of type -1 with associated function $(-\zeta + ip/\mu)$. However, calling $(u - iv)$ bi-analytic has the advantage of preserving the analogy with the theory of perfect fluids.

We now sketch the lines of development of the paper.

Chapter I first considers various forms of the equations defining bi-analytic functions and shows their equivalence. These considerations lead to the theorem that every biharmonic function is the real or imaginary part of a bi-analytic function with $k \neq 0$. (The case $k = 0$ is obviously very special since the real and imaginary parts of a bi-analytic function of type 0 are harmonic.) This has been proved by Kriszten for the case $k=-2$ but only when the domain is rectangular [2]. By using contour integrals we have removed this restriction on the domain.

Next, the problem of constructing a bi-analytic function with a given associated function is solved.

Then, some simple formulae are obtained for converting bi-analytic functions of type K into functions of a different type, k. The first kind of formula does this by taking a linear combination of two functions of type k. The result obtained shows that functions of type -1 arise as the residue at $k=-1$, of families of bi-analytic functions of type k where $k\neq -1$. The second kind of formula converts a function of type K into one of type k by adding a harmonic function to the real or imaginary part of the function of type k.

Finally the integral and derivative of a bi-analytic function are defined and shown to have the properties connoted by those words. "Cauchy's" theorem, that the integral of a bi-analytic function around a simple closed curve is zero provided the function is bi-analytic inside the curve, is an immediate consequence of the definition of the integral. The analogues of Morera's theorem and Weierstrass' double-series theorem easily follow. This section is clearly motivated by the paper of Bers and Gelbart on a generalization of the Cauchy-Riemann equations [5].

Chapter II discusses the algebra of bi-analytic functions.

It is first shown that the only transformations of the independent variables x and y that transform bi-analytic functions into bi-analytic functions are linear. It is also shown that linear transformations of the dependent variables u and v are the only such transformations permitted.

However, in addition to these purely negative results, we have made a fundamental discovery showing how to "multiply" a bi-analytic function with an analytic function so as to obtain a bi-analytic function. The product function has as associated function the product of the original associated function and the multiplying analytic function. This "multiplying" is not strictly algebraic since it involves two integrals. However, these integrals are only integrals of analytic functions. The special nature of functions of type -2 becomes apparent here.

The product is not unique; it depends on the lower limits of the integrals and on two real-valued parameters. By choosing these parameters in certain ways we obtain four particular products which are called the left and right cross products and the left and right dot products. Finally the associative law and the usual rule for differentiating products are shown to hold provided the above mentioned lower limits are chosen in a suitable manner.

Chapter III constructs bi-analytic functions denoted by $Z^{(\alpha)}$, $E(z)$, $L(z)$, $C(z)$ and $S(z)$ that have properties similar to those of the analytic functions z^{α}, e^{z}, $\log z$, $\cos z$ and $\sin z$ respectively. These functions are obtained by multiplying the corresponding analytic function with a simple bi-analytic function whose associated function is 1.

Some of these "elementary" functions give examples of the fact that the zeros of bi-analytic functions are not isolated. This is what is to be expected, of course, from the behaviour of elastic bodies.

We next give, in Chapter IV, very simple derivations of results, for bi-analytic functions, similar to Taylor's and Laurent's theorems for analytic functions. These results are obtained without using anything corresponding to Cauchy's Integral Formula.

However, we then go on to give a form of "Cauchy's" formula, which determines the values of a bi-analytic function inside a simple closed curve in terms of the boundary values of both the function and the associated function.

This formula is then applied to various problems. First, the existence of all the derivatives of a bi-analytic function is proved. Then a uniformly convergent series of bi-analytic functions is shown to be differentiable term-by-term. This result is then used to obtain simple formulae for the coefficients in "Taylor's" series. We thus obtain the Taylor's series of the elementary functions.

Finally we attempt to classify the isolated singularities of the functions. We are left with some unsolved problems on essential singularities.

Chapter V views some of the classical results of elasticity theory from the stand-point of the theory developed in the previous chapters. In particular our "Cauchy's" formula is shown to be obtainable from Betti's reciprocal theorem by integrating by parts.

In Chapter VI we first show that the equations of viscous fluid flow can be put into the form already indicated in this introduction.

It is planned to discuss some more general systems of equations, with nonconstant coefficients, in Part II.

Chapter I. Preliminaries: Derivative and integral

1. Fundamental relations. We first give exact definitions of bi-analytic functions:

Definition I. *A complex-valued function $f(z)=u(x, y)+iv(x, y)$ of the variable $z=x+iy$ will be said to be bi-analytic of type $k(\neq -1)$ in a domain D of the z-plane provided u and v have continuous derivatives of the first order in D satisfying*

$$(1.1)\qquad \begin{aligned} u_x - v_y &= \theta, \\ u_y + v_x &= \omega \end{aligned}$$

where $(k+1)\theta - i\omega$ is an analytic function of z in D, that is, θ and ω are continuous and have first derivatives in D satisfying the Cauchy-Riemann equations

$$(1.1)\qquad \begin{aligned}(k+1)\theta_x + \omega_y &= 0,\\ (k+1)\theta_y - \omega_x &= 0.\end{aligned}$$

DEFINITION II. *A complex-valued function* $f(z)=u(x, y)+iv(x, y)$ *of* $z=x+iy$ *will be called bi-analytic of type* -1 *in a domain* D *of the* z*-plane provided* u *and* v *have continuous derivatives of the first order in* D *satisfying*

$$(1.2)\qquad \begin{aligned}u_x - v_y &= \theta,\\ u_y + v_x &= 0\end{aligned}$$

and provided that there exists a definite function ω *such that*

$$(1.2)\qquad \begin{aligned}\theta_x - \omega_y &= 0,\\ \theta_y + \omega_x &= 0.\end{aligned}$$

In Definition II, we might have proceeded differently and used the condition $\Delta\theta=0$. However then ω would only be determined up to a constant. From the present viewpoint all functions with different ω's are considered as different functions even if the ω's differ only by a constant.

For convenience, a bi-analytic function of type k will be denoted by B.A.F. (k). $(k+1)\theta-i\omega$ or, when k is -1, $\theta+i\omega$, is called the associated function of $f(z)$ and will be denoted by A.F. The real and imaginary parts u and v of $f(z)$ will be said to be biharmonic conjugates, or sometimes, k-conjugates.

The main arguments of the paper will be conducted with functions of type k with $k\neq -1$ and the results for functions of type -1 will, for the most part, obtain only passing notice. We have adopted this procedure since most of the proofs when k is -1 follow on the same lines as when $k\neq -1$; whenever this is not so we shall note the differences.

It will be assumed from the start that u and v have continuous derivatives of all orders. This will be proved in the chapter on "Cauchy's" integral; this is a valid procedure since that chapter does not depend on preceding work.

From (1.1) u or v can be eliminated by differentiating. In this manner we get the following set of equations:

$$(1.3)\qquad \begin{cases}\Delta u = -\,k\theta_x,\\ \Delta v = +\,k\theta_y,\\ \;\theta = u_x - v_y.\end{cases}$$

These are the more usual equations of elasticity in the cases of plane strain and generalized plane stress.

One immediate consequence of (1.3) is that u and v are harmonic when $k=0$. Otherwise, since θ and so θ_x and θ_y are harmonic, we conclude that u and v are biharmonic, that is

$$\Delta(\Delta u) = \Delta^2 u = 0, \quad \Delta^2 v = 0. \tag{1.4}$$

When equations (1.3) are differentiated and subtracted we find

$$\Delta(u_x - v_y) = -k(\theta_{xx} + \theta_{yy})$$

or

$$(k + 1)\Delta\theta = 0.$$

Thus, if $k+1\neq 0$, equations (1.3) alone show that θ is harmonic. When $k=-1$, the equation $\Delta\theta=0$ should be added to (1.3).

We now show that not only does (1.3) follow from (1.1) but that the converse statement is also true.

THEOREM 1. *If* (1.3) *holds, then there exists a function* ω *such that* (1.1) *is true.*

Proof. Since, as has already been shown, (1.3) implies that $\Delta\theta=0$, there exists an ω, determined to within a constant, such that $(k+1)\theta-i\omega$ is analytic.

We shall show that $u_y+v_x=\omega$. Now,

$$\begin{aligned}\omega_x &= (k + 1)\theta_y \\ &= (k + 1)(u_{xy} - v_{yy}) \\ &= (u_{xy} + v_{xx}) + k(u_{xy} - v_{yy}) - (v_{xx} + v_{yy}) \\ &= (u_y + v_x)_x + k\theta_y - \Delta v \\ &= (u_y + v_x)_x.\end{aligned}$$

Similarly,

$$(u_y + v_x)_y = \omega_y.$$

Therefore,

$$\omega = (u_y + v_x) + \text{constant}.$$

Since ω is only determined to within a constant, this constant can be adjusted so that

$$u_y + v_x = \omega \qquad \text{Q.E.D.}$$

Another way of proceeding is to define ω immediately as $\omega=u_y+v_x$. Then from this and $u_x-v_y=\theta$ we have

$$\Delta u = \theta_x + \omega_y \quad \text{and so} \quad \theta_x + \omega_y = -k\theta_x \quad \text{or} \quad (k + 1)\theta_x + \omega_y = 0.$$

Similarly $(k+1)\theta_y-\omega_x=0$.

This last proof is due to Love [6]. When $k=-1$, and so the equation $\Delta\theta=0$ is added to the set (1.3), we can not proceed in Love's manner. However, the proof given above does show that $u_y+v_x=$ a constant $=a$, and so we can conclude that $(u-ay)+iv$ is B.A. (-1). Also if u and v in addition to satisfying this modification of (1.3) inside D satisfy $\oint_c udx-vdy=0$ where c is the boundary of D then we can conclude that $a=0$. For,

$$a\iint_D dxdy = \iint_D (u_y + v_x)dxdy$$
$$= \oint_c - udx + vdy = 0$$

and so

$$a = 0.$$

The condition $\oint_c udx-vdy=0$ is not artificial; it occurs in Lauricella's formulation of the elastic plate problem.

We now show that any biharmonic function is the real part of a B.A.F. with $k\neq 0$, or -1. In the proof we freely use both equations (1.1) and their equivalent, (1.3).

THEOREM 2. *If u is any function satisfying $\Delta^2 u=0$ inside D then there exists a biharmonic function v such that $u+iv$ is B.A. (k) with $k\neq 0$ or -1.*

Proof. Given u, set $\phi=\Delta u$. (1) Therefore, $\Delta\phi=0$. Therefore, there exists a function ψ such that $(\phi+i\psi)$ is analytic.

Let

$$-k\theta + i\frac{k}{k+1}\omega = \int_{z_0}^{z}(\phi + i\psi)dz$$

(2)
$$= \int_{z_0}^{z}\phi dx - \psi dy$$
$$+ i\int_{z_0}^{z}\psi dx + \phi dy$$

so that $\{(k+1)\theta-i\omega\}$ is analytic. This definition is suggested by equations (1.3).

The function v is now defined by

(3)
$$v = \int_{z_1}^{z}(-u_y + \omega)dx + (u_x - \theta)dy.$$

We show that this is a proper definition by showing that this integral is independent of the path of integration, that is, that

$$(-u_y + \omega)_y = (u_x - \theta)_x$$

or

$$-u_{yy} + \omega_y = u_{xx} - \theta_x$$

or

$$\Delta u = \theta_x + \omega_y$$

or

$$\phi = \theta_x + \omega_y.$$

Now, by our definition of θ and ω,

$$\theta_x = \frac{1}{k}\phi,$$

$$\omega_y = \frac{k+1}{k}\phi.$$

Therefore $\theta_x + \omega_y = \phi$.

It is now easily verified that $u+iv$ is B.A. For from (3),

$$v_x = -u_y + \omega \quad \text{or} \quad u_y + v_x = \omega$$

and

$$v_y = u_x - \theta \quad \text{or} \quad u_x - v_y = \theta,$$

while we have already seen that $\{(k+1)\theta - i\omega\}$ is analytic. Q.E.D.

A similar proof shows that starting with a biharmonic function v, a function u can be found such that $(u+iv)$ is B.A.

If the domain D is multiply-connected then, in general, the conjugate function is multi-valued, as in the case of harmonic functions. This gives a method of constructing multi-valued biharmonic functions; for if u is a function with an isolated singularity, the function v will usually turn out to be non-singlevalued. Some types of multi-valued functions are important in elasticity theory where they are realized as dislocations.

As an example let $u = x \log r$ where $r^2 = x^2 + y^2$. Then the method of Theorem 2 shows that

$$v = \frac{2+k}{k} y \log r + \frac{2(k+1)}{k} x \arctan \frac{y}{x} - \frac{2+k}{k} y.$$

If $k=0$, the conclusion of Theorem 2 cannot be true since the real and imaginary parts of a B.A.F. (0) are merely harmonic. If k is -1 the result is still true but the above proof breaks down; for this case a proof on entirely different lines has been given in [4].

The conjugate v of a biharmonic function u is not uniquely determined by u. This is shown up in the proof of Theorem 2. There, ψ is determined only to within an additive constant. Since z_0 is arbitrary this means that a term $-\beta + (A/(k+1))y$ can be added to θ and a term $\alpha + Ax$ to ω where

α, A, β are arbitrary constants. Again, since z_1 is arbitrary, v, as defined by (3), can have a term

$$\gamma + \alpha x + \beta y - \frac{1}{2}\frac{A}{k+1}y^2 + \frac{1}{2}Ax^2$$

added to it, where A, α, β, γ are arbitrary constants.

We now show that this is the most general form possible for v, that is, if v_1 and v_2 are any two conjugates of u then

$$v_1 = v_2 + \gamma + \alpha x + \beta y - \frac{1}{2}\frac{A}{k+1}y^2 + \frac{1}{2}Ax^2,$$

for some real constants α, β, γ, A. The converse statement, that if v_2 is conjugate to u the function v_1 defined as in the above equation is also conjugate, is fairly obvious.

To prove our assertion, set $v = v_1 - v_2$. Then, v is the conjugate of 0, that is,

$$-v_y = \theta, \qquad v_x = \omega$$

where

$$0 = (k+1)\theta_x + \omega_y = (k+1)(-v_{yx}) + v_{xy} = -kv_{yx}$$

and

$$0 = (k+1)\theta_y - \omega_x = -(k+1)v_{yy} - v_{xx}.$$

From the first equation, since we have assumed $k \neq 0$, $v = f(x) + g(y)$. Therefore $(k+1)g''(y) = -f''(x) = -A$ where A is a constant. Therefore

$$f(x) = \frac{1}{2}Ax^2 + \alpha x + \gamma_1$$

and

$$g(y) = -\frac{1}{2}\frac{A}{k+1}y^2 + \beta y + \gamma_2$$

and so

$$v = \gamma + \alpha x + \beta y - \frac{1}{2}\frac{A}{k+1}y^2 + \frac{1}{2}Ax^2, \qquad \text{Q.E.D.}$$

The result for $k = -1$ is that $v = \alpha + \beta y + \gamma y^2$. This has been shown in [4].

By the same method it can be shown that if u_1 and u_2 are two real parts of B.A.F.'s with the same imaginary part then

$$u_1 = u_2 + \gamma + \alpha x + \beta y - \frac{1}{2}\frac{A}{k+1}x^2 + \frac{1}{2}Ay^2$$

provided $k \neq -1$. When k is -1, the result is that

$$u_1 = u_2 + \alpha + \beta x + \gamma x^2.$$

We next consider a natural question. Given an analytic function $(k+1)\theta - i\omega$, can a B.A.F. (k) be constructed with this function as its A.F.? Clearly if the problem has a solution the solution is not unique since if any analytic function is added to a B.A.F. the sum is a B.A.F. with the same A.F. In the next theorem we give one solution of the problem.

Theorem 3. *If* $\{(k+1)\theta - i\omega\}$ *is any analytic function then* $(u+iv)$ *is a B.A.F.* (k) *with* $\{(k+1)\theta - i\omega\}$ *as its A.F., where*

$$(1.5) \qquad \begin{aligned} u &= -\frac{k}{2}x\theta, \\ v &= \int_{z_0}^{z}\left(\frac{k}{2}x\theta_y + \omega\right)dx - \left(\frac{k}{2}x\theta_x + \frac{k+2}{2}\theta\right)dy, \end{aligned}$$

provided $k \neq -1$.

Proof. We first verify that v is well-defined, that is, that the above integral is independent of the path of integration. The condition for this is that

$$\left[\frac{k}{2}x\theta_y + \omega\right]_y + \left[\frac{k}{2}x\theta_x + \frac{k+2}{2}\theta\right]_x = 0$$

or

$$\frac{k}{2}x\Delta\theta + (k+1)\theta_x + \omega_y = 0$$

and this is obviously true. Next,

$$u_x = -\frac{k}{2}\theta - \frac{k}{2}x\theta_x, \qquad u_y = -\frac{k}{2}x\theta_y,$$

$$v_x = \frac{k}{2}x\theta_y + \omega, \qquad v_y = -\frac{k}{2}x\theta_x - \frac{k+2}{2}\theta,$$

and so

$$u_x - v_y = \theta,$$
$$u_y + v_x = \omega. \qquad \text{Q.E.D.}$$

Again, if the domain is multiply connected the v defined in (1.5) is, generally, multi-valued even when θ and ω are single-valued.

When k is -1, (1.5) is still valid, provided it is modified by setting $k=-1$ and $\omega=0$; that is $(u+iv)$ is a B.A.F. (-1) with $\theta+i\omega$ as its A.F. where

$$u = \frac{1}{2} x\theta,$$

$$v = \int_{z_0}^{z} \left(-\frac{1}{2} x\theta_y\right) dx + \left(\frac{1}{2} x\theta_x - \frac{1}{2}\theta\right) dy.$$

When $k=0$, (1.5) still holds and shows that

$$u = 0, \qquad v = \int_{z_0}^{z} \omega dx - \theta dy = \operatorname{Im} \int_{z_0}^{z} (\theta - i\omega) dz$$

are the real and imaginary parts of a B.A.F. (0) with A.F. $\theta - i\omega$. This shows that every B.A.F. (0) is of the form

$$u + iv = (\text{an analytic function}) + i\,(\text{a harmonic function}).$$

This emphasizes the relatively trivial nature of functions of type 0.

2. **Conversion.** A natural question to ask is the following: In what way does a B.A.F. (k) depend upon the parameter k? No very precise answer can be given to this question but we have found some simple formulae that give some information on the subject. First, we have the following result:

THEOREM 4. *Suppose $f(z)$ is a B.A.F. (K) whose A.F. is $(K+1)\,\theta - i\omega$ and suppose also that $g(z)$ is a B.A.F.(K) whose A.F. is $i[(K+1)\theta - i\omega]$. Then*

$$F(z) = \frac{Kk + K + k}{K - k} f(z) + ig(z) \tag{1.6}$$

is a B.A.F. of type k whose A.F. is

$$(k+1)\Theta - i\Omega = \frac{K(K+2)(k+1)}{(K-k)(K+1)} [(K+1)\theta - i\omega]$$

provided $K \neq -1$ or k, and $k \neq -1$.

Proof. This result is easily checked by a direct computation. Q.E.D.

For $K=0$ and -2 the function $F(z)$ in (1.6) is analytic.

If k is -1, (1.6) can be modified to

$$F(z) = -\frac{1}{K+1} f(z) + ig(z)$$

whose A.F. is

$$\Theta + i\Omega = \frac{K(K+2)}{(K+1)^2}\left\{(K+1)\theta - i\omega\right\} \cdot$$

A result similar to Theorem 4 is the following:

THEOREM 5. *If $f(z)$ and $g(z)$ have the same meanings as in Theorem 4, then*

$$F(z) = -if(z) + \frac{Kk+K+k}{K-k}\, g(z) \tag{1.7}$$

is a B.A.F. (k) whose A.F. is

$$(k+1)\Theta - i\Omega = \frac{K(K+2)(k+1)}{(K-k)(K+1)}\, i\left\{(K+1)\theta - i\omega\right\}$$

provided $K \neq -1$ or k, and $k \neq -1$.

If k is -1 the formula can be modified to

$$F(z) = -if(z) - \frac{1}{K+1}\, g(z)$$

whose A.F. is

$$\Theta + i\Omega = \frac{K(K+2)}{(K+1)^2}\, i\left\{(K+1)\theta - i\omega\right\}.$$

We shall also omit the proof here and go on to the case $K=-1$.

THEOREM 6. *Suppose $f(z)$ is a B.A.F. (-1) whose A.F. is $\theta+i\omega$ and that $g(z)$ is a B.A.F. (-1) whose A.F. is $i(\theta+i\omega)$. Then*

$$F(z) = \frac{1}{k+1} f(z) + ig(z) \tag{1.8}$$

is a B.A.F. (k) whose A.F. is $\theta+i\omega$. Also

$$G(z) = -if(z) + \frac{1}{k+1}\, g(z) \tag{1.9}$$

is a B.A.F. (k) whose A.F. is $i(\theta+i\omega)$.

Here, k is not equal to -1.

The proof again is direct and simple.

Formula (1.8) shows the following: If $f(z)$ is a B.A.F. (-1) which is an integral function of a parameter k then $f(z, k=-1)$ is the residue, at $k=-1$, of a B.A.F. (k). Conversely, if $F(z)$ is a B.A.F. (k) whose A.F. does not depend on k, then $F(z)$ has, modulo an analytic function of z, a simple pole at $k=-1$ with residue a B.A.F. (-1).

The above results are very interesting but are not examples of what we shall call "conversion." This term will be used for the process of changing a B.A.F. (K) into a B.A.F. (k) by adding a harmonic function to the function of type K, this harmonic function being related to the integral of the A.F. Our result in this direction is the following:

THEOREM 7. *If $f(z)$ is a B.A.F. (K), $K\neq -1$, whose A.F. is $(K+1)\theta-i\omega$, then*

$$F(z) = f(z) + \Theta \tag{1.10}$$

where

$$(K+1)\Theta - i\Omega = \frac{K-k}{k}\int_{z_0}^{z}[(K+1)\theta - i\omega]dz,$$

is a B.A.F. (k), provided k is not -1, whose A.F. is

$$\frac{K(k+1)}{k(K+1)}[(K+1)\theta - i\omega].$$

Again the proof is trivial and will not be given.

The formula for the A.F. of $F(z)$ shows that (1.10) can not be used to convert a B.A.F. (K) into a function of type 0. It is easily seen that the only case in which a harmonic function can be added to a B.A.F. (K) to give a B.A.F. (0) is when $K=0$ and then the harmonic function can be quite arbitrary.

If K is -1, then (1.10) still holds provided

$$\Theta + i\Omega = -\frac{k+1}{k}\int_{z_0}^{z}(\theta + i\omega)dz$$

where $(\theta+i\omega)$ is the A.F. of $f(z)$. The associated function of $F(z)$ is now

$$-\frac{k+1}{k}(\theta + i\omega).$$

In the case $k=-1$, $K\neq -1$, (1.10) is again valid provided Θ is defined by the equation

$$(K+1)\Theta - i\Omega = -(K+1)\int_{z_0}^{z}[(K+1)\theta - i\omega]dz.$$

The A.F. of $F(z)$ is now $[-K/(K+1)][(K+1)\theta-i\omega]$.

In the future we shall say we are converting $f(z)$, a B.A.F. (K), not only when we form the function $F(z)$ defined in (1.10) but also when we add a function of the form

$$(\text{constant})\int_{z_0}^{z}[(K+1)\theta - i\omega]dz \quad \text{to this } F(z).$$

The resulting function is, of course, still a B.A.F. (k) with the A.F. we have indicated.

As an example of conversion, consider $f(z) = \{(\log r + x^2/r^2) + i(-xy/r^2)\}$, where $r^2 = x^2 + y^2$. This is a B.A.F. (-1); in fact, it is one of the "fundamental" functions used by Lauricella. Its A.F. is $2/z$. Thus $\Theta + i\Omega = -2((k+1)/k) \log z$ when $z_0 = 1$. Therefore $F(z) = \{(-((k+2)/k) \log r + x^2/r^2) + i(-xy/r^2)\}$ is a B.A.F. (k) whose A.F. is $[-2(k+1)/k](1/z)$, provided $k \neq 0, -1$. The $F(z)$ we have found here is known from elasticity theory; it gives the displacements due to a finite force acting at the origin, in the x-direction. Similarly, starting with Lauricella's other function $\{(xy/r^2) - i(\log r + y^2/r^2)\}$ we find the B.A.F. (k),

$$\left\{\frac{xy}{r^2} + i\left(\frac{k+2}{k} \log r - \frac{y^2}{r^2}\right)\right\}.$$

This gives the elastic displacements due to a finite force acting at the origin, in the y-direction.

3. Derivatives and integrals.

Definition III. *If $f(z) = u + iv$ is B.A. (k), then the derivative of $f(z)$ is*

$$\frac{df}{dz} = u_x + iv_x. \tag{1.11}$$

Theorem 8. *If $f(z)$ is B.A. (k) then df/dz is also a B.A.F. (k) whose A.F. is the derivative of the A.F. of $f(z)$.*

Proof. This is easily shown by a direct calculation provided it is assumed that u and v have continuous derivatives of the second order. This last assumption will be made throughout this chapter and will be proved independently in Chapter IV. Q.E.D.

Theorem 8 still holds when $k = -1$ provided ω_x and not (ω_x+a constant) is called the A.F. of df/dz. This convention will be followed in future.

Definition III and Theorem 8 are valid even in the special cases $k = 0$ and $k = -1$.

Theorem 9. *If $k \neq -1$, the only B.A.F. (k) whose derivative is identically zero in a domain is*

$$u + iv = (a + ib) + (c + id)y. \tag{1.12}$$

When $k = -1$, the corresponding result is

$$u + iv = (a + ib) + i(cy + dy^2) \tag{1.13}$$

where a, b, c and d are real constants.

Proof. This is fairly obvious, the calculations being very similar to those giving our results on the nonuniqueness of a conjugate function. Q.E.D.

The lack of symmetry with respect to x and y in (1.11), (1.12) and (1.13) is very striking. The fundamental reason for this is that the equations defining B.A.F.'s are themselves nonsymmetrical.

Formally (1.11) is the same as when $f(z)$ is analytic. If we try using the expression $(v_y - iu_y)$ as the derivative of $f(z)$ we find that the derivative is bianalytic of type $-k/(k+1)$ when $f(z)$ is B.A. (k). Thus only in the case $k=-2$ is this "derivative" of the same type as the original function, $f(z)$, and even in this case the two derivatives, (u_x+iv_x) and (v_y-iu_y), are not equal as they are for analytic functions.

We now define the integral.

DEFINITION IV. *Let $f(z)=u+iv$ be a B.A.F. (k) whose A.F. is $\{(k+1)\theta-i\omega\}$. Let $\{(k+1)\Theta-i\Omega\}$ be the analytic function defined by*

$$\{(k+1)\Theta - i\Omega\} = \int_{z_0}^{z} [(k+1)\theta - i\omega]dz.$$

Then the integral of $f(z)$ from z_0 to z along some continuous rectifiable path is

$$F(z) = U + iV = \int_{z_0}^{z} udx + (-v + \Omega)dy + i\int_{z_0}^{z} vdx + (u - \Theta)dy$$

(1.14)

$$= \int_{z_0}^{z} (u + iv)dz - i(\Theta + i\Omega)dy.$$

We denote the integral by

$$F(z) = \int_{z_0}^{z} f(z)\cdot dz.$$

The dot here does not denote multiplication. In Chapter II a dot product and other types of products will be defined; however, none of the products defined there enable us to define the integrand in our integral as the product of $f(z)$ and dz.

When $k=-1$, (1.14) must be modified by setting $\Omega=0$. Θ is then defined by $\Theta+i\Omega=\int_{z_0}^{z}(\theta+i\omega)dz$.

THEOREM 10 ("CAUCHY'S THEOREM"). *If C is a simple closed continuous, rectifiable curve inside which $f(z)=u+iv$ is B.A. (k) and such that u, v, θ, Ω are continuous inside and on C, then*

$$\oint_C f(z)\cdot dz = 0.$$

Proof. We first show that inside C the integrability conditions

$$u_y = (-v + \Omega)_x,$$
$$v_y = (u - \Theta)_x$$

hold.

These conditions reduce to the forms

$$u_y + v_x = \Omega_x = \omega,$$
$$v_y - u_x = -\Theta_x = -\theta,$$

which are obviously true.

Since we have assumed that u and v have continuous first derivatives inside C, the theorem now follows in the form

$$\oint_{C'} f(z)\cdot dz = 0$$

where C' is a curve inside C and homologous to C.

By letting $C' \to C$ the theorem now follows in the usual way. The case $k = -1$ is treated similarly. Q.E.D.

Next we show that the indefinite integral is bi-analytic.

THEOREM 11. *If $f(z)$ is a B.A.F. (k) then*

$$F(z) = \int_{z_0}^{z} f(z)\cdot dz$$

is also a B.A.F. (k) whose A.F. is

$$\{(k+1)\Theta - i\Omega\} = \int_{z_0}^{z} \{(k+1)\theta - i\omega\} dz$$

where $\{(k+1)\theta - i\omega\}$ is the A.F. of $f(z)$.

Proof. Setting $F(z) = U + iV$, we have, from (1.14)

$$U_x = u, \qquad V_x = v,$$
$$U_y = v + \Omega, \qquad V_y = u - \Theta,$$

and so

$$U_x - V_y = \Theta,$$
$$U_y + V_x = \Omega. \qquad \text{Q.E.D.}$$

Again, the theorem is true for $k = -1$ and is proved in a similar manner. Finally we show that differentiation and integration are inverse processes.

THEOREM 12.

(a) *Let $f(z)$ be a B.A.F. (k) whose A.F. is $\{(k+1)\theta - i\omega\}$. Then the derivative of $F(z) = \int_{z_0}^{z} f(z)\cdot dz$ is $f(z)$.*

$$\text{(1.15) (b)} \qquad \int_{z_0}^{z} \frac{df}{dz}\cdot dz = f(z) - f(z_0) + i(y - y_0)(\theta_0 + i\omega_0)$$

when $k \neq -1$. *When* $k = -1$, (1.15) *must be modified by setting* $\omega_0 = 0$. $(\theta_0 + i\omega_0)$ *is the value of* $(\theta + i\omega)$ *at* $z = z_0$.

Proof. Part (a) has already been proved, essentially, in Theorem 11 since it was shown there that if $F(z) = U + iV$ then U and V have derivatives and $U_x = u$ and $V_x = v$.

To prove part (b) we could use our results (1.12) and (1.13) on functions with zero-derivatives. However, it is simpler to proceed directly. Thus, $df/dz = u_x + iv_x$ has $[(k+1)\theta_x - i\omega_x]$ as its A.F. and so the integral of its A.F. is

$$\int_{z_0}^{z} [(k + 1)\theta_x - i\omega_x]dz = (k + 1)(\theta - \theta_0) - i(\omega - \omega_0).$$

Therefore,

$$\int_{z_0}^{z} \frac{df}{dz}\cdot dz = \int_{z_0}^{z} u_x dx + [-v_x + (\omega - \omega_0)]dy + i\int_{z_0}^{z} v_x dx + [u_x - (\theta - \theta_0)]dy$$
$$= (u + iv) - (u_0 + iv_0) - \omega_0(y - y_0) + i\theta_0(y - y_0).$$

The case $k = -1$ is handled similarly, remembering that the derivative of $f(z)$ is a function whose A.F. is $\theta_x + i\omega_x$ not $\theta_x + i(\omega_x + \text{a constant})$. For this reason the term $-f(z_0) + i(y - y_0)\theta_0$ is not the most general type of term with a zero derivative. Q.E.D.

Having shown that the derivative and integral have all the above simple properties, we now give some results which will be used later. First we have

THEOREM 13 ("MORERA'S THEOREM"). *Suppose D is a domain of the z-plane inside which* $[(k+1)\theta - i\omega]$ *is analytic. Also,*

$$(k + 1)\Theta - i\Omega = \int_{z_0}^{z} [(k + 1)\theta - i\omega]dz$$

where z_0 *is any point in D.*

Suppose also that u and v are two continuous functions such that

$$\int_C u dx + (-v + \Omega)dy = 0,$$

$$\int_C v dx + (u - \Theta)dy = 0$$

for every simple, closed, continuous rectifiable curve C in D. Then $(u + iv)$ *is a B.A.F.* (k) *in D having* $[(k+1)\theta - i\omega]$ *for its A.F.*

Proof. Define functions U and V by the equations

$$U = \int_{z_0}^{z} udx + (-v + \Omega)dy,$$

$$V = \int_{z_0}^{z} vdx + (u - \Theta)dy.$$

It follows from the hypotheses that U and V are functions of z. Also,

$$U_x = u, \qquad U_y = -v + \Omega,$$
$$V_x = v, \qquad V_y = u - \Theta,$$

and so

$$U_x - V_y = \Theta, \qquad U_y + V_x = \Omega,$$

and so

$$(U + iV) \text{ is B.A.}(k) \text{ with A.F.}[(k + 1)\Theta - i\Omega].$$

Therefore $(U+iV)$ has a derivative which is itself B.A. (k); in fact $U_x+iV_x=u+iv$ and so $(u+iv)$ is B.A. (k) whose A.F. is $(d/dz)\{(k+1)\Theta-i\Omega\} =(k+1)\theta-i\omega$. Q.E.D.

This result will now be used to prove the following analogue of Weierstrass' double-series theorem.

THEOREM 14. *Suppose* (u_n+iv_n), $n=1, 2, \cdots$, *is a uniformly convergent sequence of B.A.F.'s* (k) *in* D, *whose sequence of A.F.'s* $[(k+1)\theta_n-i\omega_n]$ *also converges uniformly in* D. *Then* $u+iv=\lim(u_n+iv_n)$ *is a B.A.F.* (k) *whose A.F. is* $[(k+1)\theta-i\omega]=\lim[(k+1)\theta_n-i\omega_n]$.

Proof. $[(k+1)\theta_n-i\omega_n]$ is a sequence of analytic functions converging uniformly in D. Therefore

$$(k + 1)\theta - i\omega = \lim[(k + 1)\theta_n - i\omega_n]$$

is an analytic function in D. Also setting

$$(k + 1)\Theta_n - i\Omega_n = \int_{z_0}^{z} [(k + 1)\theta_n - i\omega_n]dz$$

where z_0 is some point in D, we easily see that $[(k+1)\Theta_n-i\Omega_n]$ converges uniformly to $\{(k+1)\Theta-i\Omega\} =\int_{z_0}^{z}[(k+1)\theta-i\omega]dz$.

Now, by Theorem 10,

$$\oint_C (u_n + iv_n)\cdot dz = \oint_C u_ndx + (-v_n + \Omega_n)dy + i\oint_C v_ndx + (u_n - \Theta_n)dy$$
$$= 0,$$

when C is any simple, closed, continuous rectifiable curve in D.

Since the sequences $\{u_n\}$, $\{v_n\}$, $\{\Theta_n\}$ and $\{\Omega_n\}$ are uniformly convergent we can take the limit of the above equation by taking limits inside the integral sign.

Therefore

$$\oint_C udx + (-v + \Omega)dy + i\oint_C vdx + (u - \Theta)dy = 0.$$

The hypotheses of Theorem 13 are now seen to be satisfied and so the result follows. Q.E.D.

Theorems 13 and 14 are obviously true for all k including the singular case $k = -1$.

Chapter II. Algebra

1. **Changes of the independent variables.** The problem we shall now state and answer is the following:

Suppose $(\phi + i\psi)$ is any B.A.F. (k_1) of $z = x + iy$ and suppose $\{u=f(x, y), v=g(x, y)\}$ is a reversible transformation and such that $[(k_2+1)f-ig]$ is an analytic function of z. Under what conditions on k_1, k_2, k_3, $f(x, y)$ and $g(x, y)$ is $\phi+i\psi$ a B.A.F. (k_3) of $u+iv$?

First suppose $k_1 \neq 0, -1$ or -2. Then the only transformations satisfying all the above conditions are

(a) $$x + iy = a(u + iv) + (c + id)$$

where a, b and c are real constants. In this case $k_3 = k_1$.

(b) $$x + iy = ib(u + iv) + (c + id)$$

where b, c and d are real constants and now $k_3 = -k_1/(k_1+1)$. This last condition can be expressed by saying that $i(\phi+i\psi) = -\psi+i\phi$ is a B.A.F. (k_1).

The fact that the above transformations have the results stated is easily checked. It is much more difficult to prove that they are the only transformations that work. We shall not prove this here; the proof we have is so long that its presence would make this paper appreciably longer and the result is not of great importance.

When k_1 is -2, the only admissible transformations are

$$z = (a + ib)(u + iv) + (c + id)$$

where a, b, c and d are real. In this case $k_3 = k_1$.

The results for $k_1 = -1$ are the same as those that have been stated for $k_1 \neq 0, -1, -2$ except that in case (b), k_3 is ∞.

When $k_1 = 0$, then $(f+ig)$ can be an arbitrary analytic function of either $z = x+iy$ or $\bar{z} = x - iy$ and k_3 is, in all cases, equal to 0. This result is to be expected, of course, since we have already seen that all functions of type 0 are of the form (an analytic function) $+i$ (a harmonic function).

The result of all this is that, generally speaking, the technique of conformal mapping or anything similar is not available for the study of bi-analytic functions.

There are, however, different ways of applying conformal mapping to B.A.F.'s. Theorem 3, for instance, shows that any B.A.F. (k) is of the form

$$u + iv = \left\{\left(-\frac{k}{2}x\theta\right) + i\int_{z_0}^{z}\left(\frac{k}{2}x\theta_y + \omega\right)dx - \left(\frac{k}{2}x\theta_x + \frac{k+2}{2}\theta\right)dy\right\} + (u' + iv') \tag{2.1}$$

where $(u'+iv')$ is an analytic function and $\{(k+1)\theta - i\omega\}$ is the A.F. of $(u+iv)$.

Thus, we can make a conformal mapping of the (x, y) plane into the (ξ, η) plane and so send $[(k+1)\theta - i\omega]$ and $(u'+iv')$ into other analytic functions of $(\xi + i\eta)$. We can then apply (2.1) in the (ξ, η) plane where θ, ω, u' and v' are the functions of $(\xi + i\eta)$ obtained in the above manner. This will give us a bi-analytic function of $(\xi + i\eta)$.

Unfortunately the dissection of a B.A.F. (k) given by (2.1) is not unique; there are other ways of breaking up a B.A.F., $u+iv$, into a sum of an analytic function and a B.A.F. determined in terms of the A.F. of $u+iv$.

In fact we shall find another method of doing this in Chapter III. This technique of conformal mapping, therefore, will probably not be very useful; certainly no use has been found for it in the present paper.

2. **Changes of the dependent variables.** The possibilities in making transformations of the dependent variables are also very limited.

Suppose $\phi + i\psi$ is any B.A.F. (k_1) and suppose $\{u = f(\phi, \psi),\ v = g(\phi, \psi)\}$ where $\{(k_2+1)f - ig\}$ is an analytic function of $\phi + i\psi$. The problem we are concerned with now is the following: Under what conditions on k_1, k_2, k_3, f and g is $(u+iv)$ a B.A.F. (k_3)?

First, suppose $k_1 \neq 0$, -1, or -2. Then, the only possibilities are

(a) $$u + iv = a(\phi + i\psi) + (c + id),$$

where a, c and d are real and $k_3 = k_1$.

(b) $$u + iv = ib(\phi + i\psi) + (c + id)$$

where b, c and d are real and $k_3 = -k_1/(k_1+1)$.

This condition again means that $i(u+iv) = -v + iu$ is B.A. (k_1).

Again, by admitting functions of type ∞, the case $k_1 = -1$ comes under this general case.

When k is -2 the possibilities are again greater, we can set

$$u + iv = (a + ib)(\phi + i\psi) + (c + id)$$

and now $k_3 = -2$.

Functions of type 0 only admit the same transformations

$$u + iv = (a + ib)(\phi + i\psi) + (c + id)$$

and $k_3 = 0$.

Again, it is easily seen that the transformations given above have the desired properties while the statement that they are the only such transformations is more difficult to prove. It can be proved on the same lines as the corresponding result in §2 but the proof will be omitted for the reasons given there.

3. **Multiplication.** If (u_1+iv_1) and (u_2+iv_2) are both B.A.F.(k) then it is clear that $(u_1+u_2)+i(v_1+v_2)$ is a B.A.F. (k). It is quite easy to see, also, that if (u_1+iv_1) is a B.A.F. (k_1) with A.F. $\{(k_1+1)\theta_1-i\omega_1\}$ and (u_2+iv_2) is a B.A.F. (k_2) with A.F. $\{(k_2+1)\theta_2=i\omega_2\}$ then $(u_1+u_2)+i(v_1+v_2)$ is not a B.A.F. at all unless either $\{(k_1+1)\theta_1-i\omega_1\}$ and $\{(k_2+1)\theta_2-i\omega_2\}$ are both constant or $k_1=k_2$.

The other simple algebraic property of bi-analytic functions is that if $(u+iv)$ is a B.A.F. (k) with A.F. $\{(k+1)\theta-i\omega\}$ then $a(u+iv)=au+iav$ is a B.A.F. with A.F. $a[(k+1)\theta-i\omega]$ when a is real.

Consider now, though, the effect of multiplying by i. Let $(u+iv)$ be a B.A.F. (k) with A.F. $\{(k+1)\theta-i\omega\}$. Under what conditions on k and k_1 is $i(u+iv)=-v+iu$ a B.A.F. (k_1)? We have

$$(-v)_x - u_y = \theta' = -\omega,$$

$$(-v)_y + u_x = \omega' = \theta.$$

Thus, our question reduces to finding when $\{(k_1+1)\theta'-i\omega'\} = -(k_1+1)\omega - i\theta$ is analytic or when

$$-(k_1 + 1)\omega_x = -\theta_y,$$

$$-(k_1 + 1)\omega_y = +\theta_x.$$

But

$$\theta_x = -\frac{1}{k+1}\omega_y,$$

$$\theta_y = \frac{1}{k+1}\omega_x.$$

Thus the condition is that

$$\frac{1}{k+1} = k_1 + 1,$$

or

$$k_1 = \frac{-k}{k+1}.$$

To summarize then: if $(u+iv)$ is a B.A.F. (k) then $i(u+iv) = -v+i\omega$ is a B.A.F. $(-k/(k+1))$ whose A.F. is $[-i/(k+1)][(k+1)\theta - i\omega]$.

The only cases in which $-k/(k+1) = k$ are $k=0$, and -2.

This means, according to our remark about adding functions of different type, that only when $k=0$ and -2 can $u+iv$, a B.A.F. (k), be multiplied by $(a+ib)$ where a and b are real nonzero constants.

Consider now the possibility of multiplying two functions $(\phi+i\psi)$ and $(u+iv)$. Let $U+iV = (\phi+i\psi)(u+iv) = (\phi u - v\psi) + i(\psi u + v\phi)$. Suppose $(u+iv)$ is a B.A.F. (k) with A.F. $\{(k+1)\theta - i\omega\}$, and that $(\phi+i\psi)$ is analytic.

Then only when $k=0$ or -2 can $(U+iV)$ possibly be bi-analytic.

It is easily seen that $k=0$ will not work but $k=-2$ does. That is, we have

THEOREM 15. *If $u+iv$ is a B.A.F. (-2) whose A.F. is $\{(k+1)\theta - i\omega\}$ then $U+iV = (\phi+i\psi)(u+iv) = (\phi u - \psi v) + i(\psi u + v\phi)$ is a B.A.F. (-2) provided $(\phi+i\psi)$ is analytic. The A.F. of $U+iV$ is $(\phi+i\psi)[(k+1)\theta - i\omega]$ [with $k=-2$]. In fact, if $(\phi+i\psi)$ is analytic and $u+iv$ is B.A. (k), then only when $k=-2$ is $(\phi+i\psi)(u+iv)$ bi-analytic.*

With this result it is possible to define a "product" of an analytic function and a B.A.F. of any type k. This can be done by first converting the B.A.F. to a function of type -2, then multiplying with the analytic function and finally converting this product back into a function of type k. Since the conversion process is not unique this gives a great variety of products. The most general form obtained by this method is shown in the following definition:

DEFINITION V. *Suppose $(u+iv)$ is a B.A.F. (k) whose A.F. is $[(k+1)\theta - i\omega]$, where $k \neq -1$. Suppose also that $(\phi+i\psi)$ is an analytic function. Then, a product of $(u+iv)$ and $(\phi+i\psi)$ is any expression of the form*

$$\begin{aligned} U + iV = (\phi + i\psi)(u + iv) - \frac{k+2}{2(k+1)}\Big\{(\phi + i\psi)\Big[a_1 \operatorname{Re} \int_{z_0}^{z} [(k+1)\theta - i\omega]dz \\ - ib_1 \operatorname{Im} \int_{z_0}^{z} [(k+1)\theta - i\omega]dz\Big] \\ - a_2 \operatorname{Re} \int_{z_1}^{z} (\phi + i\psi)[(k+1)\theta - i\omega]dz \\ + ib_2 \operatorname{Im} \int_{z_1}^{z} (\phi + i\psi)[(k+1)\theta - i\omega]dz\Big\} \end{aligned} \tag{2.2}$$

where a_1, b_1, a_2, b_2 are real numbers such that $a_1+b_1=1$ and $a_2+b_2=1$. z_0 and z_1 are any two points in the domain of definition of $(u+iv)$ and $(\phi+i\psi)$; the ordered pair (z_0, z_1) is called the lower limit of the product.

In (2.2), the expression $(\phi+i\psi)(u+iv)$ is the ordinary product of these functions, that is, $(\phi u-v\psi)+i(v\phi+\psi u)$.

It is easily checked that $(U+iV)$ is a B.A.F. (k) whose A.F. is $(\phi+i\psi)[(k+1)\theta-i\omega]$, that is, $U_x-V_y=1/(k+1)\ [(k+1)\theta\phi+\omega\psi]=\Theta$. $U_y+V_x=\phi\omega-(k+1)\psi\theta=\Omega$ so that

$$(k+1)\Theta-i\Omega=(\phi+i\psi)[(k+1)\theta-i\omega].$$

This last statement is true for all k except $k=-1$, even $k=0$. This is interesting since our conversion process does not work for $k=0$.

If $k=-1$, Equation (2.2) can be modified as follows:

$$\begin{aligned} U+iV &= (\phi+i\psi)(u+iv) \\ &\quad -\frac{1}{2}\left\{(\phi+i\psi)\left[a_1\operatorname{Re}\int_{z_0}^{z}(\theta+i\omega)dz-ib_1\operatorname{Im}\int_{z_0}^{z}(\theta+i\omega)dz\right]\right. \\ &\quad \left.-a_2\operatorname{Re}\int_{z_1}^{z}(\phi+i\psi)(\theta+i\omega)dz+ib_2\operatorname{Im}\int_{z_1}^{z}(\phi+i\psi)(\theta+i\omega)dz\right\}. \end{aligned} \tag{2.3}$$

The products defined in (2.2) and (2.3) obviously have some of the simple properties that are desirable in products. Thus,

(a) they are 0 when $\phi+i\psi=0$ and when $u+iv=0$ (assuming that the conjugate of $\theta=0$ is $\omega=0$ when $k=-1$);

(b) when $u+iv=1$, they reduce to $\phi+i\psi$ (provided we make the same assumption as in (a) when $k=-1$);

(c) in fact when $(u+iv)$ is analytic, they all reduce to the ordinary product of analytic functions;

(d) when k is -2, they all reduce to the ordinary product of $(\phi+i\psi)$ and $(u+iv)$ (provided a_1, b_1, a_2 and b_2 are chosen so as to have a definite value for $k=-2$).

One apparent disadvantage of Definition V is that it gives us an infinite number of products depending on the lower limits and the parameters a_1, a_2, b_1 and b_2. However, these parameters will later be made definite while the freedom in choosing the lower limits will turn out to be an advantage.

The expressions in (2.2) and (2.3) will be denoted by $P\{\phi+i\psi, u+iv; z_0, z_1\}$. It will now be shown that the usual rule for differentiating products holds provided the limits can be chosen properly.

THEOREM 16.

$$\begin{aligned} \frac{d}{dz}P\{\phi+i\psi, u+iv; z_0, z_1\} &= P\left\{\frac{d}{dz}(\phi+i\psi), u+v; z_0, z_2\right\} \\ &\quad + P\left\{\phi+i\psi, \frac{d}{dz}(u+iv), z_3, z_2\right\} \end{aligned} \tag{2.4}$$

provided either $k=-2$ *or if* $k\neq -2$ *or* -1,

$$a_1\theta(z_3) = b_1\omega(z_3) = a_2 \operatorname{Re}\left[(\{k+1\}\theta - i\omega)(\phi + i\psi)\right]_{z=z_3}$$
$$= b_2 \operatorname{Im}\left[(\{k+1\}\theta - i\omega)(\phi + i\psi)\right]_{z=z_3} = 0.$$

Proof. From (2.3) we have

$$\begin{aligned}
&\frac{d}{dz} P\{\phi + i\psi, u + iv; z_0, z_1\} \\
&\quad = U_z + iV_z \\
&\quad = (\phi + i\psi)(u_x + iv_x) + (\phi_x + i\psi_x)(u + iv) \\
&\qquad - \frac{k+2}{2(k+1)}\Big\{a_1(k+1)\theta(\phi + i\psi) + ib_1\omega(\phi + i\psi) \\
&\qquad - a_2[(k+1)\theta\phi + \psi\omega] + ib_2[(k+1)\theta\psi - \phi\omega] \\
&\qquad + a_1(\phi_x + i\psi_x) \operatorname{Re}\int_{z_0}^{z} [(k+1)\theta - i\omega]dz \\
&\qquad - ib_1(\phi_x + i\psi_x) \operatorname{Im}\int_{z_0}^{z} [(k+1)\theta - i\omega]dz\Big\} \\
&\quad = (u + iv)(\phi_x + i\psi_x) - \frac{k+2}{2(k+1)}\Big\{(\phi_x + i\psi_x)\Big[a_1 \operatorname{Re}\int_{z_0}^{z} [(k+1)\theta - i\omega]dz \\
&\qquad - ib_1 \operatorname{Im}\int_{z_0}^{z} [(k+1)\theta - i\omega]dz\Big] \\
&\qquad - a_2 \operatorname{Re}\int_{z_2}^{z} \left[\frac{d}{dz}(\phi + i\psi)\right][(k+1)\theta - i\omega]dz \\
&\qquad + ib_2 \operatorname{Im}\int_{z_2}^{z} \left[\frac{d}{dz}(\phi + i\psi)\right][(k+1)\theta - i\omega]dz\Big\} \\
&\qquad + (\phi + i\psi)(u_x + iv_x) - \frac{k+2}{2(k+1)}\Big\{(\phi + i\psi)[a_1(k+1)\theta + ib_1\omega] \\
&\qquad - a_2\Big[(\{k+1\}\theta\phi + \psi\omega) - \operatorname{Re}\int_{z_2}^{z} \left[\frac{d}{dz}(\phi + i\psi)\right][(k+1)\theta - i\omega]dz\Big] \\
&\qquad + ib_2\Big[(\{k+1\}\theta\psi - \phi\omega) - \operatorname{Im}\int_{z_2}^{z} \left[\frac{d}{dz}(\phi + i\psi)\right][(k+1)\theta - i\omega]dz\Big]\Big\}
\end{aligned}$$

$$= P\left\{\frac{d}{dz}(\phi + i\psi), u + iv; z_0, z_2\right\} + (\phi + i\psi)(u_x + iv_x)$$

$$- \frac{k+2}{2(k+1)}\left\{(\phi + i\psi)\left[a_1 \operatorname{Re}\int_{z_3}^{z} \frac{d}{dz}[(k+1)\theta - i\omega]dz\right.\right.$$

$$\left. - ib_1\int_{z_3}^{z} \frac{d}{dz}[(k+1)\theta - i\omega]dz\right]$$

$$- a_2 \operatorname{Re}\int_{z_2}^{z} (\phi + i\psi)\frac{d}{dz}[(k+1)\theta - i\omega]dz$$

$$\left. + ib_2 \operatorname{Im}\int_{z_2}^{z} (\phi + i\psi)\frac{d}{dz}[(k+1)\theta - i\omega]dz\right\}$$

$$- \frac{k+2}{2(k+1)}\left\{(\phi + i\psi)[a_1(k+1)\theta(z_3) + ib_1\omega(z_3)]\right.$$

$$\left. - a_2[(k+1)\theta\phi + \psi\omega]_{z_2} + ib_2[(k+1)\theta\psi - \phi\omega]_{z_2}\right\}$$

by integrating by parts.

The assertion now follows. Q.E.D.

When k is -1, (2.4) still holds under the conditions

$$a_1\theta(z_3) = b_1\omega(z_3) = a_2 \operatorname{Re}[(\theta + i\omega)(\phi + i\psi)]_{z=z_2}$$
$$= b_2 \operatorname{Im}[(\theta + i\omega)(\phi + i\psi)]_{z=z_2} = 0.$$

In general the product in Definition V is not associative and commutative with respect to multiplication by analytic functions. In order to get products with these properties we shall specify the values of the parameters suitably.

If, in (2.2), we set $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$ we get a product that will be called the left cross-product. If $a_1 = a_2 = 0$ and $b_1 = b_1 = 1$ the product will be called the right cross-product. These quantities have most of the properties desired in a product and so will be often used hereafter. For convenience, formal definitions are now given.

DEFINITION VI. *Suppose* $(u+iv)$ *is a B.A.F.* (k) *whose A.F. is* $\{(k+1)\theta - i\omega\}$ *where* $k \neq -1$, *and suppose* $(\phi + i\psi)$ *is analytic. Then, the left cross-product of* $(u+iv)$ *and* $(\phi+i\psi)$ *is*

$$(2.5)\quad \begin{aligned}(\phi + i\psi) \times (u + iv) &= (\phi + i\psi)(u + iv)\\ &- \frac{k+2}{2(k+1)}\left\{(\phi + i\psi)\operatorname{Re}\int_{z_0}^{z}[(k+1)\theta - i\omega]dz\right.\\ &\left. - \operatorname{Re}\int_{z_1}^{z}(\phi + i\psi)[(k+1)\theta - i\omega]dz\right\}.\end{aligned}$$

When k is -1, *the left cross-product is*

$$(\phi + i\psi) \times (u + iv) = (\phi + i\psi)(u + iv) - \frac{1}{2}(\phi + i\psi)\,\mathrm{Re}\int_{z_0}^{z}(\theta + i\omega)dz + \frac{1}{2}\int_{z_1}^{z}(\phi + i\psi)(\theta + i\omega)dz. \tag{2.5}$$

DEFINITION VII. *Suppose* $(u+iv)$ *is a B.A.F.* (k) *with A.F.* $\{(k+1)\theta-i\omega\}$ *where* $k\neq -1$. *Suppose also that* $(\phi+i\psi)$ *is analytic. Then, the right cross-product of* $(u+iv)$ *and* $(\phi+i\psi)$ *is*

$$(u + iv) \times (\phi + i\psi) = (\phi + i\psi)(u + iv) + \frac{k+2}{2(k+1)}i\left\{(\phi + i\psi)\,\mathrm{Im}\int_{z_0}^{z}[(k + 1)\theta - i\omega]dz - \mathrm{Im}\int_{z_1}^{z}(\phi + i\psi)[(k + 1)\theta - i\omega]dz\right\}. \tag{2.6}$$

When k is -1, *the definition is*

$$(u + iv) \times (\phi + i\psi) = (\phi + i\psi)(u + iv) + \frac{1}{2}i\left\{(\phi + i\psi)\,\mathrm{Im}\int_{z_0}^{z}(\theta + i\omega)dz - \mathrm{Im}\int_{z_1}^{z}(\phi + i\psi)(\theta + i\omega)dz\right\}. \tag{2.6}$$

We now show that these cross-products are commutative and associative with respect to multiplication by analytic functions.

THEOREM 17. *Suppose* $(u+iv)$ *is B.A.* (k) *and that* $(\phi+i\psi_1)$ *and* $(\psi_2+i\psi_2)$ *are analytic. Then*

$$(\phi_2 + i\psi_2) \times [(\phi_1 + i\psi_1) \times (u + iv)] = [(\phi_1 + i\psi_1)(\phi_2 + i\psi_2)] \times (u + iv)$$

provided that

(a) *the lower limit of* $(\phi_1+i\psi_1)\times(u+iv)$ *is* (z_2, z_0),
(b) *the lower limit of* $(\phi_2+i\psi_2)\times[(\phi_1+i\psi_1)\times(u+iv)]$ *is* (z_0, z_1),
(c) *the lower limit of* $[(\phi_1+i\psi_1)(\phi_2+i\psi_2)]\times(u+iv)$ *is* (z_2, z_1).

Proof. We give the proof for the case $k\neq -1$. The proof is exactly similar when $k=-1$.

Let the A.F. of $(u+iv)$ be $[(k+1)\theta-i\omega]$. Then the A.F. of $(\phi_1+i\psi_1)\times(u+iv)$ is $(\phi_1+i\psi_1)[(k+1)\theta-i\omega]$. Therefore,

$$
\begin{aligned}
(\phi_2 + i\psi_2) \times [(\phi_1 + i\psi_1) \times (u + iv)] \\
&= (\phi_2 + i\psi_2)\{(\phi_1 + i\psi_1) \times (u + iv)\} \\
&\quad - \frac{k+2}{2(k+1)}\left\{(\phi_2 + i\psi_2) \operatorname{Re} \int_{z_0}^{z} [(k+1)\theta - i\omega](\phi_1 + i\psi_1)dz \right. \\
&\quad \left. - \operatorname{Re} \int_{z_1}^{z} [(k+1)\theta - i\omega](\phi_1 + i\psi_1)(\phi_2 + i\psi_2)dz \right\} \\
&= (\phi_2 + i\psi_2)(\phi_1 + i\psi_1)(u + iv) \\
&\quad - \frac{k+2}{2(k+1)}\left\{(\phi_2 + i\psi_2)(\phi_1 + i\psi_1) \operatorname{Re} \int_{z_2}^{z} ([k+1)\theta - i\omega]dz \right. \\
&\quad \left. - (\phi_2 + i\psi_2) \operatorname{Re} \int_{z_0}^{z} (\phi_1 + i\psi_1)[(k+1)\theta - i\omega]dz \right\} \\
&\quad - \frac{k+2}{2(k+1)}\left\{(\phi_2 + i\psi_2) \int_{z_0}^{z} [(k+1)\theta - i\omega](\phi_1 + i\psi_1)dz \right. \\
&\quad \left. - \operatorname{Re} \int_{z_1}^{z} [(k+1)\theta - i\omega](\phi_1 + i\psi_1)(\phi_2 + i\psi_2)dz \right\} \\
&= (\phi_2 + i\psi_2)(\phi_1 + i\psi_1)(u + iv) \\
&\quad - \frac{k+2}{2(k+1)}\left\{(\phi_2 + i\psi_2)(\phi_1 + i\psi_1) \operatorname{Re} \int_{z_2}^{z} [(k+1)\theta - i\omega]dz \right. \\
&\quad \left. - \operatorname{Re} \int_{z_1}^{z} [(k+1)\theta - i\omega](\phi_1 + i\psi_1)(\phi_2 + i\psi_2)dz \right\} \\
&= [(\phi_1 + i\psi_1)(\phi_2 + i\psi_2)] \times (u + iv). \qquad \text{Q.E.D.}
\end{aligned}
$$

A similar result holds for the right cross-product.

THEOREM 18. *If* $(u+iv)$, $(\phi_1+i\psi_1)$ *and* $(\phi_2+i\psi_2)$ *have the same meanings as in Theorem* 17, *then*

$$[(u + iv) \times (\phi_1 + i\psi_1)] \times (\phi_2 + i\psi_2) = (u + iv) \times [(\phi_1 + i\psi_1)(\phi_2 + i\psi_2)]$$

provided

(a) *the lower limit of* $(u+iv)\times(\phi_1+i\psi_1)$ *is* (z_2, z_0),
(b) *the lower limit of* $[(u+iv)\times(\phi_1+i\psi_1)]\times(\phi_2+i\psi_2)$ *is* (z_0, z_1),
(c) *the lower limit of* $(u+iv)\times[(\phi_1+i\psi_1)(\phi_2+i\psi_2)]$ *is* (z_2, z_1).

The proof is exactly similar to that of Theorem 17 and so will be omitted. We could also consider a mixed product

$$(\phi_2 + i\psi_2) \times [(u + iv) \times (\phi_1 + i\psi_1)].$$

It turns out that this is not anything very much simpler.

Another property of cross products is that if in (2.5) and (2.6), $(\phi+i\psi)$ is a real constant, c, and $z_0=z_1$ these products both reduce to $c(u+iv)$. However this choice of the lower limit will sometimes be inconvenient.

Other types of products, dot products, will now be defined. If, in Definition V, $a_1=b_2=1$ and $a_2=b_1=0$, the resultant product will be called a left dot product while if $a_1=b_2=0$ and $a_2=b_1=1$, the result is the right dot product. Again, formal definitions will be given.

DEFINITION VIII. *If* $(\phi+i\psi)$ *and* $(u+iv)$ *have the same meanings as in Definitions* VI *and* VII, *then, when* $k\neq -1$,

$$(2.7)\quad \begin{aligned}(\phi + i\psi)\cdot(u + iv) = {} & (\phi + i\psi)(u + iv) \\ & - \frac{k+2}{2(k+1)}\left\{(\phi + i\psi) \operatorname{Re} \int_{z_0}^{z} \{(k+1)\theta - i\omega\} dz \right. \\ & \left. + i \operatorname{Im} \int_{z_1}^{z} (\phi + i\psi)[(k+1)\theta - i\omega]dz \right\}.\end{aligned}$$

When $k=-1$, *the definition is*

$$(2.7)\quad \begin{aligned}(\phi + i\psi)\cdot(u + iv) = {} & (\phi + i\psi)(u + iv) - \frac{1}{2}(\phi + i\psi) \operatorname{Re} \int_{z_0}^{z} (\theta + i\omega)dz \\ & - \frac{1}{2}(\phi + i\psi) i \operatorname{Im} \int_{z_1}^{z} (\phi + i\psi)(\theta + i\omega)dz.\end{aligned}$$

DEFINITION IX. *When* $k\neq -1$,

$$(2.8)\quad \begin{aligned}(u + iv)\cdot(\phi + i\psi) = {} & (\phi + i\psi)(u + iv) \\ & + \frac{k+2}{2(k+1)}\left\{(\phi + i\psi) i \operatorname{Im} \int_{z_0}^{z} [(k+1)\theta - i\omega] dz \right. \\ & \left. + \operatorname{Re} \int_{z_1}^{z} (\phi + i\psi)[(k+1)\theta - i\omega]dz \right\}.\end{aligned}$$

When $k=-1$,

$$(2.8)\quad \begin{aligned}(u + iv)\cdot(\phi + i\psi) = {} & (\phi + i\psi)(u + iv) + \frac{1}{2}(\phi + i\psi) i \operatorname{Im} \int_{z_0}^{z} (\theta + i\omega)dz \\ & + \frac{1}{2} \operatorname{Re} \int_{z_1}^{z} (\phi + i\psi)(\theta + i\omega)dz.\end{aligned}$$

Dot products are not associative. Thus if $z_0=z_1$,

$$i\cdot[i\cdot(u + iv)] = -(u + iv) + \frac{k+2}{k+1}\int_{z_0}^{z} [(k+1)\theta - i\omega]dz$$

while

$$-1\cdot(u+iv) = -(u+iv) + \frac{k+2}{2(k+1)}\int_{z_0}^{z}[(k+1)\theta - i\omega]dz$$

and so

$$i\cdot i\cdot(u+iv) \neq (i\cdot i)\cdot(u+iv) \quad \text{since} \quad i\cdot i = -1.$$

A similar result holds for right dot-products.

However there are some simple results on repeated products which we give in the following theorems.

THEOREM 19. *Suppose* $(\phi_1+i\psi_1)$ *and* $(\phi_2+i\psi_2)$ *are two analytic functions and suppose* $f(z)=u+iv$ *is a B.A.F.* (k). *Then*

$$(2.9) \qquad [(\phi_1+i\psi_1)\cdot f(z)]\cdot(\phi_2+i\psi_2) = [(\phi_1+i\psi_1)(\phi_2+i\psi_2)]\times f(z)$$

and

$$(2.10) \qquad (\phi_2+i\psi_2)\cdot[f(z)\cdot(\phi_1+i\psi_1)] = f(z)\times[(\phi_1+i\psi_1)(\phi_2+i\psi_2)]$$

provided that, in (2.9), *the lower limit of*

(a) $[(\phi_1+i\psi_1)\cdot f(z)]\cdot(\phi_2+i\psi_2)$ *is* (z_0, z_1);
(b) $(\phi_1+i\psi_1)\cdot f(z)$ *is* (z_2, z_0);
(c) $(\phi_1+i\psi_1)(\phi_2+i\psi_2)\times f(z)$ *is* (z_2, z_1);

and that, in (2.10), *the lower limit*

(a) *of* $(\phi_2+i\psi_2)\cdot[f(z)\cdot(\phi_1+i\psi_1)]$ *is* (z_0, z_1);
(b) *of* $f(z)\cdot(\phi_1+i\psi_1)$ *is* (z_2, z_0);
(c) *of* $f(z)\times(\phi_1+i\psi_1)(\phi_2+i\psi_2)$ *is* (z_2, z_1).

THEOREM 20.

$$(2.11) \qquad \begin{aligned}[(\phi_1+i\psi_1)\cdot f(z)]\times(\phi_2+i\psi_2) &= (\phi_1+i\psi_1)\cdot[(\phi_2+i\psi_2)\times f(z)] \\ &= [(\phi_1+i\psi_1)(\phi_2+i\psi_2)]\cdot f(z)\end{aligned}$$

provided that the lower limit of

(a) $(\phi_2+i\psi_2)\times f(z)$ *and* $(\phi_1+i\psi_1)\cdot f(z)$ *is* (z_2, z_0);
(b) $(\phi_1+i\psi_1)\cdot[(\phi_2+i\psi_2)\times f(z)]$ *and* $[(\phi_1+i\psi_1)\cdot f(z)]\times(\phi_2+i\psi)$ *is* (z_0, z_1);
(c) $[(\phi_1+i\psi)(\phi_2+i\psi_2)]\cdot f(z)$ *is* (z_2, z_1).

Also,

$$\begin{aligned}(\phi_2+i\psi_2)\times[f(z)\cdot(\phi_1+i\psi)] &= [f(z)\times(\phi_2+i\psi_2)]\cdot(\phi_1+i\psi_1) \\ &= f(z)\cdot[(\phi_1+i\psi_1)(\phi_2+i\psi_2)]\end{aligned}$$

provided that the lower limit of

(a) $f(z)\times(\phi_2+i\psi_2)$ *and* $f(z)\cdot(\phi_1+i\psi_1)$ *is* (z_2, z_0),
(b) $(\phi_2+i\psi_2)\times[f(z)\cdot(\phi_1+i\psi_1)$ *and* $[f(z)\times(\phi_2+i\psi_2)]\cdot(\phi_1+i\psi_1)$ *is* (z_0, z_1),
(c) $f(z)\cdot[(\phi_1+i\psi_1)(\phi_2+i\psi_2)]$ *is* (z_2, z_1).

The proofs of Theorems 19 and 20 will not be given since they are similar to that of Theorem 17.

A mnemonic can be given for all these results, Theorems 17, 18, 19 and 20; namely, two cross-products and two dot are equivalent to a cross-product, while a dot and cross-product is equivalent to a dot-product. The methods of choosing the lower limits in all these results are very similar.

Chapter III. The Elementary Functions

1. **Preliminary considerations.** By means of the multiplication formulae it is possible to generate a great variety of bi-analytic functions. Some such functions will be constructed by this method in this chapter and a number of their properties will be demonstrated.

First, a B.A.F. whose A.F. is 1 is easily found. There are an infinite number of such functions, of course, all differing from each other by analytic functions. The one that will be continually used in the following work is

$$Z^{(0)}(k; z) = \frac{k+2}{2(k+1)} x + i \frac{k}{2(k+1)} y. \tag{3.1}$$

We shall also use the following function, whose A.F. is i:

$$i \cdot Z^{(0)}(k, z) = -\frac{k}{2(k+1)} y - i \frac{k+2}{2(k+1)} x. \tag{3.2}$$

In constructing (3.2) from (3.1) the lower limit (0, 0) has been used.

If $(u+iv)$ is a B.A.F. (k), we shall use the notation $a(u+iv)$ for the B.A.F. $(au)+i(av)$ where a is a real number. Also, if a and b are real, the notation $(a+ib)\cdot(u+iv)$ will be used for the function $a(u+iv)+b\{i\cdot(u+iv)\}$. It should be noted that if $(u+iv)$ has $f(z)$ as its A.F., then the A.F. of $(a+ib)\cdot(u+iv)$ is $(a+ib)f(z)$.

Thus,

$$(a+ib)\cdot Z^{(0)}(k, z) = aZ^{(0)}(k, z) + bi\cdot Z^{(0)}(k, z)$$

is a B.A.F. (k) whose A.F. is $(a+ib)$.

Now,

$$\frac{dZ^{(0)}}{dz} = \frac{k+2}{2(k+1)}$$

and

$$\frac{di\cdot Z^{(0)}}{dz} = -i \frac{k+2}{2(k+1)}$$

and so

$$(3.3)\qquad \frac{d}{dz}\{(a+ib)\cdot Z^{(0)}\} = \frac{k+2}{2(k+1)}\{a-ib\}.$$

When k is -1, the functions used will be

$$Z^{(0)}(-1;z) = \frac{1}{2}x - \frac{1}{2}iy$$

and

$$(3.4)\qquad i\cdot Z^{(0)}(-1;z) = +\frac{1}{2}y - \frac{1}{2}ix.$$

Suppose, now, that $f(z)$ is any analytic function. Then, $f(z)\times Z^{(0)}(k,z)$ is a B.A.F. with A.F. $f(z)$. Also, $i\cdot Z^{(0)}(k,z)\times f(z) = i\cdot\{f(z)\times Z^{(0)}\}$ is a B.A.F. with A.F. $if(z)$ and so $(a+ib)\cdot\{f(z)\times Z^{(0)}\}$ is a B.A.F., with A.F. $(a+ib)f(z)$.

By setting $f(z)$ equal to z^n, $\log z$, e^z, $\sin z$ and $\cos z$ the above formulae gives us B.A.F.'s which will be denoted by $(a+ib)\cdot Z^{(n)}(k;z)$, $(a+ib)\cdot L(k;z)(a+ib)\cdot E(k;z)$, $(a+ib)\cdot S(k;z)$ and $(a+ib)\cdot C(k;z)$ respectively.

It turns out that many of the properties of these functions are formally very similar to the properties of their A.F.'s, provided the formalism developed in this paper is used.

In all the above and in the rest of the chapter, the variable z could be replaced by $z-z_0$, where z_0 is any complex number. In this way the special role of $z=0$ can be transferred to an arbitrary point $z=z_0$. We have preferred to write the results with z instead of $z-z_0$ merely for conciseness.

2. **The generalized powers $Z^{(n)}(k;z)$.** Suppose, first, that $k\neq -1$. Then,

$$(3.5)\qquad Z^{(n)}(k;z) = z^n\times Z^{(0)} = \frac{k}{2(k+1)}iyz^n + \frac{k+2}{2(k+1)(n+1)}\operatorname{Re}(z^{n+1})$$

provided $n\neq -1$. When $n=-1$, we have

$$(3.6)\qquad Z^{(-1)}(k;z) = z^{-1}\times Z^{(0)} = \frac{k}{2(k+1)}iyz^{-1} + \frac{k+2}{2(k+1)}\log|z|.$$

Also,

$$(3.7)\qquad i\cdot Z^{(n)}(k;z) = -\frac{k}{2(k+1)}yz^n - i\frac{k+2}{2(k+1)(n+1)}\operatorname{Re}(z^{n+1}) = i\cdot Z^{(0)}\times z^n,\qquad n\neq -1$$

while

$$(3.8)\qquad i\cdot Z^{(-1)}(k;z) = -\frac{k}{2(k+1)}yz^{-1} - i\frac{k+2}{2(k+1)}\log|z| = i\cdot Z^{(0)}\times z^{-1}.$$

When $\mathrm{Re}(n) > -1$, the lower limit used in the cross product was (0, 0), when $\mathrm{Re}(n) < -1$ and also when $\mathrm{Re}(n) = -1$, $\mathrm{Im}(n) \neq 0$, the lower limit was $(0, \infty)$ while when $n = -1$, the lower limit was (0, 1). This choice of limits makes the results somewhat simpler than any other choice.

In the above, n is any complex number. If n is not a positive or negative integer the functions are, of course, multi-valued. Also, when n is a positive integer, $Z^{(n)}$ and $i \cdot Z^{(n)}$ are homogeneous polynomials of degree $(n+1)$ in x and y.

We now can easily see that

$$\begin{aligned} Z^{(m+p)}(k; z) &= z^{(m+p)} \times Z^{(0)} \\ &= z^m \times Z^{(p)} \end{aligned}$$

and

$$\begin{aligned} i \cdot Z^{(m+p)}(k; z) &= i \cdot Z^{(0)} \times z^{m+p} \\ &= i \cdot Z^{(p)} \times z^m \end{aligned} \tag{3.9}$$

provided the lower limits are chosen so that the contribution from the lower limits in the integrals is always 0.

Suppose, now, $\mathrm{Re}(n) > 0$. Then,

$$Z^{(n)} = z^n \times Z^{(0)} = z^{(n-1)} \times (z \times Z^{(0)})$$

and so

$$\begin{aligned} \frac{dZ^{(n)}}{dz} &= \left(\frac{dz^{n-1}}{dz} \times Z^{(1)}\right) + \left(z^{n-1} \times \frac{dZ^{(1)}}{dz}\right) \\ &= ((n-1) \times Z^{(n-1)}) + Z^{(n-1)} \end{aligned}$$

since

$$\frac{dZ^{(1)}}{dz} = Z^{(0)}.$$

Therefore

$$\frac{dZ^{(n)}}{dz} = n \times Z^{(n-1)}. \tag{3.10}$$

Similarly

$$\frac{di \cdot Z^{(n)}}{dz} = i \cdot Z^{(n-1)} \times n. \tag{3.10}$$

We had to proceed in the above manner since $dZ^{(0)}/dz \neq 0$. Also, it can be seen that the restriction $\mathrm{Re}(n) > 0$ is not essential and the formulae (3.10) really hold for all $n \neq 0$. If k is -1, the definitions of the generalized powers are

$$Z^{(n)}(-1;z) = -\frac{1}{2} iyz^n + \frac{1}{2(n+1)} \operatorname{Re}(z^{n+1}), \tag{3.10}$$

provided $n \neq -1$, while

$$Z^{(-1)}(-1;z) = \frac{1}{2} \log |z| - \frac{1}{2} iyz^{-1}. \tag{3.11}$$

Also,

$$i\cdot Z^{(n)}(-1;z) = \frac{1}{2} yz^n - \frac{i}{2(n+1)} \operatorname{Re}(z^{n+1}) \tag{3.12}$$

for $n \neq -1$, and

$$i\cdot Z^{(-1)}(-1;z) = \frac{1}{2} yz^{-1} - \frac{1}{2} i \log |z|. \tag{3.13}$$

Equations (3.9) and (3.10) hold even when $k = -1$.

It should be noted that if $k \neq -1$,

$$Z^{(n)}(k;z) = \frac{1}{k+1} Z^{(n)}(-1;z) + i\{i\cdot Z^{(n)}(-1;z)\}$$

and

$$i\cdot Z^{(n)}(k;z) = -i\{Z^{(n)}(-1;z)\} + \frac{1}{k+1} i\cdot Z^{(n)}(-1;z). \tag{3.14}$$

These formulae are an illustration of Theorem 6. We obtain the "polar form" of $Z^{(n)}$ by setting $z = re^{i\theta}$. Thus, when $k \neq -1$,

$$Z^{(n)}(k;z) = \frac{1}{2(n+1)(k+1)} r^{n+1}\{[(k+2)\cos\theta\cos n\theta - (nk+2k+2)\sin\theta\sin n\theta] + i(n+1)k\sin\theta\cos n\theta\},$$

for $n \neq -1$, and

$$Z^{(-1)}(k;z) = \left\{\frac{k}{2(k+1)}\sin^2\theta + \frac{k+2}{2(k+1)}\log r\right\} + i\frac{k}{2(k+1)}\cos\theta\sin\theta.$$

The results for $k = -1$ are obvious from these formulae.

Similarly,

$$i\cdot Z^{(n)}(k;z) = -\frac{1}{2(n+1)(k+1)} r^{n+1}\{(n+1)k\sin\theta\cos n\theta + i[(nk-2)\sin\theta\sin n\theta + (k+2)\cos\theta\cos n\theta]\},$$

and

$$i\cdot Z^{(-1)}(k;z) = -\frac{k}{2(k+1)}\sin\theta\cos\theta + i\left\{\frac{k}{2(k+1)}\sin^2\theta - \frac{k+2}{2(k+1)}\log r\right\}.$$

If n is real, these formulae show that for $n \neq -1$,

$$|Z^{(n)}(-1;z)| \leqq \frac{r^{n+1}}{2|n+1|}\{|n| + |1| + |n+1|\}$$

$$|i\cdot Z^{(n)}(-1;z)| \leqq \frac{r^{n+1}}{2|n+1|}\{1 + |n+1| + |n+2|\}$$

$$|Z^{(n)}(k;z)| \leqq \frac{r^{n+1}}{2|n+1|\,|k+1|}\{|k+2| + |nk+2k+2| + |n+1|\,|k|\}$$

$$|i\cdot Z^{(n)}(k;z)| \leqq \frac{r^{n+1}}{2|n+1|\,|k+1|}\{|k+2| + |nk-2| + |n+1|\,|k|\}.$$

Even cruder estimates, easily obtainable from the above, will serve our purpose later, namely,

$$\begin{aligned} |Z^{(n)}(-1;z)| &\leqq 2r^{n+1}, \\ |i\cdot Z^{(n)}(-1;z)| &\leqq 4r^{n+1}, \\ |Z^{(n)}(k;z)| &\leqq \left\{3 + \frac{1}{|k+1|}\right\} r^{n+1}, \\ |i\cdot Z^{(n)}(k;z)| &\leqq \left\{1 + \frac{3}{|k+1|}\right\} r^{n+1} \end{aligned} \tag{3.15}$$

all valid for $|n| \geqq 2$.

It is interesting to consider the zeros of $Z^{(n)}(k;z)$ since in this respect, the properties of the bi-analytic functions are quite different from those of the corresponding analytic functions z^n.

Suppose, first that k is -1. Then, $Z^{(-1)}(-1;z)=0$ for $z=\pm 1$ and $z=\pm ie$. Also, if $n=(2p+1)/(2q)$ where p and q are integers then $Z^{(n)}(-1,z)=0$ for $z=re^{i\pi}$ (q odd) and $z=re^{2i\pi}$ (q even). Thus, in this case, $Z^{(n)}(-1,z)$ has a whole line of zeros going through the origin. The only other cases in which $Z^{(n)}(-1,z)$ is zero are when $\mathrm{Re}(n) > -1$, in which cases $z=0$ is a zero.

Consider now, the situation when k is -2. First, $Z^{(-1)}(-2;z)=0$ whenever y is 0. For all other n, $Z^{(n)}(-2;z)=0$ for $z=re^{ip\pi}$ where p is an integer. Thus, in every case, $Z^{(n)}(-2;z)$ has a line of zeros. Now, suppose k is not 0. -1 or -2. Then, $Z^{(-1)}(k;z)=0$ for $z=\pm 1$ and $z=\pm ie^{-k/(k+2)}$. If $n=-2(k+1)/k$, $Z^{(n)}(k,z)=0$ for $z=re^{i([2p+1]/2q)\pi}$ where p and q are integers.

Again, if $n=(2p+1)/2q$, $Z^{(n)}(k;z)=0$ for $z=re^{i\pi}$ or $re^{2i\pi}$, as in the case $k=-1$. If n is not -1, $-2(k+1)/k$ or $(2p+1)/2q$ then the only time the $Z^{(n)}(k;z)=0$ is when $\operatorname{Re}(n)>-1$ and $z=0$.

All the above statements are easily proved using the "polar form" of $Z^{(n)}(k;z)$.

3. **The logarithmic function,** $L(k;z)$. Again, we shall first consider the case of $k\neq -1$.

Using the lower limit (0, 0) we find

$$L(k;z) = (\log z) \times Z^{(0)}(k;z) = \left\{\frac{k+2}{2(k+1)} x (\log |z| - 1) - y \operatorname{Im}(\log z)\right\} + \frac{k}{2(k+1)} iy \log |z| . \tag{3.16}$$

We easily see that

$$\frac{dL(k;z)}{dz} = Z^{(-1)}(k;z). \tag{3.17}$$

$L(k;z)$, like the corresponding analytic function, is an infinite-valued function, with $z=0$ as the branch-point. However, unlike $\log z$, $L(k;z)\to 0$ for $z\to 0$. Also, $L(k;z)$ has other zeros, namely $z=\pm e$, and the points on $|z|=1$ where $\arg z=\pm n\pi\mp\epsilon_n$ $(\epsilon_n>0)$ the numbers $\theta=\pm n\pi+\epsilon_n$ being the roots of $\tan\theta=\theta/2$.

Next,

$$i\cdot L(k;z) = i\cdot Z^{(0)} \times (\log z) = -\frac{k}{2(k+1)} y \log |z| + i\left\{\frac{1}{k+1} y \operatorname{Im}(\log z) + \frac{k+2}{2(k+1)} x (1 - \log |z|)\right\} \tag{3.18}$$

and

$$\frac{di\cdot L(k;z)}{dz} = i\cdot Z^{(-1)}(k;z). \tag{3.19}$$

Again, $i\cdot L(k;z)$ is infinite valued and has $z=0$ as its branch-point. The zeros of $i\cdot L(k;z)$ are the same as those of $L(k;z)$.

Coming now to the case $k=-1$, we have

$$\begin{aligned} L(-1;z) &= (\log z) \times Z^{(0)}(-1;z) \\ &= \frac{1}{2}(x \log |z| - x) - \frac{1}{2} iy \log |z| , \end{aligned}$$

$$\frac{dL(-1;z)}{dz} = Z^{(-1)}(-1;z)$$

and

$$L(-1;z) = 0 \text{ for } z = 0,\ z = \pm e, \text{ and } z = \pm i.$$

It is remarkable that $L(-1;z)$ is single-valued, although, of course, $z=0$ is still a singularity of the function in the sense that the function is not bi-analytic at this point.

Finally,

$$\begin{aligned} i\cdot L(-1;z) &= i\cdot Z^{(0)}(-1;z) \times (\log z) \\ &= \frac{1}{2} y \log |z| + \frac{1}{2} i\{-x \log |z| + x + 2y \operatorname{Im}(\log z)\} \end{aligned}$$

and

$$\frac{d\, i\cdot L(-1;z)}{dz} = i\cdot Z^{(-1)}(-1;z).$$

This function is, again, infinite valued. Its zeros are $z=0$, $z=\pm e$, and, once more, the points on $|z|=1$ where $\arg z = \pm n\pi \mp \epsilon_n$.

The relations between the functions of type k, $k \neq -1$, and the functions of type -1 is the same as for the generalized powers, namely

$$L(k;z) = \frac{1}{k+1} L(-1;z) + i\{i\cdot L(-1;z)\}$$

$$i\cdot L(k;z) = -iL(-1;z) + \frac{1}{k+1} i\cdot L(-1;z).$$

The author has not been able to find any simple analogue of the functional equation $\log z_1 z_2 = \log z_1 + \log z_2$.

4. **The exponential function,** $E(k;z)$. Using $(0, -\infty + i(0))$ as the lower limit we find, when k is not -1,

$$\begin{aligned} E(k;z) &= e^z \times Z^{(0)}(k;z) \\ &= \left(\frac{k+2}{2(k+1)} e^x \cos y - \frac{k}{2(k+1)} e^x y \sin y\right) + i \frac{k}{2(k+1)} y e^x \cos y. \end{aligned} \tag{3.20}$$

This function is clearly single-valued and bi-analytic for all z. Also it is never 0, except when k is -2, when $y=0$ is a line of zeros.

We easily see that

$$\frac{dE(k;z)}{dz} = E(k;z). \tag{3.21}$$

Also,

$$(3.22)\quad \begin{aligned} i\cdot E(k;z) = i\cdot Z^{(0)}(k;z)\times e^z = -\frac{k}{2(k+1)}ye^x\cos y \\ + ie^x\left[\frac{-k}{2(k+1)}y\sin y - \frac{k+2}{2(k+1)}\cos y\right] \end{aligned}$$

and

$$(3.23)\quad \frac{d\, i\cdot E(k;z)}{dz} = i\cdot E(k;z).$$

This function also is a single-valued bi-analytic function for all z which, when k is not -2, is never equal to 0. Again $y=0$ is a line of zeros in the case $k=-2$.

When k is -1, we have

$$\begin{aligned} E(-1;z) &= e^z \times Z^{(0)}(-1;z) \\ &= \frac{1}{2}e^x(\cos y + y\sin y) - \frac{1}{2}iye^x\cos y, \\ \frac{dE(-1;z)}{dz} &= E(-1;z), \\ i\cdot E(-1;z) &= i\cdot Z^{(0)}(-1;z)\times e^z \\ &= \frac{1}{2}ye^x\cos y - \frac{1}{2}ie^x(\cos y - y\sin y). \end{aligned}$$

$E(-1;z)$ and $i\cdot E(-1;z)$ are never zero. Once again,

$$\begin{aligned} E(k;z) &= \frac{1}{k+1}E(-1;z) + i\{i\cdot E(-1;z)\}, \\ i\cdot E(k;z) &= -iE(-1;z) + \frac{1}{k+1}i\cdot E(-1;z). \end{aligned}$$

The functional equation of the ordinary exponential e^z has an analogue. Thus

$$e^{\zeta(z)}\times E(z) = e^{\zeta}\times(e^z\times Z^{(0)}) = e^{\zeta+z}\times Z^{(0)}$$

and so

$$E(z_1+z_2) = e^{\zeta(z_1)}\times E(z_1)$$

where $\zeta(z_1)=z_2$. The same result holds for $i\cdot E(k;z)$.

In the case $k=0$,

$$E(z + 2n\pi i) = E(z),$$
$$i\cdot E(z + 2n\pi i) = i\cdot E(z),$$

where n is an integer. When $k \neq 0$, $E(k;\, z)$ has no periodic properties.

5. **The trigonometric functions,** $S(k;\, z)$ **and** $C(k;\, z)$. Once again we consider the case $k \neq -1$ first. We define

$$\begin{aligned} S(k;\, z) &= (\sin z) \times Z^{(0)}(k;\, z) \\ &= -\cos x\left(\frac{k+2}{2(k+1)}\cosh y + \frac{k}{2(k+1)}\, y\sinh y\right) \\ &\quad + i\,\frac{k}{2(k+1)}\, y \sin x \cosh y, \end{aligned} \tag{3.24}$$

$$\begin{aligned} C(k;\, z) &= \cos z \times Z^{(0)}(k;\, z) \\ &= \sin x\left(\frac{k}{2(k+1)}\, y\sinh y + \frac{k+2}{2(k+1)}\cosh y\right) \\ &\quad + i\,\frac{k}{2(k+1)}\, y \cos x \cosh y. \end{aligned} \tag{3.25}$$

Most of the familiar properties of $\sin z$ and $\cos z$ hold for these functions. Thus,

$$\frac{dC(k;\, z)}{dz} = -\, S(k;\, z), \qquad \frac{dS(k;\, z)}{dz} = +\, C(k;\, z),$$
$$C(z + 2\pi) = C(z), \qquad S(z + 2\pi) = S(z),$$
$$C(z + \pi) = -\, C(z), \qquad S(z + \pi) = -\, S(z),$$
$$S(z + \pi/2) = C(z), \qquad C(z + \pi/2) = -\, S(Z).$$

However, $S(z)$ is even while $C(z)$ is odd:

$$S(-z) = S(z), \qquad C(-z) = -\, C(z),$$

and so

$$S(\pi/2 - z) = -\, C(z), \qquad C(\pi/2 - z) = -\, S(z),$$
$$S(\pi - z) = -\, S(z), \qquad C(\pi - z) = +\, C(z).$$

There also exist forms of Euler's formulae. If we define $E(i\cdot z) = e^{iz} \times Z^{(0)}$, we have $C(z) + [i \times S(z)] = E(i\cdot z)$ and

$$C(z) = \frac{1}{2}\left\{E(i\cdot z) + E(-i\cdot z)\right\},$$

$$S(z) = \frac{1}{2i} \times \left\{E(i\cdot z) - E(-i\cdot z)\right\}.$$

Also,

$$C(n\cdot z) + [i \times S(n\cdot z)] = E(ni\cdot z)$$

where

$$C(n\cdot z) = \cos nz \times Z^{(0)}$$

and

$$S(n\cdot z) = \sin nz \times Z^{(0)}.$$

Next,

$$\begin{aligned} i\cdot S(k; z) &= i\cdot Z^{(0)}(k; z) \times \sin z \\ &= -\frac{k}{2(k+1)} y \sin x \cosh y \\ &\quad + i \cos x\left(\frac{k+2}{2(k+1)} \cosh y - \frac{k}{2(k+1)} y \sinh y\right), \end{aligned} \tag{3.26}$$

$$\begin{aligned} i\cdot C(k; z) &= i\cdot Z^{(0)}(k; z) \times \cos z \\ &= -\frac{k}{2(k+1)} y \cos x \cosh y \\ &\quad - i \sin x\left(\frac{k+2}{2(k+1)} \cosh y - \frac{k}{2(k+1)} y \sinh y\right). \end{aligned} \tag{3.27}$$

Again we have the results

$$\frac{d\, i\cdot C(k; z)}{dz} = -\, i\cdot S(k; z),$$

$$\frac{d\, i\cdot S(k; z)}{dz} = +\, i\cdot C(k; z),$$

$$\begin{aligned}
i\cdot C(z + 2\pi) &= i\cdot C(z), & i\cdot S(z + 2\pi) &= i\cdot S(z),\\
i\cdot C(z + \pi) &= -\, i\cdot C(z), & i\cdot S(z + \pi) &= -\, i\cdot S(z),\\
i\cdot S(z + \pi/2) &= i\cdot C(z), & i\cdot C(z + \pi/2) &= -\, i\cdot S(z),\\
i\cdot S(-z) &= i\cdot S(z), & i\cdot C(-z) &= -\, i\cdot C(z),\\
i\cdot S(\pi/2 - z) &= -\, i\cdot C(z), & i\cdot C(\pi/2 - z) &= -\, i\cdot S(z),\\
i\cdot S(\pi - z) &= -\, i\cdot S(z), & i\cdot C(\pi - z) &= +\, i\cdot C(z).
\end{aligned}$$

Consider now, the zeros of these functions.

When $k=0$, $C(z)$ and $i\cdot C(z)$ have lines of zeros, $x=p\pi$ while $S(z)$ and $i\cdot S(z)$ have lines of zeros, $x=(p+1/2)\pi$, where p is an integer.

When k is -2, $y=0$ is a line of zeros of $C(z)$, $i\cdot C(z)$, $S(z)$ and $i\cdot S(z)$.

Suppose, now, that k is not 0, -1 or -2. Then, $y=0$, $x=p\pi$ are zeros of $C(z)$ and $i\cdot C(z)$. If, in addition, $-2<k<0$, then $C(z)$ has zeros at $x=(p+1/2)\pi$, $y=\pm Y$, where Y is the positive root of $\tanh y=-((k+2)/k)(1/y)$. If $k>0$ or $k<-2$ then the only zeros of $C(z)$ are $y=0$, $x=p\pi$. However, if $k>0$ or <-2, then $i\cdot C(z)$ has zeros at $x=(p+1/2)\pi$ and $y=\pm Y'$ where Y' is the positive root of $\tanh y=+((k+2)/k)(1/y)$ while if $-2<k<0$, $y=0$, $x=p\pi$ are the only zeros.

If $-2<k<0$, $(k\neq -1)$ then the zeros of $S(z)$ are $y=0$, $x=(p+1/2)\pi$ and $x=p\pi$, $y=\pm Y$ while $y=0$, $x=(p+1/2)\pi$ are the only zeros if $k>0$ or $k<-2$. Finally, if $-2<k<0$, the only zeros of $i\cdot S(z)$ are $y=0$, $x=(p+1/2)\pi$ while if $k>0$ or <-2, it has additional zeros,

$$x = p\pi, \qquad y = \pm Y'.$$

In the case $k=-1$,

$$S(-1;z) = -\frac{1}{2}\cos x(\cosh y - y\sinh y) - \frac{1}{2} iy \sin x \cosh y,$$

$$C(-1;z) = \frac{1}{2}\sin x(\cosh y - y\sinh y) - \frac{1}{2} iy \cos x \cosh y,$$

$$i\cdot S(-1;z) = \frac{1}{2} y \sin x \cosh y + \frac{1}{2} i \cos x(y\sinh y + \cosh y),$$

$$i\cdot C(-1;z) = \frac{1}{2} y \cos x \cosh y - \frac{1}{2} i \sin x(y\sinh y + \cosh y).$$

All the properties of these functions in the cases $k\neq -1$ carry over to this case, $k=-1$. Even the discussion of the zeros is valid provided Y is taken to be the positive root of $\tanh y=+(1/y)$ (while Y', of course, does not exist).

Chapter IV: Taylor's and Laurent's Theorems; Cauchy's Integral Formula

1. **Taylor's and Laurent's Theorems.** We shall now show that if $f(z)$ is B.A. (k) in a circle of radius R and centre z_0, then

$$f(z) = \sum_0^\infty a_n\cdot Z^{(n)}(k;z-z_0) + \sum_0^\infty b_n(z-z_0)^n$$

where the series converge absolutely for $|z-z_0|<R$ and uniformly for $|z-z_0|\leq R_0$, R_0 being any positive number less than R. a_n and b_n are, of course, constants.

The proof given here does not rely on any analogue of Cauchy's integral formula but proceeds by reducing the result to Taylor's Theorem for analytic functions. (Of course, the fact that the derivative of a B.A.F. exists and is B.A. is used and this depends on Cauchy's integral.) The idea behind the proof has already been indicated in [4] where the result for B.A.F.'s of type -1 was obtained with, however, the restriction that the expansions were valid only in a circle of radius R' where R'/R was sufficiently small. We are now able to remove this restriction.

Suppose, then, that $f(z)$ is a B.A.F. (k) in $|z-z_0|<R$, with $k\neq -1$, whose A.F. is $[(k+1)\theta - i\omega]$.

Then

$$(k+1)\theta - i\omega = \sum_0^\infty a_n(z-z_0)^n$$

for $|z-z_0|<R$ and where, by Cauchy's inequality, $|a_n|\leq M/R^n$.

Now, $a_n\cdot Z^{(n)}(k;\, z-z_0)$ is a B.A.F. whose A.F. is $a_n(z-z_0)^n$. Also, if $a_n=\alpha_n+i\beta_n$, we have by equations (3.15)

$$\begin{aligned}|a_n\cdot Z^{(n)}(k,z-z_0)| &= |\alpha_n Z^{(n)}(k;z-z_0)+\beta_n i\cdot Z^{(n)}(k,z-z_0)|\\ &\leq \frac{4M}{R^n}\left\{1+\frac{1}{|k+1|}\right\}|z-z_0|^{n+1} = \frac{M'|z-z_0|^{n+1}}{R^n}\end{aligned}$$

for $n\geq 2$ and all $(z-z_0)$.

Therefore, $\sum_0^\infty a_n\cdot Z^{(n)}(k;\, z-z_0)$ is a series of B.A.F.'s which converges uniformly for all R_0 such that $|z-z_0|\leq R_0<R$ and absolutely for $|z-z_0|<R$.

Thus the hypotheses of our version of Weierstrass' double-series theorem (Theorem 14) are all satisfied and so $\sum_0^\infty a_n\cdot Z^{(n)}(k;\, z-z_0)$ is a B.A.F. (k) in $|z-z_0|<R$ with A.F. $[(k+1)\theta-i\omega]$.

Therefore, $\{f(z)-\sum_0^\infty a_n\cdot Z^{(n)}(k;\, z-z_0)\}$ is a B.A.F. (k) whose A.F. is 0, that is, it is an analytic function in $|z-z_0|<R$. Therefore

$$f(z) - \sum_0^\infty a_n\cdot Z^{(n)}(k;z-z_0) = \sum_0^\infty b_n(z-z_0)^n.$$

We note, finally that the argument is still valid, with unessential changes in the case $k=-1$; the major difference is that M' must be replaced by $6M$.

We have thus proved

THEOREM 21 ("TAYLOR'S THEOREM"). *Suppose $f(z)$ is B.A. for $|z-z_0|<R$. Then, for all z such that $|z-z_0|<R$,*

$$f(z) = \sum_0^\infty a_n\cdot Z^{(n)}(k;z-z_0) + \sum_0^\infty b_n(z-z_0)^n \tag{4.1}$$

where the series converge absolutely for $|z-z_0|<R$ and uniformly for $|z-z_0|\leq R_0$ where R_0 is any positive number less than R. a_n is determined uniquely by the A.F. of $f(z)$ and b_n uniquely by the a_n's and $f(z)$.

Suppose next, that $f(z)$ is B.A. (k) for $R_1<|z-z_0|<R_2$. Then, its A.F.

$(k+1)\theta - i\omega = \sum_0^\infty a_n(z-z_0)^n + \sum_0^\infty a_{-n}(z-z_0)^{-n}$ where the first series converges absolutely for $|z-z_0| < R_2$ and the second for $|z-z_0| > R_1$. Therefore,

$$|a_n| \leqq \frac{M}{R_1^n}, \qquad |a_{-n}| \leqq MR_2^n.$$

Therefore, by the type of argument used in proving Taylor's Theorem, $\sum_0^\infty a_n \cdot Z^{(n)}(k; z-z_0)$ and $\sum_0^\infty a_{-n} \cdot Z^{(-n)}(k; z-z_0)$ are B.A.F. (k), in $|z-z_0| < R_1$ and $|z-z_0| > R_2$ respectively, and have respective A.F.'s $\sum_0^\infty a_n(z-z_0)^n$ and $\sum_0^\infty a_{-n}(z-z_0)^{-n}$.

Carrying through the argument as in Taylor's Theorem we get the following result:

THEOREM 22 ("LAURENT'S THEOREM"). *Suppose $f(z)$ is B.A. for $R_1 < |z-z_0| < R_2$. Then, for all z in this domain*

$$f(z) = \sum_{-\infty}^{\infty} a_n \cdot Z^{(n)}(k; z - z_0) + \sum_{-\infty}^{\infty} b_n(z - z_0)^n. \tag{4.2}$$

The series of positive powers converge absolutely for $|z-z_0| < R_2$ and uniformly for $|z-z_0| \leqq R_{20} < R_2$ while the series of negative powers converge absolutely for $|z-z_0| > R_1$ and uniformly for $\infty > R_{1\infty} \geqq |z-z_0| \geqq R_{10} > R_1$. a_n is determined uniquely by the A.F. of $f(z)$ and b_n uniquely by the a_n's and $f(z)$.

2. **Cauchy's integral formula.** When any of the products, defined in Chapter II, of a B.A.F. and the analytic function $1/(z-\zeta)$ are formed it is found that the resulting functions are multi-valued. Therefore the analogue of Cauchy's integral formula can not be obtained in this way.

To circumvent this difficulty, we shall form an expression involving two B.A.F.'s of type k, one of which has $[1/(z-\zeta)]^2$ as its A.F.

Suppose, then, that $F_1(z) = \phi_1 + i\psi_1$ is a B.A.F. (k) whose A.F. is $[(k+1)\theta_1 - i\omega_1]$ and that $F_2(z) = \phi_2 + i\psi_2$ is another B.A.F. (k) with A.F. $[(k+1)\theta_2 - i\omega_2]$.

Let

$$[(k+1)\theta_1' - i\omega'] = \int_{z_1}^{z} [(k+1)\theta_1 - i\omega_1]dz,$$

$$[(k+1)\theta_2' - i\omega_2'] = \int_{z_2}^{z} [(k+1)\theta_2 - i\omega_2]dz.$$

Consider the expression

$$\{(\phi_1 + i\psi_1) \times [(k+1)\theta_2' - i\omega_2']\} + \{(\phi_2 + i\psi_2) \times [(k+1)\theta_2' - i\omega_2']\},$$

where the first product has (z_1, z_2) as its lower limit and the second product (z_2, z_2) as lower limit.

This is a B.A.F. (k) and a little algebra shows that it is equal to

$$[(k+1)\theta_2' - i\omega_2'](\phi_1 + i\psi_1) + [(k+1)\theta_1' - i\omega_1'](\phi_2 + i\psi_2) - \frac{k+2}{k+1}\omega_1'\omega_2',$$

that is, the integrals disappear.

If, now, we form the dot product of $-i$ and this expression, we obtain the following:

LEMMA.

$$\begin{aligned} &-i\cdot\{(\phi_1 + i\psi_1) \times [(k+1)\theta_2' - i\omega_2'] + (\phi_2 + i\psi_2) \times [(k+1)\theta_1' - i\omega_1']\} \\ (4.3) \qquad &= -i\{[(k+1)\theta_2' - i\omega_2'](\phi_1 + i\psi_1) + [(k+1)\theta_1' - i\omega_1'](\phi_2 + i\psi_2) \\ &\quad - (k+1)(k+2)\theta_1'\theta_2'\} \end{aligned}$$

is a B.A.F. (k) whose A.F. is

$$\begin{aligned} &-i\{[(k+1)\theta_2' - i\omega_2'][(k+1)\theta_1 - i\omega_1] + [(k+1)\theta_1' - i\omega_1'][(k+1)\theta_2 - i\omega_2]\} \\ &\qquad = \frac{d}{dz}\{-i[(k+1)\theta_2' - i\omega_2'][(k+1)\theta_1' - i\omega_1']\}. \end{aligned}$$

The lower limit of the first cross-product is (z_1, z_2) and of the second is (z_2, z_2) while that of the dot product is either (z_1, z_1) or (z_2, z_2).

An analogue of Cauchy's integral can now be obtained by setting $\phi_2 + i\psi_2 = -Z^{(-2)}(k; z-\zeta)$ in (4.3) and applying "Cauchy's Theorem" (Theorem 10) to the resulting B.A.F. The formula is given in the following theorem.

THEOREM 23 ("CAUCHY'S INTEGRAL FORMULA"). *Suppose $f(z) = u + iv$ is B.A.F. (k), $k \neq -1$, in a domain D bounded by a simple, closed continuous rectifiable contour, C. Let the A.F. of $f(z)$ be $[(k+1)\theta - i\omega]$ and let $[(k+1)\theta' - i\omega'] = \int_{z_1}^{z}[(k+1)\theta - i\omega]dz$ where z_1 is any point in D.*

Suppose also that $f(z)$ and $[(k+1)\theta' - i\omega']$ are continuous in the closure of D. Then, if ζ is any point in D,

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi}\oint_C \left\{-i\cdot\left[\left(f(z) \times \frac{1}{z-\zeta}\right) - Z^{(-2)}(k, z-\zeta) \times ((k+1)\theta' - i\omega')\right]\right\}\cdot dz \\ &= \frac{1}{2\pi i}\oint_C \left\{\frac{f(z)}{z-\zeta} - [(k+1)\theta' - i\omega']Z^{(-2)}(k; z-\zeta) \right. \\ (4.4) \qquad &\qquad \left. - (k+2)\theta' \operatorname{Re}\left(\frac{1}{z-\zeta}\right)\right\} dz \\ &\quad - i\left\{\left[(-k+1)\theta' - \frac{i}{k+1}\omega'\right]\operatorname{Re}\left(\frac{1}{z-\zeta}\right) \right. \\ &\qquad \left. + i(\theta' + i\omega')\operatorname{Im}\left(\frac{1}{z-\zeta}\right)\right\} dy. \end{aligned}$$

Proof. Applying "Cauchy's Theorem" to the function shown in (4.4), the integral in (4.4) is seen to be equal to the integral of the same function taken around any small circle Γ with ζ as centre. Let $z-\zeta=re^{i\theta}$ be such a circle. We shall show that as $r\to 0$, the integral approaches $f(\zeta)$.

Now, the first term, $(1/2\pi i)\oint f(z)/(z-\zeta)dz$ gives $f(\zeta)$ in the limit, by the same argument used to prove the result for analytic functions. It is only necessary, then, to show that the limit of the other terms is 0.

Now,

$$Z^{(-2)}(k, z-\zeta) = \frac{-r^{-1}}{2(k+1)}\left\{[(k+2)\cos\theta\cos 2\theta + 2\sin\theta\sin 2\theta] - ik\sin\theta\cos 2\theta\right\}.$$

Therefore

$$\oint_\Gamma \operatorname{Re}\left\{Z^{(-2)}(k; z-\zeta)\right\}dz = \int_0^{2\pi} -\frac{1}{2(k+1)}[(k+2)\cos\theta\cos 2\theta + 2\sin\theta\sin 2\theta]e^{i\theta}id\theta = -\frac{k+4}{4(k+1)}\pi i$$

and

$$\oint_\Gamma \operatorname{Im}\left\{Z^{(-2)}(k; z-\zeta)\right\}dz = +\int_0^{2\pi}\frac{ki}{2(k+1)}\sin\theta\cos 2\theta e^{i\theta}d\theta = \frac{k\pi}{4(k+1)}.$$

Therefore, the terms $-(1/2\pi i)\oint_P[(k+1)\theta' - i\omega']Z^{(-2)}(k; z-\zeta)dz$ give, in the limit, $(1/2(k+1))[(k+1)\theta'(\zeta) - i\omega'(\zeta)]$.

Next,

$$\oint_\Gamma \operatorname{Re}\left(\frac{1}{z-\zeta}\right)dz = \pi i,$$

$$\oint_\Gamma \operatorname{Im}\left(\frac{1}{z-\zeta}\right)dy = 0,$$

$$\oint_\Gamma \operatorname{Re}\left(\frac{1}{z-\zeta}\right)dy = \pi$$

and so the other terms give, in the limit,

$$-\frac{k+2}{2}\theta'(\zeta)+\frac{1}{2}\left[(k+1)\theta'(\zeta)+\frac{i}{k+1}\omega'(\zeta)\right]$$
$$=-\frac{1}{2(k+1)}\left[(k+1)\theta'(\zeta)-i\omega'(\zeta)\right].$$

Thus the total contribution form the integral around a small circle is $f(\zeta)$. Q.E.D.

The corresponding formula in the case $k=-1$ is

$$f(\zeta)=\frac{1}{2\pi}\oint_C\left\{-i\cdot\left[\left(f(z)\times\frac{1}{z-\zeta}\right)-Z^{(-2)}(-1;z-\zeta)\times(\theta'+i\omega')\right]\right\}\cdot dz$$

$$(4.5)\qquad =\frac{1}{2\pi i}\oint_C\left\{\frac{f(z)}{z-\zeta}-(\theta'+i\omega')Z^{(-2)}(-1;z-\zeta)-\theta'\operatorname{Re}\left(\frac{1}{z-\zeta}\right)\right\}dz$$
$$+\left\{\omega'\operatorname{Re}\left(\frac{1}{z-\zeta}\right)+\theta'\operatorname{Im}\left(\frac{1}{z-\zeta}\right)\right\}dy.$$

We note that the only one of our previous results used in the above proof was Cauchy's Theorem. Therefore the only hypotheses needed for $(u+iv)$ are those given in our original definitions, Definition I and Definition II.

If other types of dot and cross-products are used in the integrand, the integrand will be found to differ from the above by an additive term proportional to

$$\frac{1}{2\pi i}\;\frac{(k+2)[(k+1)\theta'-i\omega']}{(k+1)(z-\zeta)}$$

(with a real constant of proportionality). Thus the integral will have the value

$$f(\zeta)+(\text{real constant})\,\frac{k+2}{k+1}\left[(k+1)\theta'(\zeta)i\omega'(\zeta)\right].$$

The integral in (4.5) is always a B.A.F. (k) for ζ inside or outside C, provided u, v, θ' and ω' are any functions continuous on C. This is not immediately obvious since the integrand, regarded as a function of ζ, is not one of the products we have defined in Chapter III. However, a little algebra shows that it is B.A. and its A.F. is

$$\frac{1}{2\pi i}\oint_C\frac{(k+1)\theta'-i\omega'}{(z-\zeta)^2}\,dz.$$

This, of course, is $(d/dz)[(k+1)\theta'-i\omega']=[(k+1)\theta-i\omega]$ when $[(k+1)\theta'-i\omega']$ is analytic inside C but, in any case, it is analytic. This A.F. does not, of course, involve $f(z)$ since the terms involving $f(z)$ are obviously analytic.

It is clear that, as in the case of analytic functions, the domain D does not have to be simply-connected but can be bounded and multiply-connected.

As a first consequence of Cauchy's formula we see that all derivatives of $f(z)$ exist inside D; this follows as for analytic functions.

Cauchy's formula will now be applied to the problem of differentiating series term-by-term.

THEOREM 24. *Suppose that* $\{f_1(z), f_2(z), \cdots, f_n(z)\}$ *is a sequence of B.A.F. (k), inside a domain D and that* $\sum_1^\infty f_n(z)$ *is uniformly convergent in every closed region D′ interior to D. Suppose also that the series of A.F.'s* $\sum_1^\infty [(k+1)\theta_n - i\omega_n]$ *is uniformly convergent in D′. Then*

(a) $$f(z) = \sum_1^\infty f_n(z)$$

is bi-analytic of type k inside D and its A.F. is

$$[(k+1)\theta - i\omega] = \sum_1^\infty [(k+1)\theta_n - i\omega_n].$$

(b) $$\frac{d}{dz} f(z) = \sum_1^\infty \frac{d}{dz} f_n(z)$$

in D′.

The A.F. $(d/dz)f(z)$ *is*

$$d/dz[(k+1)\theta - i\omega] = \sum_1^\infty d/dz[(k+1)\theta_n - i\omega_n].$$

Also, the differentiated series is uniformly convergent in D′ and has a uniformly convergent A.F.

Proof. (a) This has been proved previously. A different proof will now be given, using Cauchy's integral.

Let C be a continuous simple closed rectifiable curve or a set of such curves, bounding D'. Then, $f_n(\zeta)$ is a Cauchy integral taken over C. Since θ_n', ω_n' and $f_n(z)$ are uniformly convergent on C, we can sum such integrals by summing inside the integral sign.

Thus, $f(\zeta) = \sum_1^\infty U_n(\zeta)$ is a Cauchy integral and so, by our previous remark, B.A. (k) with A.F.

$$\frac{1}{2\pi i} \oint \frac{(k+1)\theta' - i\omega'}{(z-\zeta)^2} dz = [(k+1)\theta - i\omega]$$

since $((k+1)\theta' - i\omega')$ is analytic inside C. Q.E.D.

(b) We have

$$\frac{d}{d\zeta}f(\zeta) = \frac{1}{2\pi i}\oint_C \frac{d}{d\zeta}\left\{\left[\frac{f(z)}{z-\zeta} - [(k+1)\theta' - i\omega']Z^{(-2)}(k, z-\zeta)\right.\right.$$

$$\left. - (k+2)\theta' \operatorname{Re}\left(\frac{1}{z-\zeta}\right)\right]dz - i\left[\left(-\{k+1\}\theta' - \frac{i}{k+1}\omega'\right)\operatorname{Re}\left(\frac{1}{z-\zeta}\right)\right.$$

$$\left.\left. + i(\theta' + i\omega)\operatorname{Im}\left(\frac{1}{z-\zeta}\right)\right]dy\right\}$$

$$= \frac{1}{2\pi i}\oint \frac{d}{d\zeta}\left\{\left[\frac{\sum f_n(z)}{z-\zeta} - (\sum\{(k+1\}\theta_n' - i\omega_n')Z^{(-2)}(k, z-\zeta)\right.\right.$$

$$\left. - (k+2)\theta' \operatorname{Re}\left(\frac{1}{z-\zeta}\right)\right]dz - i\left[\left(-\{k+1\}\theta' - \frac{i}{k+1}\omega'\right)\operatorname{Re}\left(\frac{1}{z-\zeta}\right)\right.$$

$$\left.\left. + i(\theta' + i\omega')\operatorname{Im}\left(\frac{1}{z-\zeta}\right)\right]dy\right\}$$

$$= \sum_1^\infty \frac{1}{2\pi i}\oint \frac{d}{d\zeta}\left\{\left[\frac{f_n(\zeta)}{z-\zeta} - [(k+1)\theta_n' - i\omega_n']Z^{(-2)}(k, z, \zeta)\right.\right.$$

$$\left. - (k+2)\theta' \operatorname{Re}\left(\frac{1}{z-\zeta}\right)\right]dz - i\left[\left(-\{k+1\}\theta' - \frac{i}{k+1}\omega'\right)\operatorname{Re}\left(\frac{1}{z-\zeta}\right)\right.$$

$$\left.\left. + i(\theta' + i\omega')\operatorname{Im}\left(\frac{1}{z-\zeta}\right)\right]dy\right\}$$

$$= \sum_1^\infty \frac{df_n(\zeta)}{d\zeta}.$$

That the A.F. of $(d/dz)f(z)$ is $\sum_1^\infty (d/dz)[(k+1)\theta_n - i\omega_n]$ follows from the above and that this is $(d/dz)[(k+1)\theta - i\omega]$ follows from the result on analytic functions.

That the differentiated series $\sum_1^\infty (d/dz)f_n(z)$ is uniformly convergent follows as for analytic functions [7]. Q.E.D.

By repeating the above process, we see that a uniformly convergent series of B.A.F.'s can be differentiated term-by-term any number of times.

It is sufficient for the validity of Theorem 24 to assume that $\sum f_n(z)$, $\sum \theta_n$ and $\sum \omega_n$ are uniformly convergent on some simple closed rectifiable contour C; the above proof then goes through and the same conclusions follow.

Using Theorem 24 it is possible to obtain simple formulae for the coefficients a_n and b_n in equation (4.1). a_n, of course, is determined by the derivative of the A.F. of $f(z)$. Also, $b_0 = f(z_0)$. To find b_1, differentiate equation (4.1) term-by-term. Thus,

$$\sum_{1}^{\infty} nb_n(z - z_0)^{n-1} = \frac{d}{dz}\left\{f(z) - \sum_{0}^{\infty} a_n \cdot Z^{(n)}(k; z - z_0)\right\}.$$

Now, this derivative in the sense of analytic functions can be written as

$$\frac{df}{dz} - \frac{d}{dz}\sum_{0}^{\infty} a_n \cdot Z^{(n)}(k; z - z_0)$$

where the differentiation is in the sense of bi-analytic functions since bi-analytic differentiation reduces to analytic differentiation whenever the differentiated function is analytic.

Therefore,

$$\sum_{1}^{\infty} nb_n(z - z_0)^{n-1} = \frac{df}{dz} - \sum_{1}^{\infty} na_n \cdot Z^{(n-1)}(k, z - z_0) - \frac{d}{dz} a_0 \cdot Z^{(0)}(k, z - z_0),$$

and so, putting $z = z_0$ in this equation,

$$b_1 = \left.\frac{df}{dz}\right|_{z=z_0} - \bar{a}_0 \frac{k + 2}{2(k + 1)}$$

where $\bar{a}_n$ is the complex conjugate of a_n. Repeating this process of differentiating term-by-term, we get, finally,

(4.6)
$$\begin{aligned} a_0 &= [(k + 1)\theta(z_0) - i\omega(z_0)], \\ a_n &= \frac{1}{n!}\frac{d^n}{dz^n}[(k + 1)\theta - i\omega]\Big|_{z=z_0}, \\ b_0 &= f(z_0), \\ b_n &= \frac{1}{n!}\left\{\frac{d^n}{dz^n}f(z)\Big|_{z=z_0} - (n - 1)!\,\frac{k + 2}{2(k + 1)}\,\bar{a}_{n-1}\right\}. \end{aligned}$$

The result when $k = -1$ is

(4.7)
$$\begin{aligned} a_0 &= \theta(z_0) + i\omega(z_0), \\ a_n &= \frac{1}{n!}\frac{d^n}{dz^n}(\theta + i\omega)\Big|_{z=z_0}, \\ b_0 &= f(z_0), \\ b_n &= \frac{1}{n!}\left\{\frac{d^n f(z)}{dz^n}\Big|_{z=z_0} - \frac{(n - 1)!}{2}\,\bar{a}_{n-1}\right\}. \end{aligned}$$

The Taylor's series of the elementary functions are now easily obtainable. The results are

$$E(k; z) = \frac{k+2}{2(k+1)} + \sum_0^\infty \frac{1}{n!} Z^{(n)}(k; z) \qquad (0! = 1), \tag{4.8}$$

$$S(k; z) = -\frac{k+2}{2(k+1)} + \sum_0^\infty \frac{(-1)^n}{(2n+1)!} Z^{(2n+1)}(k; z), \tag{4.9}$$

$$C(k; z) = \sum_0^\infty \frac{(-1)^n}{(2n)!} Z^{(2n)}(k; z), \tag{4.10}$$

$$L(k; 1+z) = -\frac{k+2}{2(k+1)} + \sum_1^\infty \frac{(-1)^{n-1}}{n} Z^{(n)}(k; z). \tag{4.11}$$

Finally, if m and z_0 are real, $m \neq -1$, we have the following analogue of the Binomial Theorem:

$$\begin{aligned} Z^{(m)}(k; z+z_0) = {} & \frac{k+2}{2(k+1)} \frac{z_0^{m+1}}{m+1} \\ & + \sum_0^\infty \frac{m(m-1)\cdots(m-n+1)}{n!} z_0^{m-n} \cdot Z^{(n)}(k; z). \end{aligned} \tag{4.12}$$

If m is -1, the first term must be replaced by $((k+2)/2(k+1)) \log |z_0|$.

The above formulae are obviously valid wherever the corresponding formulae for analytic functions hold.

The results when $k = -1$ are easily written down, from the above.

3. **Bi-analytic functions with isolated singularities.** The example $Z^{(-n)}(k; z)$, where n is a positive integer, shows that it is possible to have single-valued functions which are bi-analytic except at one point. We shall say that a function $f(z)$ which is bi-analytic for $0 < |z - z_0| < R$ has an isolated singularity at $z = z_0$. An attempt will be made in this section to classify the types of isolated singularities.

We shall assume that both the B.A.F. and its A.F. are single-valued. At least when k is -1 the example $L(-1; z)$ shows that it is possible for a function to be single-valued but to have a multi-valued A.F. We shall not consider such functions.

Suppose, then, that $f(z)$ is B.A. for $0 < |z - z_0| \leqq R$. Then, Laurent's Theorem shows that

$$f(z) = \sum_{-\infty}^{\infty} a_n \cdot Z^{(n)}(k; z - z_0) + \sum_{-\infty}^{\infty} b_n (z - z_0)^n \quad \text{for} \quad 0 < |z - z_0| \leqq R.$$

Let $g(z) = \sum_{-\infty}^{\infty} b_n (z - z_0)^n$, $f_1(z) = \sum_{-\infty}^{\infty} a_n \cdot Z^{(n)}(k; z - z_0)$.

Now, $\{a_n\}$ and so $f_1(z)$ is completely determined by the A.F. of $f(z)$, say $h(z)$. Therefore, in classifying the singularities of $f(z)$, attention must be paid to the singularities of the A.F., $h(z)$. The difficulties of the problem lie in the question as to what types of behavior in $f(z)$ are possible when $h(z)$ behaves in a given manner; as will be seen we have not been completely successful in answering this question.

The problem will be discussed under three headings, corresponding to the different possible singularities of $h(z)$.

(A) *Suppose $h(z)$ is bounded in the neighbourhood of z_0.*

Then, $h(z)$ has at most a removable singularity and so therefore its integral $\int_{z_1}^{z} h(z)dz$ is regular, where z, is any point in the neighbourhood of z_0. Thus $f_1(z) = \sum_0^\infty a_n \cdot Z^{(n)}(k; z-z_0)$.

(A.1) Suppose $f(z)$ is bounded in the neighbourhood of z_0.

Then it can be shown that $f(z)$ is equal in the neighbourhood of z_0 but not necessarily at z_0, to a B.A.F. which is B.A. not only in the neighbourhood of z_0 but at z_0 itself, that is, for all z, in $|z-z_0| \leqq R$. In this case, $f(z)$ is said to have a removable singularity.

This is proved, as with analytic functions, by representing $f(z)$ as a Cauchy integral taken over $|z-z_0| = R$ and a small circle $|z-z_0| = \rho < R$. It is easily seen that as $\rho \to 0$, the latter integral $\to 0$ and since the values of $f(z)$ do not depend on ρ, $f(z)$ is equal, except at z_0, to a Cauchy integral taken over $|z-z_0| = R$. Q.E.D.

Thus,

$$f(z) = \sum_0^\infty a_n \cdot Z^{(n)}(k; z - z_0) + \sum_0^\infty b_n(z - z_0)^n \tag{4.13}$$

for $|z-z_0| \leqq R$.

(A.2) $$|f(z)| \to \infty \quad \text{as} \quad z \to z_0.$$

Then

$$f(z) = \sum_0^\infty a_n \cdot Z^{(n)}(k; z - z_0) + \sum_{-p}^\infty b_n(z - z_0)^n \tag{4.14}$$
$$\text{with } b_{-p} \neq 0, \quad \text{for} \quad |z - z_0| \leqq R.$$

For, $f_1(z) \to 0$ as $z \to z_0$ and there can not be an infinite number of terms with n negative in $g(z)$, since $g(z)$ would not approach ∞ if there were, and so $f(z)$ would not approach ∞.

This means that for some integer p, $|z-z_0|^p f(z)$ is bounded.

(A.3) $f(z)$ is unbounded for z near z_0 but $|f(z)|$ does not approach ∞ as $z \to z_0$.

Then,

$$f(z) = \sum_0^\infty a_n \cdot Z^{(n)}(k; z - z_0) + \sum_{-\infty}^\infty b_n(z - \zeta)^n \quad \text{for } 0 < |z - z_0| \leqq R. \tag{4.15}$$

Thus $f(z)$ comes close to every value for z near z_0 (Weierstrass) and takes on every value, with at most one exception an infinite number of times (Picard).

Next, consider the situation when $h(z)$ has a pole.

(B) $h(z)\to\infty$ *as* $z\to z_0$

Thus, $f_1(z)=\sum_{-p}^{\infty} a_n\cdot Z^{(n)}(k;\, z-z_0)$, $a_{-p}\neq 0$, and so $|z-z_0|^{p-1}f_1(z)$ is bounded for some integer p.

(B.1) $f(z)$ is bounded near z_0. We shall show that this is impossible.

Now, $f_1(z)\to\infty$ as $z\to z_0$. Therefore, if $f(z)$ is to be bounded,

$$f(z) = f_1(z) + \sum_{-p+1}^{\infty} b_n(z - z_0)^n, \qquad b_{-p+1} \neq 0.$$

The sum in $g(z)$ must start at $(-p+1)$ since if it started at any other value, $g(z)$ would approach ∞ at a different order than $f_1(z)$ and so $f(z)$ could not be bounded.

By the same argument, the highest order terms $a_{-p}\cdot Z^{-p}(k;\, z-z_0)$ and $b_{-p+1}(z-z_0)^{-p+1}$ must add up to zero if $f(z)$ is to be bounded.

Consider the real part of such an equation and put the functions in the polar form. Then, there is a factor r^{-p}, $p\neq -1$, which can be cancelled and so we get a trigonometric series that adds up to zero. All the coefficients must therefore be 0 and so it can be seen then that a_{-p} and b_{-p+1} must each be 0.

If p is -1, then $|Z^{(-1)}(k;\, z-z_0)|=0$ (log r) while $|z-z_0|^{-p+1}=0(1)$ and so the argument is simpler. Q.E.D.

(B.2) $f(z)\to\infty$ as $z\to z_0$.

$$(4.16)\qquad f(z) = \sum_{-p}^{\infty} a_n\cdot Z^{(n)}(k; z - z_0) + \sum_{-\infty}^{\infty} b_n(z - z_0)^n.$$

We have not been able to discover whether a finite or an infinite number of the b_{-n}'s $(n>0)$ differ from 0 in this case.

(B.3) $f(z)$ is unbounded for z near z_0 but $|f(z)|$ does not approach ∞ as $z\to z_0$.

In this case it is obvious that

$$(4.17)\qquad f(z) = \sum_{-p}^{\infty} a_n\cdot Z^{(n)}(k; z - z_0) + \sum_{-\infty}^{\infty} b_n(z - z_0)^n$$

where there are an infinite number of the b_n's for $n<0$ which are nonzero.

Finally we have

(C) $h(z)$ *has an essential singularity.*

Then

$$f_1(z) = \sum_{-\infty}^{\infty} a_n\cdot Z^{(n)}(k; z - z_0).$$

It would appear plausible that a function like this $f_1(z)$ would approach indefinitely close to any given number as $z \to z_0$. However, we have not been able to prove any theorem on the behavior of such functions.

Consequently we are not able to give any further results in this case.

CHAPTER V: ELASTICITY THEORY

1. **Stress functions.** The usual method of relating problems of plane strain and generalized plane stress to the theory of biharmonic functions is by means of a stress function [8]. We shall now relate this procedure to the theory of bi-analytic functions.

Suppose, then, that $(u+iv)$ is bi-analytic function of type k with associated function $(k+1)\theta - i\omega$ $(k>0)$.

Let

$$\begin{aligned} E_{11} &= (k+1)u_x - (k-1)v_y, \\ E_{12} &= u_y - v_x, \\ E_{22} &= (k-1)u_x - (k+1)v_y. \end{aligned} \tag{5.1}$$

These functions E_{ij} when multiplied by Lamé's constant μ give the elastic stresses when u and $-v$ are elastic displacements.

Then,

$$\begin{aligned} \frac{\partial E_{11}}{\partial x} + \frac{\partial E_{12}}{\partial y} &= u_{xx} + u_{yy} + k\theta_x = 0, \\ \frac{\partial E_{12}}{\partial x} + \frac{\partial E_{22}}{\partial y} &= k\theta_y - v_{xx} - v_{yy} = 0, \end{aligned} \tag{5.2}$$

$$E_{11} + E_{22} = 2k\theta, \quad \text{and} \quad \text{so} \quad \Delta(E_{11} + E_{22}) = 0.$$

From (5.2), it follows that there exists a function Φ such that

$$\begin{aligned} E_{11} &= 2\Phi_{yy}, & E_{12} &= -2\Phi_{xy}, \\ E_{22} &= 2\Phi_{xx}, & &\text{and } \Delta(\Delta\Phi) = 0. \end{aligned} \tag{5.3}$$

This function Φ is the usual Stress-function [8].

We shall now express u and v in terms of Φ.

Let $(\Phi+i\Psi)$ be B.A. (k) with A.F. $[(k+1)\Theta - i\Omega]$.

Then, $2k\theta = E_{11}+E_{22} = 2\Delta\Phi = -2k\Theta_x$ and so $\Theta_x = -\theta$. Also

$$\begin{aligned} E_{11} &= (k+1)u_x - (k-1)v_y \\ &= (k+1)u_x - (k-1)(u_x - \theta) \\ &= 2u_x + (k-1)\theta \end{aligned}$$

and so

$$u_x = \frac{1}{2} E_{11} + \frac{k-1}{2} \Theta_x. \tag{5.4}$$

Similarly,

$$v_y = -\frac{1}{2} E_{22} - \frac{k-1}{2} \Theta_x. \tag{5.5}$$

Let $\phi + i\psi = d/dz(\Phi + i\Psi)$.
Then, from (5.4),

$$\begin{aligned} u_x &= \Phi_{yy} + \frac{k-1}{2} \Theta_x \\ &= (\Omega - \psi_x)y + \frac{k-1}{2} \Theta_x = -\frac{k+3}{2} \Theta_x - \psi_y \\ &= -\frac{k+3}{2} \Theta_x - (\phi_x - \Theta_x) \\ &= -\left(\frac{k+1}{2} \Theta + \phi\right)_x. \end{aligned}$$

Similarly

$$v_y = \left(\frac{1}{2} \Omega - \psi\right)_y.$$

Thus

$$\left[u + \frac{1}{2}(k+1)\Theta + \phi\right]_x = 0, \qquad \left(v - \frac{1}{2}\Omega + \psi\right)_y = 0.$$

Now,

$$\left[u + \frac{1}{2}(k+1)\Theta + \phi\right] \text{ and } \left(v - \frac{1}{2}\Omega + \psi\right)$$

are the real and imaginary parts of a bi-analytic function. It is therefore easy to see that

$$u + iv = -(\phi + i\psi) - \frac{1}{2}\left[(k+1)\Theta - i\Omega\right] + (ay + b + icx + id)$$

where a, b, c and d are real constants. However, since $E_{12} = u_y - v_x = -2\Phi_{xy}$, it follows that $c = a$. Further ψ is only determined by Φ to within a function of the form

$$\frac{1}{2}Ax^2 - \frac{1}{2}\frac{A}{k+1}y^2 + \alpha x + \beta y + \gamma$$

and so it is easily seen that $(\phi+i\psi)+(1/2)[(k+1)\Theta-i\Omega]$ is determined only to within a function of the form $(ay+b)+i(ax+d)$.

Finally, then, we can write

$$u + iv = -\frac{d}{dz}(\Phi + i\Psi) - \frac{1}{2}[(k+1)\Theta - i\Omega]. \tag{5.6}$$

This result shows that the introduction of a stress function in elasticity theory is analogous to the introduction of a complex potential in the theory of perfect fluids; in fact, when $(u+iv)$ is analytic, (5.6) reduces to

$$u + iv = -\frac{d}{dz}(fz)$$

where $f(z)$ is analytic.

What has been shown above is that if $(u+iv)$ is B.A. (k) then it can be expressed in the form (5.6). The converse statement is also easily seen to be true: If $(\Phi+i\psi)$ is B.A. (k) with A.F. $[(k+1)\Theta-i\Omega]$, then $(u+iv)$ defined by (5.6) is B.A. (k). Also, if E_{ij} are then defined by (5.1), then (5.3) holds.

Finally, we note that the terms of the form $(ay+b)+i(ax+d)$ in (5.6) represent a rigid body displacement.

2. **Betti's Reciprocal Theorem.** Starting with the equations defining bi-analytic functions, we shall now obtain some forms of Green's Identities. These will correspond to those given in [4] for the case $k=-1$. With them we shall obtain a uniqueness theorem and Betti's Reciprocal Theorem. We shall then obtain "Cauchy's Integral Formula" (Theorem 23) from Betti's Theorem by an integration-by-parts technique. This result corresponds to the fact that Cauchy's Formula for analytic functions can be obtained, by integrating by parts, from the Green's representation of harmonic functions

$$u(x, y) = \frac{1}{2\pi}\int_c \left(u\frac{\partial \log r}{\partial n} - \frac{\partial u}{\partial n}\log r\right) ds.$$

First, then, we have

LEMMA I. *Suppose* $f(z)=u+iv$ *is B.A.* (k) *in a domain D and whose A.F. is* $(k+1)\theta-i\omega$. *Suppose also that U and V are functions possessing continuous first derivatives in D and let*

$$\Theta = U_x - V_y, \qquad \Omega = U_y + V_x.$$

Let θ, ω, *U and V be continuous in* $D+C$, *where C is the simple closed continuous rectifiable boundary of D.*

Then

$$\iint_D \theta\Theta dxdy + \frac{1}{k+1}\iint_D \omega\Omega dxdy = \oint_C \left[U\left(\theta n_1 + \frac{1}{k+1}\omega n_2\right) - V\left(\theta n_2 - \frac{1}{k+1}\omega n_1\right)\right]ds, \tag{5.7}$$

where n_1 and n_2 are the x and y components of the outward drawn normal to C.

Proof. This is essentially the same as the proof of Theorem 8 in [4]. Q.E.D.

COROLLARY I. As a corollary of this we see that *if $(u+iv)$ is B.A. (k) and u, v, θ and ω are continuous in $D+C$, then*

$$\iint_D \left(\theta^2 + \frac{1}{k+1}\omega^2\right)dxdy = \oint_C \left[u\left(\theta n_1 + \frac{1}{k+1}\omega n_2\right) - v\left(\theta n_2 - \frac{1}{k+1}\omega n_1\right)\right]ds. \tag{5.8}$$

COROLLARY II. From (5.8) we see that *if, in addition, $u=v=0$, then $u=v=0$ in $D+C$, provided $k+1>0$.*

For then the right hand side of (5.8) is 0 and so $\theta^2+(1/(k+1))\omega^2=0$ in D and so $\theta=\omega=0$ in D. Therefore $(u+iv)$ is an analytic function with 0 boundary values and so is identically 0. Q.E.D.

LEMMA II. *With the same assumptions as in Lemma* I,

$$\iint_D \{U(\theta_x + 2\omega_y) + V(2\omega_x - \theta_y)\}dxdy = \frac{2k+1}{k+1}\oint_C \omega(Un_2 + Vn_1)ds - \frac{2k+1}{k+1}\iint_D \omega\Omega dxdy. \tag{5.9}$$

Proof.

$$\begin{aligned}
&\iint_D \{U(\theta_x + 2\omega_y) + V(2\omega_x - \theta_y)\}dxdy \\
&= \frac{2k+1}{k+1}\iint_D (U\omega_y + V\omega_x)dxdy \\
&= \frac{2k+1}{k+1}\iint \{(U\omega)_y + (V\omega)_x\}dxdy - \frac{2k+1}{k+1}\iint_D (\omega U_y + \omega V_x)dxdy \\
&= \frac{2k+1}{k+1}\oint_C \omega(Un_2 + Vn_1)ds - \frac{2k+1}{k+1}\iint_D \omega\Omega dxdy. \qquad \text{Q.E.D.}
\end{aligned}$$

We now make the following definitions, valid for any pair of functions, (u, v) having continuous first derivatives in $(D+C)$:

$$A(u, v) = \frac{1}{2}(u_x + v_y)n_1 + u_y n_2, \tag{5.10}$$
$$B(u, v) = v_x n_1 + \frac{1}{2}(v_y + u_x)n_2.$$

LEMMA III. *Let*

$$\theta = u_x - v_y, \qquad \omega = u_y + v_x,$$
$$\Theta = U_x - V_y, \qquad \Omega = V_y + V_x,$$

where u, v, U, V *are continuous and have continuous first derivatives in* $D+C$, *and* θ, ω, Θ, Ω *have continuous second derivatives in* D.

Then

$$\begin{aligned} \oint_C \{UA(u,v) + VB(u,v)\}ds - \frac{1}{2}\iint_D \{U(\theta_x + 2\omega_y) + V(2\omega_x - \theta_y)\}dxdy \\ = \oint_C \{uA(U, V) + vB(U, V)\}ds \\ - \frac{1}{2}\iint \{u(\Theta_x + 2\Omega_y) + v(2\Omega_x - \Theta_y)\}dxdy. \end{aligned} \tag{5.11}$$

Proof. This is very similar to the proof of Theorem 9 of [4].

Let

$$M(u, v) \equiv u_{xx} + 2u_{yy} + v_{yx} = \theta_x + 2\omega_y,$$
$$\overline{M}(u, v) \equiv 2v_{xx} + v_{yy} + u_{yx} = 2\omega_x - \theta_y.$$

Then

$$\iint_D \{UM(u, v) + V\overline{M}(u, v) - uM(U, V) - v\overline{M}(U, V)\}dxdy$$
$$= \iint_D \{(Uu_x)_x + 2(Uu_y)_y + (Uv_y)_x + 2(Vv_x)_x + (Vv_y)_y + (Vu_x)_y\}dxdy$$
$$- \iint_D \{(uU_x)_x + 2(uU_y)_y + (uV_y)_x + 2(vV_x)_x + (vV_y)_y + (vV_x)_y\}dxdy$$
$$= 2\oint_C \{UA(u, v) + VB(u, v) - uA(U, V) - vB(U, V)\}ds$$
$$= \iint_D \{U(\theta_x + 2\omega_y) + V(2\omega_x - \theta_y) - u(\Theta_x + 2\Omega_y) - v(2\Omega_x - \Theta_y)\}dxdy.$$

Q.E.D.

DEFINITION.

$$(5.12)\quad \begin{aligned} X(u, v) &= A(u, v) - \frac{1}{2}\,\omega n_2 + \frac{1}{2}\,k\theta n_1 \\ &= \frac{1}{2}\,[u_x + v_y + k(u_x - v_y)]n_1 - \left(u_y - \frac{1}{2}\,\omega\right)n_2; \\ Y(u, v) &= B(u, v) - \frac{1}{2}\,\omega n_1 + \frac{1}{2}\,k\theta n_2 \\ &= \left(v_x - \frac{1}{2}\,\omega\right)n_1 + \frac{1}{2}\,[u_x + v_y - k(u_x - v_y)]n_2. \end{aligned}$$

THEOREM 25 (BETTI'S RECIPROCAL THEOREM).

$$(5.13)\quad \oint_C \{UX(u, v) + VY(u, v)\}ds = \oint \{uX(U, V) + vY(U, V)\}ds$$

where $(u+iv)$ and $(U+iV)$ are B.A. (k) and (u, v), (U, V) are continuous and have continuous first derivatives in $D+C$.

Proof. Starting with (5.11) and substituting from (5.9) for the double integrals we get

$$(\alpha)\quad \begin{aligned} &\oint_C \left\{U\left[A(u, v) - \frac{2k+1}{2(k+1)}\,\omega n_2\right] + V\left[B(u, v) - \frac{2k+1}{2(k+1)}\,\omega n_1\right]\right\} ds \\ &= \oint_C \left\{u\left[A(U, V) - \frac{2k+1}{2(k+1)}\,\Omega n_2\right] + v\left[B(U, V) - \frac{2k+1}{2(k+1)}\,\Omega n_1\right]\right\} ds. \end{aligned}$$

Now, from Lemma I,

$$(\beta)\quad \begin{aligned} &\oint_C \left\{U\left(\theta n_1 + \frac{1}{k+1}\,\omega n_2\right) - V\left(\theta n_2 - \frac{1}{k+1}\,\omega n_1\right)\right\} ds \\ &= \oint_C \left\{u\left(\Theta n_1 + \frac{1}{k+1}\,\Omega n_2\right) - v\left(\Theta n_2 - \frac{1}{k+1}\,\Omega n_1\right)\right\} ds. \end{aligned}$$

Adding $k(\beta)/2$ to (α) we get the required result. Q.E.D.

If

$$X_x = \frac{1}{2}\,\{(k+1)u_x - (k-1)v_y\},$$

$$X_y = \frac{1}{2}\,(u_y - v_x),$$

$$Y_y = \frac{1}{2}\,\{(k-1)u_x - (k+1)v_y\},$$

then

$$X(u, v) = X_x n_1 + X_y n_2,$$
$$-Y(u, v) = X_y n_1 + Y_y n_2.$$

Now, X_x, X_y, Y_y are the usual stresses of elasticity theory divided by 2μ, μ being Lamé's constant and so Theorem 25 is the usual form of Betti's Reciprocal Theorem.

This could be obtained more directly of course, but we wished to do it starting with the basic equations defining bi-analytic functions.

It is intended, later on, to integrate by parts in (5.13). To do this we first prove

LEMMA IV.

$$(5.14)\qquad \begin{aligned} X(u, v) &= \frac{\partial}{\partial s}\left(v - \frac{1}{2}\omega'\right), \\ Y(u, v) &= \frac{\partial}{\partial s}\left(u - \frac{k+1}{2}\theta'\right), \end{aligned}$$

where $\theta = \theta_x'$, $\omega = \omega_x'$, *so that*

$$(k+1)\theta - i\omega = \frac{d}{dz}\left[(k+1)\theta' - i\omega'\right].$$

Proof.

$$\begin{aligned} X(u, v) &= \frac{1}{2}\left[u_x + v_y + k(u_x - v_y)\right]n_1 + \left[u_y - \frac{1}{2}(u_y + v_x)\right]n_2 \\ &= \frac{1}{2}\left[v_y n_1 - v_x n_2\right] + \frac{1}{2}u_y n_2 + \frac{1}{2}(k+1)u_x n_1 - \frac{1}{2}k v_y n_1 \\ &= \frac{1}{2}\left[v_y n_1 - v_x n_2\right] + \frac{1}{2}(\omega - v_x)n_2 + \frac{1}{2}(k+1)(\theta + v_y)n_1 - \frac{1}{2}k v_y n_1 \\ &= (v_y n_1 - v_x n_2) + \frac{1}{2}\left[\omega n_2 + (k+1)\theta n_1\right] \\ &= (v_y n_1 - v_x n_2) - \frac{1}{2}(-\omega_x' n_2 + \omega_y' n_1) \\ &= \frac{\partial}{\partial s}\left(v - \frac{1}{2}\omega'\right). \end{aligned}$$

The second result follows similarly. Q.E.D

LEMMA V. *Under the hypotheses of Theorem* 25,

$$\oint_C \left\{ u \frac{\partial \Omega'}{\partial s} - (k+1)v \frac{\partial \Theta'}{\partial s} + \omega' \frac{\partial U}{\partial s} - (k+1)\theta' \frac{\partial V}{\partial s} \right\} ds = 0, \tag{5.15}$$

where

$$\frac{d}{dz}\left[(k+1)\Theta' - i\Omega\right] = (k+1)\Theta - i\Omega,$$

$$\frac{d}{dz}\left[(k+1)\theta' - i\omega'\right] = (k+1)\theta - i\omega.$$

Proof. This follows immediately from Theorem 25 and Lemma IV, by integrating by parts in the integral $\oint [UX(u, v) + YY(u, v)]ds$. It is assumed, of course, that u, v, U, V, θ', ω' are single valued on C. Q.E.D.

LEMMA VI.

$$u(\xi\eta) = -\frac{1}{2\pi}\oint_C \left[u \frac{\partial \Omega_1'}{\partial s} - (k+1)v \frac{\partial \Theta_1'}{\partial s} + \omega' \frac{\partial U_1}{\partial s} - (k+1)\theta' \frac{\partial V_1}{\partial s} \right] ds,$$

(5.16)

$$v(\xi\eta) = +\frac{1}{2\pi}\oint \left[u \frac{\partial \Omega_2'}{\partial s} - (k+1)v \frac{\partial \Theta_2'}{\partial s} + \omega' \frac{\partial U_2}{\partial s} - (k+1)\theta' \frac{\partial V_2}{\partial s} \right] ds,$$

where

$$U_1 = \frac{k}{2(k+1)} \frac{(y-n)^2}{r^2} + \frac{k+2}{2(k+1)} \log r,$$

$$V_1 = \frac{k}{2(k+1)} \frac{(x-\xi)(y-n)}{r^2},$$

$$(k+1)\Theta_1' - i\Omega_1' = \log(z-\xi),$$

where $\xi = \xi + in$, $z = x + iy$, $r^2 = (x-\xi)^2 + (y-n)^2$ *and where*

$$U_2 = -\frac{k}{2(k+1)} \frac{(x-\xi)(y-n)}{r^2},$$

$$V_2 = \frac{k}{2(k+1)} \frac{(y-n)^2}{r^2} - \frac{k+2}{2(k+1)} \log r,$$

$$(k+1)\Theta_2' - i\Omega_2' = i \log(z-\xi).$$

Proof. Lemma V is applied to a contour consisting of the boundary C and a small circle Γ about ξ.

When $U+iV = Z^{(-1)}(k, z-\xi)$ the first result in (5.13) is obtained; when $U+iV = i\cdot Z^{(-1)}(k, z-\xi)$ the second result is obtained. Q.E.D.

THEOREM 26. *If $(u+iv)$ is B.A. (k) in D, and u and v have continuous derivatives of the first order in $D+C$ and if C is a continuous rectifiable contour $x=x(s)$, $y=y(s)$ with $x'(s)$ and $y'(s)$ continuous $(x'^2+y'^2>0)$, then*

$$
\begin{aligned}
u(\xi) - iv(\xi) = \frac{1}{2\pi i}\oint \Bigg\{\left[\frac{u+iv}{z-\xi} - [(k+1)\theta' - i\omega']Z^{(-2)}(k, z-\xi)\right] dz \\
- i[(k+1)\theta' - i\omega'](\theta + i\omega)dy - (k+2)\theta' \operatorname{Re}\left(\frac{dz}{z-\xi}\right)\Bigg\}
\end{aligned}
\tag{5.17}
$$

when $(k+1)\theta_1 - i\omega_1 = 1/(z-\xi)$.

Proof. From Lemma VI,

$$
\begin{aligned}
u(\xi) + iv(\xi) &= \frac{1}{2\pi}\int_C \Bigg\{-(u+iv)\frac{\partial\Omega!}{\partial s} - (k+1)\frac{\partial\Theta!}{\partial s}(-v+iu) \\
&\quad - \frac{\partial U_1}{\partial s}[\omega' + i(k+1)\theta'] + \frac{\partial V_1}{\partial s}[(k+1)\theta' - i\omega'] \\
&\quad + (k+1)\theta' i \frac{k+2}{k+1}\frac{\partial \log r}{\partial s}\Bigg\} ds \\
&= \frac{1}{2\pi i}\oint_C \Bigg\{(u+iv)\left([k+1]\frac{\partial\Theta_1}{\partial s} - i\frac{\partial\Omega_1}{\partial s}\right) \\
&\quad + [(k+1)\theta' - i\omega']\left(\frac{\partial U_1}{\partial s} + i\frac{\partial V_1}{\partial s}\right) - (k+2)\theta'\frac{\partial \log r}{\partial s}\Bigg\} ds \\
&= \frac{1}{2\pi i}\oint_C \Bigg\{\frac{u+iv}{z-\xi}dz + \Bigg[(\{k+1\}\theta' - i\omega')\left(\frac{\partial U_1}{\partial s} + i\frac{\partial V_1}{\partial s}\right) \\
&\quad - (k+2)\theta'\frac{\partial \log r}{\partial s}\Bigg] ds.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\left(\frac{\partial U_1}{\partial s} + i\frac{\partial V_1}{\partial s}\right) ds &= \left(\frac{\partial U_1}{\partial x}dx + i\frac{\partial V_1}{\partial x}dx\right) + \left(\frac{\partial U_1}{\partial y} + i\frac{\partial V_1}{\partial y}\right) dy \\
&= \frac{d}{dz}(U_1 + iV_1)dx + \left[\left(\omega_1 - \frac{\partial V_1}{\partial x}\right) + i\left(\frac{\partial U_1}{\partial x} - \theta_1\right)\right] dy \\
&= \left\{\frac{d}{dz}Z^{(-1)}(k, z-\xi)\right\} dx + \left(\frac{\partial U_1}{\partial x} + i\frac{\partial V_1}{\partial x}\right) idy + (\omega_1 - i\theta_1)dy \\
&= \left[\frac{d}{dz}Z^{(-1)}(k, z-\xi)\right] dz - i(\theta_1 + i\omega_1)dy.
\end{aligned}
$$

Thus,

$$u(\xi) + iv(\xi) = \frac{1}{2\pi i}\oint_C \left\{\frac{u+iv}{z-\xi} - [(k+1)\theta' - i\omega']Z^{(-2)}(k, z-\xi)\right\} dy$$

$$- i[(k+1)\theta' - i\omega'](\theta_1 + i\omega_1)dy - (k+2)\theta' \frac{\partial \log r}{\partial s} ds.$$

Next,

$$\frac{\partial \log r}{\partial s} ds = \frac{\partial \log r}{\partial x} dx - \frac{\partial \operatorname{Im} \log (z-\xi)}{\partial x} dy$$

$$= \operatorname{Re}\left[\frac{\partial \log r}{\partial x} + \frac{\partial}{\partial x} i \operatorname{Im}(\log\{z-\xi\})\right][dx + idy]$$

$$= \operatorname{Re}\left(\frac{d \log (z-\xi)}{dz} dz\right) = \operatorname{Re}\left(\frac{dz}{z-\xi}\right).$$

Finally then, we get (5.17). Q.E.D.

The result is, of course, the same as in Theorem 23 except that these less restrictive assumptions had to be made on the smoothness of the function and the contour.

Chapter VI: Viscous Fluids

1. **The form of the equations.** Consider the viscous flow of a fluid in the $x-y$ plane. The motion of the fluid is determined by two functions $u(x, y)$ and $v(x, y)$, the components of the velocity. If the velocities are sufficiently slow for nonlinear terms to be neglected the equations of motion are

$$\text{(6.1)} \qquad \begin{cases} \mu\Delta u = p_x, \\ \mu\Delta v = p_y, \\ u_x + v_y = 0, \end{cases}$$

where $p(x, y)$ is the pressure and μ a constant, is the coefficient of viscosity.

Let $\zeta = u_y - v_x$, so that ζ is the vorticity.

Then,

$$0 = \mu\Delta(u_x + v_y) = \Delta p.$$

Also, $\mu\Delta(u_y - v_x) = (p_{xy} - p_{yx}) = 0$.

Thus,

$$\text{(6.2)} \qquad \begin{cases} u_x + v_y = 0, \\ u_y - v_x = \zeta, \\ \Delta\zeta = 0. \end{cases}$$

Equations (6.1), therefore, imply (6.2). Conversely, however, starting from (6.2) we can get (6.1).

For, since $\Delta\zeta = 0$,

$$\Delta(u_y - v_x) = 0 \quad \text{or} \quad (\Delta u)_y = (\Delta v)_x.$$

Thus, there exists a function p such that

$$\Delta u = \frac{1}{\mu} p_x, \qquad \Delta v = \frac{1}{\mu} p_y. \qquad \text{Q.E.D.}$$

Also, $[p/\mu + i\zeta]$ is an analytic function, since

$$\zeta_x = u_{yx} - v_{xx} = -v_{yy} - v_{xx} = -\Delta v = -\frac{1}{\mu} p_y,$$

$$\zeta_y = u_{yy} - v_{xy} = u_{yy} + u_{xx} = \Delta u = \frac{1}{\mu} p_x,$$

which are the Cauchy-Riemann equations.

Thus, $(v + iu)$ is a bi-analytic function of type -1, whose A.F. is $[-\zeta + i(p/\mu)]$. We could also say, of course, that $(u - iv)$ is a bi-analytic function of type ∞ whose A.F. is $[(1/\mu)p + i\zeta]$.

All our results on bi-analytic functions can thus be applied to the flow of viscous fluids.

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