ALGEBRAIC ASPECTS OF THE THEORY OF DIFFERENTIAL EQUATIONS

BY

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The theory of systems of algebraic equations is one of the most highly developed chapters of algebra. At the center of this theory stands the concept of polynomial in one or more letters. The formal properties of polynomials offer themselves first for investigation. One of the first results is that on the representation of a polynomial as a product of irreducible factors. Out of the factorization theorem grows, by a profound generalization, the theory of ideals of polynomials. There is furnished thus a basis for studying many aspects of algebraic manifolds, whose complete theory constitutes algebraic geometry. By the side of the theory of manifolds stands the theory of elimination, with algorithms for solving systems of equations and theorems for counting solutions. In elimination theory, resultants of systems of polynomials play a fundamental rôle. The key to the general theory of algebraic systems is Hilbert's famous theorem on the existence of finite bases of infinite systems of polynomials.

Algebraic equations may be regarded as differential equations of order zero, that is, as differential equations in which derivatives do not appear effectively. It is thus proper to ask whether the theory of differential equations can be developed in such a way as to yield, by specialization, the classic results on systems of algebraic equations. One would expect to be able to carry over certain algebraic results into the field of differential equations and, in the process, to find, for differential equations, properties with no counterpart in the algebraic theory. Naturally, one would begin with algebraic differential equations, proceeding from them to more general types.

The application of algebraic methods to differential equations is justified by more than the expectation of creating new links between algebra and analysis and of discovering novel problems in differential equations. For want of proper algebraic viewpoints, questions which date from the beginnings of mathematical analysis remained without systematic treatment. Thus, the older literature contains only fragmentary and heuristic indications on the elimination theory of a system of differential equations. The problem of the number of arbitrary constants in the solution of a system, that is, the problem of the *order* of a system, failed to receive even a sound formulation. The most striking condition of all is perhaps that in the literature on singular solutions. The greatest source of light on the nature of the singular solutions of an algebraic differential equation is probably the fact that the solutions of the equation separate, in a unique manner, into collections analogous to irreducible algebraic manifolds. The description of this decomposition, for a given equation, is the first step in the classification of the singular solutions of the equation. In writings of Laplace, Lagrange and Poisson on singular solutions,* one can discern a groping towards such a decomposition theorem as we have just mentioned. When they wrote, the time was not ripe for the exact investigation of such questions. It was a time when even proofs of the uniqueness of the representation of an integer as a product of primes and of the uniqueness of the representation of a polynomial as a product of irreducible polynomials were not generally known. Certainly, it was too much to expect a systematic investigation of the algebra of differential equations. It is in the light of what is now known that one can understand the direction of the efforts of the great analysts named above.

The construction of an algebraic theory of differential equations has, during the past several years, occupied the present writer and his colleagues H. W. Raudenbush, E. Gourin, and W. C. Strodt.† For systems, ordinary or partial, which are algebraic in the unknowns and their derivatives, the nucleus of a general theory has been secured and applications to special problems, such as the problem of singular solutions, have been made. The manifold theory for algebraic differential equations has already taken definite shape. Raudenbush has supplied a purely algebraic proof of the counterpart, for differential polynomials, of the Hilbert basis theorem. He has also constructed a restricted theory of ideals of differential polynomials. A theory of non-algebraic systems is already in process of development.

In what follows, we shall describe some of the principal results which have been secured and shall indicate some of the methods of proof. An analysis will be made of such early work, bearing on our topic, as has come to our attention. We shall restrict our discussion to ordinary differential equations; previous publications‡ will inform one as to what has been done for several independent variables. We shall leave untouched also the theory of algebraic difference equations which, reconnoitered in papers of J. L. Doob, F. Herzog, W. C. Strodt, and the present writer, still awaits intensive development.

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* See §§17, 21, 22 below.

† One will understand that we are speaking of a theory constructed from the viewpoints of algebraic geometry and of the theory of elimination. Analogies between differential equations and algebraic equations have frequently been subjects of investigation. Similarities between linear differential equations and algebraic equations were observed by the early analysts. To this older work is related that of Landau, Loewy, Blumberg, Emmy Noether, Schmeidler, and Ore, who have studied the factorization of linear differential expressions. The Lie theory of differential equations was inspired by the Galois theory. The researches of Koenigsberger, Picard, Drach, and Vessiot have transformed the Lie theory into a closer analogue of the Galois theory.

‡ A bibliography is given at the end of the paper.

FORMS AND BASES

1. In constructing a theory of algebraic ordinary differential equations, it is natural to begin by studying the formal properties of polynomials in a set of unknown functions and their derivatives of various orders, the coefficients in the polynomials being functions of the independent variable. The nature of the coefficients should be specified with a view towards simplicity and generality for the resulting theory.

2. Let $\mathfrak{A}$ be an open region in the plane of the complex variable x. Let $\mathfrak{F}$ be a set of functions each of which is meromorphic in $\mathfrak{A}$ and at least one of which is not identically zero. We shall call $\mathfrak{F}$ a field if $\mathfrak{F}$ is closed with respect to rational operations and to differentiation. If f and g are functions in a field $\mathfrak{F}$, $f+g$, $f-g$ and fg are in $\mathfrak{F}$. If g is not identically zero, f/g is in $\mathfrak{F}$. Again, if f is any function in $\mathfrak{F}$, the derivative of f is in $\mathfrak{F}$. Thus is made explicit the meaning of the two types of closure.

3. Let n be any positive integer. Let $y_1, \cdots, y_n$ be unknown functions of the independent variable x. We denote by y_{ij} the jth derivative of y_i, $j=1, 2, \cdots$. We write, frequently, $y_i=y_{i0}$ and refer to y_i as its own derivative of order zero. By a *differential polynomial*, we shall mean a polynomial in a certain (eo ipso finite) number of the y_{ij}, with functions meromorphic in $\mathfrak{A}$ as coefficients. As a rule, we shall substitute the briefer term *form* for *differential polynomial.*

In every discussion, all forms which appear will be understood to have coefficients belonging to a given field $\mathfrak{F}$.

Raudenbush has employed *forms* in which the coefficients are not meromorphic functions, but rather abstract elements for which a differentiation process can be defined. This is certainly an excellent procedure from the point of view of abstract algebra. Nevertheless, the case of meromorphic coefficients contains most of what is vital for analysis, and allows the algebraic problems to be seen in their full proportions.

4. We shall use capital italic letters to denote forms, and large Greek letters to denote systems of forms. All forms will involve a definite set of unknowns $y_1, \cdots, y_n$.

One might conjecture, on the basis of Hilbert's theorem relative to the existence of finite bases for infinite systems of polynomials, that, in every infinite system Σ of forms, there is a finite system such that every form of Σ is a linear combination of the forms of the finite system, and of their derivatives of various orders, with forms for coefficients. Such a result does not hold. There exists, however, for infinite systems of forms, a theorem which may be regarded as an extension of Hilbert's theorem.

Let Σ be an infinite system of forms. Let Φ be a finite subset of the forms in Σ. We shall call Φ a *basis* of Σ if, for every form G in Σ, there exists a positive integer p, which depends on G, such that G^p is a linear combination of the forms in Φ and of derivatives of various orders of the forms in Φ, such that the coefficients in the linear combination are all of them forms.*

We are now ready to consider the fundamental theorem:†

THEOREM I. *Given a set of unknowns* $y_1, \cdots, y_n$, *every infinite system of forms in* $y_1, \cdots, y_n$ *has a basis.*

Theorem I is, for the case of forms with meromorphic coefficients, a consequence of two theorems which we shall proceed to explain.

By a *solution* of a system Σ of forms, we shall mean a solution of the system of equations obtained by equating the forms of Σ to zero. The totality of the solutions of a system of forms will be called the *manifold* of the system. The first of our theorems is the following:

THEOREM I′. *Let* Σ *be an infinite system of forms in* $y_1, \cdots, y_n$. *The system* Σ *contains a finite subsystem whose manifold is identical with that of* Σ.

The second theorem is as follows:

THEOREM II. *Let a form G vanish for every solution in the manifold of a finite system* $H_1, \cdots, H_r$. *Then some power of G is a linear combination of the* H_i *and of their derivatives of various orders, the coefficients in the linear combination being forms.*

Theorem I implies I′, and I′ and II together imply I. Theorem II will be recognized as an analogue of the well known *Nullstellensatz* of Hilbert and Netto. I′ and II are due to the present writer, who gave an algebraic proof of I′ and a proof of II depending on methods of analysis. Raudenbush gave a purely algebraic proof of I and also such a proof of an abstract version of II.

With respect to Theorem I, a first question which arises is the following. If $H_1, \cdots, H_r$ is a basis of Σ, does a positive integer p exist such that, for every form G in Σ, G^p is a linear combination of the H_i and their derivatives? Raudenbush furnished a negative answer to this question [12]. Let Σ be the totality of forms in the single unknown y which admit the solution $y=0$. By Theorem II, y^3 is a basis for Σ. Raudenbush showed that no p as above exists for the basis y^3.‡ It is proper now to ask whether there does not exist, for every infinite system Σ, some basis for which a single p as above can be found. This question is open at the present time.

5. The proof of Theorem I is much simpler for the case of a single unknown than for the case of many unknowns, particularly if one assumes the Hilbert basis theorem. It will possibly not be out of place for us to

* As was indicated in §3, all forms mentioned are understood to have their coefficients in a given field.

† See Ritt [14, p. 161], for a discussion of related work of Tresse. Our remarks on Tresse's work apply also to Drach's discussion mentioned below.

‡ A similar result holds for y^2, as Raudenbush has shown. The proof for y^3 is simpler.

present a proof for the case of one unknown here. Although I implies I′, the relative simplicity of the proof of I′ will justify starting with that theorem and passing from it to I.

6. We consider forms in a single unknown y, representing the jth derivative of y by y_j and y itself, at times, by y_0. Let A be a form which involves effectively one or more y_j, $(j=0, 1, 2, \cdots)$. By the *order* of A, we shall mean the greatest j for which y_j appears effectively in A. A form which is merely a function of x will be considered to have the order zero.*

Let A be a form involving one or more y_j effectively. Let B be any form. If the order of A exceeds that of B, we shall say that A is *higher* than B. If A and B have the same order p and if the degree† in y_p of A is higher than that of B, we shall say, again, that A is higher than B. Two forms for which no difference in rank is created by what precedes will be said to be of the same rank. Thus, any two functions of x in $\mathfrak{F}$ are of the same rank.

Let A be any form which is not a function of x, and let A be of order p. We shall call the form $\partial A/\partial y_p$ the *separant* of A.

Let A be any form which is not a function of x. If G is a form whose order exceeds that of A, and if j is the difference of the orders of G and A, the jth derivative‡ A_j of A will have the same order as G. If p is the common order of G and A_j, A_j will involve y_p linearly, with S, the separant of A, for coefficient of y_p. Thus, if we multiply G by a suitable power S^g of S, and subtract a suitable multiple of A_j from S^gG, the remainder G_1 will be of lower order than G. If G_1 is of higher order than A, we give it the treatment accorded to G. We see thus that there is a positive integer q such that, when a suitable linear combination of derivatives of A is subtracted from S^qG, the remainder R is not of higher order than A. Using as small as possible a power of S at each stage of the reduction, we are led to a unique R. We call this R the *residue of G with respect to A*.

If G is given with an order not exceeding that of A, we take G itself as the residue of G with respect to A.

Considering now an infinite system Σ, let us call Σ *complete* if there is a finite subsystem of Σ whose manifold is that of Σ, and *incomplete* if there is no such finite subsystem.

Every system of forms in y which contains nonzero forms contains a nonzero form which is not higher than any other nonzero form of the system. We call any such nonzero form of lowest rank a *first form* of the system.

Suppose now that there exist incomplete systems. We consider the totality of such systems and select from them one, call it Σ, whose first forms are not higher than the first forms of any other incomplete system.

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* Of course, there are forms of order zero which involve y_0.

† If $p=0$ and if B is identically zero, the degree of B in y_0 is taken here as zero.

‡ In differentiating A, we consider y as a function of x.

Let A be a first form of Σ. Then A is not free of the y_j; otherwise, A and Σ would both be devoid of solutions, and this, by what one should read into the definition of completeness, would mean that Σ is complete. Let S represent the separant of A.

For every form G in Σ which is distinct from A, let a residue R with respect to A be found. The system Ω composed of A and of all the R involves no y_j higher than the highest y_j in A. By Hilbert's basis theorem, Ω has a finite subset Φ such that every form in Ω is a linear combination of the forms in Φ. The manifold of Ω is thus the manifold of Φ. We may and shall assume that Φ contains A.

Every R in Ω is obtained, by a subtraction, from some S^qG, where G is in Σ. Let Λ be the system composed of A and of the totality of the forms S^qG. It is easy to see that Λ is complete. In short, if Ψ is the finite system composed of A and of those S^qG in Λ which furnish the R in Φ, Λ has the same manifold as Ψ.

Now, let

$$S^{q_1}M_1, \cdots, S^{q_r}M_r, \tag{1}$$

where the M_i are forms in Σ, be any finite subsystem of Λ with the same manifold as Λ. We note that A is not prevented from figuring in (1). As Σ is incomplete, there is a K in Σ which does not have the property of vanishing for every solution of the system $M_1, \cdots, M_r$. Now some S^qK vanishes for every solution of (1). This means that certain solutions of the system $S, M_1, \cdots, M_r$ are not solutions of K. Now the $M_1, \cdots, M_r$ in (1) may be taken so as to include any given finite subset of Σ, since the adjunction, to any set (1), of further forms of Λ produces a set with the same manifold as Λ. It follows that the set $\Sigma+S$, obtained by adjoining S to Σ, is incomplete.* Now S is lower than A. This contradicts the assumption that Σ is an incomplete system with lowest first forms, so that Theorem I′ is proved for the case of one unknown.

7. Following Raudenbush, we shall now modify the foregoing proof so as to obtain Theorem I. We shall write

$$P \equiv Q \qquad (C_1, \cdots, C_r),$$

if $P-Q$ is a linear combination of the r forms C_i and their derivatives of various orders, with forms for coefficients. We prove the following lemma:

LEMMA. *If, for some positive integer* g,

$$(PQ)^g \equiv 0 \qquad (C_1, \cdots, C_r), \tag{2}$$

then, if P' *is the derivative of* P,

$$(P'Q)^{2g} \equiv 0 \qquad (C_1, \cdots, C_r). \tag{3}$$

* Note that, if $\Sigma+S$ were complete, any finite subset of $\Sigma+S$ with the same manifold as $\Sigma+S$ could be enlarged by the addition of any finite number of other forms of $\Sigma+S$.

We find from (2), by differentiation,*

$$gP^{g-1}P'Q^g + gP^gQ^{g-1}Q' \equiv 0. \tag{4}$$

Multiplying (4) by Q/g, we have

$$P^{g-1}P'Q^{g+1} \equiv 0. \tag{5}$$

If $g=1$, we have (3). If $g>1$, we differentiate (5) and multiply through by $P'Q$. We find (3) to hold for $g=2$. We continue to higher values of g, performing a differentiation and a multiplication by $P'Q$ at each step.

Let us suppose now that there exist infinite systems without bases. Let Σ be such a system whose first forms are as low as possible. We obtain Ω and Λ as above. By Hilbert's theorem, Λ has a basis. If this basis is taken so as to include A, the set of corresponding forms in Λ is easily seen to be a basis of Λ.

We say now that $\Sigma+S$ has no basis. Let this be false. Let (1) with the M_i in Σ, be a basis for Λ, and let us assume, as we evidently may, that $S, M_1, \cdots, M_r$ is a basis for $\Sigma+S$. Let K be any form in Σ. Then, for some g, we have

$$K^g = LS + PM_1 + \cdots, \tag{6}$$

where the unwritten terms involve derivatives of S, or the M_i or their derivatives. Taking a positive integer t, we consider the expression for K^{tg} obtained from (6). If t is large, every term in this expression which is free of the M_i and their derivatives will involve S, or one of its derivatives appearing in (6), to a high power. Let S_j be any derivative of S which appears in (6). Because (1) is a basis for Λ, some power of SK is linear in the forms of (1) and their derivatives. By the lemma proved above, some power of S_jK is linear in the forms of (1) and their derivatives. Thus, if we take the expression for K^{tg} with t large and multiply it by an appropriate power of K, we shall secure a linear combination of the M_i and their derivatives. Then $M_1, \cdots, M_r$ is a basis of Σ. Thus $\Sigma+S$ has no basis, an absurdity which implies the truth of Theorem I for the case of one unknown.

The decomposition theorem

8. We deal with systems of forms in the n unknowns $y_1, \cdots, y_n$. Given two systems, Σ_1 and Σ_2, we shall say that Σ_1 *holds* Σ_2 if every solution of Σ_2 is a solution of Σ_1. A system Σ will be said to be equivalent to a finite set of systems, $\Sigma_1, \cdots, \Sigma_r$, if Σ holds each Σ_i and if every solution of Σ is a solution of some Σ_i.

A system Σ will be called *reducible* if there exist two forms G and H such that GH holds Σ while neither G nor H does. A system which is not reducible will be called *irreducible*. We now state the fundamental theorem:

* All congruences are with respect to $C_1, \cdots, C_r$.

THEOREM III. *Every system Σ is equivalent to a finite set of irreducible systems.*

Let a system Σ which contradicts our statement exist. Then Σ is reducible. Let G_1H_1 hold Σ, while neither G_1 nor H_1 does. Σ is equivalent to the set $\Sigma+G_1$, $\Sigma+H_1$. Thus, one of the latter systems must lack the property of being equivalent to a finite set of irreducible systems. Let $\Sigma+G_1$ not have the property. We find now a G_2, which does not hold $\Sigma+G_1$, such that $\Sigma+G_1+G_2$ does not have the mentioned property. We continue, finding a G_n for every n. We shall show that the system

$$\Sigma + G_1 + G_2 + \cdots + G_n + \cdots \tag{7}$$

is incomplete. Let Φ be any finite subsystem of (7). For some n, Φ is contained in $\Sigma+G_1+ \cdots +G_n$; Φ thus holds the latter system. Hence G_{n+1} does not hold Φ. The incompleteness of (7) proves our theorem.

The pattern of the work in the present section is identical with that which Emmy Noether used in constructing her well known theory of ideals. The chief component of the pattern is a basis theorem. Actually, the idea of employing a basis theorem was derived by us from an earlier source, namely, from Drach's dissertation of 1898.* On the other hand, Raudenbush, in developing his theory of ideals which will be discussed below, was greatly influenced by Emmy Noether's work.

9. The decomposition of Σ into irreducible systems is unique in the following sense. From any finite set of systems to which Σ is equivalent, we can, by suppressing systems, obtain a set of systems, equivalent to Σ, in which no system holds any other. Now let Σ be equivalent to the set of irreducible systems $\Sigma_1, \cdots, \Sigma_r$, in which Σ_i does not hold Σ_j if $j\neq i$. If $\Omega_1, \cdots, \Omega_s$ is a second decomposition of Σ into irreducible systems none of which holds any other, then $r=s$ and each Σ_i is equivalent to some Ω_j. We omit the proof, which is perfectly simple.

10. We call the manifold of an irreducible system an *irreducible manifold*. Thus, *the manifold of a system of differential polynomials is composed of a finite number of irreducible manifolds.*

The analogy of this result to the fundamental theorem for algebraic manifolds is clear. However, differences between the two theories arise early. For instance, the manifold of an irreducible algebraic polynomial is an irreducible algebraic manifold. On the other hand, the manifold of an algebraically irreducible differential polynomial may easily be reducible. Let us consider, for instance, the equation

$$y_1^2 - 4y = 0 \tag{8}$$

in the single unknown y. Differentiating (8), we find

* Drach [2, pp. 292–296]. At other places in our work, we were guided by the literature inspired by Emmy Noether, especially by the writings of van der Waerden. This can be seen in Ritt [14].

$$2y_1(y_2 - 2) = 0.$$

The manifold of (8) is given by $y=(x+c)^2$, c constant. The form y_2-2 is annulled by the solutions $(x+c)^2$, but not by $y=0$. The form y_1 is annulled by $y=0$, but by no other solution of (8). Thus y_1^2-4y, which cannot be factored in any domain of rationality, is a reducible system. It is equivalent to a set of two irreducible systems, the first system composed of y_1^2-4y and y_2-2, the second system composed of y_1^2-4y and y_1.

11. Examples of algebraically irreducible differential polynomials which constitute reducible systems are plentiful in the first chapters of the formal theory of differential equations. In treating equations of the type

$$y = f\left(x, \frac{dy}{dx}\right), \tag{9}$$

one differentiates (9) and considers the resulting equation as a differential equation of the first order for the unknown function dy/dx. Often the derived equation permits of a factorization. In many cases, such a factorization implies a reducibility from the point of view of differential equation theory. An illustration of this was seen in §10. As another example, consider the Clairaut equation

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2. \tag{10}$$

We find, by differentiation,

$$\frac{d^2y}{dx^2}\left(x + 2\frac{dy}{dx}\right) = 0. \tag{11}$$

The first factor in (11) is annulled by the non-singular solutions of (10) and the second factor by the singular solution.

What precedes is a sufficient hint of the bearing of the reducibility notion on the theory of singular solutions. We shall have more to say on this question later.

12. The simple examples treated above might lead one to suspect that the decomposition of a finite system of forms into irreducible systems can be effected by differentiating the forms a certain number of times and then performing algebraic operations. This is actually so. Only, no general principle exists at present for deciding how many differentiations of the given forms are necessary before algebraic combinations can be effected which will produce the irreducible systems. Even for a system composed of an algebraically irreducible form in one unknown, this is a complicated problem as soon as the order of the form exceeds unity. The complete treatment of this special case will apparently require a detailed study of the singular solutions of algebraic differential equations.

THEORY OF IDEALS

13. Behind the manifold theory which we have just considered, stands Raudenbush's elegant theory of ideals of differential polynomials [10].

A system Σ of forms in $y_1, \cdots, y_n$ is called an *ideal* if Σ has the following properties:

(a) If $A_1, \cdots, A_p$ is any finite set of forms of Σ, every linear combination of the A_i, with forms for coefficients, is contained in Σ.

(b) Given any form in Σ, the derivative of the form is contained in Σ.

Condition (a) states that Σ is an ideal in the sense in which such entities are defined in algebra. Condition (b) asks, in addition, that Σ be closed with respect to differentiation. As usual, we are dealing with forms whose coefficients lie in a given field.

An ideal Σ is called *perfect* by Raudenbush if, whenever a form G is such that some positive integral power of G is contained in Σ, G is contained in Σ. An ideal Σ is called *prime* if, for every pair of forms G and H with GH in Σ, at least one of G and H is in Σ.

We may now state the theorem of Raudenbush:

THEOREM IV. *Every perfect ideal of forms in $y_1, \cdots, y_n$ is the intersection of a finite set of prime ideals.*

Let a perfect ideal Σ exist for which our statement does not hold. Let AA', but neither A nor A', be contained in Σ. Let $H_1, \cdots, H_r$ be any basis of Σ. Let G be any form such that, for some g,

$$G^g \equiv 0 \qquad (H_1, \cdots, H_r, A). \tag{12}$$

It is easy to see that the totality Σ_1 of such forms G is a perfect ideal of which Σ is a proper part. Using A' in place of A, we obtain similarly a perfect ideal Σ_1'. We shall now show that Σ is the intersection of Σ_1 and Σ_1'. We have to prove that every form G common to Σ_1 and Σ_1' is in Σ. Considering such a G, let g be such that one has (12) and also

$$G^g \equiv 0 \qquad (H_1, \cdots, H_r, A'). \tag{13}$$

We multiply the two expressions for G^g which are indicated by (12) and (13). We secure an expression for G^{2g} in which every term free of the H_i and their derivatives contains some term A_iA_j', where subscripts indicate differentiation. A high power of G^{2g} will contain, in addition to terms involving the H_i or their derivatives, terms in which some A_iA_j' appears to a high power. By the lemma of §7, we have, for t large,

$$G^{2gt} \equiv 0 \qquad (H_1, \cdots, H_r, AA'),$$

which means that G is in Σ.

This settled, we observe that at least one of Σ_1, Σ_1' is not the intersection of a finite set of prime ideals. Let this be the case for Σ_1. We treat Σ_1

as Σ was treated and continue, forming an infinite sequence of perfect ideals,

$$\Sigma, \Sigma_1, \cdots, \Sigma_m, \cdots, \tag{14}$$

each a proper part of its successor. Let Ω be the logical sum of the ideals (14), and let $H_1, \cdots, H_r$ be a basis of Ω. For some m, Σ_m contains every H_i. Such a Σ_m will contain Ω. This absurdity proves the decomposition theorem.

As to the uniqueness of the representation of Σ, it is easy to prove that Σ is the intersection of prime ideals none of which is contained in any other, and that this representation is unique.

The theorem of Raudenbush is adequate for those phases of the theory of manifolds which exist at present. One may properly hope, however, for a theory which will treat ideals of the most general type.

14. Let us see now, for the case of forms in one unknown y, how Raudenbush applies his ideal theory to the proof of Theorem II of §4. Let G hold the finite system $H_1, \cdots, H_r$, and suppose that no power of G is congruent to 0 for $(H_1, \cdots, H_r)$. We limit ourselves, as we evidently may, to the case in which the H_i are not all zero. The totality Σ of forms J for which a congruence as just described exists is a perfect ideal, and G is not in Σ. Hence, in the representation of Σ as an intersection of prime ideals, there is some prime ideal, call it Σ_1, not containing G. We consider the first forms of Σ_1. They cannot be functions of x, else Σ_1 would contain every form. As Σ_1 is prime, we can evidently find a first form of Σ_1 which is algebraically irreducible. Let A be such a first form of Σ_1, and let S be the separant of A. We shall show that every solution of A which does not annul S is a solution of Σ_1. Let K be any form in Σ_1. Let L be the residue of K with respect to A. Then L is in Σ_1. Now, if A is of order p in y, and if I is the coefficient of the highest power of y_p in A, we have a relation

$$I^i L = PA + Q, \tag{15}$$

with Q, which is in Σ_1, lower than A. Thus $Q=0$, so that, because A is algebraically irreducible, and not a factor of I, L is divisible by A. This means that SK holds A, so that every solution of A which does not annul S is a solution of K, and indeed, of Σ_1. Now, because S, which is lower than A, is not in Σ_1, the residue H of G, with respect to A, is not in Σ_1. Hence H is not divisible by A. The form SH, whose order in y does not exceed that of A, is thus not divisible by A. Then SH does not hold A. This means that SG does not hold A. Then certain solutions of A which do not annul S are not solutions of G. Thus G does not hold Σ_1. But Σ holds Σ_1 and Σ is held by $H_1, \cdots, H_r$. This contradiction proves Theorem II for the case of forms in one unknown.

On the basis of Theorem II, we see that, if Σ is a prime ideal, Σ contains every form which holds Σ. The manifold of a prime ideal is therefore irreducible. The theorem on the decomposition of a system of forms into irreducible systems is thus seen to be contained in Raudenbush's ideal theory.

GENERAL SOLUTIONS

15. To arrive at a thorough understanding of the nature of an irreducible manifold, we must acquaint ourselves first with the notion of the *general solution* of a single differential equation. Here we shall find ourselves developing an idea which existed in a rudimentary state in the work of Lagrange.

Let A be a form in the unknowns $y_1, \cdots, y_n$ which is algebraically irreducible; that is, A is not a function of x alone and A is not the product of two forms, with coefficients in the underlying field, neither of which is a function of x alone. Let y_j be an unknown such that A involves effectively either y_j or some derivative of y_j. Let S_j be the partial derivative of A with respect to the highest derivative of y_j (which may be y_j itself) appearing effectively in A. Let

$$\Sigma_1, \cdots, \Sigma_s \tag{16}$$

be a decomposition of A into prime ideals none of which contains any other. It can be shown that *there is one and only one of the* Σ_i *which is not held by* S_j. The Σ_i which is thus determined is the same for every y_j described as above. Let us suppose that it is Σ_1 which is not held by the S_j. We shall call the manifold of Σ_1 *the general solution of* A.

For instance if A is the form $y_1^2 - 4y$ in the single unknown y, which appears in (8), the general solution of A is the set of solutions $(x+c)^2$. The manifold of A contains a second irreducible manifold, composed of the single solution $y=0$.

We consider two other examples [19, part 2] which involve a single unknown y, the mth derivative of y being denoted by y_m.

Let

$$A = \left[\frac{d}{dx}(y_1^2 - y^3)\right]^2 - (y_1^2 - y^3).$$

In this case, A yields two irreducible manifolds, the general solution and the manifold given by

$$y = \frac{4}{(x+c)^2} \tag{17}$$

with c constant. The manifold (17) is the solution of $y_1^2 - y^3$.

Again let

$$F = y(yy_2 + yy_1 - 2y_1)^2 - (y_1 - y)^2,$$

and let A be the algebraically irreducible form defined by

$$yA = y_1F^2 - \prod_{i=0}^{4}\left(y_1 - y + \frac{y^2}{x+i}\right).$$

Here are the general solution and five other irreducible manifolds, the manifolds being the forms

$$(18) \qquad y_1 - y + \frac{y^2}{x+i}, \qquad i = 0, \cdots, 4.$$

The manifold of each form (18) contains the solution $y=0$, which also belongs to the general solution.

16. The phenomenon of the separation of the manifold of a single algebraically irreducible form A into a set of irreducible manifolds, one of which is the general solution of A, is possibly one of the most interesting effects encountered in the general theory which we are discussing. It reveals the great interest possessed by differential polynomials as algebraic entities. As has already been indicated, we have here a radical difference from what is found in the theory of algebraic polynomials, where an irreducible polynomial furnishes a single irreducible manifold. The algebraic structure of differential polynomials is a topic for investigation which may easily prove fruitful. The Newton polygons to which differential polynomials lead should figure prominently in such a study. The algebraic properties of a form become visible in the forms obtained from the given one by successive differentiations.

17. The concept of the general solution appears not to have caught the attention of mathematicians during the era of great progress which followed the appearance of the fundamental existence theorems of Cauchy. There is therefore all the more interest in putting the notion of general solution into a form which shows that some idea of its nature was possessed by Lagrange.

Let us consider again an algebraically irreducible form A and the various S_j associated with it. Let us call a solution of A *singular* if every S_j vanishes for the solution, and *non-singular* if at least one S_j does not vanish for it. The general solution contains all of the non-singular solutions. There arises the question: *Which singular solutions belong to the general solution?* The answer is: *A singular solution belongs to the general solution if and only if every form which vanishes for every non-singular solution vanishes for the singular solution.*

It follows directly from this that a singular solution $\bar{y}_1(x), \cdots, \bar{y}_n(x)$ of A certainly belongs to the general solution if the singular solution can be approximated uniformly in some area, with arbitrary closeness, by non-singular solutions. For instance, the non-singular solutions of

$$\left(\frac{dy}{dx}\right)^2 - y^3 = 0$$

are given by $y=4(x+c)^{-2}$. Thus, the singular solution $y=0$, which is approached uniformly by non-singular solutions, is contained in the general solution.

We turn now to work of Lagrange [6, §13, p. 25] published in 1774. Lagrange considers a differential equation (not necessarily algebraic) in the unknown y,

$$V\left(x, y, \frac{dy}{dx}\right) = 0, \tag{19}$$

and a one-parameter family of solutions $y=f(x, a)$ which is supposed to have been determined for (19). Such a one-parameter family he calls the *complete integral* of (19). Referring to any special solution $y(x)$ of (19), he formulates conditions under which $y(x)$ will satisfy not only (19), but also "*à toutes les équations des ordres ultérieurs qui en seraient derivées.*"

The satisfaction by $y(x)$ of "all equations of higher orders" derived from (19) is offered, more or less, as a basis for considering $y(x)$ to be contained in the "complete integral."

The idea of "derived equation of higher order" is not made precise. It seems that one is expected to secure the derived equations from (19) by differentiations and eliminations. To secure precision, one would have to specify the nature of V in (19). One might, for instance, take V as an algebraically irreducible form. The derived equations might then be defined as those algebraic differential equations which are satisfied by all non-singular solutions of (19). A definition in algebraic terms would require the general theory which is the subject of the present article. One would be led to take all equations $U=0$ where U is a form such that, S being the separant of A, some power of SU is linear in V and its derivatives, with forms for coefficients.

We thus see the concept of general solution, as presented here, taking form in Lagrange's work. The notion of irreducible manifold is missing, but there is present the idea of singular solutions which have the formal properties of the totality of non-singular solutions.

18. The remark made above on singular solutions which can be approximated uniformly by non-singular solutions suggests the investigation of the manner in which a singular solution which is contained in the general solution can be approximated by non-singular solutions. An answer to this question, useful for many purposes, is given by the following theorem [14, p. 101; 20, part 4]:

Let A be an algebraically irreducible form in $y_1, \cdots, y_n$. For a singular solution $\bar{y}_1, \cdots, \bar{y}_n$ of A to belong to the general solution of A, it is necessary and sufficient that there exist a value a of x, at which the $\bar{y}_i$ are analytic, such that, given any $\epsilon>0$ and any positive integer m, A has a non-singular solution $\tilde{y}_1, \cdots, \tilde{y}_n$, analytic at a, such that, for $i=1, \cdots, n$, each of the first $m+1$ coefficients in the Taylor expansion at a of $\tilde{y}_i-\bar{y}_i$ is less than ϵ in modulus.

The existence of a single point a as above is shown to imply the existence of a set of such points which is dense in the area in which the $\bar{y}_i$ are analytic. Strodt has shown that if there are points at which the $\bar{y}_i$ cannot be approximated by $\tilde{y}_i$ as above, those points are either finite in number or countably infinite [23].

Whether the $\bar{y}_i$ of a singular solution in the general solution can always be approximated uniformly in some area by non-singular $\tilde{y}_i$ is not known at present. It is known that the $\bar{y}_i$ may fail to be analytically embedded in a one-parameter family of $\tilde{y}_i$ [20, part 4].

Classification of singular solutions

19. What precedes is enough to show the importance of the manifold theory for the study of the singular solutions of a differential equation in a single unknown function. A number of the early continental analysts, notably Euler, Lagrange, Laplace, and Poisson, undertook the study of such singular solutions. They succeeded, by heuristic methods, in revealing certain interesting situations, but rigorous results, giving a complete description of the phenomena involved, were beyond their reach. Let us say at once that, for equations of the first order, quite a good insight can be obtained into the nature of the singular solutions without the manifold theory. The older analysts secured a complete heuristic survey of the situation and a rigorous theory was supplied by Hamburger [4] in 1893. It is chiefly with equations of orders higher than the first that we shall be occupied here. We shall give an account of our principal results and shall compare them with the older work.

20. Let Σ be an irreducible system of forms in the single unknown y. Let us assume that Σ has solutions and contains at least one non-zero form. It is easy to show that the manifold of Σ is the general solution of an algebraically irreducible form.

On this basis, if A is an algebraically irreducible form in y, the manifold of A is composed of the general solutions of the forms of a certain finite set. As a first problem in the study of the singular solutions of A, one might propose the problem of determining a set of forms whose general solutions make up the manifold of A. There may be singular solutions which are contained in more than one of these general solutions. One would thus suggest as a second problem the determination, for any given singular solution of A, of those general solutions to which that singular solution belongs.

For the form A, one can obtain, by processes of elimination, a set of algebraically irreducible forms $B_1, \cdots, B_p$ whose general solutions make up the manifold of A [14, chap. 5]. It then becomes a matter of determining, for a given B_i, whether or not its general solution is an *essential* irreducible manifold in the manifold of A. By an *essential* irreducible manifold, we mean one which is not a proper part of some other irreducible manifold in the manifold of A.

This determination can be made with the help of a finite number of rational operations and differentiations [18, part 1]. In the special and interesting case of a B_i equal to y, a simple statement is possible. It is to the following effect. *Let A be an algebraically irreducible form in y, of order n in y. Let A vanish for $y=0$. For $y=0$ to be an essential irreducible manifold in the manifold of A, it is necessary and sufficient that A, considered as a polynomial in $y, y_1, \cdots, y_n$, contain a term in y alone, that is, a term free of $y_1, \cdots, y_n$, which is of lower degree than every other term of A.*

Thus, $y=0$ is an essential manifold for y_1y_2-y but not for yy_3-y_2 or for $y_2y_3-y^2$.

The problem of determining those irreducible manifolds to which a given singular solution of A belongs, has been solved for forms A of the second order in y [19, part 2]. In this problem, the following special question turns out to be of particular moment. *Let A, an algebraically irreducible form of the second order, vanish for $y=0$. It is required to determine whether $y=0$ belongs to the general solution of A.* This question can always be answered after there are performed a finite number of operations in which one examines certain polygons, of the Newton type, associated with A.

21. In a paper published in 1772, Laplace [7] considered a differential equation in y, of any order n. Laplace uses the term *general integral* to designate, apparently, a family of functions, satisfying the equation, which depends on n arbitrary constants. By a *solution* of the given equation, he understands an equation of order lower than n which "satisfies" the given equation. The precise meaning of "satisfy" is not given; nothing is said, for instance, in regard to the singular solutions of the second equation. A *particular integral* is a solution "contained in" the general integral and a *particular solution* is a solution which is not so contained. Laplace sets the following two problems:

Being given a differential equation of any order

(1) *to determine whether an equation of lower order which satisfies it is contained or not contained in the general integral*;

(2) *to determine all of the particular solutions of the given equation.*

Most of Laplace's work deals with his problem (2), which is a vague and incomplete formulation of the question, raised in §20, of determining whether the general solution of a B_i is essential. Laplace seeks to give a meaning to the concept, mentioned above, of particular integral. The point of view which he takes seems to be clear if one confines oneself to some simple examples, suitably chosen, but his ideas are far from being precise enough for the purposes of a general theory. His definition of particular integral suggests, in a distant way, the condition given in §18 for a singular solution to belong to the general solution.

What is possibly most interesting, in Laplace's paper, is a discussion of equations of the second order in which he treats "particular solutions without differences." [7, §9] The counterparts of these, in the theory of manifolds, are essential irreducible manifolds composed of one solution. Having regard to this relationship, one may attribute to Laplace the heuristic discovery of portions of the theorem of §20 which deals with the case of $B_i = y$. An example treated by Laplace suggests somewhat that his faith in his theoretical speculations was derived from calculations (not contained in his paper), which for that example, and for similar ones, would reveal reducibility in the sense in which we have been using that term.

Fragmentary as Laplace's work on this subject may be, one can see that he had derived, from a study of examples, the principle that the solutions of a differential equation separate into coherent families.

22. A paper by Poisson [8] in 1806 treats, in a somewhat different manner, the questions Laplace raised. Poisson's method is most easily understood from his discussion of Laplace's "particular solutions without differences," which Poisson calls "algebraic particular solutions." [8, §3]

Poisson considers that it is proper to call a solution $y(x)$ of a differential equation of any order an "algebraic particular solution" if and only if the equation does not have a one-parameter family of solutions $y(x)+\alpha z$, with α an arbitrary constant and z a function of x and α. More or less, an algebraic particular solution is, for Poisson, one which cannot be analytically embedded in a one-parameter family of solutions. With this definition, Poisson is able to state, for certain classes of equations, necessary and sufficient conditions for a given solution to be an algebraic particular solution. The results of Poisson, like those of Laplace, may be regarded as heuristic equivalents of portions of the theorem of §20. For instance, Poisson concludes that $y=0$ is a particular solution of

$$\left(\frac{dy}{dx}\right)^m \frac{d^2y}{dx^2} = y^n$$

if and only if $m \geqq n$. Actually, Poisson might have applied his method to perfectly general algebraic differential equations; his failure to do so would indicate that polynomials in several variables did not enjoy, in his day, the docility which is attributed to them now.

Poisson's ingenious formal work neglects convergence questions. His definition of "algebraic particular solution" is partially validated by the theory of manifolds. If a solution $\bar{y}$ of a form A is not an essential irreducible manifold, A is satisfied formally by a series

$$y = \bar{y} + \alpha\phi_1 + \alpha^2\phi_2 + \cdots$$

with α an arbitrary constant and the ϕ_i analytic functions of x. However, the series may diverge for every α distinct from 0.

One of Poisson's enunciations is even heuristically unsound [8, p. 89, lines 21–24]. It is a question of judging whether a solution of an equation of the second order is embedded in a one-parameter family or in a two-parameter family. Poisson's criterion would require one to conclude, for instance, that $y=0$ does not belong to the general solution (as defined here) of

$$\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + y = 0.$$

Actually, $y=0$ is so contained. The difficulty arises out of the stipulation which Poisson makes for the manner in which a given solution is to be embedded among other solutions. This stipulation is too strong for a sound description of phenomena, even on a heuristic basis.

One may conclude, from the work of Lagrange, Laplace and Poisson, that they had sensed the existence of such an algebraic theory of differential equations as we have been discussing. If they did not develop that theory, it was because a great drift in algebra and in analysis had to be awaited before the question became susceptible to treatment.

Irreducible manifolds

23. The totality of sets of n analytic functions $y_1, \cdots, y_n$ is an irreducible manifold, the manifold of the form zero. Every other irreducible manifold in $y_1, \cdots, y_n$ may be considered as a birational map of the general solution of some form in n or fewer unknowns. This fact, which epitomizes the theory of irreducible manifolds, will now be discussed.

24. We shall say that a prime ideal Σ in $y_1, \cdots, y_n$ is *non-trivial* if Σ contains a form distinct from 0 and if Σ does not contain every form. Let Σ be a non-trivial prime ideal. It may be that, for every i from 1 to n, Σ contains a non-zero form involving only y_i. Let us suppose that this is not so. Let y_j be some unknown such that Σ has no non-zero form in y_j alone. We designate y_j by u_1. There may exist an unknown distinct from u_1 such that Σ contains no non-zero form involving only that unknown and u_1. If such unknowns distinct from u_1 exist, we choose any one of them and call it u_2. Continuing, we find a set $u_1, \cdots, u_q$, with $q<n$, such that Σ contains no non-zero form in the u_i alone, but contains, for every other unknown y_j, a non-zero form involving only y_j and the u_i. We now let $p=n-q$, and, changing the notation if necessary, designate the unknowns distinct from the u_i by $y_1, \cdots, y_p$. The unknowns $u_1, \cdots, u_q$ will be called *a set of arbitrary unknowns for* Σ.

The u_i are "arbitrary" in the following sense. One can take the u_i as any set of q analytic functions which does not annul a certain form, depending on Σ, which involves only the u_i. A set of differential equations then determines $y_1, \cdots, y_p$ in succession, to within arbitrary constants, so as to furnish a solution of Σ.

The integer q does not depend on the manner in which the u_i are selected. Σ may have many distinct sets of arbitrary unknowns, but all of the sets will contain the same number of unknowns.

25. We assume, as may be done without loss of generality, that the field underlying our work does not consist purely of constants. It is then possible to form a rational combination w of the* u_{ij} and y_{ij}, such that the system of equations obtained by equating the forms in Σ to zero determines each y_i, $(i=1, \cdots, p)$, as a rational combination of $w, u_1, \cdots, u_q$ and their derivatives of various orders. The coefficients in all of the rational combinations mentioned are functions of x in the underlying field.

Let us make clearer the nature of the birational relations just described. The expression for w is of the type Q/P, where P and Q are forms in the u_i and y_i. Let us now consider the system Λ of forms, in the $n+1$ letters u_i, y_i, and w, composed of Σ and of $Pw-Q$. We are now considering w as a new unknown, rather than as the rational combination mentioned above. Let Ω be the totality of forms, in the letters of Λ, which vanish for those solutions of Λ for which $P \neq 0$. It is easy to prove that Ω is a prime ideal. For every i from 1 to p, Ω will contain a form $R_i y_i + T_i$, where R and T involve only the u_i and w. This is the sense in which the birational relations exist.

Ω contains an algebraically irreducible form in w and the u_i alone, which is unique if one makes abstraction of multiplication by a function of x. Let A be such a form in Ω. We call the equation $A=0$ a *resolvent* of Σ.

Given any solution

$$\bar{u}_1, \cdots, \bar{u}_q; \bar{y}_1, \cdots, \bar{y}_p; \bar{w}$$

of Ω, $\bar{u}_1, \cdots, \bar{u}_q; \bar{w}$ belongs to the general solution of A considered as a form in the u_i and w. This permits the manifold of Σ to be considered as a birational image of the general solution of A.

We have spoken as if the u_i exist. With a proper suppression of details, the discussion applies to the case where there are no u_i.

The rational combination for w may be chosen with great freedom. A prime ideal thus has many resolvents. It is a noteworthy fact, however, that the order, with respect to w, of the resolvent, that is, the order of the highest derivative of w present in A, depends only on Σ and on $u_1, \cdots, u_q$—not on the particular way in which w is formed.

26. The order of the resolvent in w is a measure of the number of arbitrary constants upon which the manifold of Σ depends after the arbitrary unknowns are selected.

We may thus consider the dimensionality of an irreducible manifold

* As in §3, the second subscript denotes differentiation.

to be measured by two numbers, the number q of arbitrary unknowns and the order h in w of the resolvent. The number q is unique for a given irreducible manifold, while h depends on the particular choice of the arbitrary unknowns.

27. In connection with the dimensionality of irreducible manifolds, Gourin [3] has obtained a theorem which parallels the theorem of algebraic geometry stating that if $\mathfrak{M}$ and $\mathfrak{N}$ are irreducible algebraic manifolds with $\mathfrak{N}$ a proper part of $\mathfrak{M}$, the dimensionality of $\mathfrak{M}$ exceeds that of $\mathfrak{N}$. Gourin's theorem is as follows. *Let Σ_1 and Σ_2 be non-trivial prime systems with the manifold of Σ_2 a proper part of that of Σ_1. Let $u_1, \cdots, u_q$ be a set of arbitrary unknowns for Σ_2. Then either Σ_1 has a set of arbitrary unknowns of which $u_1, \cdots, u_q$ is a proper subset, or $u_1, \cdots, u_q$ is a set of arbitrary unknowns for Σ_1 and the order in w of the resolvent of Σ_1, for $u_1, \cdots, u_q$, exceeds the corresponding order for Σ_2.*

Further work on the dimensionality of manifolds will be found in [16], [17], and [19].

With this we close our discussion. We have sought to point out some directions in which advances may be made. A study of the publications listed in the bibliography will reveal many others.

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