

ABSTRACT DIFFERENTIAL ALGEBRA AND THE ANALYTIC CASE

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In Chapter VI, *Analytical considerations*, of his book *Differential algebra* [4], J. F. Ritt states the following theorem:

THEOREM. Let $Y_i^{p_i} + F_i$, $i=1, \dots, n$, be differential polynomials in $L\{Y_1, \dots, Y_n\}$ with each p_i a positive integer and each F_i either identically zero or else composed of terms each of which is of total degree greater than p_i in the derivatives of the Y_j . Then $(0, \dots, 0)$ is a component of the system $Y_i^{p_i} + F_i = 0$, $i=1, \dots, n$.

Here the field L is not an arbitrary differential field but is an ordinary differential field appropriate to the "analytic case."¹ While it lies at hand to conjecture the theorem for an arbitrary (ordinary) differential field L of characteristic 0, the type of proof given by Ritt does not place the question beyond doubt.² The object of the present note is to establish a rather general principle, analogous to the well-known "Principle of Lefschetz" [2], whereby it will be seen that any theorem of the above type, more or less, which holds in the analytic case also holds in the abstract case. The main point of this principle is embodied in the *Embedding Theorem*, which tells us that any field finite over the rationals is isomorphic to a field of meromorphic functions. The principle itself can be precisely formulated as follows.

PRINCIPLE. If a theorem $T(L)$ obtains for the field L provided it obtains for the subfields of L finite over the rationals, and if it holds in the analytic case, then it holds for arbitrary L .

To illustrate, consider the theorem above. The conclusion can be formulated as follows: there exists a polynomial $G(Y) \in L\{Y_1, \dots, Y_n\}$ with $G(0) = 1$ and an integer r such that $Y_i^r G(Y) \equiv 0 [Y_i^{p_i} + F_i, \dots, Y_n^{p_n} + F_n]$, $i=1, \dots, n$. The theorem is thus seen really to refer to the field K which is obtained by adjoining to the rationals the coeffi-

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¹ Consider all the functions defined in an open region A , each of which can be expanded into a convergent power-series in a neighborhood of each point of A . Because of the connectedness of A , these functions form an integral domain. Any subfield of its quotient-field is a field of meromorphic functions. The "analytic case" refers to the case that the base field is a field of meromorphic functions.

² The theorem has been established by a direct, algebraic argument in [3; p. 553]. Incidentally, we have verified that the theorem also holds in the partial differential case.

cients of the F_i . Identifying K with a field of meromorphic functions and applying the analytic case, we obtain a polynomial $G(Y)$ of the desired form, with, moreover, $G(Y) \in K\{Y\}$.

The notation is the standard notation of Ritt's book, and all page references are to that book.

EMBEDDING THEOREM. *Let R be the rational number-field, K a finite partial differential extension-field of R with m types of differentiation. Then K is isomorphic to a field F of meromorphic functions of m complex variables.*

The following lemma, though nothing but a simple computational rule, will prove useful in the given formulation.

LEMMA. *Let R be the rational number-field, $K = R\langle u_1, \dots, u_n \rangle$ a finite ordinary differential extension-field of R . Let $\sigma: u_{ij} \rightarrow c_{ij}$ be an arbitrary abstract-field isomorphism of K onto a field K_1 . Let the field of formal power-series $K_1((z))$ be converted into a differential field by placing $(\sum a_j z^j / j!)' = \sum a_j z^{j-1} / (j-1)!$. Then the assignment $u_i \rightarrow \bar{u}_i = \sum c_{ij} z^j / j!$ defines a differential-field isomorphism of K onto $R\langle \bar{u}_1, \dots, \bar{u}_n \rangle$. A similar statement holds for partial differential fields.*

PROOF. Let $H(U) \in R\{U_1, \dots, U_n\}$, where the U_i are indeterminates. Then $H(\bar{u}) = \sum H^{(j)}(c) z^j / j!$, where $H^{(j)}$ is the j th derivative of H and $H^{(j)}(c) = H^{(j)}(\bar{u})$ evaluated at $z=0$; or, what comes to the same thing, $H^{(j)}(c) = H_j(c_{kl})$, where $H^{(j)}(U) = H_j(U_{kl})$. Hence $H(\bar{u}) = 0$ if and only if $H(u) = 0$. This completes the proof in the ordinary case, and the proof for the partial case follows similarly.

PROOF OF THE THEOREM. The proof merely applies the lemma in such a way that the power-series converge. For the sake of exposition, we consider first the ordinary case ($m=1$). We may assume that the differential degree of transcendency of K/R is ≥ 1 , for we may always adjoin a transcendental quantity to K . Using the theorem of the primitive element,³ we may therefore assume that K is of the form $R\langle u_1, \dots, u_n \rangle$, where u_1, \dots, u_{n-1} form a transcendency basis of K/R . Let u_n satisfy an equation of order r , but no less, over $R\langle u_1, \dots, u_{n-1} \rangle$: say $G(U_1, \dots, U_n) = 0$ is an equation of the type specified, with $G \in R\{U\}$ and G irreducible. Let complex numbers c_{ij} algebraically independent over R be assigned to the U_{ij} occurring in G with the exception of U_{nr} , and let c_{nr} be determined from the condition $G(c) = 0$. We then have $S(c) \neq 0$, where $S = \partial G / \partial U_{nr}$. Hence we

³ For a simple proof of this theorem, which also holds for arbitrary m , see [5]; see also [1]. The use here of the primitive element is, however, merely a matter of convenience in exposition; the theorem is not used in the proof below for $m > 1$.

can solve $G(U) = 0$ formally for U_{nr} to obtain

$$U_{nr} - c_{nr} = T(U_{ij} - c_{ij}; U_{n0} - c_{n0}, \dots, U_{nr-1} - c_{nr-1}) \quad (i < n),$$

where T is a formal power-series (over the rationals) in the differences indicated. Moreover it is known that T converges in some neighborhood of the specified values of the U_{ij} . For each $i, i = 1, \dots, n-1$, we now construct a power-series $\sum c_{ij}z^i/j!$, where we use the c_{ij} already considered and fill in the other c_{ij} with arbitrary values subject to the following conditions: first we want all the $c_{ij}, i = 1, \dots, n-1, j = 0, 1, \dots$ and c_{n0}, \dots, c_{nr-1} to be algebraically independent over the rationals—this condition can certainly be met since the degree of transcendency of the complexes over the rationals is the cardinal number of the complexes; and second, we want the power-series to be convergent in some neighborhood of $z = 0$ —this is easily accomplished, say by making the c_{ij} approach 1. These power-series are then substituted into T (with U_{ij} being replaced by the j th derivative of U_i) to obtain a function $T(z; U_{n0} - c_{n0}, \dots, U_{nr-1} - c_{nr-1})$, which will be analytic in some neighborhood of $z = 0, U_{n0} = c_{n0}, \dots, U_{nr-1} = c_{nr-1}$. We now regard $U_{nr} - c_{nr} = T(z; U_{n0} - c_{n0}, \dots, U_{nr-1} - c_{nr-1})$ as a differential equation with specified initial conditions. It is well-known that there exists a solution $\sum c_{nj}z^j/j!$, where c_{n0}, \dots, c_{nr} are as assigned. Let $v_i = \sum c_{ij}z^i/j!$. Since $\sigma: u_{ij} \rightarrow c_{ij}$ clearly determines an abstract-field isomorphism, the assignment $u_i \rightarrow v_i$ determines, by the lemma, a differential isomorphism of $R\langle u_1, \dots, u_n \rangle$ onto $R\langle v_1, \dots, v_n \rangle$. This completes the proof (for $m = 1$).

REMARK. The above argument has points of contact with the argument of Ritt on pp. 28–29.

The proof for the partial differential case proceeds on the same lines as for the ordinary case, but is more complicated due to the involved character of the necessary existence theorem for partial differential equations. Here we rely on Ritt's exposition of Riquier's existence theorem (Chapter VIII), and on the argument of pp. 172–174, which can be easily adapted to our purpose.

Let $K = R\langle u_1, \dots, u_n \rangle$, let u_1, \dots, u_n determine the prime ideal P in $R\{U_1, \dots, U_n\}$, and let G_1, \dots, G_s be a characteristic set for P . Let p_i be the leader and S_i the separant of G_i . Consider the U_{ij} (where j abbreviates $j_1 \dots j_m$) which actually occur in the G_k . Assign complex numbers c_{ij} algebraically independent over R to the U_{ij} with exception of the p_k . Then for these $ij, R(\dots, u_{ij}, \dots) \cong R(\dots, c_{ij}, \dots)$; and this isomorphism can be extended to include all the ij . In this way we have complex numbers c_{ij} which satisfy

the system $G_1=0, \dots, G_s=0, \pi S_i \neq 0$. Solving for the p_i successively from $G_1=0, \dots, G_s=0$ in the neighborhood of the c_{ij} , we get $p_i = g_i(U_{jk} - c_{jk})$, where on the right occur only parametric derivatives. Then essentially by Riquier's existence theorem (see, however, also the argument of Ritt on pp. 172-174), the system $p_i = g_i$, considered as a system of differential equations, has a unique solution for any given initial determinations. The initial determination of U_i is a power-series in which the terms $c_{ij_1 \dots j_m} z_1^{j_1} \dots z_m^{j_m} / j_1! \dots j_m!$ correspond to the parametric derivatives, and in which the c_{ij} are arbitrary subject to the condition that the series converge in a neighborhood of $z_1 = \dots = z_m = 0$. We take the c_{ij} subject to this condition and in such way that they are algebraically independent over R . For the corresponding solution $v_i = \sum c_{ij_1 \dots j_m} z_1^{j_1} \dots z_m^{j_m} / j_1! \dots j_m!$, one finds (as for $m=1$) that $R\{u_1, \dots, u_n\}$ is isomorphic to $R\{v_1, \dots, v_n\}$. This completes the proof.

Let K^* be a field of meromorphic functions with domain A , and let B be an open sub-region of A ; then the functions of K^* restricted to B form a field K_1^* isomorphic to K^* . The two fields K^*, K_1^* are frequently tacitly identified; and, in fact, this is so in the following corollary.

COROLLARY. *Let K, K_1 be finite extensions of the rational number-field $R, K \subseteq K_1$, and let $\tau: K \rightarrow K^*$ be an isomorphism of K with a field of meromorphic functions K^* . Then τ can be extended to an isomorphism $\tau: K_1 \rightarrow K_1^*$ of K_1 with a field of meromorphic functions K_1^* .*

PROOF. One could get a proof by modifying the above proof, letting K take over the role of R . The result would resemble considerably Ritt's proof that a polynomial ideal $\neq (1)$, in the analytic case, has an analytic zero (see pp. 28-29, 172-174, 176). We can do perhaps a little better as follows. Let $\tau: u_i \rightarrow u_i^*$, where $K = R\langle u_1, \dots, u_n \rangle$; we select some point P , which we may assume to be the origin, at which the u_i^* are analytic (and not merely meromorphic). Continuing now first for $m=1$, the power-series $u_i^* = \sum c_{ij} z^j / j!$ define an (abstract) isomorphism $\sigma: u_{ij} \rightarrow c_{ij}$ of K into the complex number-field. Let $K_1 = K\langle v_1, \dots, v_m \rangle$ with v_1, \dots, v_{m-1} algebraically independent over K and v_m a primitive element of K_1 over

$$R\langle u_1, \dots, u_{n-1}, v_1, \dots, v_{m-1} \rangle.$$

Clearly σ can be extended to K_1 , $\sigma: v_{ij} \rightarrow d_{ij}$, with $v_i^* = \sum d_{ij} z^j / j!$, $i=1, \dots, m-1$, convergent by choice and $v_m^* = \sum d_{mj} z^j / j!$ convergent by the theorem. By the lemma, $\tau': u_i \rightarrow u_i^*, v_i \rightarrow v_i^*$ determines an isomorphism of K_1 with $R\langle u_1^*, \dots, v_m^* \rangle$, and clearly τ' induces τ .

This completes the proof for $m = 1$. The proof for $m > 1$ is similar.

The following corollary is, as just mentioned, well-known.

COROLLARY. *A polynomial ideal $A \neq (1)$, in the analytic case, has an analytic zero.*

PROOF. We take a finite basis for the ideal A and work over the field K generated by the coefficients. There exists an abstract zero v_1, \dots, v_m . By the last corollary we identify $K_1 = K\langle v_1, \dots, v_m \rangle$ with a field of meromorphic functions. Any sub-region B (of the region for K_1) in which the v_i are analytic together with the v_i define an analytic zero of A .

REMARKS. (1) Essentially on the basis of the above principle, the following results, established by Ritt in the analytic case, can now be considered to obtain in the abstract case (characteristic 0):

(a) the result on pp. 129–131,

(b) the result on pp. 144–145,

(c) the results on p. 132, §12, provided we take as definition Ritt's characterization of a limited manifold as one which is held by a differential polynomial of the form $y^p + F$, where F either is zero or is of degree less than p .

The application of the principle to these results is not absolutely automatic, but as there is no essential difficulty, we will retain our proofs in manuscript form.

(2) It would not be without significance to go over all of Ritt's results, whether established only in the analytic case or not, in order to make clear the logical character of the principle, but as the considerations can not be supported with technical difficulties, we omit them. In this connection, the reader may wish to decide whether Lefschetz's Principle applies to Lüroth's Theorem. It does apply to Hilbert's Nullstellensatz, though this point was overlooked in [6; p. 373].

(3) The above principle is not merely more difficult to establish than the corresponding principle for Algebraic Geometry, but it actually has a different character. In Algebraic Geometry, the type of theorem we have been considering can in principle be decided in a finite number of steps. That is not so in Differential Algebra: for example, given a polynomial G in $R\{Y\}$, it is not known how to write down finite sets of equations for the components of $G = 0$ (see pp. 177–178).

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SOME NODAL NONCOMMUTATIVE JORDAN ALGEBRAS¹

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1. **Introduction.** An algebra A over a field F is called a noncommutative Jordan algebra if A is flexible and if the associated algebra A^+ is a Jordan algebra. In a recent study by Schafer [2] of noncommutative Jordan algebras of characteristic p , such an algebra A is called a nodal algebra if it is finite dimensional, has a unity element 1, and has the form $A = 1F + N$ where every element of N is nilpotent but N is not a subalgebra of A . It is known [1] that nodal noncommutative Jordan algebras of characteristic zero cannot exist.

Since a noncommutative Jordan algebra is flexible, it satisfies

$$(1) \quad (x, y, z) + (z, y, x) = 0,$$

where $(x, y, z) = (xy)z - x(yz)$. This is the linearized form of the flexible law $(xy)x = x(yx)$. The associated algebra A^+ is obtained from the algebra A by redefining multiplication by $x \cdot y = (xy + yx)/2$. (When the characteristic of A is 2 we use $x \cdot y = xy + yx$.)

In this paper we give a construction for a class of nodal noncommutative Jordan algebras for every characteristic $p \neq 2$. A subclass of these algebras consists of simple algebras and a description of ideals is given for the algebras which are not simple. For the sake of completeness, we construct a nodal algebra of characteristic 2. These examples all have the stronger property that A^+ is an associative algebra.

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