

BLASCHKE PRODUCTS AND NONIDEAL IDEALS IN HIGHER ORDER LIPSCHITZ ALGEBRAS

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*To Victor Petrovich Havin, with admiration (best phrased as a palindrome):
ВОТ ПЕДАГОГ АДЕПТОВ!*

ABSTRACT. We investigate certain ideals (associated with Blaschke products) of the analytic Lipschitz algebra A^α , with $\alpha > 1$, that fail to be “ideal spaces”. The latter means that the ideals in question are not describable by any size condition on the function’s modulus. In the case where $\alpha = n$ is an integer, we study this phenomenon for the algebra $H_n^\infty = \{f : f^{(n)} \in H^\infty\}$ rather than for its more manageable Zygmund-type version. This part is based on a new theorem concerning the canonical factorization in H_n^∞ .

§1. INTRODUCTION AND RESULTS

Let H^∞ stand for the algebra of bounded analytic functions on the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Recall that a function $\theta \in H^\infty$ is said to be *inner* if $|\theta(\zeta)| = \lim_{r \rightarrow 1^-} |\theta(r\zeta)| = 1$ at almost all points ζ of $\mathbb{T} := \partial\mathbb{D}$. Given $f \in H^\infty$ and an inner function θ , we trivially have $f\theta^k \in H^\infty$ for every k in $\mathbb{N} := \{1, 2, \dots\}$. Our plan is to show how little of this remains true — and to discuss the subtleties that arise — when H^∞ gets replaced by a smaller algebra whose members are suitably smooth up to \mathbb{T} , especially when the order of smoothness exceeds 1. Specifically, we shall be concerned with the analytic *Lipschitz spaces*, to be defined in a moment.

For $\alpha > 0$, we write A^α for the set of all $f \in H^\infty$ that satisfy

$$(1.1) \quad |f^{(m)}(z)| = O((1 - |z|)^{\alpha - m}), \quad z \in \mathbb{D},$$

for some (any) integer m with $m > \alpha$; here $f^{(m)}$ is the m th order derivative of f .

It is well known that (1.1) does not actually depend on the choice of m , as long as $m > \alpha$, except for the constant in the O -condition. Also, for $0 < \alpha < 1$, a classical theorem of Hardy and Littlewood tells us that A^α is formed by precisely those analytic functions f on \mathbb{D} that obey the Lipschitz condition of order α , i.e.,

$$(1.2) \quad |f(z) - f(w)| \leq C|z - w|^\alpha, \quad z, w \in \mathbb{D},$$

with some fixed $C = C_f > 0$. Similarly, in the case of $\alpha \in (0, \infty) \setminus \mathbb{N}$, an analytic function f will be in A^α if and only if $f^{(n)}$ satisfies the Lipschitz condition of order $\alpha - n$, where $n = [\alpha]$ is the integral part of α . The space A^1 is known as the analytic *Zygmund class*; the higher order Zygmund classes A^n (i.e., the A^α -spaces with $\alpha = n \in \mathbb{N}$) are related to it by the formula $A^n = \{f : f^{(n-1)} \in A^1\}$.

2010 *Mathematics Subject Classification*. Primary 30J10, 30H10, 46J15, 46J20.

Key words and phrases. Inner functions, Blaschke products, Lipschitz spaces, ideals.

Supported in part by grant MTM2008-05561-C02-01 from El Ministerio de Ciencia e Innovación (Spain) and grant 2009-SGR-1303 from AGAUR (Generalitat de Catalunya).

Furthermore, we are concerned with the algebras

$$H_n^\infty := \{f \in H^\infty : f^{(n)} \in H^\infty\}, \quad n \in \mathbb{N}.$$

Of course, H_1^∞ coincides with the set of all $f \in H^\infty$ that satisfy the Lipschitz condition (1.2) with $\alpha = 1$. We also recall that H_n^∞ is properly contained in A^n , for each n .

Next, we need the “real variable” Lipschitz–Zygmund spaces $\Lambda^\alpha = \Lambda^\alpha(\mathbb{T})$, which actually consist of complex-valued functions and can be defined by $\Lambda^\alpha := A^\alpha + \bar{A}^\alpha$. An equivalent, and more traditional, definition is as follows: for $0 < \alpha < \infty$, the space Λ^α consists of the functions $f \in C(\mathbb{T})$ such that

$$\|\Delta_h^m f\|_\infty = O(|h|^\alpha), \quad h \in \mathbb{R},$$

where $\|\cdot\|_\infty$ is the sup-norm on \mathbb{T} , m is an integer with $m > \alpha$, and Δ_h^m stands for the m th order difference operator with step h . (As usual, the difference operators Δ_h^k are defined by induction: one puts $(\Delta_h^1 f)(\zeta) := f(e^{ih}\zeta) - f(\zeta)$ and $\Delta_h^k f = \Delta_h^1 \Delta_h^{k-1} f$.) Finally, observe that Λ^α is an algebra for each $\alpha > 0$, and so is $A^\alpha = \Lambda^\alpha \cap H^\infty$.

Now, let $\alpha \in (0, \infty)$ and fix a function $h \in H^\infty$. This done, consider the set

$$(1.3) \quad \mathcal{I} = \mathcal{I}(\alpha, h) := \{f \in A^\alpha : fh \in A^\alpha\}.$$

Clearly, \mathcal{I} is an *ideal* of the algebra A^α . Indeed, it is a linear (possibly nonclosed) subspace thereof, and $fg \in \mathcal{I}$ whenever $f \in \mathcal{I}$ and $g \in A^\alpha$. Our aim is to study the ideals $\mathcal{I}(\alpha, h)$ that arise when $h = \theta^k$, with θ inner and $k \in \mathbb{N}$. Later on, we shall also look at similar ideals in H_n^∞ , but let us stick to the A^α case for the time being.

Two questions will be addressed. The first of these concerns the relationship between $\mathcal{I}(\alpha, \theta^k)$ and $\mathcal{I}(\alpha, \theta^l)$ for $k \neq l$. Secondly, we ask whether the functions f in $\mathcal{I}(\alpha, \theta^k)$ can be nicely described in terms of their moduli. After all, since a nontrivial inner function θ is highly discontinuous at some points of \mathbb{T} , one feels that the inclusion $f\theta^k \in A^\alpha$ can only be true if $|f|$ becomes appropriately small near the singular set of θ . Precisely speaking, we want to know if/when $\mathcal{I}(\alpha, \theta^k)$ is an “ideal space”, or rather an *ideal subspace* of A^α , a (fairly standard) notion that we are going to recall now. Note, however, the new meaning attached to the word “ideal”.

Suppose X is a subspace, not necessarily closed, of a function space Y . We say that X is an *ideal subspace* of Y if, for any $f \in X$ and any $g \in Y$ with $|g| \leq |f|$, we have $g \in X$. Roughly speaking, this means that the elements of X are describable, among all functions in Y , by a certain “size condition” on the function’s modulus. Of course, the inequality $|g| \leq |f|$ in the above definition is assumed to hold everywhere — or perhaps almost everywhere — on the underlying set. Throughout this paper, the bigger space Y is taken to be either A^α or H_n^∞ , so the set in question is \mathbb{D} .

There seems to be no chance of confusion between the adjective “ideal”, as used in the preceding paragraph, and the noun “ideal” that appears elsewhere, e.g., in the sentence following (1.3). Moreover, we shall repeatedly refer to *ideal ideals* in A^α or in H_n^∞ (by definition, these are ideals of the corresponding algebra that are also ideal subspaces thereof) and to *nonideal ideals* (i.e., those that fail to be ideal subspaces). In particular, “nonideal” is always an adjective.

The following theorem, to be found in [2, 3], provides a criterion for a function $f \in A^\alpha$ to be multipliable or divisible by (a power of) an inner function θ . See also [5, 6, 9] for alternative versions and approaches. The criterion will be stated in terms of a decrease condition to be satisfied by f along the set

$$\Omega(\theta, \varepsilon) := \{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}, \quad 0 < \varepsilon < 1.$$

Theorem A. *Let $0 < \alpha < \infty$ and let m be an integer with $m > \alpha$. Given $f \in A^\alpha$ and an inner function θ , the following conditions are equivalent.*

- (i.A) $f\theta^m \in A^\alpha$.
- (ii.A) $f\theta^k \in \Lambda^\alpha$ for all $k \in \mathbb{Z}$.
- (iii.A) $f/\theta^m \in \Lambda^\alpha$.
- (iv.A) For some $\varepsilon \in (0, 1)$, we have

$$|f(z)| = O((1 - |z|)^\alpha), \quad z \in \Omega(\theta, \varepsilon).$$

Yet another equivalent condition is obtained from (iv.A) upon replacing the word “some” by “each”; see [2] or [3].

In what follows, we shall restrict our attention to the A^α -spaces with $\alpha \notin \mathbb{N}$. The Zygmund classes A^n will not reappear (except, very briefly, in the proof of Theorem 1.3). For integral values of the smoothness exponent, we are going to consider the H_n^∞ spaces instead; we shall return to this in a while.

We know from Theorem A that, for $k \in \mathbb{N} \cap (\alpha, \infty)$, the set $\mathcal{I}(\alpha, \theta^k)$ does not depend on the choice of k and is an ideal ideal of A^α . Indeed, if f and g are A^α -functions with $|g| \leq |f|$ on \mathbb{D} , then g is sure to satisfy (iv.A) whenever f does. Now what about $\mathcal{I}(\alpha, \theta^k)$ in the range $k \in \mathbb{N} \cap (0, \alpha)$? Of course, the question is meaningful for $\alpha > 1$ only, and this time a different picture comes into view. Namely, we shall soon see that the ideals $\mathcal{I}(\alpha, \theta^k)$ with $1 \leq k \leq [\alpha]$ may happen to be pairwise distinct, all of them strictly larger than $\mathcal{I}(\alpha, \theta^{[\alpha]+1})$, and also nonideal.

Meanwhile, we observe that one always has

$$\mathcal{I}(\alpha, \theta^k) \subset \mathcal{I}(\alpha, \theta^l) \quad \text{for } k \geq l \geq 1.$$

This is due to the so-called *f-property* (or *division property*) of A^α that was established by Havin in [11]: whenever $F \in H^\infty$ and I is an inner function with $FI \in A^\alpha$, we actually have $F \in A^\alpha$. We mention in passing that the *f-property*, and in fact a certain stronger property involving coanalytic Toeplitz operators, was verified in [11] for a large number of important “smooth analytic classes”. This laid the foundation for much of the subsequent research in the field; see [14] for an overview and further developments.

Now we recall that every inner function θ can be factored canonically as $\theta = \lambda BS$, where λ is a unimodular constant, B is a *Blaschke product*, and S is a *singular inner function*; see [10, Chapter II]. More explicitly, the factors involved are of the form

$$B(z) = B_{\{z_j\}}(z) := \prod_j \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z},$$

where $\{z_j\} \subset \mathbb{D}$ is a sequence (possibly finite or empty) with $\sum_j (1 - |z_j|) < \infty$, and

$$S(z) = S_\mu(z) := \exp \left\{ - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right\},$$

where μ is a (nonnegative) singular measure on \mathbb{T} . The closure of the set $\{z_j\} \cup \text{supp } \mu$ is called the *spectrum of θ* ; we shall denote it by $\text{spec } \theta$.

If $\theta = S$ is a singular inner function, then we still have

$$\mathcal{I}(\alpha, \theta) = \mathcal{I}(\alpha, \theta^2) = \dots$$

for each $\alpha \in (0, \infty)$, precisely as it happens for an arbitrary inner function in the range $\alpha \in (0, 1)$. Moreover, the ideal $\mathcal{I}(\alpha, \theta)$ is then ideal, as a subspace of A^α . To verify these claims, write $\theta = S_0^m$, where S_0 is another singular inner function and m is an integer with $m > \alpha$; then apply Theorem A to S_0 .

However, Blaschke products — and hence generic inner functions — may exhibit a different type of behavior. This was discovered by Shirokov (see [12] or [14, Chapter I]) who came up with an ingenious construction of a function $g \in A^\infty := \bigcap_{0 < \alpha < \infty} A^\alpha$ and a Blaschke product B such that $g/B \in A^\infty$, but $gB \notin \bigcup_{\alpha > 1} A^\alpha$. It follows then, for any

fixed $\alpha > 1$, that the function $f := g/B$ lies in $\mathcal{I}(\alpha, B) \setminus \mathcal{I}(\alpha, B^2)$. In addition, the ideal $\mathcal{I}(\alpha, B)$ is now nonideal: indeed, we have $f \in \mathcal{I}(\alpha, B)$ and $g \in A^\alpha \setminus \mathcal{I}(\alpha, B)$, whereas $|g| \leq |f|$ on \mathbb{D} .

Our current purpose is a more detailed analysis of the (possibly) nonideal ideals of the form $\mathcal{I}(\alpha, B^k)$, with $1 \leq k \leq n$. Here and below, α takes values in $(1, \infty) \setminus \mathbb{N}$ and $n = [\alpha]$ is the integral part of α . Furthermore, a special class of Blaschke products will be singled out and dealt with. Namely, it will be assumed that $B = B_{\{z_j\}}$ is an A^α -interpolating Blaschke product, in the sense that its zeros z_j are all simple and their closure $E := \text{clos}\{z_j\}$ is an A^α -interpolating set. The latter means, in its turn, that we can freely prescribe the values of an A^α -function and its successive derivatives (of order at most n) on E , within a certain natural class of data. The formal definition and a geometric description of such sets will be recalled in §2 below. In particular, it turns out that the class of A^α -interpolating sets (and hence Blaschke products) does not depend on α . The sets in question are thus the same as A^β -interpolating sets for some (any) $\beta \in (0, 1)$, and these are easy to define: we call E an A^β -interpolating set if every function $\varphi : E \rightarrow \mathbb{C}$ satisfying

$$(1.4) \quad |\varphi(z) - \varphi(w)| \leq \text{const} \cdot |z - w|^\beta, \quad z, w \in E,$$

can be written as $\varphi = f|_E$ for some $f \in A^\beta$.

Once α and the Blaschke product B are fixed, we write $\mathcal{J}_k := \mathcal{I}(\alpha, B^k)$, so that

$$\mathcal{J}_k = \{f \in A^\alpha : fB^k \in A^\alpha\}, \quad k \in \mathbb{N}.$$

In view of the above discussion, we always have

$$(1.5) \quad \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_n \supset \mathcal{J}_{n+1}$$

and

$$(1.6) \quad \mathcal{J}_{n+1} = \mathcal{J}_{n+2} = \cdots$$

The ideals in (1.6) are ideal subspaces of A^α , as we know, but the preceding ones (i.e., $\mathcal{J}_1, \dots, \mathcal{J}_n$) may well be nonideal. Our first result provides a criterion for this to happen, and also for the inclusions in (1.5) to be proper (strict). When stating it, and later on, we shall use the notation d_j for the quantity $\text{dist}(z_j, \{z_l\}_{l \neq j})$ associated with the zero sequence $\{z_j\}$ of B . In other words,

$$d_j := \inf \{|z_j - z_l| : l \in \mathbb{N} \setminus \{j\}\}.$$

Theorem 1.1. *Let $n \in \mathbb{N}$, $n < \alpha < n+1$, and suppose B is an A^α -interpolating Blaschke product with zeros $\{z_j\}$. The following are equivalent.*

- (i.1) *The ideals $\mathcal{J}_1, \dots, \mathcal{J}_n$ are all nonideal (as subspaces of A^α).*
- (ii.1) *The inclusions $\mathcal{J}_1 \supset \cdots \supset \mathcal{J}_n \supset \mathcal{J}_{n+1}$ are all proper.*
- (iii.1) *\mathcal{J}_n is nonideal.*
- (iv.1) *$\mathcal{J}_{n+1} \neq \mathcal{J}_n$.*
- (v.1) *We have*

$$(1.7) \quad \sup_j \frac{d_j}{1 - |z_j|} = \infty.$$

To get an example of an A^α -interpolating Blaschke product $B = B_{\{z_j\}}$ that satisfies (1.7), take $z_j = (1 - a^j) \exp(ib^j)$, $j \in \mathbb{N}$, where a and b are any fixed numbers with $0 < a < b < 1$. On the other hand, the Blaschke product with zeros $z_j = 1 - a^j$, where $a \in (0, 1)$, is A^α -interpolating and violates (1.7).

A restricted version of Theorem 1.1 appeared as Theorem 1 in [4]. However, none of the current conditions (i.1)–(iii.1) were discussed in that paper, nor was the notion of a (non)ideal ideal introduced.

Now we turn to the case of an integral order of smoothness, an issue that was not touched upon in [4] either. This time, the algebras to be dealt with are $H_N^\infty = \{f : f^{(N)} \in H^\infty\}$. The phenomenon we are interested in may only occur when $N \geq 2$, so it will be convenient to write $N = n + 1$ with $n \in \mathbb{N}$. Given n and a Blaschke product B , we put

$$I_k := \{f \in H_{n+1}^\infty : fB^k \in H_{n+1}^\infty\}, \quad k \in \mathbb{N}.$$

Our intention is to study the I_k 's, as ideals/subspaces of H_{n+1}^∞ , in the same spirit as the \mathcal{J}_k 's above. As before, we have

$$(1.8) \quad I_1 \supset \cdots \supset I_n \supset I_{n+1}$$

and

$$(1.9) \quad I_{n+1} = I_{n+2} = \cdots .$$

Here, the inclusions (1.8) are due to the fact that H_{n+1}^∞ has the f -property (i.e., division by inner factors preserves membership in H_{n+1}^∞), as was shown by Shirokov in [12]. The identities (1.9) can likewise be deduced from Shirokov's results; see [12] or Lemma 3.6 below.

The H_N^∞ counterpart of Theorem 1.1 reads as follows.

Theorem 1.2. *Let $n \in \mathbb{N}$, and suppose B is an H_1^∞ -interpolating Blaschke product with zeros $\{z_j\}$. The following are equivalent.*

- (i.2) *The ideals I_1, \dots, I_n are all nonideal (as subspaces of H_{n+1}^∞).*
- (ii.2) *The inclusions $I_1 \supset \cdots \supset I_n \supset I_{n+1}$ are all proper.*
- (iii.2) *I_n is nonideal.*
- (iv.2) *$I_{n+1} \neq I_n$.*
- (v.2) *The sequence $\{z_j\}$ satisfies (1.7).*

Of course, by saying that $B = B_{\{z_j\}}$ is an H_1^∞ -interpolating Blaschke product we mean that its zeros are simple and their closure $E := \text{clos}\{z_j\}$ is an H_1^∞ -interpolating set in the natural sense. That is, every function $\varphi : E \rightarrow \mathbb{C}$ satisfying (1.4) with $\beta = 1$ should be representable as $\varphi = f|_E$ for some $f \in H_1^\infty$.

While the proof of Theorem 1.1 relies heavily on Theorem A, we could expect to prove Theorem 1.2 by following the same pattern, should the appropriate factorization theorem (similar to Theorem A) exist in the H_N^∞ setting. Specifically, we would need to have some variant of condition (iv.A) at our disposal, playing a similar role. Unfortunately, no such thing seems to be readily available, and the next result is intended to fill that gap. In addition to H_N^∞ , also involved in the statement below is the space

$$\text{BMOA}_N := \{f \in \text{BMOA} : f^{(N)} \in \text{BMOA}\},$$

which is mentioned for the sake of completeness. Here, as usual, BMOA stands for the analytic subspace of $\text{BMO} = \text{BMO}(\mathbb{T})$, the class of functions of bounded mean oscillation on \mathbb{T} ; see [10, Chapter VI].

Theorem 1.3. *Let m and N be positive integers with $m \geq N$. Given $f \in H_N^\infty$ and an inner function θ , the following are equivalent.*

- (i.3) *$f\theta^m \in H_N^\infty$.*
- (ii.3) *$f\theta^k \in H_N^\infty$ for all $k \in \mathbb{N}$.*
- (iii.3) *$f\theta^k \in \text{BMOA}_N$ for all $k \in \mathbb{N}$.*
- (iv.3) *For some $\varepsilon \in (0, 1)$, we have*

$$|f(z)| = O((1 - |z|)^N), \quad z \in \Omega(\theta, \varepsilon).$$

This last result is somewhat subtler than Theorem A and calls for a new method of proof. Indeed, neither duality arguments (as in [2, 3]) nor the pseudoanalytic extension approach (as in [9]) that worked for A^α carry over to H_N^∞ . Our proof will be accomplished by combining some of Shirokov’s techniques from [12] with those developed by the author. Finally, we remark that the equivalence between (ii.3) and (iii.3) reflects an amusing “self-improving property” (i.e., an automatic increase in smoothness), a phenomenon discussed in greater generality in [7].

Going back to the A^α setting, we wish to discuss yet another aspect of the problem, namely, the construction of an “ideal hull” for a nonideal ideal. Suppose, under the hypotheses of Theorem 1.1, that (1.7) is fulfilled; assume also that $1 \leq k \leq n$. We know that the corresponding ideals \mathcal{J}_k are then properly contained in each other and are nonideal. Thus, for $f \in A^\alpha$, no kind of “ideal” smallness condition on $|f|$ — in particular, no reasonable size condition on $|f(z_j)|$ — can possibly be necessary and sufficient in order that $f \in \mathcal{J}_k$. At the same time, one feels that the values $|f(z_j)|$ *must* become appropriately small, as $j \rightarrow \infty$, whenever $f \in \mathcal{J}_k$. This motivates our search for a *necessary* condition, ideal in nature and involving the decrease rate of $|f(z_j)|$, that should be fulfilled for each $f \in \mathcal{J}_k$. Moreover, we want our condition to be sensitive enough to distinguish between \mathcal{J}_k and \mathcal{J}_{k-1} .

We have been able to find such a necessary condition in the case where $n/2 < k \leq n$. This is provided by part (a) of the theorem below, while part (b) shows that the condition is optimal. The latter deals with the full range $1 \leq k \leq n$, and we strongly believe that the former should also extend to all of these k ’s.

Theorem 1.4. *Let $n \in \mathbb{N}$, $n < \alpha < n+1$, and suppose B is an A^α -interpolating Blaschke product with zeros $\{z_j\}$.*

(a) *If k is an integer with $n/2 < k \leq n$, then every $f \in \mathcal{J}_k$ satisfies*

$$(1.10) \quad |f(z_j)| \leq \text{const} \cdot d_j^{\alpha-k} (1 - |z_j|)^k, \quad j \in \mathbb{N}.$$

(b) *For each $k = 1, \dots, n$, there is a function $f \in \mathcal{J}_k$ with*

$$(1.11) \quad |f(z_j)| \asymp d_j^{\alpha-k} (1 - |z_j|)^k, \quad j \in \mathbb{N}.$$

Here and throughout, the notation $U \asymp V$ means that the ratio U/V lies between two positive constants. Those (hidden) constants in (1.11) are of course independent of j , as is the constant in (1.10).

If $n = 1$, then the only possible value of k is 1 and the inequality $n/2 < k$ is automatic. Now if $1 \leq n/2 < k \leq n$ and if (1.7) is true, then Theorem 1.4 tells us that the ideal

$$\tilde{\mathcal{J}}_k := \{f \in A^\alpha : |f(z_j)| = O(d_j^{\alpha-k} (1 - |z_j|)^k), \quad j \in \mathbb{N}\}$$

contains \mathcal{J}_k but not \mathcal{J}_{k-1} . Indeed, (1.7) says that the quantity $d_j^{\alpha-k} (1 - |z_j|)^k$ is essentially smaller than

$$(1.12) \quad d_j^{\alpha-k+1} (1 - |z_j|)^{k-1},$$

whereas statement (b) of the theorem produces a function $f \in \mathcal{J}_{k-1}$ for which $|f(z_j)|$ has the same order of magnitude as (1.12). Thus, we have constructed an *ideal envelope*, namely $\tilde{\mathcal{J}}_k$, of the nonideal ideal \mathcal{J}_k without making it “too much fatter”. In fact, no smaller ideal ideal resulting from a stronger decrease condition on $|f(z_j)|$ would do.

In conclusion, we point out a corollary to Theorem 1.4 that establishes a relationship between \mathcal{J}_k , with k as above, and the ideal

$$\mathcal{J}_{-1} := \{f \in A^\alpha : f/B \in \Lambda^\alpha\}.$$

This last result will also rely on the following characterization of \mathcal{J}_{-1} , which appeared in [2, Corollary 4.3]: for a function $f \in A^\alpha$ ($0 < \alpha < \infty$) and an interpolating Blaschke product $B = B_{\{z_j\}}$,

$$(1.13) \quad f \in \mathcal{J}_{-1} \iff |f(z_j)| = O((1 - |z_j|)^\alpha).$$

(Here and below, “interpolating” stands for “ H^∞ -interpolating”, meaning that $\{z_j\}$ is an interpolating sequence for H^∞ ; cf. [10, Chapter VII]. It is known that every A^α -interpolating Blaschke product is H^∞ -interpolating.) As a consequence of (1.13), we see that, under the current conditions, \mathcal{J}_{-1} is an ideal ideal of A^α .

Corollary 1.5. *Let $n \in \mathbb{N}$, $n < \alpha < n + 1$, and suppose B is an A^α -interpolating Blaschke product with zeros $\{z_j\}$. Given an integer k with $n/2 < k \leq n$, we have $\mathcal{J}_k \subset \mathcal{J}_{-1}$ if and only if*

$$(1.14) \quad \sup_j \frac{d_j}{1 - |z_j|} < \infty.$$

To prove the “if” part, we combine (1.14) with (1.10) to get

$$|f(z_j)| \leq \text{const} \cdot (1 - |z_j|)^\alpha, \quad j \in \mathbb{N},$$

for every $f \in \mathcal{J}_k$. Then we invoke (1.13) to conclude that $\mathcal{J}_k \subset \mathcal{J}_{-1}$. Conversely, assuming (1.7) and taking an $f \in \mathcal{J}_k$ with property (1.11), we obtain

$$\sup_j \frac{|f(z_j)|}{(1 - |z_j|)^\alpha} = \infty.$$

Therefore, this specific f is not in \mathcal{J}_{-1} , by (1.13) again, and the “only if” part follows.

The remaining part of the paper contains some preliminary material on interpolating sets for A^α and H_{n+1}^∞ , a few lemmas, and finally the proofs of our main results.

§2. PRELIMINARIES ON FREE INTERPOLATION IN A^α AND H_N^∞

Let n be a nonnegative integer. Given $g \in A^\alpha$, with $n < \alpha < n + 1$, we have

$$(2.1) \quad \left| g^{(s)}(z) - \sum_{m=0}^{n-s} \frac{g^{(s+m)}(w)}{m!} (z - w)^m \right| \leq C |z - w|^{\alpha-s} \quad (z, w \in \mathbb{D}, \quad s = 0, \dots, n),$$

where $C = C_g$ is a constant independent of z and w . If $g \in H_{n+1}^\infty$, then (2.1) is true with $\alpha = n + 1$.

A closed set $E \subset \text{clos } \mathbb{D}$ is said to be A^α -interpolating if every interpolation problem

$$(2.2) \quad g|_E = \varphi_0, \quad g'|_E = \varphi_1, \quad \dots, \quad g^{(n)}|_E = \varphi_n$$

has a solution $g \in A^\alpha$, provided that the data $\varphi_s : E \rightarrow \mathbb{C}$ satisfy

$$(2.3) \quad \left| \varphi_s(z) - \sum_{m=0}^{n-s} \frac{\varphi_{s+m}(w)}{m!} (z - w)^m \right| \leq C |z - w|^{\alpha-s} \quad (z, w \in E, \quad s = 0, \dots, n)$$

with some fixed $C > 0$.

Similarly, we call E an H_{n+1}^∞ -interpolating set if every interpolation problem (2.2) has a solution $g \in H_{n+1}^\infty$ whenever the data $\varphi_s : E \rightarrow \mathbb{C}$ ($0 \leq s \leq n$) satisfy (2.3) for $\alpha = n + 1$ and for some constant $C > 0$.

Of course, (2.3) means simply that the φ_s obey the necessary conditions coming from (2.1). The validity of (2.3), with $n < \alpha \leq n + 1$, will also be expressed by saying that the $(n + 1)$ -tuple $(\varphi_0, \dots, \varphi_n)$ is an α -admissible jet on E .

The following characterization of A^α -interpolating sets was given by Dyn’kin in [8]; see also [9, §3]. We shall use the notation $\rho(\cdot, \cdot)$ for the pseudohyperbolic distance on \mathbb{D} , so that $\rho(z, w) := |z - w|/|1 - \bar{z}w|$.

Theorem B. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, and let E be a closed subset of $\text{clos } \mathbb{D}$. Then E is an A^α -interpolating set if and only if it has the two properties below.*

(1.B) *The set $E \cap \mathbb{D}$ is separated, in the sense that*

$$\inf\{\rho(z, w) : z, w \in E \cap \mathbb{D}, z \neq w\} > 0.$$

(2.B) *There is a constant $c > 0$ such that every arc $I \subset \mathbb{T}$ satisfies*

$$\sup_{\zeta \in I} \text{dist}(\zeta, E) \geq c|I|,$$

where $\text{dist}(\zeta, E) := \inf_{z \in E} |\zeta - z|$ and $|I|$ is the length of I .

Subsequently, Shirokov [13] extended Theorem B to a larger scale of Lipschitz-type spaces involving general moduli of continuity. As a special case (namely, for the modulus of continuity $\omega(t) = t$), his results provide a description of H_{n+1}^∞ -interpolating sets, which can be stated as follows.

Theorem C. *Suppose n is a nonnegative integer and E is a closed subset of $\text{clos } \mathbb{D}$. Then E is an H_{n+1}^∞ -interpolating set if and only if it satisfies (1.B)₂ and (2.B), where (1.B)₂ is the condition that $E \cap \mathbb{D}$ be a union of two separated sets.*

A detailed discussion of the geometric condition (1.B)&(2.B) can be found in [8, §5]. In particular, it was shown there that if E is an A^α -interpolating set, then $E \cap \mathbb{D}$ is an interpolating set for H^∞ , i.e.,

$$(2.4) \quad \inf_j \prod_{l:l \neq j} \rho(z_j, z_l) > 0$$

for any enumeration $\{z_j\}$ of $E \cap \mathbb{D}$. Thus, every A^α -interpolating Blaschke product is also H^∞ -interpolating (or simply “interpolating”, in standard terminology), as we mentioned before. Consequently, an H_{n+1}^∞ -interpolating (or equivalently, H_1^∞ -interpolating) Blaschke product B can always be written as $B = B_1 B_2$, where the two factors are interpolating Blaschke products.

Finally, we remark that each A^α - or H_{n+1}^∞ -interpolating set E satisfies the *Beurling–Carleson condition*

$$\int_{\mathbb{T}} \log \text{dist}(\zeta, E) |d\zeta| > -\infty$$

and is, therefore, a nonuniqueness set for A^α (see [1]).

§3. SOME LEMMAS

Given a sequence $\{z_j\} \subset \mathbb{D}$, we recall the notation

$$d_j := \inf \{|z_j - z_l| : l \in \mathbb{N} \setminus \{j\}\}.$$

Lemma 3.1. *Let $0 \leq k \leq n < \alpha \leq n + 1$, where k and n are integers. Assume also that $\{z_j\} \subset \mathbb{D}$ is a sequence satisfying $z_j \neq z_l$ for $j \neq l$ and having no accumulation points in \mathbb{D} . Finally, write $E := \text{clos}\{z_j\}$ and define the functions φ_s , $s = 0, \dots, n$, on E by putting*

$$\varphi_s \equiv 0 \quad \text{for } s \neq k, \quad \varphi_k(z_j) = d_j^{\alpha-k} \quad (j = 1, 2, \dots), \quad \text{and } \varphi_k|_{E \cap \mathbb{T}} = 0.$$

Then $(\varphi_0, \dots, \varphi_n)$ is an α -admissible jet on E .

Proof. Since the functions φ_s are continuous on E , it suffices to check that

$$(3.1) \quad \left| \varphi_s(z_l) - \sum_{m=0}^{n-s} \frac{\varphi_{s+m}(z_j)}{m!} (z_l - z_j)^m \right| \leq C |z_l - z_j|^{\alpha-s} \quad (j, l \in \mathbb{N}, j \neq l)$$

for $s = 0, \dots, n$ and for some fixed $C > 0$. (This is precisely (2.3) with $z = z_l$ and $w = z_j$.) We let LHS stand for the left-hand side of (3.1), and we estimate it by considering three cases as follows.

If $0 \leq s < k$, then

$$\text{LHS} = \left| \frac{\varphi_k(z_j)}{(k-s)!} (z_l - z_j)^{k-s} \right| = \frac{1}{(k-s)!} d_j^{\alpha-k} |z_l - z_j|^{k-s} \leq |z_l - z_j|^{\alpha-s},$$

where the final inequality is due to the obvious facts that $(k-s)! \geq 1$ and $d_j \leq |z_l - z_j|$.

If $s = k$, then

$$\text{LHS} = |\varphi_k(z_l) - \varphi_k(z_j)| = |d_l^{\alpha-k} - d_j^{\alpha-k}| \leq |z_l - z_j|^{\alpha-k} = |z_l - z_j|^{\alpha-s},$$

because, clearly, $\max(d_l, d_j) \leq |z_l - z_j|$.

Finally, if $k < s \leq n$, then LHS = 0. Thus, in all cases, (3.1) is true with $C = 1$. \square

Lemma 3.2. *Suppose f is an analytic function on \mathbb{D} , B is a Blaschke product with zeros $\{z_j\}$, and m is a nonnegative integer. Then*

$$(fB^m)^{(m)}(z_j) = f(z_j) \cdot (B^m)^{(m)}(z_j), \quad j \in \mathbb{N}.$$

Proof. Indeed,

$$\begin{aligned} (fB^m)^{(m)}(z_j) &= \sum_{l=0}^m \binom{m}{l} f^{(m-l)}(z_j) (B^m)^{(l)}(z_j) \\ &= f(z_j) \cdot (B^m)^{(m)}(z_j), \end{aligned}$$

because $(B^m)^{(l)}(z_j) = 0$ for $0 \leq l \leq m - 1$. \square

Lemma 3.3. *Suppose B is an interpolating Blaschke product with zeros $\{z_j\}$, and let $\delta = \delta(B)$ be the value of the infimum in (2.4). Then, for $m \in \mathbb{N}$,*

$$|(B^m)^{(m)}(z_j)| \asymp (1 - |z_j|)^{-m}, \quad j \in \mathbb{N},$$

where the constants involved depend only on m and δ .

Proof. The inequality

$$|(B^m)^{(m)}(z_j)| \leq C_m (1 - |z_j|)^{-m}$$

is clearly true because B^m is an H^∞ -function of norm 1.

To prove the reverse inequality,

$$(3.2) \quad |(B^m)^{(m)}(z_j)| \geq c_{m,\delta} (1 - |z_j|)^{-m},$$

we proceed by induction. When $m = 1$, (3.2) is a well-known restatement of the fact that $\{z_j\}$ is an interpolating sequence. Next, assuming that (3.2) is established for some value of m , we note that

$$(B^{m+1})^{(m+1)}(z_j) = (m+1) \cdot (B^m B')^{(m)}(z_j) = (m+1) \cdot (B^m)^{(m)}(z_j) \cdot B'(z_j);$$

to check the last step, we apply Lemma 3.2 with $f = B'$. Thus,

$$|(B^{m+1})^{(m+1)}(z_j)| = (m+1) |(B^m)^{(m)}(z_j)| |B'(z_j)|,$$

and now the desired estimate

$$|(B^{m+1})^{(m+1)}(z_j)| \geq \text{const} \cdot (1 - |z_j|)^{-m-1}$$

follows from the inductive hypothesis (3.2) combined with the $m = 1$ case. \square

The next two lemmas are borrowed from [2]; see Lemma 2.2 and Theorem 2.4 of that paper.

Lemma 3.4. *Suppose that f is analytic on \mathbb{D} , θ is an inner function, $0 < \varepsilon < 1$, $\alpha > 0$, and $k \in \mathbb{N}$. If*

$$|f(z)| = O((1 - |z|)^\alpha), \quad z \in \Omega(\theta, \varepsilon),$$

then

$$|f^{(k)}(z)| = O((1 - |z|)^{\alpha-k}), \quad z \in \Omega(\theta, \varepsilon/2).$$

Lemma 3.5. *Suppose that f is a nonnull function in H_N^∞ with $N \in \mathbb{N}$, and θ is an inner function. The following are equivalent.*

(i) *For some $\varepsilon \in (0, 1)$, we have*

$$|f(z)| = O((1 - |z|)^N), \quad z \in \Omega(\theta, \varepsilon).$$

(ii) *The set $\text{spec } \theta \cap \mathbb{T}$ has Lebesgue measure 0, and*

$$|f(\zeta)| = O(1/|\theta'(\zeta)|^N), \quad \zeta \in \mathbb{T} \setminus \text{spec } \theta.$$

Finally, we list some of Shirokov’s results from [12]. In particular, the next lemma comprises Theorems 1 and 3 of [12], when specialized to the H_N^∞ case.

Lemma 3.6. *Let $g \in H_N^\infty$, where $N \in \mathbb{N}$, and suppose I is an inner function such that $g/I \in H^\infty$. Then $g/I \in H_N^\infty$. If, moreover, the zeros of I in \mathbb{D} (if any) are all of multiplicity at least N , then we also have $gI \in H_N^\infty$ (and hence $gI^k \in H_N^\infty$ for all $k \in \mathbb{N}$).*

The last statement in parentheses was not mentioned by Shirokov explicitly, but it follows readily from the preceding one by induction. We conclude by citing a restricted version of [12, Lemma 4].

Lemma 3.7. *Given an inner function θ and a point $\zeta \in \mathbb{T} \setminus \text{spec } \theta$, write*

$$d_\theta(\zeta) := \text{dist}(\zeta, \text{spec } \theta) \quad \text{and} \quad \tau_\theta(\zeta) := \min(d_\theta(\zeta), 1/|\theta'(\zeta)|).$$

Then, for every $l \in \mathbb{N}$, there is a constant $c_l > 0$ such that

$$|\theta^{(l)}(\zeta)| \leq c_l \cdot \{\tau_\theta(\zeta)\}^{-l}, \quad \zeta \in \mathbb{T} \setminus \text{spec } \theta.$$

§4. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. The implications (i.1) \implies (iii.1) and (ii.1) \implies (iv.1) are obvious. It is also clear that (iii.1) implies (iv.1), because \mathcal{J}_{n+1} is an ideal subspace of A^α (by Theorem A).

(iv.1) \implies (i.1). Let $f \in \mathcal{J}_n \setminus \mathcal{J}_{n+1}$, so that

$$(4.1) \quad fB^n \in A^\alpha \quad \text{but} \quad fB^{n+1} \notin A^\alpha.$$

It follows that, for $k = 1, \dots, n$, the function $g_k := fB^{n-k}$ is in \mathcal{J}_k , while $Bg_k = fB^{n-k+1}$ is in $A^\alpha \setminus \mathcal{J}_k$. Since $|Bg_k| \leq |g_k|$ in \mathbb{D} , we conclude that \mathcal{J}_k is nonideal.

(iv.1) \implies (ii.1). Once again, let $f \in \mathcal{J}_n \setminus \mathcal{J}_{n+1}$ and $g_k = fB^{n-k}$. For $k = 1, \dots, n$, (4.1) tells us that $g_k \in \mathcal{J}_k \setminus \mathcal{J}_{k+1}$, so the inclusion $\mathcal{J}_{k+1} \subset \mathcal{J}_k$ is proper.

Now we know that (i.1) \iff (ii.1) \iff (iii.1) \iff (iv.1), so it is the equivalence between (iv.1) and (v.1) that remains to be proved.

(iv.1) \implies (v.1). Suppose (v.1) fails, so that

$$(4.2) \quad \sup_j \frac{d_j}{1 - |z_j|} < \infty,$$

and let $f \in \mathcal{J}_n$. The function $g := fB^n$ is then in A^α , and

$$g(z_j) = g'(z_j) = \dots = g^{(n-1)}(z_j) = 0$$

for all j . Therefore, inequality (2.1) with $s = 0$ and $w = z_j$ yields

$$(4.3) \quad \left| g(z) - \frac{g^{(n)}(z_j)}{n!} (z - z_j)^n \right| \leq C |z - z_j|^\alpha, \quad z \in \mathbb{D}.$$

In particular, for any fixed j and for $z = z_l$ ($l \neq j$), this gives

$$\frac{|g^{(n)}(z_j)|}{n!} \leq C |z_l - z_j|^{\alpha-n}.$$

Taking the infimum over $l \in \mathbb{N} \setminus \{j\}$ and recalling (4.2), we get

$$(4.4) \quad \frac{|g^{(n)}(z_j)|}{n!} \leq C d_j^{\alpha-n} \leq \tilde{C} (1 - |z_j|)^{\alpha-n},$$

with a suitable constant $\tilde{C} > 0$.

Now suppose $z \in \Omega(B, \varepsilon)$, where $\varepsilon > 0$ is appropriately small. Since B is an interpolating Blaschke product, it follows (cf. [10, Chapter X, Lemma 1.4]) that there is a number $\lambda \in (0, 1)$ and a zero z_j of B such that $\rho(z, z_j) < \lambda$. Here, both ε and λ can be taken to depend only on the ‘‘Carleson constant’’ $\delta = \delta(B)$, defined as the infimum in (2.4). Then inequality $\rho(z, z_j) < \lambda$ implies that

$$(4.5) \quad |z - z_j| \leq c(1 - |z|) \quad \text{and} \quad 1 - |z_j| \leq c(1 - |z|)$$

for some $c = c(\delta) > 0$. By (4.3), we have

$$|g(z)| \leq \frac{|g^{(n)}(z_j)|}{n!} |z - z_j|^n + C |z - z_j|^\alpha,$$

and combining this with (4.4) and (4.5), we finally obtain

$$|g(z)| \leq \text{const} \cdot (1 - |z|)^\alpha, \quad z \in \Omega(B, \varepsilon).$$

Theorem A now tells us that $gB^k \in A^\alpha$ for all $k \in \mathbb{Z}$. In particular, $gB = fB^{n+1} \in A^\alpha$ and so $f \in \mathcal{J}_{n+1}$.

Thus, we have checked that (4.2) implies the inclusion $\mathcal{J}_n \subset \mathcal{J}_{n+1}$, whence $\mathcal{J}_n = \mathcal{J}_{n+1}$, contradicting (iv.1).

(v.1) \implies (iv.1). Since $\text{clos}\{z_j\}$ is an A^α -interpolating set, we can apply Lemma 3.1 with $k = n$ to find a function $g \in A^\alpha$ satisfying

$$(4.6) \quad g(z_j) = g'(z_j) = \dots = g^{(n-1)}(z_j) = 0, \quad g^{(n)}(z_j) = d_j^{\alpha-n}$$

for all j . This g is then divisible by B^n , so that $g = fB^n$ with $f \in H^\infty$. In fact, we have $f \in A^\alpha$ (because A^α enjoys the f -property; see §1), and hence $f \in \mathcal{J}_n$. Next, we use Lemma 3.2 to rewrite the last condition in (4.6) as

$$f(z_j) \cdot (B^n)^{(n)}(z_j) = d_j^{\alpha-n}.$$

In view of (the trivial part of) Lemma 3.3, or simply because $B^n \in H^\infty$, it follows that

$$|f(z_j)|(1 - |z_j|)^{-n} \geq \text{const} \cdot d_j^{\alpha-n}.$$

Consequently,

$$\sup_j \frac{|f(z_j)|}{(1 - |z_j|)^\alpha} \geq \text{const} \cdot \sup_j \left(\frac{d_j}{1 - |z_j|} \right)^{\alpha-n} = \infty,$$

where we have also used (v.1). Finally, we invoke Theorem A (specifically, the (i.A) \implies (iv.A) part with $\theta = B$ and $m = n + 1$) to conclude that $fB^{n+1} \notin A^\alpha$. Thus, $f \in \mathcal{J}_n \setminus \mathcal{J}_{n+1}$, which yields (iv.1) and completes the proof. \square

To prove Theorem 1.2, we may proceed in quite a similar fashion, as soon as we have Theorem 1.3 at our disposal. Taking the latter result for granted and postponing its verification to the next section, we now describe the passage from the proof of Theorem 1.1 above to that of Theorem 1.2. Basically, this reduces to the following adjustments. Throughout, change the tags (i.1), . . . , (v.1) to (i.2), . . . , (v.2), respectively; replace \mathcal{J}_k by I_k , A^α by H_{n+1}^∞ , and α (except in A^α) by $n + 1$; instead of Theorem A, refer to Theorem 1.3 with $N = n + 1$. Finally, a minor modification is needed to check that the set $\Omega(B, \varepsilon)$ is again contained in $\bigcup_j \{z : \rho(z, z_j) < \lambda\}$ for some ε and λ in $(0, 1)$. This time, we write $B = B_1 B_2$, where B_1 and B_2 are interpolating Blaschke products (see §2), and note that

$$\Omega(B, \varepsilon) \subset \Omega(B_1, \sqrt{\varepsilon}) \cup \Omega(B_2, \sqrt{\varepsilon}).$$

§5. PROOF OF THEOREM 1.3

(i.3) \implies (ii.3). This implication is a consequence of Shirokov’s results that are collected in Lemma 3.6. Indeed, the last assertion of the lemma, when applied to $g := f\theta^m$ and $I := \theta^m$, tells us that $f\theta^{km} \in H_N^\infty$ for all $k \in \mathbb{N}$. Together with the fact that H_N^∞ enjoys the f -property (which is also contained in Lemma 3.6), this yields (ii.3).

(ii.3) \implies (iii.3). This is obvious, because $H_N^\infty \subset \text{BMOA}_N$.

(iii.3) \implies (iv.3). Clearly, from (iii.3) it follows that $f\theta^{N+1}$ lies in BMOA_N and hence, *a fortiori*, in the (higher order) Zygmund class

$$A^N = \left\{ f \in H^\infty : |f^{(N+1)}(z)| = O((1 - |z|)^{-1}), \quad z \in \mathbb{D} \right\}.$$

To verify the inclusion $\text{BMOA}_N \subset A^N$, as used here, recall that BMOA is contained in the Bloch space \mathcal{B} (see [10, Chapter VI]). Thus, we have checked that $f\theta^{N+1} \in A^N$; this done, we apply the (i.A) \implies (iv.A) part of Theorem A, with $\alpha = N$ and $m = N + 1$, to arrive at (iv.3).

(iv.3) \implies (i.3). First, we prove that (iv.3) implies that $f\theta \in H_N^\infty$. We want to show that the function

$$(5.1) \quad (f\theta)^{(N)} = \sum_{l=0}^N \binom{N}{l} f^{(N-l)} \theta^{(l)}$$

is bounded, and our plan is to check this for (the boundary values of) each summand involved.

Using (iv.3) and Lemma 3.4, we obtain

$$(5.2) \quad |f^{(N-l)}(z)| = O((1 - |z|)^l), \quad z \in \Omega(\theta, \varepsilon/2),$$

for $l = 0, \dots, N$. Since $f^{(N-l)} \in H_l^\infty$, Lemma 3.5 enables us to rewrite the preceding estimate as

$$(5.3) \quad |f^{(N-l)}(\zeta)| = O(1/|\theta'(\zeta)|^l), \quad \zeta \in \mathbb{T} \setminus \text{spec } \theta.$$

Of course, we may assume that $f \not\equiv 0$, so Lemma 3.5 tells us also that $\mathbb{T} \setminus \text{spec } \theta$ is a set of full measure on \mathbb{T} .

Now let $\zeta \in \mathbb{T} \setminus \text{spec } \theta$, and let $\zeta^* \in \text{spec } \theta$ be a point with

$$|\zeta - \zeta^*| = \text{dist}(\zeta, \text{spec } \theta) =: d_\theta(\zeta).$$

Then we have

$$(5.4) \quad \left| f^{(N-l)}(\zeta) - \sum_{j=0}^{l-1} \frac{f^{(N-l+j)}(\zeta^*)}{j!} (\zeta - \zeta^*)^j \right| \leq C|\zeta - \zeta^*|^l = C d_\theta^l(\zeta),$$

whence

$$(5.5) \quad |f^{(N-l)}(\zeta)| \leq C d_\theta^l(\zeta) + \sum_{j=0}^{l-1} \frac{|f^{(N-l+j)}(\zeta^*)|}{j!} d_\theta^j(\zeta).$$

(The admissible values of l are again $0, 1, \dots, N$. When $l = 0$, it is understood that the sums in (5.4) and (5.5) equal 0; the estimates then reduce to saying that $|f^{(N)}(\zeta)| \leq C$ for almost all $\zeta \in \mathbb{T}$, in accordance with the hypothesis that $f \in H_N^\infty$.)

Recalling that $\zeta^* \in \text{spec } \theta \subset \text{clos } \Omega(\theta, \varepsilon/2)$, we now combine (5.2) with the obvious inequality $1 - |\zeta^*| \leq d_\theta(\zeta)$ to get

$$(5.6) \quad |f^{(N-l+j)}(\zeta^*)| \leq \tilde{C} (1 - |\zeta^*|)^{l-j} \leq \tilde{C} \cdot d_\theta^{l-j}(\zeta), \quad j = 0, \dots, l-1,$$

where \tilde{C} is a suitable constant. Substituting the resulting estimate from (5.6) into (5.5), we see that

$$|f^{(N-l)}(\zeta)| \leq \text{const} \cdot d_\theta^l(\zeta).$$

Comparing this with (5.3) gives

$$(5.7) \quad |f^{(N-l)}(\zeta)| \leq \text{const} \cdot \tau_\theta^l(\zeta), \quad \zeta \in \mathbb{T} \setminus \text{spec } \theta,$$

where

$$\tau_\theta(\zeta) := \min(d_\theta(\zeta), 1/|\theta'(\zeta)|).$$

In conjunction with Lemma 3.7, this shows that the products $f^{(N-l)}\theta^{(l)}$ appearing on the right-hand side of (5.1) are all essentially bounded on \mathbb{T} . Therefore, $f\theta$ is indeed in H_N^∞ , as desired.

Finally, proceeding by induction, we use the already established implication

$$(5.8) \quad (\text{iv.3}) \implies f\theta \in H_N^\infty$$

to deduce that (iv.3) actually implies (ii.3). To this end, we apply (5.8) with f successively replaced by $f\theta, f\theta^2$, etc. And since (ii.3) trivially implies (i.3), we are done.

§6. PROOF OF THEOREM 1.4

(a) Let $f \in \mathcal{J}_k$, where $n/2 < k \leq n$, and set $g := fB^k$. Then we have $g \in A^\alpha$ and

$$(6.1) \quad g(z_j) = g'(z_j) = \dots = g^{(k-1)}(z_j) = 0, \quad j \in \mathbb{N}.$$

Inequalities (2.1), applied with $z = z_l$ and $w = z_j$ ($j \neq l$), yield

$$\left| g^{(s)}(z_l) - \sum_{m=s}^n \frac{g^{(m)}(z_j)}{(m-s)!} (z_l - z_j)^{m-s} \right| \leq C |z_l - z_j|^{\alpha-s}$$

for every integer s in $[0, n]$. Now, for $s = 0, \dots, n-k$, this further reduces to

$$(6.2) \quad \left| \sum_{m=k}^n \frac{g^{(m)}(z_j)}{(m-s)!} (z_l - z_j)^{m-s} \right| \leq C |z_l - z_j|^{\alpha-s},$$

in view of (6.1). (Note that the current values of s do not exceed $k-1$, thanks to the assumption $k > n/2$.) Multiplying both sides of (6.2) by $|z_l - z_j|^{s-k}$, we get

$$(6.3) \quad \left| \sum_{m=k}^n \frac{g^{(m)}(z_j)}{(m-s)!} (z_l - z_j)^{m-k} \right| \leq C |z_l - z_j|^{\alpha-k} \quad (s = 0, \dots, n-k).$$

Next, keeping j and l fixed (with $j \neq l$), we write

$$(6.4) \quad \sum_{m=k}^n \frac{g^{(m)}(z_j)}{(m-s)!} (z_l - z_j)^{m-k} =: R_s \quad (s = 0, \dots, n-k).$$

We shall view (6.4) as a system of $n - k + 1$ linear equations with the “unknowns”

$$g^{(k)}(z_j), \quad g^{(k+1)}(z_j) \cdot (z_l - z_j), \quad \dots, \quad g^{(n)}(z_j) \cdot (z_l - z_j)^{n-k}$$

and with “constant terms” R_0, \dots, R_{n-k} . The coefficient matrix that arises, say \mathcal{M} , is nonsingular. Indeed,

$$\mathcal{M} = \mathcal{M}(k, n) := \begin{pmatrix} \frac{1}{k!} & \frac{1}{(k+1)!} & \cdots & \frac{1}{n!} \\ \frac{1}{(k-1)!} & \frac{1}{k!} & \cdots & \frac{1}{(n-1)!} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(2k-n)!} & \frac{1}{(2k-n+1)!} & \cdots & \frac{1}{k!} \end{pmatrix},$$

and one verifies that $\det \mathcal{M} \neq 0$ (e.g., computing the determinant explicitly) by induction on $n - k$. By Cramer’s rule,

$$(6.5) \quad g^{(k)}(z_j) = \frac{\det \mathcal{M}_1}{\det \mathcal{M}},$$

where \mathcal{M}_1 is obtained from \mathcal{M} by replacing its first column with $(R_0, \dots, R_{n-k})^T$.

Since

$$|R_s| \leq \text{const} \cdot |z_l - z_j|^{\alpha-k} \quad (s = 0, \dots, n - k),$$

as ensured by (6.3), while the entries of \mathcal{M} depend only on k and n , it follows that $\det \mathcal{M}_1$ admits a similar estimate. (To see why, expand the determinant along the first column.) Consequently, by (6.5), we also have

$$|g^{(k)}(z_j)| \leq \text{const} \cdot |z_l - z_j|^{\alpha-k},$$

with a constant not depending on j and l . Taking the infimum over $l \in \mathbb{N} \setminus \{j\}$ and noting that

$$g^{(k)}(z_j) = f(z_j) \cdot (B^k)^{(k)}(z_j)$$

(by Lemma 3.2), we deduce that

$$|f(z_j)| \cdot |(B^k)^{(k)}(z_j)| \leq \text{const} \cdot d_j^{\alpha-k}, \quad j = 1, 2, \dots$$

Finally, we combine this with the estimate

$$|(B^k)^{(k)}(z_j)| \geq \text{const} \cdot (1 - |z_j|)^{-k}$$

(from Lemma 3.3) to arrive at (1.10).

(b) Since $\text{clos}\{z_j\}$ is an A^α -interpolating set, Lemma 3.1 enables us to solve the interpolation problem

$$(6.6) \quad g^{(s)}(z_j) = 0 \quad (0 \leq s \leq n, \quad s \neq k), \quad g^{(k)}(z_j) = d_j^{\alpha-k}, \quad j \in \mathbb{N},$$

with a function $g \in A^\alpha$. This g is therefore divisible by B^k , so that $g = fB^k$ with $f \in H^\infty$. We know that f is actually in A^α and hence in \mathcal{J}_k . Furthermore, by Lemma 3.2, the condition $g^{(k)}(z_j) = d_j^{\alpha-k}$ from (6.6) takes the form

$$f(z_j) \cdot (B^k)^{(k)}(z_j) = d_j^{\alpha-k}.$$

This, in conjunction with Lemma 3.3, implies that

$$|f(z_j)| \asymp d_j^{\alpha-k} (1 - |z_j|)^k,$$

as desired.

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Received 14/JAN/2009

Originally published in English