MINIMAL PROBLEMS IN AIRPLANE PERFORMANCE*

BY

BORIS GARFINKEL

Ballistic Research Laboratory, Aberdeen Proving Ground

Abstract. We develop here the theory of operating an airplane so as to minimize an arbitrary function of the end-values of the generalized coordinates. A propeller-driven airplane is treated as a particle in equilibrium, subject to the forces of drag, lift, thrust, and gravity. We assume that the specific fuel consumption is a function of the power only, and that the available power is independent of the altitude.

The problem is shown to be of the Bolza type in the Calculus of Variations, with the complications arising from the presence of inequalities, discontinuities, and variables whose derivatives do not enter the problem explicitly. The Euler-Lagrange equations are derived and discussed.

Notation. A subscript will sometimes denote an index, at other times the argument of partial differentiation. A superscript dot will indicate differentiation with respect to the parameter t. The Summation Convention will be observed. In referring to equations decimals may be used; e.g. (59.4) is the fourth equation of the set (59). δ_{ij} is the Kronecker delta.

1. Introduction. In the absence of lateral wind we shall treat the airplane as a point, P, in a four-space, specified by the coordinates (T, X, Y, m). Here T is the time, X the length of arc of a great circle of the earth, Y the altitude; and m the mass of the airplane. The end-conditions prescribe the point of departure, P_1, as $T_1 = 0$, $X_1 = 0$, $Y_1 = 0$, $m_1 = m(0)$, and some, but not all, of the coordinates of the destination, P_2. We seek to minimize some prescribed function, G, of the remaining coordinates of P_2. The following types of problem are of obvious practical significance:

1) $G = T_2$, minimizing the time of flight,
2) $G = -X_2$, maximizing the range,
3) $G = -Y_2$, maximizing the altitude,
4) $G = -m_2$, minimizing the fuel consumption,
5) $G = -X_2(m_2 - m_{\min})$, maximizing the "transport",
6) $G = -(a + m_2)/T_2$, maxmizing the "profit".

Regardless of the nature of G, and the end-conditions, all problems lead to the same set of the Euler-Lagrange equations. We shall derive the latter from the physical laws governing the motion of an airplane.

2. The physics of the problem. The dynamical laws governing the steady motion of an airplane are

$$\tau = D + mg \sin \phi, \qquad L = mg \cos \phi \tag{1}$$

where τ, D, L are the thrust, the drag, and the lift, respectively, m the mass, g the accelera-

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tion of gravity, and ϕ the inclination of the trajectory to the horizon. The drag coefficient, C_D and the lift coefficient, C_L, are defined by

$$\left.\begin{aligned} D &= \tfrac{1}{2}C_D S\rho v^2, \\ L &= \tfrac{1}{2}C_L S\rho v^2 \end{aligned}\right\}, \tag{2}$$

S being the "characteristic" area, ρ the density of the air, and v the air speed. C_D and C_L are connected by a parabolic relation[1]

$$C_D = A + C_L^2/B, \tag{3}$$

A and B being a pair of constants.

The power delivered by the propeller is

$$N\epsilon\Pi = \tau v, \tag{4}$$

where N is the number of engines, ϵ the "propulsive" efficiency, and Π the power developed per engine. ϵ is assumed to be an empirical function of the relative air density, σ,

$$\epsilon = \epsilon(\sigma). \tag{5}$$

The specific fuel consumption, C, defined by

$$\frac{dm}{dT} = -NC\Pi, \tag{6}$$

will be assumed to be an empirical function of Π only;

$$C = C(\Pi). \tag{7}$$

We may impose the requirement that the lean fuel mixture be used if $\Pi < \Pi^*$, and the rich mixture if $\Pi > \Pi^*$. At the transition point, Π^*, the function $C(\Pi)$, generally, has a discontinuity.

The distribution of air density will be assumed to obey the exponential law

$$\sigma = \rho/\rho_0 = e^{-Y/\beta}, \tag{8}$$

β being a constant. The effect of wind and the effect of cowl-flaps have been treated in a separate paper, and will not be considered here. A zero wind implies

$$\frac{dX}{dT} = v \cos\phi. \tag{9}$$

The variables Y and Π are bounded by the inequalities:

$$Y \geq 0, \qquad \Pi_{\min} \leq \Pi \leq \Pi_{\max}. \tag{10}$$

The following simplifying assumptions are made:

1) The use of a supercharger makes the range of available power independent of the mode of operation.

2) The trajectory has a gentle slope: $\cos\phi \simeq 1$.

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[1]R. von Mises, *Theory of Flight*, first edition, 142.

3. Choice of variables. We define dimensionless quantities ω, θ, π, c, E:

$$-\omega = \log(m/m_0), \qquad \theta = \log[C_L/(AB)^{1/2}],$$
$$\pi = \Pi/\Pi_0, \qquad c = C/C_0, \qquad E = \epsilon/\epsilon_0. \tag{11}$$

Here m_0 is the initial mass, θ is related to the angle of attack, ϵ_0 is the value of ϵ at $Y = 0$, and (Π_0, C_0) is a pair of values satisfying (7). If C has a single minimum, as is generally the case, we take

$$C_0 = C_{\min}. \tag{12}$$

It is convenient to use the logarithmic variables ξ, η, and the derived variables η', g_0 defined by

$$\xi = \log \pi, \qquad \eta = \log c\pi,$$
$$\eta' = d\eta/d\xi, \qquad (1 - 2g_0)\eta' = 1. \tag{13}$$

We note that at $\pi = 1$, or $\xi = 0$

$$c = 1, \qquad dc/d\pi = 0, \qquad d^2c/d\pi^2 > 0,$$

$$\eta' = 1, \qquad g_0 = 0$$

and

$$\eta' > 0, \qquad g_0 < 1/2 \qquad \text{for all } \xi. \tag{14}$$

The latter follows from the experimental fact that $|\, dm/dT \,|$ in (6) is an increasing function of π.

Dimensionless t, x, y are defined by introducing the scale factors β_0, β_1, β_2:

$$T = \beta_0 t, \qquad X = \beta_1 x, \qquad Y = \beta_2 y. \tag{15}$$

It is convenient to choose

$$\beta_0 = \frac{m_0 g}{N\epsilon_0\Pi_0}\beta,$$

$$\beta_1 = \frac{1}{2}\left(\frac{B}{A}\right)^{1/2}\beta, \tag{16}$$

$$\beta_2 = \beta.$$

Two dimensionless parameters represent the aerodynamic characteristics and the engine characteristics of the airplane:

$$a = gC_0\beta/\epsilon_0$$
$$b = (2m_0g)^{3/2}(S\rho_0)^{-1/2}A^{1/4}B^{-3/4}/N\epsilon_0\Pi_0. \tag{17}$$

Now the system of 11 equations (1-9) in 13 variables X, Y, m, v, Π, C, ϵ, σ, τ, D, L, C_D, C_L reduces to a system of three equations,

$$E\pi e^{\omega} = \dot{x} \cosh \theta + \dot{y},$$

$$\dot{x} = be^{(-\omega+y-\theta)/2}, \tag{18}$$

$$\dot{\omega} = ac\pi e^{\omega},$$

where $E(y) \sim 1$, and can, generally, be represented as

$$E = e^{\kappa y}, \qquad \kappa = \text{const.} \ll 1. \tag{19}$$

In the logarithmic form (18) is equivalent, in view of (13), (19), to

$$\phi_1 = \tfrac{1}{2}(-\omega + y - \theta) - \log \dot{x} + \text{const.} = 0,$$

$$\phi_2 = -\omega - \eta + \log \dot{\omega} + \text{const.} = 0, \tag{20}$$

$$\phi_3 = \log (\dot{x} \cosh \theta + \dot{y}) + \eta - \xi - \kappa y - \log \dot{\omega} = 0,$$

$\eta(\xi)$ being a known function. Thus the five generalized coordinates, y_i, where

$$y_1 = x, \qquad y_2 = y, \qquad y_3 = \omega, \qquad y_4 = \theta, \qquad y_5 = \xi, \tag{21}$$

specify the state of the airplane in terms of its position, mass, the angle of attack, and the power as functions of the time, $y_0 = t$. Three non-holonomic equations of constraint, (20), leave us with two degrees of freedom. These can be realized physically by an arbitrary choice of θ, controlled by the "elevator", and an arbitrary choice of ξ, controlled by the throttle. Among the transfinite set of the pairs of functions $(\theta(t), \xi(t))$ we seek that pair which minimizes a prescribed function $G(t_2, x_2, y_2, \omega_2)$, subject to prescribed end-conditions, and satisfying the differential equations (20).

4. Variational approach. We identify our problem with the Problem of Bolza[2] in the Calculus of Variations in the special form: "Required the arc $y_i(t)$ satisfying the equations

$$\phi_\beta(t; y_i, \dot{y}_i) = 0; \qquad i = 1, \cdots, n; \qquad \beta, j = 1, \cdots, m < n, \tag{22}$$

and the end-conditions

$$\Phi_\alpha(t_2; y_j(t_2)) = 0; \qquad \alpha = 1, \cdots, r \leq m,$$
$$t_1 = \text{const.}, \qquad y_j(t_1) = \text{const.}, \tag{23}$$

and minimizing the function $G(t_2, y_j(t_2))$".

In later sections we shall consider the effect of inequalities of the form $\psi(t, y_i) \geq 0$, and discontinuities of ϕ_β with respect to y_i.

The solution of the problem is obtained by introducing a set of variable Lagrangian multipliers, $\lambda_\beta(t)$, a set of constant multipliers, μ_α, and constructing the auxiliary functions F, Γ, J as

$$F = \lambda_\beta \phi_\beta, \qquad \Gamma = G + \mu_\alpha \Phi_\alpha,$$
$$J = \Gamma + \int_{t_1}^{t_2} F \, dt. \tag{24}$$

[2]G. A. Bliss, *Lectures in the Calculus of Variations*, 189, e.f.

By differentiation we obtain

$$dJ = [(\Gamma_t + F - \dot{y}_i F_{\dot{y}i})\, dt + (\Gamma_{y_i} + F_{\dot{y}i})\, dy_i]_{t_2} + \int_{t_1}^{t_2} \left(F_{yi} - \frac{d}{dt} F_{\dot{y}i}\right) \delta y_i\, dt, \quad (25)$$

since P_1 is fixed, and $\delta y = dy - \dot{y}\, dt$. Our problem is equivalent to that of minimizing J; i.e. satisfying

$$dJ = 0, \qquad d^2J > 0. \quad (26)$$

Thus (25) splits into n Euler-Lagrange equations,[3]

$$F_{yi} = \frac{d}{dt} F_{\dot{y}} \quad (27)$$

and $m + 1$ Transversality Conditions at t_2 ,

$$\Gamma_{y_j} + p_j = 0, \qquad j = 0, 1, \cdots, m, \quad (28)$$

where we had set

$$p_j = F_{\dot{y}j}, \qquad j = 1, \cdots, m,$$
$$p_0 = F - \dot{y}_j F_{\dot{y}j}, \qquad y_0 = t. \quad (29)$$

We shall further assume that the end-conditions at P_2 separate the variables y_j ; i.e.

$$\Phi_\alpha = y_\gamma - \text{const.} = 0, \qquad \alpha = 1, \cdots, r \leq m \quad (30)$$

γ assuming r values in the range $0 \leq r \leq m$. Then μ_α exist, and can be eliminated from (28), yielding $m + 1 - r$ equations

$$G_{y_j} + p_j = 0, \qquad j \neq \gamma, \quad (31)$$

involving only the "free" variables; i.e., such y_j as do not enter the end-conditions at t_2 .

The $n + m$ unknowns (y_i , λ_β) are determined by the system of differential equations

$$F_{yi} = \frac{d}{dt} F_{\dot{y}i}, \qquad i = 1, \cdots, n = 5,$$
$$\phi_\beta = 0, \qquad \beta = 1, \cdots, m = 3, \quad (32)$$

whose order is generally $2n + m$. In our problem, however, the order is depressed by three circumstances: 1) $\dot{y}_j$ in ϕ_β are soluble for y_i , 2) there are $n - m$ equations $F_{yk} = 0$, $k = m + 1, \cdots, n$, of (32.1), for the y_i whose derivatives are absent in F, 3) λ_m can be eliminated, since F is a linear homogeneous function of λ_β . The resulting order is $2m - 1$, so that $2m - 1$ initial constants must be furnished. m such constants are $y_j(0) = 0$; $m - 1$ additional constants may be chosen as $g_\beta(0)$, $\beta = 1, \cdots, m - 1$, where we define

$$g_\beta = \lambda_\beta / 2\lambda_m . \quad (33)$$

These parameters. $g_\beta(0)$, of our family of extremals can be determined from the conditions of the particular problem. For, there exist r end-conditions, (30), and $m + 1 - r$ Transversality Conditions, (31), connecting the $m + 1$ quantities t_2 , $\lambda_m(t_2)$, $g_\beta(0)$.

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[3] *ibidem*, 202, e.f.

5. Euler-Lagrange equations. An extremal, generally, is compounded of arcs lying in the interior of the admissible region $\psi \geq 0$, and of arcs lying in the boundary $\psi = 0$. In view of the inequalities (10), prescribed in our problem, we must distinguish arcs of the following types:

A) general case, $\pi \not\equiv$ const., $y \not\equiv 0$;
B) flight under constant power, $\pi \equiv$ const., $y \not\equiv 0$;
C) level flight, $\pi \not\equiv$ const., $y \equiv 0$.

These arcs are joined together in accord with the Corner-Condition, discussed in section 8.

In order to take into account inequalities $\psi \geq 0$ we augment F so as to include the constraints $\psi = 0$, by writing

$$\phi_4 = \begin{cases} \xi - \xi_{\min} , \\ \xi_{\max} - \xi \end{cases} \qquad \phi_5 = \tfrac{1}{2}y, \tag{34}$$

$$\lambda_4\phi_4 = 0, \qquad \lambda_5\phi_5 = 0,$$

and construct $\overline{F} = \lambda_\beta\phi_\beta$, $\beta = 1, \cdots, 5$, with the aid of (20), (33):

$$\begin{aligned} \overline{F} = \{(g_1 + g_5 - \kappa)y - (g_1 + 2g_2)\omega - g_1(\theta + 2 \log \dot{x}) + \\ \cdot \log (\dot{x} \cosh \theta + \dot{y}) + (1 - 2g_2)(\eta - \log \dot{\omega}) - (1 \mp 2g_4)\xi\} \end{aligned} \tag{35}$$

The Euler-Lagrange equation in θ is

$$g_1(\dot{x} \cosh \theta + \dot{y}) = \dot{x} \sinh \theta, \tag{36}$$

which, in vew of (18), is equivalent to

$$p = be^{(-3\omega + y)/2}/E, \qquad g_1 = \frac{p}{\pi} e^{-\theta/2} \sinh \theta \tag{37}$$

$p(\omega, y)$ being defined by (37.1). The Euler-Lagrange equations in x, y, ω, ξ can be simplified by eliminating from them $\dot{x}, \dot{y}, \dot{\omega}$ with the aid of (18), and then making use of (37) and (13). On introducing the constant

$$\alpha = \tfrac{1}{2} \log 3 = \coth^{-1} 2, \tag{38}$$

the final result can be written as

$$\begin{aligned} \lambda_3 \sinh (\theta - \alpha)/E\pi e^{\omega} &= \text{const.}, \\ \frac{d}{d\omega} \log (\lambda_3/E\pi e^{\omega}) &= \frac{E}{ac} (g_1 + g_5 - \kappa), \\ \frac{d}{dt} [\lambda_3(1 - 2g_2)/ac\pi e^{\omega}] &= \lambda_3(g_1 + 2g_2), \\ \eta'(g_0 - g_2) \pm g_4 &= 0. \end{aligned} \tag{39}$$

Moreover, since F does not contain t explicitly, there exists the integral $\bar{F} - \dot{y}_i \bar{F}_{\dot{y}_i} =$ const.; i.e.,

$$\lambda_3(g_1 - g_2) = \text{const.}, \tag{40}$$

which we shall use in place of (39.3).

The Lagrange multipliers can be eliminated from (39) as follows. First λ_3 is eliminated among (40), (39.1, 2); next g_4, g_5 are eliminated with the aid of (34). Then (39) becomes

$$\begin{aligned} g_2 &= g_1 + K \sinh(\theta - \alpha)/E\pi e^{\omega}, \\ (g_2 - g_0)(\pi - \text{const.}) &= 0, \\ y\left\{\frac{d}{d\omega} \log \sinh(\theta - \alpha) + \frac{E}{ac}(g_1 - \kappa)\right\} &= 0, \end{aligned} \tag{41}$$

K being a constant. Finally, g_1 and g_2 are eliminated from (41.1,2), with the result

$$(g_0\pi - pe^{-\theta/2} \sinh \theta - K \sinh(\theta - \alpha)/Ee^{\omega})(\pi - \text{const.}) = 0. \tag{42}$$

Equations (18) can be written, in view of (37), as

$$\begin{aligned} \frac{dt}{d\omega} &= \frac{1}{ac\pi e^{\omega}}, \\ \frac{dx}{d\omega} &= \frac{Epe^{-\theta/2}}{ac\pi}, \\ \frac{dy}{d\omega} &= \frac{E}{ac}\left(1 - \frac{pe^{-\theta/2}}{\pi} \cosh \theta\right). \end{aligned} \tag{43}$$

6. Computational procedure. The airplane is represented by three constants a, b, κ and the function $c(\pi)$. These automatically define the functions

$$c = c(\pi), \qquad g_0 = g_0(\pi), \qquad E = E(y), \tag{44}$$

in view of (13), (19). We recall that g_1, p are known functions of y_i:

$$p = be^{-(3\omega + \nu)/2}/E(y), \qquad g_1 = \frac{p}{\pi} e^{-\theta/2} \sinh \theta. \tag{37}$$

Thus (41.3), (42), (43) is a definitive system of five equations, and determines t, x, y, θ, π as functions of ω, provided five initial constants are furnished. Three such constants are $t(0) = 0$, $x(0) = 0$, $y(0) = 0$, prescribed by the end-conditions; two additional constants may be chosen as $\theta(0)$ and K, leading to a two-parameter family of extremals

$$y_i = y_i(t; \theta(0), K); \qquad i = 1, \cdots, 5$$

Since t, x do not enter the system explicitly, the corresponding equations (43.1, 2) can be split off, and done by quadrature after the solution for y, θ, π has been obtained.

Thus in the general case, A, $\pi \not\equiv$ const., $y \not\equiv 0$, our system of equations consists of (37), (44), and

$$\frac{dy}{d\omega} = \frac{E}{ac}\left(1 - \frac{pe^{-\theta/2}}{\pi} \cosh \theta\right),$$

$$\frac{d}{d\omega} \log \sinh (\theta - \alpha) + \frac{E}{ac}(g_1 - \kappa) = 0, \tag{45}$$

$$g_0\pi = pe^{-\theta/2} \sinh \theta + K \sinh (\theta - \alpha)/Ee^{\omega}.$$

A choice of $\theta(0)$ and K determines in succession $\pi(0)$ and $g_1(0)$ from (45.3) and (37), so that the integration may proceed.

In case B (45.3) is replaced by $\pi \equiv$ const., and K can be discarded. In case C, $y \equiv 0$, we have $g_1 = \tanh \theta$, and

$$p = be^{-3\omega/2},$$

$$\pi = pe^{-\theta/2} \cosh \theta, \tag{46}$$

$$g_0\pi = pe^{-\theta/2} \sinh \theta + Ke^{-\omega} \sinh (\theta - \alpha).$$

Here a choice of K determines θ and π when y and ω are known. In both the special cases, therefore, the number of initial parameters is reduced to one.

The parameters $\theta(0)$, K for a particular problem can be determined from the end-conditions and the Transversality Condition at t_2. In order to make use of the latter it is necessary to consider first the Sufficiency Condition and the Corner Condition.

7. Sufficiency condition. In the notation of Bliss the sufficiency conditions for a weak relative minimum are I, III′, IV′. Condition I is met by the solution of the Euler-Lagrange equations. The strengthened Legendre-Clebsch Condition,[4] III′, is equivalent to

$$F_{z_\lambda z_\mu}\, \delta z_\lambda\, \delta z_\mu > 0, \tag{47}$$

where $z = (\dot{y}_j, y_k)$; $j = 1, \cdots, m$, $k = m + 1, \cdots, n$; i.e. the set of the highest derivatives entering F, and δz satisfy the differentiated equations of constraint,

$$\phi_{\beta z_i}\, \delta z_i = 0, \qquad \beta = 1, \cdots, m; i = 1, \cdots, n. \tag{48}$$

Applying (47) and (48) to (35) and (20), respectively, and eliminating $\delta\dot{y}_j$, we are led to the requirement that

$$\lambda_3\left\{\frac{pe^{\theta/2}}{4\pi}(1 + 3e^{-2\theta})\, \delta\theta^2 + 2\eta' \frac{d}{d\pi}(g_0\pi)\, \delta\xi^2\right\} > 0 \tag{49}$$

be a positive-definite quadratic form. Since p, π, η' are positive, we deduce

$$\left.\begin{aligned} \lambda_3 &> 0, \\ \frac{d}{d\pi}(g_0\pi) &> 0 \end{aligned}\right\}. \tag{50}$$

(50.1) is equivalent to

$$\lambda_3(t_2) > 0. \tag{51}$$

[4] *ibidem*, 235.

For, y_i and $\dot{y}_i$ are bounded by physical considerations, and g_1, g_2 are bounded in view of (37) and (41.1). Hence, (39.3) implies that λ_3 cannot change its sign.

(50.2) restricts the range of π. Generally, $g_0\pi$ has a single minimum at some value $\bar{\pi} < 1$. Then we construct a new lower bound, $\pi'_{\min}$ as

$$\pi'_{\min} = \max(\pi_{\min}, \bar{\pi}). \tag{52}$$

As a numerical example, let us take

$$\begin{aligned} c &= 4/(\pi + 1)(3 - \pi), && \text{if} \quad \pi < \pi^* = 1.5 \quad \text{(lean mixture)}, \\ c &= 0.5(\pi + 1), && \text{if} \quad \pi > 1.5 \quad \text{(rich mixture)}, \end{aligned} \tag{53}$$

$$\pi_{\min} = 0.5, \qquad \pi_{\max} = 2.0.$$

Then

$$\begin{aligned} g_0 &= \pi(\pi - 1)/(\pi^2 + 3), & \frac{dg_0}{d\pi} &= (\pi^2 + 6\pi - 3)/(\pi^2 + 3)^2 & \text{if} \quad \pi < 1.5, \\ g_0 &= \pi/2(1 + 2\pi), & \frac{dg_0}{d\pi} &= 1/2(1 + 2\pi)^2 & \text{if} \quad \pi > 1.5, \end{aligned} \tag{54}$$

and (50.2) becomes

$$\pi^2 + 9\pi - 6 > 0$$

so that $\bar{\pi} = 0.638$, and $\pi'_{\min} = 0.638$.

Jacobi's Condition,[5] IV′ has received in our treatment only an empirical verification, in the fact that the family of extremals covers simply the admissible region of space.

8. The Corner condition. Consider an extremal compounded of the branches E_{10} and E_{02} joined at $t = t_0$. If the junction point lies in the surface $\psi(y_i) = 0$, then its coordinates satisfy

$$\psi_{y_i}\, dy_i = 0. \tag{55}$$

Now applying (25) to the two branches, respectively, and adding the results we obtain the additional term

$$\Delta J_0 = -[(\Delta p_i)\, dy_i]_{t_0}, \qquad j = 0, 1, \cdots, m, \tag{56}$$

where we define the "jump" in any function, f, as

$$\Delta f = \lim_{\epsilon \to 0} [f(t_0 + \epsilon) - f(t_0 - \epsilon)] = f_+ - f_- \tag{57}$$

If J is minimized it is necessary that $\Delta J_0 \geq 0$, subject to (55). This leads to $\Delta J_0 = 0$, and

$$\begin{aligned} \Delta p_j &= \mu \psi_{y_j}, \\ 0 &= \mu \psi_{y_k}, \end{aligned} \tag{58}$$

μ being a constant, and $p_k \equiv 0$. (58) can be regarded as an extension of the Weierstrass-

[5] *ibidem*, 258.

Erdman Corner Condition.[6] If we now impose the requirement $\Delta y_i = 0$, dictated by physical considerations, we shall have the system of equations

$$\begin{aligned} \Delta p_j &= \mu\psi_{\dot{y}j}, \qquad j = 0, \cdots, m, \\ 0 &= \mu\psi_{\dot{y}k}, \\ \Delta y_i &= 0, \\ \phi_\beta &= 0, \qquad \beta = 1, \cdots, m, \\ F_{\dot{y}k} &= 0, \qquad k = m+1, \cdots, n. \end{aligned} \tag{59}$$

The last line gives the Euler-Lagrange equations in the variables y_k.

Suppose E_{10} lies in the admissible region $\psi \geq 0$, and E_{02} lies in the boundary $\psi = 0$. If F is of class C^2 at t_0, and satisfies the Legendre-Clebsh Condition, it can be shown that the only solution of (59) is the trivial solution

$$\Delta y_i = 0, \qquad \Delta \dot{y}_j = 0, \qquad \Delta\lambda_\beta = 0, \qquad \mu = 0. \tag{60}$$

This is the, so-called, Tangency Condition, which requires that all variables and first derivatives appearing explicitly in F be continuous at the junction. Moreover, for the branch E_{02} we have

$$\psi_{\dot{y}i}\, \delta y_i \geq 0, \tag{61}$$

which, in view of (25) and $dJ \geq 0$, leads to the Convexity Condition,

$$\begin{aligned} F_{yi} - \frac{d}{dt} F_{\dot{y}i} + \lambda\psi_{yi} &= 0, \\ \lambda(t) &\leq 0. \end{aligned} \tag{62}$$

$\lambda(t)$ can be readily identified with the Lagrange multiplier of the constraint $\psi = 0$, entering the augmented function $\mathfrak{F} = F + \lambda\psi$. The point at which $\lambda = 0$ may be termed the point of "inflection".

In our problem there are two inequalities $\psi \geq 0$; (34) gives

$$\psi_1 = \phi_4, \qquad \psi_2 = \phi_5.$$

Then (62.2) requires that $\lambda_4 \leq 0$, $\lambda_5 \leq 0$. Since $\lambda_3 > 0$ by (50), we obtain from (33)

$$g_4 \leq 0, \qquad g_5 \leq 0. \tag{63}$$

Now (39.4), in view of (14), yields

$$\begin{aligned} g_0 &\geq g_2 \qquad \text{if } \pi = \pi_{\min}, \\ g_0 &= g_2 \qquad \text{if } \pi_{\min} < \pi < \pi_{\max}, \\ g_0 &\leq g_2 \qquad \text{if } \pi = \pi_{\max}, \end{aligned} \tag{63}$$

[6] *ibidem*, 12, 203.

while (39.2, 1) leads to

$$\frac{1}{ac}(g_1 - \kappa) + \frac{d}{d\omega} \log \sinh (\theta - \alpha) \geq 0 \qquad \text{if} \qquad y \equiv 0. \tag{65}$$

The latter determines the point of inflection, beyond which minimal level flight cannot proceed.

Application of the Corner Condition to the discontinuity at $\xi = \xi^*$ will be made in the Appendix.

9. Transversality condition. There are two inequalities affecting y_i at t_2: $y \geq 0$, $\omega_{max} - \omega \geq 0$. Thus the function G in (31) is to be augmented by writing

$$\begin{aligned} &\bar{G} = G + \mu_\alpha \Phi_\alpha , && \alpha = 1, 2 \\ &\Phi_1 = y, && \Phi_2 = \omega_{max} - \omega \\ &\mu_1 y = 0, && \mu_2(\omega_{max} - \omega) = 0. \end{aligned} \tag{66}$$

Then (31) becomes

$$\begin{aligned} G_t - 2\lambda_3 K \sinh (\theta - \alpha)/E\pi e^\omega &= 0, \\ G_x - \lambda_3 \sinh (\theta - \alpha)/E\pi e^\omega \sinh \alpha &= 0, \\ y(G_y + \lambda_3/E\pi e^\omega) &= 0, \\ [G_\omega - \lambda_3(1 - 2g_2)/ac\pi e^\omega](\omega_{max} - \omega) &= 0, \end{aligned} \tag{67}$$

upon elimination of μ_1, μ_2 with the aid of (66).

We shall now consider problems of the type $G = \pm y_r$, r assuming any one of the values 0, $\cdots m = 3$. Then $G_{y_i} = \pm \delta_{ri}$. Observing that a, α, ω, π, c, E are positive, we deduce from (67) and (51) the following Table:

Transversality Condition at t_2

Free Variable	If $G = y_i$	If $G = -y_i$	If $G_{y_i} = 0$
$y_0 = t$	$K(\theta - \alpha) > 0$	$K(\theta - \alpha) < 0$	$K(\theta - \alpha) = 0$
$y_1 = x$	$\theta > \alpha$	$\theta < \alpha$	$\theta = \alpha$
$y_2 = y$	—	—	$y = 0$
$y_3 = \omega$	$g_2 < \frac{1}{2}$	$g_2 > \frac{1}{2}$	$g_2 = \frac{1}{2}$ or $\omega = \omega_{max}$.

In the last column the variable y_i is ignored at t_2; i.e., it is an argument neither of G nor of the end-conditions at t_2.

If the entire extremal is of the type B, $\pi \equiv$ const., then dt and $d\omega$ in (25) are no longer independent, but connected by $d\omega = ac\pi e^{\omega}\, dt$. As a result, the first and last rows of the Table coalesce into

$$\left.\begin{aligned} y_0 &= t \\ y_3 &= \omega \end{aligned}\right\} \qquad g_1 < \tfrac{1}{2}, \qquad\qquad g_1 > \tfrac{1}{2}, \qquad\qquad g_1 = \tfrac{1}{2} \text{ or } \omega = \omega_{\max}. \tag{68}$$

In applying the Table we make use of the following Remarks concerning minimal curves:

1) "θ and g_1 have like signs"
2) "$K(\theta - \alpha) = 0$ implies $g_1 = g_2$, and conversely"
3) "$\theta \leq \alpha$ and $g_1 > \frac{1}{2}$ implies $\dot{y} < 0$"
4) "$g_2 \geq \frac{1}{2}$ implies $\pi = \pi_{\max}$"
5) "g_2 decreases in passing through the value $\frac{1}{2}$ provided $g_1 > -1$"
6) "The quantities λ_3, $\theta - \alpha$ and $g_1 - g_2$ cannot change their signs"

The proofs depend on (37), (41.1), (43) and (38), (64) and (14), (39.3), (39.1) and (41.1), respectively. In virtue of the Remarks 6 and 2 the first two lines of the Table hold for all $t < t_2$.

We shall next illustrate the use of the Table by considering a few special cases.

10. Special cases. 1) *Problems not involving the time.*

Here $G_t = 0$, so that $g_1 = g_2$, and $K(\theta - \alpha) = 0$ for all t, there being but a single parameter, say $\pi(0)$. In the case A (45.3) becomes

$$g_0\pi = pe^{-\theta/2} \sinh \theta. \tag{69}$$

In the case C, $y \equiv 0$, (46) reduces to

$$\begin{aligned} \pi &= be^{-(3\omega+\theta)/2} \cosh \theta, \\ g_0 &= \tanh \theta. \end{aligned} \tag{70}$$

Here

$$g_0 = g_1 = g_2, \tag{71}$$

so that the number of parameters is reduced to zero; the family shrinks to a single curve, which can be calculated by quadratures. For, eliminating θ in (70) and (43) gives

$$\begin{aligned} \omega &= \frac{1}{6} \log \pi^4(1 + g_0)^3(1 - g_0) + \log b^{2/3} \equiv \Omega(\pi) + \log b^{2/3}, \\ t &= b^{-2/3}\tau(\pi) + k_1, \\ x &= \mathrm{X}(\pi) + k_2. \end{aligned} \tag{72}$$

τ, X are functionals of $c(\pi)$, and can be tabulated as functions of π. k_1, k_2 are determined from the initial conditions.

Let us next consider the problem of maximizing the range, x, when t, y, ω are ignored. Then $G = -x$, and the Table gives

$$\theta < \alpha, \qquad K = g_1 - g_2 = 0 \qquad \text{for all } t,$$

$$y = 0, \qquad \omega = \omega_{\max} \qquad \text{or} \qquad g_2 = \tfrac{1}{2} \qquad \text{at } t_2.$$

The alternative $g_2 = g_1 = \frac{1}{2}$ at t_2 must be ruled out if $y(0) = 0$. For, then $g_2 = g_1 > \frac{1}{2}$ at $t < t_2$ by Remark 5, and hence $\dot{y} < 0$ by Remark 3, so that the inequality $y \geq 0$ is violated. Thus our solution is the curve for which $y = 0$ when $\omega = \omega_{\max}$.

2) *Problems not involving the range.*

Here $G_x = 0$, so that $\theta = \alpha$ for all t. Then (45) is to be replaced by

$$\frac{dy}{d\omega} = \frac{E}{ac}\left[1 - 2(3)^{-3/4}\frac{p}{\pi}\right],$$

$$(\pi - \pi_{\text{extr}})(g_0\pi - 3^{-3/4}p - K'/Ee^{\omega}) = 0, \tag{73}$$

where the constant $K' = K \sinh(\theta - \alpha)$ need not vanish.

Consider the problem of the most economical climb to a given altitude. We set $G = \omega$, $G_t = G_x = 0$, and note that $\theta = \alpha$, $K' = 0$ for all t. Hence (73.2) becomes

$$(g_0\pi - 3^{-3/4}p)(\pi - \pi_{\text{extr}}) = 0. \tag{74}$$

The first factor has only one root, π_1, since π is a single-valued function of $g_0\pi$ by (50.2). The solution of (74), subject to (64) is the point in the interval $(\pi_{\min}, \pi_{\max})$ nearest to π_1. Having determined π, we proceed with the integration of (73.1), obtaining the unique solution of the problem.

Similarly, the problem of the fastest climb is solved by setting $G = t$, $G_x = G_\omega = 0$, and noting that $\theta = \alpha$, $K' > 0$ for all t, and $g_2 = \frac{1}{2}$ at t_2. Then from the Remarks 1, 5, 4 and the equation (68) it follows that for all t we must have $\pi = \pi_{\max}$, $g_1 < \frac{1}{2}$.

11. Summary. The Euler-Lagrange equations, derived in section 5, can be readily integrated by the procedure of section 6, leading to a two-parameter family of curves. Such a calculation was carried out, for a typical airplane, by the Differential Analyzer at the Ballistic Research Laboratories in 1948, using ten integrators. The complications arising from the presence of inequalities and a discontinuity were easily resolved by applying the Corner Condition, discussed in sections 8, 12.

The family of curves so constructed represents the totality of solutions of all possible problems under consideration. The process of selecting from the family the curve that solves a particular problem is carried out by scanning the end-conditions and the Transversality Condition at t_2. The latter is discussed and illustrated in sections 9, 10. The Sufficiency Conditions for a weak relative minimum can be, generally, satisfied, as shown in section 7.

APPENDIX

12. Effect of discontinuities. Applying the results of section 8 we consider the case in which E_{10} lies in the region $\psi \geq 0$ where $\psi = 0$ is a surface of discontinuity of F with respect to the arguments of ψ. Two possibilities arise:

a) ψ is not a function of y_k, so that $\psi_{y_k} = 0$, and (59.2) drops out. Then the system (59) reduces to $n + 2m + 2$ equations among the $n + 2m + 2$ unknowns $(\Delta y_i, \Delta \dot{y}_j, \Delta\lambda_\beta, \mu)$, and may be expected to have a non-trivial solution at a given $t = t_0$.

b) ψ is a function of y_k, so that (59.2) contributes $\mu = 0$, over-determining the system by one equation. Therefore, at t_0 there exists, generally, only the trivial solution (60). Then the extremal must enter the surface $\psi = 0$ by a continuous transition, and remain in $\psi = 0$ until it reaches some point $t = t_0^*$, where a non-trivial solution exists. At that point there occurs a discontinuous transition, which could be termed "delayed refraction", or "reflection", according as $\psi = 0$ is crossed or not.

In our problem a discontinuity occurs at the surface

$$\psi = \xi^* - \xi = 0$$

Since $\dot{\xi}$ does not appear explicitly in F, we are dealing here with a transition of type b. The system of equations (59), in view of (75) and (35), reduces to

$$\begin{gathered} \Delta\theta = \Delta(g_1\pi) = \Delta(g_2\pi) = \Delta[(1 - 2g_2)/c] = 0, \\ (g_2 - g_0)(\pi - \pi_{\text{extr}}) = 0. \end{gathered} \tag{76}$$

From the last three equations of (76) we deduce

$$\begin{gathered} \left\{(\pi_-)\frac{c_+ - c_-}{\pi_+ - \pi_-} - \left(\pi\frac{dc}{d\pi}\right)_+\right\}(\pi - \pi_{\text{extr}}) = 0, \\ 2(g_2)_+ = (\pi_-)\frac{c_+ - c_-}{(c\pi)_+ - (c\pi)_-}, \\ (g_2\pi)_+ = (g_2\pi)_- . \end{gathered} \tag{77}$$

which, in conjunction with (64), determines π_+, $(g_2)_+$, $(g_2)_-$. These three values are invariant, being dependent only on the function $c(\pi)$.

As an example, let us suppose that in (53) the discontinuity at π^* is approached from the "lean" branch of the curve. Then $\pi_- = \pi^* = 1.5$, $c_- = 1.067$, and the first factor of (77.1), equated to zero, is

$$1.5\,\frac{0.5(\pi + 1) - 1.067}{\pi - 1.5} - 0.5\pi = 0, \qquad \text{if} \qquad \pi > 1.5,$$

whose root, $\pi = 2.24$, falls outside the admissible range of π. The second factor of (77.1) leads to two solutions:

1) $\pi_+ = \pi_{\max} = 2, \quad c_+ = 1.5, \quad (g_2)_+ = 0.232, \quad (g_0)_+ = 0.200,$

$\pi_+ = \pi_{\min} = 0.5, \quad c_+ = 1.067, \quad (g_2)_+ = 0, \quad (g_0)_+ = 0.077,$

the values of c and g_0 being obtained from (53) and (54). Since only the first one of these solutions satisfies (64), we take

$$\pi_+ = 2, \qquad (g_2)_+ = 0.232, \qquad (g_2)_- = 0.309.$$

After entering the surface $\pi = \pi^* = 1.5$ (lean), the extremal remains there until g_2 reaches the value $(g_2)_-$. At this point, which can be observed by means of (40.1) during the computation, a delayed refraction occurs, π jumping to $\pi_+ = 2$.