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A Mathematician comes of age

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For Charles Paine,
a good friend and a great teacher.
He was the first to give me a
glimpse of mathematical maturity.

Contents

Preface

The process of learning mathematics, more precisely of learning to be a mathematician, is a long and exacting one. It requires special discipline, and peculiar determination. It also requires a certain level of intelligence, but we will see in the discussions of the present book that intelligence is *not* the primary or determining factor.

As with any serious task, one should not embark on it unless one fully realizes what one is getting into. The purpose of this book is to explore what the task entails, who should engage in it, and what the rewards are.

The centerpiece of the mathematical education of any student is the intellectual development of that student. In grade school the child learns arithmetic and other basic mathematical operations. In middle school and high school there begins an exposure to algebra and other more abstract mathematical ideas. Geometry, trigonometry, and the theory of functions (that is, what is a function, and what does it do, and how do we manipulate functions?) follow in good order.

In today's world, however, the American K–12 student passing through this standard curriculum gets little or no exposure to rigor or to the concept of proof. Sophisticated problem-solving and analytical skills are not developed. As a result, the students that we have in our freshman calculus classes do not know what a proof is or what serious problem-solving is. Their problem-solving skills are nascent at best. They have rarely seen a proof, and are not equipped to create one.

If a student is to be a mathematics major and to become a practicing mathematician, then that student must become familiar with the traditional notions of mathematical rigor. This demand means that the tyro must learn about logic, set theory, axiomatics, the construction of the number systems, and proofs. The student must be able to read and evaluate proofs, but also must move on to being able to *create* proofs.

It is a considerable leap to develop from the textbook problem-solving state of mind so typical of lower-division mathematics courses to the theoretical, analytical, definition-theorem-proof state of mind that is typical of real analysis and abstract algebra (and beyond). Many colleges and universities now have a *transitions course* to help effect this intellectual change (see [DAW] for one of the most innovative books for such a course, and [KRA3] for another). Teaching such a course is both a pleasure and a challenge; for one must determine how to get students over this hump. How does one teach a student to put down the old iPhone and think hard about a nontrivial mathematical proof?

The milieu described in the last paragraph raises the question of *mathematical maturity*. Every mathematician grows up hearing, at least in conversation, about mathematical maturity. It does not appear in the dictionary, or in a standard reference work. But it exists. It is an idea that most mathematicians accept and profess to understand. We frequently make statements like, "My course on elliptic functions requires a certain degree of mathematical maturity." or "The goal of our mathematics major is to develop students who have some mathematical maturity." We know what we mean when we say these things, but we would be hard put to define our terms precisely.

It is curious that other disciplines do not speak of "maturity" as we do. One does not hear of "English literature maturity," or "chemistry maturity," or even of "physics maturity". Other disciplines do not have the strict vertical structure of mathematics (perhaps "tree-like structure" would be more accurate), so their values and their vocabulary are bound to be different. In history, literature, and philosophy there is nothing to prove. In chemistry, physics, and biology, the exernal world is the arbiter of truth. In mathematics we must rely on our own minds to determine the truth. There is no other judge. That is what sets mathematics apart.

Mathematical maturity consists of the ability to:

- handle increasingly abstract ideas
- generalize from specific examples to broad concepts
- work out concrete examples
- master mathematical notation
- communicate mathematical concepts
- formulate problems and reduce difficult problems to simpler ones
- analyze what is required to solve a problem
- recognize a valid proof and detect incorrect reasoning
- recognize mathematical patterns
- work with analytical, algebraic, and geometrical concepts

- move from the intuitive to the rigorous
- learn from mistakes
- construct proofs, often by pursuing incorrect paths and adjusting the plan of attack accordingly
- use approximate truths to find a path to a genuine truth

It requires a talented teacher, and also a good deal of drive and discipline on the part of the student, to achieve the goal of attaining mathematical maturity. How can we inspire our students to follow this path (the path that, presumably, we have all successfully followed)? How can we write texts and construct courses that will facilitate this transition?

These are the questions that we intend to tackle in the present book: What is mathematical maturity? How does the idea of mathematical maturity set us apart from other intellectual disciplines? How can we identify students who have mathematical maturity, or who are achieving mathematical maturity? How can we aid in the process?

You cannot succeed at anything in life unless you know what it is you are trying to achieve. In the case of being a successful and effective mathematics instructor, the nub of the matter is to help your students to achieve mathematical maturity. Any college teacher who takes the task seriously must address this issue, and find answers that work—consistent with that teacher's style of teaching.

This brief text will explore all aspects of the mathematical maturity question and endeavor to present some answers. A novice teacher will want to give careful thought to the issues presented here, and will want to internalize those issues as part of becoming an effective mathematics instructor. Senior faculty will also find ideas of interest here. Our presentation should cause even seasoned veterans to rethink what they do, and how they approach the teaching game. Resetting and adjusting our goals is part of development as a scholar.

A lot has been said about mathematical maturity in the context of informal coffee-table discussions. Not enough has been said about it in a rigorous, scholarly fashion. This book will be a first attempt to fill that void.

— SGK
St. Louis, Missouri

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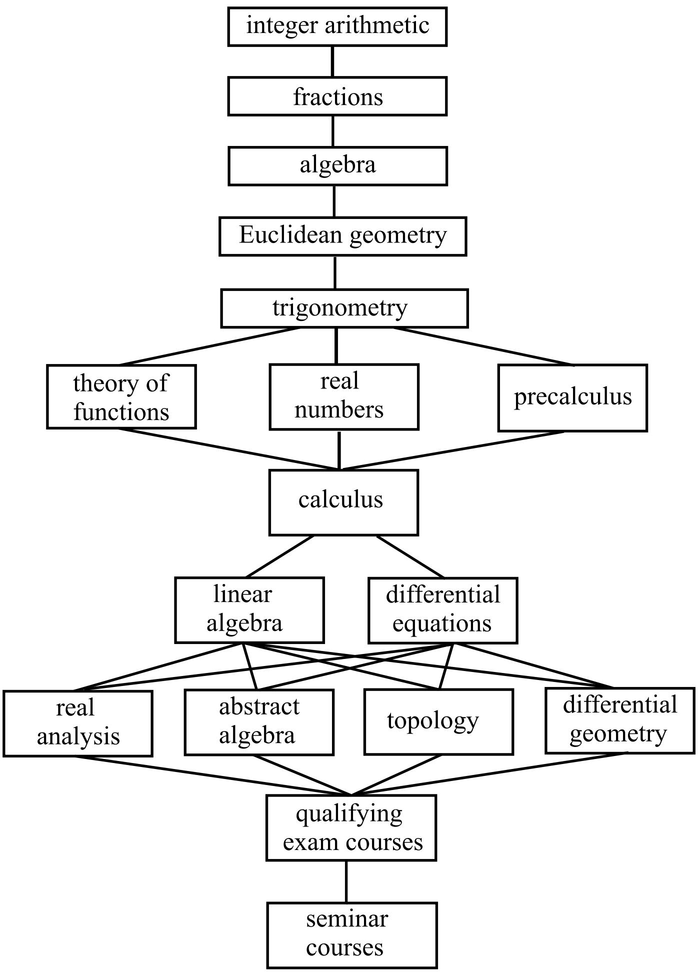
I would like to thank Harold Boas, John P. D'Angelo, Deborah K. Nelson, Robert Palais, Hung-Hsi Wu, and Doron Zeilberger for sharing their thoughts on mathematical maturity. Robert Burckel, David Collins, Roger Cooke, Jerry Folland, Jeremy Gray, Marvin J. Greenberg, Mark Saul, and James Walker read the entire manuscript with particular care, and contributed many incisive remarks.

As always, Don Albers was a supportive and vigorous editor who helped to bring the project along. He engaged an especially talented copy editor who helped me to sharpen and polish the prose. It is always a pleasure to work with Carol Baxter, Beverly Joy Ruedi and the other MAA book people. My thanks to all.

The Tree of Mathematical Maturity



Etymology of the Word "Maturity"

The word "maturity" is derived from the Old French word *maturite* and from the Latin words *maturitas* (ripeness) and *maturus* (early, speedy, ripe)

Cognate words are the Dutch *matse*, English *matzo*, French *maturité*, German *Matura*, Italian *maturità*, Yiddish *matse*.

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About the Author

Steven G. Krantz was born in San Francisco, California in 1951. He received the B.A. degree from the University of California at Santa Cruz in 1971 and the Ph.D. from Princeton University in 1974.

Krantz has taught at UCLA, Penn State, Princeton University, and Washington University in St. Louis. He served as Chair of the latter department for five years.

Krantz has published more than 60 books and more than 160 scholarly papers. He is the recipient of the Chauvenet Prize and the Beckenbach Book Award of the MAA. He has received the UCLA Alumni Foundation Distinguished Teaching Award and the Kemper Award. He has directed 18 Ph.D. theses and 9 Masters theses.

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This book is about the concept of mathematical maturity. Mathematical maturity is central to a mathematics education. The goal of a mathematics education is to transform the student from someone who treats mathematical ideas empirically and intuitively to someone who treats mathematical ideas analytically and can control and manipulate them effectively. Put more directly, a mathematically mature person is one who can read, analyze, and evaluate proofs. And, most significantly, he/she is one who can create proofs. For this is what modern mathematics is all about: coming up with new ideas and validating them with proofs. The book provides background, data, and analysis for understanding the concept of mathematical maturity. It turns the idea of mathematical maturity from a topic for coffee-room conversation to a topic for analysis and serious consideration.

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