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Research article

On meromorphic solutions of certain differential-difference equations

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Abstract: In this article, we mainly use Nevanlinna theory to investigate some differential-difference equations. Our results about the existence and the forms of solutions for these differential-difference equations extend the previous theorems given by Wang, Xu and Tu [19].

Keywords: entire functions; differential-difference equations; value distribution; finite-order
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1. Introduction and main results

We assume that the reader is familiar with the basic notions of Nevanlinna theory (see [4, 6, 22]). Recently, a number of papers (including [1–3, 5, 7–21, 23]) have focused on solvability and existence of meromorphic solutions of difference equations or differential-difference equations in complex plane. In 2009, Liu [10] obtianed the Fermat type equation $l(z)^2 + [l(z+c) - l(z)]^2 = 1$ has a nonconstant entire solution of finite order. In 2012, Liu et al. [11] proved that $l(z)^2+l(z+c)^2 = 1$ has a transcendental entire solution of finite order. In 2018, Zhang [23] obtained the difference equations $l(z)^2 + [l(z+c) - l(z)]^2 = R(z)$ has no finite order transcendental meromorphic solutions with finitely many poles. In 2020, Wang et al. [18] further discussed the existence and the forms of the solutions for some differential-difference equations, they obtained

Theorem A. Let c be a nonzero constant, $R(z)$ be a nonzero rational function, and $\alpha,\beta \in \mathbb{C}$ satisfy $\alpha^2 - \beta^2 \neq 1$. Then the following difference equation of Fermat-type

$$l(z)^2 + [\alpha l(z+c) - \beta l(z)]^2 = R(z),$$

has no finite order transcendental meromorphic solutions with finitely many poles.

Theorem B. Let $c(\neq 0)$, $\alpha(\neq 0)$, $\beta \in \mathbb{C}$, and $P(z)$, $Q(z)$ be nonzero polynomials satisfying one of two following cases:

(i) $deg_z P(z) \geq 1$, $deg_z Q(z) \geq 1$;

(ii) $P(z)$, $Q(z)$ are two constants and $P^2(\alpha^2-\beta^2)\neq 1$. Then the following Fermat-type difference equation

$$l(z)^2+P^2(z)[\alpha l(z+c)-\beta l(z)]^2=Q(z),$$

has no transcendental entire solutions with finite order.

For further study, we continue to discuss the existence and the forms of solutions for certain differential-difference equations with more general forms than the previous forms by Liu et al. [10, 11, 18, 23] and obtain the following results.

Theorem 1.1. Let $c_j(j=1,2,\cdots,m)$ be distinct constants, $a\in\mathbb{C}\backslash\{0\}$, $\varrho_i\in\mathbb{C}$ $(i=1,2,\cdots,m)$, $R(z)$ be a nonzero rational function, and $\sum_{i=1}^{m}\varrho_i(exp^{ac_i}+exp^{-ac_i})\neq 0$. Then the following difference equation

$$l(z)^2+[\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m)]^2=R(z) \tag{1.1}$$

has no finite order transcendental meromorphic solutions with finitely many poles.

Theorem 1.2. Let $c_j(j=1,2,\cdots,m)$ be distinct constants, $a\in\mathbb{C}\backslash\{0\}$, $\varrho_i\in\mathbb{C}$ $(i=1,2,\cdots,m)$, and $P(z)$, $Q(z)$ be nonzero polynomials satisfying one of two following cases:

(i) $deg_z P(z)\geq 1$;

(ii) P is a constant and $P^2[\sum_{i=1}^{m}\varrho_i exp^{ac_i}\sum_{i=1}^{m}\varrho_i exp^{-ac_i}]\neq 1$. Then the following difference equation

$$l(z)^2+P(z)^2[\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m)]^2=Q(z) \tag{1.2}$$

has no transcendental entire solutions with finite order.

Theorem 1.3. Let $c_j(j=1,2,\cdots,m)$ be distinct constants, $a\in\mathbb{C}\backslash\{0\}$, $\varrho_i\in\mathbb{C}$ $(i=1,2,\cdots,m)$. Let $l(z)$ be a transcendental finite order meromorphic solution of difference-differential equation

$$l'(z)^2+[\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m)]^2=R(z), \tag{1.3}$$

where $R(z)$ is a nonzero rational function. If $l(z)$ has finitely many poles, and $\sum_{j=1}^{m}c_j{}^2\varrho_j exp^{ac_j}\sum_{j=1}^{m}c_j{}^2\varrho_j exp^{-ac_j}\neq 0$, then $R(z)$ is a nonconstant polynomial with $deg_z R(z)\leqslant 2$, and $\sum_{j=1}^{m}c_j\varrho_j exp^{ac_j}\sum_{j=1}^{m}c_j\varrho_j exp^{-ac_j}=1$. Furthermore,

(i) If $R(z)$ is a nonconstant polynomial with $deg_z R(z)\leqslant 2$, and $\sum_{i=1}^{m}\varrho_i\neq 0$, then we have

$$l(z)=\frac{s_1(z)exp^{az+b}+s_2(z)exp^{-(az+b)}}{2},$$

where $R(z)=(m_1+as_1(z))(m_2-as_2(z)), a\neq 0, b\in\mathbb{C}$ and a,b,c_j,ϱ_i satisfy $i(\varrho_1 exp^{ac_1}+\cdots+\varrho_m exp^{ac_m})=a$ and $i(\varrho_1 exp^{-ac_1}+\cdots+\varrho_m exp^{-ac_m})=a$, where $s_j(z)=m_jz+n_j, m_j, n_j\in\mathbb{C}(j=1,2)$.

(ii) If $R(z)$ is a nonzero constant, and $\sum_{i=1}^{m}\varrho_i\neq 0$, then

$$l(z)=\frac{n_1 exp^{az+b}+n_2 exp^{-(az+b)}}{2},$$

where $R(z) = -a^2 n_1 n_2, a \neq 0, b \in \mathbb{C}$.

Theorem 1.4. Let $c_j (j = 1, 2, \cdots, m)$ be distinct constants, $a \in \mathbb{C} \backslash \{0\}$, $\varrho_i \in \mathbb{C}$ $(i = 1, 2, \cdots, m)$. Let $l(z)$ be a transcendental meromorphic solution of the following difference-differential equation

$$l''(z)^2 + [\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m)]^2 = R(z), \tag{1.4}$$

where $R(z)$ is a nonzero rational function.

(i) If $\sum_{i=1}^{m} \varrho_i exp^{ac_i} + \sum_{i=1}^{m} \varrho_i exp^{-ac_i} \neq 0$, then (1.4) has no finite order transcendental meromorphic solution with finitely many poles.

(ii) If $\sum_{j=1}^{m} ic_j \varrho_j exp^{ac_j} \neq 2a$, $\sum_{j=1}^{m} ic_j \varrho_j exp^{-ac_j} \neq 2a$, and (1.4) has a finite order transcendental meromorphic solution $l(z)$ with finitely many poles, then $R(z)$ is a constant. Furthermore if $\sum_{i=1}^{m} \varrho_i \neq 0$, then we have

$$l(z) = \frac{t_1 exp^{az+b} + t_2 exp^{-(az+b)}}{2},$$

where $a, b, t_1, t_2, \varrho_i, c_j$ satisfy $\sum_{i=1}^{m} \varrho_i exp^{ac_i} + \sum_{i=1}^{m} \varrho_i exp^{-ac_i} = 0$, $R(z) = a^4 t_1 t_2$, $b \in \mathbb{C}$.

2. Preliminary lemmas

The following two lemmas play an important role in the proof of our results.

Lemma 2.1. ([22]) *Suppose that $f_1, f_2, \cdots, f_n (n \geq 2)$ are meromorphic functions and $g_1, g_2, \cdots, g_n$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j exp^{g_j} \equiv 0$;

(ii) *$g_j - g_k$ are not constants for $1 \leq j < k \leq n$;*

(iii) *For $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, exp^{g_h - g_k})\} (r \to \infty, r \notin E)$, where E is a set of $r \in (0, \infty)$ with finite linear measure.*

Then $f_j \equiv 0 (j = 1, 2, \cdots, m)$.

Lemma 2.2. ([22]) *Let $l(z)$ be a meromorphic function of finite order $\rho(l)$. Write*

$$l(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots, (c_k \neq 0),$$

near $z = 0$ and let $\{a_1, a_2, \cdots\}$ and $\{b_1, b_2, \cdots\}$ be the zeros and poles of l in $\mathbb{C} \setminus \{0\}$, respectively. Then

$$l(z) = z^k exp^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ and $P_2(z)$ are the canonical products of l formed with the non-null zeros and poles of l, respectively, and $Q(z)$ is a polynomial of degree $\leqslant \rho(l)$.

3. Proof of Theorem 1.1

Suppose that (1.1) has a finite order transcendental meromorphic solution $l(z)$ with finitely many poles. Rewriting (1.1) as follows

$$\begin{aligned}&(l(z) + i(\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m)))(l(z) - \\ &i(\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m))) = R(z).\end{aligned} \tag{3.1}$$

Since $l(z)$ has finitely many poles, $R(z)$ is a nonzero rational function, then $l(z)+i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m))$ and $l(z)-i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m))$ both have finitely many poles and zeros. Together Lemma 2.2 with (3.1), we obtain that

$$l(z)+i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m)) = R_1(z)exp^{p(z)}, \tag{3.2}$$

and

$$l(z)-i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m)) = R_2(z)exp^{-p(z)}, \tag{3.3}$$

where $R_1(z), R_2(z)$ are two nonzero rational functions such that $R_1(z)R_2(z) = R(z)$, and $p(z)$ is a nonconstant polynomial. (3.2) and (3.3) imply that

$$l(z) = \frac{R_1(z)exp^{p(z)} + R_2(z)exp^{-p(z)}}{2}, \tag{3.4}$$

and

$$\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m) = \frac{R_1(z)exp^{p(z)} - R_2(z)exp^{-p(z)}}{2i}. \tag{3.5}$$

Substituting (3.4) into (3.5), we have

$$\begin{aligned}
&exp^{p(z)}(i\varrho_1 R_1(z+c_1)exp^{p(z+c_1)-p(z)} + i\varrho_2 R_1(z+c_2)exp^{p(z+c_2)-p(z)} + \cdots\\
&+i\varrho_m R_1(z+c_m)exp^{p(z+c_m)-p(z)} - R_1(z))+\\
&exp^{-p(z)}(i\varrho_1 R_2(z+c_1)exp^{p(z)-p(z+c_1)} + i\varrho_2 R_2(z+c_2)exp^{p(z)-p(z+c_2)} + \cdots\\
&+i\varrho_m R_2(z+c_m)exp^{p(z)-p(z+c_m)} + R_2(z)) = 0.
\end{aligned} \tag{3.6}$$

By Lemma 2.1 and (3.6), we have

$$\begin{aligned}
&i\varrho_1 R_1(z+c_1)exp^{p(z+c_1)-p(z)} + i\varrho_2 R_1(z+c_2)exp^{p(z+c_2)-p(z)}\\
&+\cdots+i\varrho_m R_1(z+c_m)exp^{p(z+c_m)-p(z)} - R_1(z) = 0,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&i\varrho_1 R_2(z+c_1)exp^{p(z)-p(z+c_1)} + i\varrho_2 R_2(z+c_2)exp^{p(z)-p(z+c_2)}\\
&+\cdots+i\varrho_m R_2(z+c_m)exp^{p(z)-p(z+c_m)} + R_2(z) = 0.
\end{aligned} \tag{3.8}$$

Since $R_1(z), R_2(z)$ are two nonzero rational functions and that $l(z)$ is of finite order, we obtain that $p(z)$ is a polynomial of degree one. If $deg_z p(z) \geq 2$, then we obtain that $deg_z[p(z+c_j) - p(z+c_i)] \geq 1$. Hence, we have $T(r, i\varrho_j R_j(z+c_j)) = S(r, exp^{p(z+c_i)-p(z+c_j)})$, Lemma 2.1 and (3.7) imply that $R_1(z) \equiv 0$. This is impossible. By the similar method as above, we also have $R_2(z) \equiv 0$, a contradiction. So we have $deg_z p(z) = 1$. Set $p(z) = az+b, a \neq 0, b \in \mathbb{C}$. By (3.7) and (3.8), we have

$$\begin{aligned}
&\lim_{|z|\to\infty} i(\varrho_1 \tfrac{R_1(z+c_1)}{R_1(z)}exp^{p(z+c_1)-p(z)} + \cdots + \varrho_m \tfrac{R_1(z+c_m)}{R_1(z)}exp^{p(z+c_m)-p(z)})\\
&= i(\varrho_1 exp^{ac_1} + \cdots + \varrho_m exp^{ac_m}) = 1,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{|z|\to\infty} i(\varrho_1 \tfrac{R_2(z+c_1)}{R_2(z)}exp^{p(z)-p(z+c_1)} + \cdots + \varrho_m \tfrac{R_2(z+c_m)}{R_2(z)}exp^{p(z)-p(z+c_m)})\\
&= i(\varrho_1 exp^{-ac_1} + \cdots + \varrho_m exp^{-ac_m}) = -1.
\end{aligned}$$

Thus, it yields that $\sum_{i=1}^{m} \varrho_i(exp^{ac_i}+exp^{-ac_i}) = 0$, this is a contradiction with the assumption of Theorem 1.1. Hence, Theorem 1.1 holds.

4. Proof of Theorem 1.2

If $l(z)$ is a transcendental entire solution with finite order of (1.2), then by the similar method as the proof of Theorem 1.1, we have

$$l(z) = \frac{Q_1(z)exp^{p(z)} + Q_2(z)exp^{-p(z)}}{2}, \tag{4.1}$$

and

$$\varrho_1 l(z+c_1) + \varrho_2 l(z+c_2) + \cdots + \varrho_m l(z+c_m) = \frac{Q_1(z)exp^{p(z)} - Q_2(z)exp^{-p(z)}}{2iP(z)}, \tag{4.2}$$

where $p(z)$ is a nonconstant polynomial and $Q_1(z)Q_2(z) = Q(z)$, $Q_1(z)$, $Q_2(z)$ are nonzero polynomials. Together (4.1) with (4.2), we have

$$\begin{aligned} &exp^{p(z)}(i\varrho_1 P(z)Q_1(z+c_1)exp^{p(z+c_1)-p(z)} + i\varrho_2 P(z)Q_1(z+c_2)exp^{p(z+c_2)-p(z)} \\ &+ \cdots + i\varrho_m P(z)Q_1(z+c_m)exp^{p(z+c_m)-p(z)} - Q_1(z)) + \\ &exp^{-p(z)}(i\varrho_1 P(z)Q_2(z+c_1)exp^{p(z)-p(z+c_1)} + i\varrho_2 P(z)Q_2(z+c_2)exp^{p(z)-p(z+c_2)} \\ &+ \cdots + i\varrho_m P(z)Q_2(z+c_m)exp^{p(z)-p(z+c_m)} + Q_2(z)) = 0. \end{aligned} \tag{4.3}$$

By Lemma 2.1 and $p(z)$ is a nonconstant polynomial, we have

$$\begin{aligned} &i\varrho_1 P(z)Q_1(z+c_1)exp^{p(z+c_1)-p(z)} + i\varrho_2 P(z)Q_1(z+c_2)exp^{p(z+c_2)-p(z)} \\ &+ \cdots + i\varrho_m P(z)Q_1(z+c_m)exp^{p(z+c_m)-p(z)} - Q_1(z) = 0, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &i\varrho_1 P(z)Q_2(z+c_1)exp^{p(z)-p(z+c_1)} + i\varrho_2 P(z)Q_2(z+c_2)exp^{p(z)-p(z+c_2)} \\ &+ \cdots + i\varrho_m P(z)Q_2(z+c_m)exp^{p(z)-p(z+c_m)} + Q_2(z) = 0. \end{aligned} \tag{4.5}$$

If $deg_z p(z) \geq 2$, then we have that $deg_z[p(z+c_j)-p(z+c_i)] \geq 1$. Hence, we have $T(r, i\varrho_j P(z)Q_1(z+c_j)) = S(r, exp^{p(z+c_i)-p(z+c_j)})$, Lemma 2.1 and (4.4) imply that $Q_1(z) \equiv 0$. A contradiction. By the similar method as above, we also obtain that $Q_2(z) \equiv 0$, this is also impossible. Hence, $deg_z p(z) = 1$. Let $p(z) = az + b, a \neq 0, b \in \mathbb{C}$. (4.4) and (4.5) imply that

$$\begin{aligned} &i\varrho_1 P(z)Q_1(z+c_1)exp^{p(z+c_1)-p(z)} + i\varrho_2 P(z)Q_1(z+c_2)exp^{p(z+c_2)-p(z)} \\ &\quad + \cdots + i\varrho_m P(z)Q_1(z+c_m)exp^{p(z+c_m)-p(z)} = Q_1(z), \end{aligned}$$

and

$$\begin{aligned} &i\varrho_1 P(z)Q_2(z+c_1)exp^{p(z)-p(z+c_1)} + i\varrho_2 P(z)Q_2(z+c_2)exp^{p(z)-p(z+c_2)} \\ &\quad + \cdots + i\varrho_m P(z)Q_2(z+c_m)exp^{p(z)-p(z+c_m)} = -Q_2(z). \end{aligned}$$

By this, we have

$$\begin{aligned} &P(z)^2[{\varrho_1}^2 Q(z+c_1) + {\varrho_2}^2 Q(z+c_1) + \cdots + {\varrho_m}^2 Q(z+c_m) + \\ &\varrho_1\varrho_2 Q_1(z+c_1)Q_2(z+c_2)exp^{ac_1-ac_2} + \cdots + \\ &\varrho_1\varrho_m Q_1(z+c_1)Q_2(z+c_m)exp^{ac_1-ac_m} + \varrho_2\varrho_1 Q_1(z+c_2)Q_2(z+c_1)exp^{ac_2-ac_1} \\ &+ \cdots + \varrho_2\varrho_m Q_1(z+c_2)Q_2(z+c_m)exp^{ac_2-ac_m} \\ &+ \cdots + \varrho_m\varrho_{m-1} Q_1(z+c_{m-1})Q_2(z+c_m)exp^{ac_m-ac_{m-1}}] = Q(z). \end{aligned} \tag{4.6}$$

Set $deg_z P(z) = p$ and $deg_z Q(z) = q$, then $p \geq 0, q \geq 0$ and $p, q \in \mathbb{N}_+$. Next we divided the following proof into four cases:

Case 1. $p \geq 1$ and $\sum_{i=1}^{m} \varrho_i exp^{ac_i} \sum_{i=1}^{m} \varrho_i exp^{-ac_i} = 0$. If $q \geq 1$, by comparing the order both sides of (4.6), we have $2p + q - 1 \leqslant q$, that is, $p \leqslant \frac{1}{2}$, this is impossible. If $q = 0$, that is, $Q(z)$ is a constant. Hence, by (4.6), we have $Q(z) = 0$, a contradiction.

Case 2. $p \geq 1$ and $\sum_{i=1}^{m} \varrho_i exp^{ac_i} \sum_{i=1}^{m} \varrho_i exp^{-ac_i} \neq 0$. If $q \geq 1$, by comparing the order both sides of (4.6), we have $2p+q = q$, that is, $p = 0$, a contradiction. If $q = 0$, that is, $Q(z)$ is a constant. Hence, by (4.6), we have $P(z)$ is a constant, this is impossible.

Case 3. $p = 0$ and $\sum_{i=1}^{m} \varrho_i exp^{ac_i} \sum_{i=1}^{m} \varrho_i exp^{-ac_i} = 0$. That is, $P(z) = K(\neq 0)$. If $q \geq 1$, we have $q - 1 = q$, this is impossible. If $q = 0$, we have $Q(z) \equiv 0$. A contradiction.

Case 4. $p = 0$ and $\sum_{i=1}^{m} \varrho_i exp^{ac_i} \sum_{i=1}^{m} \varrho_i exp^{-ac_i} \neq 0$. If $q \geq 1$, set $P(z) = K(\neq 0)$, $Q(z) = b_q z^q + b_{q-1} z^{q-1} + \cdots + b_0, b_q \neq 0, b_{q-1}, \cdots, b_0$ are constants. By comparing the coefficients of z^q both sides of (4.6), we have

$$K^2[\sum_{i=1}^{m} \varrho_i exp^{ac_i} \sum_{i=1}^{m} \varrho_i exp^{-ac_i}] = 1. \tag{4.7}$$

This is a contradiction with the condition of Theorem 1.2. If $q = 0$, then $K^2[\sum_{i=1}^{m} \varrho_i exp^{ac_i} \sum_{i=1}^{m} \varrho_i exp^{-ac_i}] = 1$, this is impossible.

Hecne, Theorem 1.2 holds.

5. Proof of Theorem 1.3

Suppose that (1.3) has a finite order transcendental meromorphic solution $l(z)$ with finitely many poles. Rewriting (1.3) as follows

$$\begin{aligned}&(l'(z) + i(\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m)))(l'(z)-\\ &i(\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m))) = R(z).\end{aligned} \tag{5.1}$$

Since $l(z)$ has finitely many poles, and $R(z)$ is a nonzero rational function, then $l'(z) + i(\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m))$ and $l'(z) - i(\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m))$ both have finitely many poles and zeros. Hence, by Lemma 2.2, (5.1) can be written as

$$l'(z) + i(\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m)) = R_1(z)exp^{p(z)}, \tag{5.2}$$

and

$$l'(z) - i(\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m)) = R_2(z)exp^{-p(z)}, \tag{5.3}$$

where $R_1(z), R_2(z)$ are two nonzero rational functions such that $R_1(z)R_2(z) = R(z)$, and $p(z)$ is a nonconstant polynomial. (5.2) and (5.3) imply that

$$l'(z) = \frac{R_1(z)exp^{p(z)} + R_2(z)exp^{-p(z)}}{2}, \tag{5.4}$$

and

$$\varrho_1 l(z + c_1) + \varrho_2 l(z + c_2) + \cdots + \varrho_m l(z + c_m) = \frac{R_1(z)exp^{p(z)} - R_2(z)exp^{-p(z)}}{2i}. \tag{5.5}$$

(5.5) implies that

$$\varrho_1 l'(z+c_1)+\varrho_2 l'(z+c_2)+\cdots+\varrho_m l'(z+c_m)=\frac{A_1(z)exp^{p(z)}-B_1(z)exp^{-p(z)}}{2i}, \tag{5.6}$$

where $A_1(z)=R_1'+R_1(z)p'$ and $B_1(z)=R_2'-R_2(z)p'$. Substituting (5.4) into (5.6), we have

$$\begin{aligned}&exp^{p(z)}(i\varrho_1 R_1(z+c_1)exp^{p(z+c_1)-p(z)}+i\varrho_2 R_1(z+c_2)exp^{p(z+c_2)-p(z)}\\&+\cdots+i\varrho_m R_1(z+c_m)exp^{p(z+c_m)-p(z)}-A_1(z))+\\&exp^{-p(z)}(i\varrho_1 R_2(z+c_1)exp^{p(z)-p(z+c_1)}+i\varrho_2 R_2(z+c_2)exp^{p(z)-p(z+c_2)}\\&+\cdots+i\varrho_m R_2(z+c_m)exp^{p(z)-p(z+c_m)}+B_1(z))=0.\end{aligned} \tag{5.7}$$

Together Lemma 2.1 with (5.7), we have

$$\begin{aligned}&i\varrho_1 R_1(z+c_1)exp^{p(z+c_1)-p(z)}+i\varrho_2 R_1(z+c_2)exp^{p(z+c_2)-p(z)}\\&+\cdots+i\varrho_m R_1(z+c_m)exp^{p(z+c_m)-p(z)}-A_1(z)=0,\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}&i\varrho_1 R_2(z+c_1)exp^{p(z)-p(z+c_1)}+i\varrho_2 R_2(z+c_2)exp^{p(z)-p(z+c_2)}\\&+\cdots+i\varrho_m R_2(z+c_m)exp^{p(z)-p(z+c_m)}+B_1(z)=0.\end{aligned} \tag{5.9}$$

Since $R_1(z), R_2(z)$ are two nonzero rational functions and $l(z)$ is of finite order, by the similar method as the proof of Theorem 1.1, we have $deg_z p(z)=1$. Let $p(z)=az+b, a\neq 0, b\in\mathbb{C}$. Substituting $p(z)$, $A_1(z)$, $B_1(z)$ into (5.8) and (5.9), as $z\longrightarrow\infty$, we have

$$\begin{aligned}&\lim_{|z|\to\infty} i(\varrho_1\tfrac{R_1(z+c_1)}{R_1(z)}exp^{p(z+c_1)-p(z)}+\cdots+\varrho_m\tfrac{R_1(z+c_m)}{R_1(z)}exp^{p(z+c_m)-p(z)})\\&=i(\varrho_1 exp^{ac_1}+\cdots+\varrho_m exp^{ac_m})=\tfrac{R'_1(z)}{R_1(z)}+a=a,\end{aligned}$$

and

$$\begin{aligned}&\lim_{|z|\to\infty} i(\varrho_1\tfrac{R_2(z+c_1)}{R_2(z)}exp^{p(z)-p(z+c_1)}+\cdots+\varrho_m\tfrac{R_2(z+c_m)}{R_2(z)}exp^{p(z)-p(z+c_m)})\\&=i(\varrho_1 exp^{-ac_1}+\cdots+\varrho_m exp^{-ac_m})=-\tfrac{R'_2(z)}{R_2(z)}+a=a.\end{aligned}$$

That is

$$\begin{aligned}&i(\varrho_1 exp^{ac_1}+\cdots+\varrho_m exp^{ac_m})=a,\\&i(\varrho_1 exp^{-ac_1}+\cdots+\varrho_m exp^{-ac_m})=a.\end{aligned} \tag{5.10}$$

According to (5.8), (5.9) and (5.10), we have

$$\begin{aligned}&i\varrho_1 exp^{ac_1}(R_1(z+c_1)-R_1(z))+i\varrho_2 exp^{ac_2}(R_1(z+c_2)-R_1(z))\\&+\cdots+i\varrho_m exp^{ac_m}(R_1(z+c_m)-R_1(z))=R_1{}'(z),\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}&i\varrho_1 exp^{-ac_1}(R_2(z+c_1)-R_2(z))+i\varrho_2 exp^{-ac_2}(R_2(z+c_2)-R_2(z))\\&+\cdots+i\varrho_m exp^{-ac_m}(R_2(z+c_m)-R_2(z))=-R_2{}'(z).\end{aligned} \tag{5.12}$$

If $R_1(z), R_2(z)$ are two nonzero constants, then (5.11) and (5.12) hold and $R_1(z)R_2(z) = R(z)$ is a constant.

We next consider the case that $R_1(z), R_2(z)$ are two nonzero rational functions. If $R_1(z)$ has a pole of multiplicity v at z_0, by (5.11), we know that there exists at least on index $l_1 \in \{1, 2, \cdots, m\}$ such that

$z_0 + c_{l_1}$ is a pole of $R_1(z)$ of multiplicity $v+1$, following the above step, we know $R_1(z)$ has a sequence of poles

$$\{\tau_n = z_0 + c_{l_1} + \cdots + c_{l_n} : n = 1, 2, \cdots\}.$$

Hence, we have $\lambda(\frac{1}{R_1(z)}) \geq 1$, this is impossible. So $R_1(z)$ is a polynomial. Using the same method as above, we know that $R_2(z)$ is also a polynomial. If $R_i(z)$ is a nonconstant polynomial with $deg_z R_i(z) \geq 2$. Let $R_i(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, then

$$R'_i(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots, \tag{5.13}$$

$$R_i(z + c_m) - R_i(z) = n a_n c_m z^{n-1} + (a_n C_n^2 c_m^2 + (n-1) a_{n-1} c_m) z^{n-2} + \cdots, \tag{5.14}$$

where $i = 1, 2$. Substituting (5.13) and (5.14) into (5.11) and (5.12), comparing the coefficients of z^{n-1}, z^{n-2}, we have $\sum_{j=1}^{m} i c_j \varrho_j exp^{ac_j} = 1$, $\sum_{j=1}^{m} {c_j}^2 \varrho_j exp^{ac_j} = 0$ and $\sum_{j=1}^{m} i c_j \varrho_j exp^{-ac_j} = -1$, $\sum_{j=1}^{m} {c_j}^2 \varrho_j exp^{-ac_j} = 0$, a contradiction with $\sum_{j=1}^{m} {c_j}^2 \varrho_j exp^{ac_j} \sum_{j=1}^{m} {c_j}^2 \varrho_j exp^{-ac_j} \neq 0$. Hence, $deg_z R_i(z) \leq 1$. So $deg_z R(z) = deg_z R_1(z) R_2(z) \leq 2$.

(i) If $R(z)$ is a nonconstant polynomial with $deg_z R(z) \leqslant 2$, then by (5.4), we have

$$l(z) = \frac{s_1(z) exp^{az+b} + s_2(z) exp^{-(az+b)}}{2} + \vartheta, \tag{5.15}$$

where $s_j(z) = m_j z + n_j, m_j, n_j \in \mathbb{C}, (j = 1, 2)$ and $\vartheta \in \mathbb{C}$;

Case 1. If $deg_z R(z) = 2$, then $m_j \neq 0, j = 1, 2$. If $\sum_{i=1}^{m} \varrho_i \neq 0$, substituting (5.15) into (5.5), we have $\vartheta \equiv 0$, $R(z) = (m_1 + a s_1(z))(m_2 - a s_2(z))$. Hence, we have

$$l(z) = \frac{s_1(z) exp^{az+b} + s_2(z) exp^{-(az+b)}}{2},$$

$R(z) = (m_1 + a s_1(z))(m_2 - a s_2(z)), a \neq 0, b \in \mathbb{C}$.

Case 2. If $deg_z R(z) = 1$, then one of m_1, m_2 is zero, we can assume that $m_1 = 0$. Substituting (5.15) into (5.5), we have $R_1(z)$ is a constant and $R_2(z)$ is a polynomial of degree one. Using the same method as case 1, we have $\vartheta \equiv 0$. Hence, we obtain that

$$l(z) = \frac{s_1(z) exp^{az+b} + s_2(z) exp^{-(az+b)}}{2},$$

$R(z) = (m_1 + a s_1(z))(m_2 - a s_2(z)), a \neq 0, b \in \mathbb{C}$.

(ii) If $R(z)$ is a nonzero constant, by (5.4), we have

$$l(z) = \frac{n_1 exp^{az+b} + n_2 exp^{-(az+b)}}{2} + d, \tag{5.16}$$

where $n_1, n_2 \in \mathbb{C}$ and $d \in \mathbb{C}$. Substituting (5.16) into (5.5), we have $d = 0$, $R(z) = -a^2 n_1 n_2$. Hence, Theorem 1.3 holds.

6. Proof of Theorem 1.4

Suppose that (1.4) has a finite order transcendental meromorphic solution $l(z)$ with finitely many poles. Rewriting (1.4) as follows

$$\begin{aligned}&(l''(z)+i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m)))(l''(z)-\\&i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m)))=R(z).\end{aligned}\tag{6.1}$$

Since $l(z)$ has finitely many poles, $R(z)$ is a nonzero rational function, then $l''(z)+i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m))$ and $l''(z)-i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m))$ both have finitely many poles and zeros. Hence, we can rewrite (6.1) as follows

$$l''(z)+i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m))=R_1(z)exp^{p(z)},\tag{6.2}$$

and

$$l''(z)-i(\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m))=R_2(z)exp^{-p(z)},\tag{6.3}$$

where $R_1(z), R_2(z)$ are two nonzero rational functions such that $R_1(z)R_2(z) = R(z)$, and $p(z)$ is a nonconstant polynomial. By (6.2) and (6.3), we obtain

$$l''(z)=\frac{R_1(z)exp^{p(z)}+R_2(z)exp^{-p(z)}}{2},\tag{6.4}$$

and

$$\varrho_1 l(z+c_1)+\varrho_2 l(z+c_2)+\cdots+\varrho_m l(z+c_m)=\frac{R_1(z)exp^{p(z)}-R_2(z)exp^{-p(z)}}{2i}.\tag{6.5}$$

(6.5) implies that

$$\varrho_1 l''(z+c_1)+\varrho_2 l''(z+c_2)+\cdots+\varrho_m l''(z+c_m)=\frac{A_2(z)exp^{p(z)}-B_2(z)exp^{-p(z)}}{2i},\tag{6.6}$$

where $A_2(z) = A_1' + A_1(z)p'$ and $B_2(z) = B_1' - B_1(z)p'$. Together (6.4) with (6.6), we obtain that

$$\begin{aligned}&exp^{p(z)}(i\varrho_1 R_1(z+c_1)exp^{p(z+c_1)-p(z)}+i\varrho_2 R_1(z+c_2)exp^{p(z+c_2)-p(z)}\\&+\cdots+i\varrho_m R_1(z+c_m)exp^{p(z+c_m)-p(z)}-A_2(z))+\\&exp^{-p(z)}(i\varrho_1 R_2(z+c_1)exp^{p(z)-p(z+c_1)}+i\varrho_2 R_2(z+c_2)exp^{p(z)-p(z+c_2)}\\&+\cdots+i\varrho_m R_2(z+c_m)exp^{p(z)-p(z+c_m)}+B_2(z))=0.\end{aligned}\tag{6.7}$$

Lemma 2.1 and (6.7) imply that

$$\begin{aligned}&i\varrho_1 R_1(z+c_1)exp^{p(z+c_1)-p(z)}+i\varrho_2 R_1(z+c_2)exp^{p(z+c_2)-p(z)}\\&+\cdots+i\varrho_m R_1(z+c_m)exp^{p(z+c_m)-p(z)}-A_2(z)=0,\end{aligned}\tag{6.8}$$

and

$$\begin{aligned}&i\varrho_1 R_2(z+c_1)exp^{p(z)-p(z+c_1)}+i\varrho_2 R_2(z+c_2)exp^{p(z)-p(z+c_2)}\\&+\cdots+i\varrho_m R_2(z+c_m)exp^{p(z)-p(z+c_m)}+B_2(z)=0.\end{aligned}\tag{6.9}$$

Since $R_1(z), R_2(z)$ are two nonzero rational functions and $l(z)$ is of finite order, using the similar method as the proof of Theorem 1.1, we know that $p(z)$ is a polynomial of degree one. Let $p(z) = az + b, a \neq 0, b \in \mathbb{C}$. Substituting $p(z), A_2(z), B_2(z)$ into (6.8) and (6.9), and as $z \longrightarrow \infty$, we have

$$\begin{aligned}&\lim_{|z|\to\infty} i(\varrho_1 \tfrac{R_1(z+c_1)}{R_1(z)} exp^{p(z+c_1)-p(z)} + \cdots + \varrho_m \tfrac{R_1(z+c_m)}{R_1(z)} exp^{p(z+c_m)-p(z)})\\ &= i(\varrho_1 exp^{ac_1} + \cdots + \varrho_m exp^{ac_m}) = \tfrac{A_1{}'(z)}{R_1(z)} + a^2 = a^2,\end{aligned}$$

and

$$\begin{aligned}&\lim_{|z|\to\infty} i(\varrho_1 \tfrac{R_2(z+c_1)}{R_2(z)} exp^{p(z)-p(z+c_1)} + \cdots + \varrho_m \tfrac{R_2(z+c_m)}{R_2(z)} exp^{p(z)-p(z+c_m)})\\ &= i(\varrho_1 exp^{-ac_1} + \cdots + \varrho_m exp^{-ac_m}) = -\tfrac{B_1{}'(z)}{R_2(z)} - a^2 = -a^2,\end{aligned}$$

that is

$$\begin{aligned}&i(\varrho_1 exp^{ac_1} + \cdots + \varrho_m exp^{ac_m}) = a^2,\\ &i(\varrho_1 exp^{-ac_1} + \cdots + \varrho_m exp^{-ac_m}) = -a^2.\end{aligned} \tag{6.10}$$

So, we have $\sum_{i=1}^{m} \varrho_i exp^{ac_i} + \sum_{i=1}^{m} \varrho_i exp^{-ac_i} = 0$.

(i) If $\sum_{i=1}^{m} \varrho_i exp^{ac_i} + \sum_{i=1}^{m} \varrho_i exp^{-ac_i} \neq 0$, this is a contradiction with $\sum_{i=1}^{m} \varrho_i exp^{ac_i} + \sum_{i=1}^{m} \varrho_i exp^{-ac_i} = 0$. Hence, Theorem 1.4 (i) holds.

(ii) If $\sum_{j=1}^{m} ic_j\varrho_j exp^{ac_j} \neq 2a$ and $\sum_{j=1}^{m} ic_j\varrho_j exp^{-ac_j} \neq 2a$. By (6.8)–(6.10), we have

$$\begin{aligned}&i\varrho_1 exp^{ac_1}(R_1(z+c_1) - R_1(z)) + i\varrho_2 exp^{ac_2}(R_1(z+c_2) - R_1(z))\\ &+ \cdots + i\varrho_m exp^{ac_m}(R_1(z+c_m) - R_1(z)) = R_1{}''(z) + 2aR_1{}'(z),\end{aligned} \tag{6.11}$$

and

$$\begin{aligned}&i\varrho_1 exp^{-ac_1}(R_2(z+c_1) - R_2(z)) + i\varrho_2 exp^{-ac_2}(R_2(z+c_2) - R_2(z))\\ &+ \cdots + i\varrho_m exp^{-ac_m}(R_2(z+c_m) - R_2(z)) = -R_2{}''(z) + 2aR_2{}'(z).\end{aligned} \tag{6.12}$$

If $R_1(z), R_2(z)$ are two nonzero rational functions, using the similar method as the proof of Theorem 1.3, we know that $R_i(z)$ is a polynomial. If $\deg_z R_i(z) \geq 2$. Let $R_i(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$, then

$$\begin{aligned}&R'_i(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots,\\ &R''_i(z) = n(n-1)a_n z^{n-2} + (n-1)(n-2)a_{n-1}z^{n-3} + \cdots,\\ &R_i(z+c_m) - R_i(z) = na_n c_m z^{n-1} + (a_n C_n^2 c_m^2 + (n-1)a_{n-1}c_m)z^{n-2} +\\ &(a_n C_n^3 c_m^3 + a_{n-1}C_{n-1}^2 c_m^2 + (n-2)a_{n-2}c_m)z^{n-3} + \cdots,\end{aligned} \tag{6.13}$$

where $i = 1, 2$. Substituting (6.13) into (6.11) and (6.12), comparing the coefficients of z^{n-1}, z^{n-2}, we have $\sum_{j=1}^{m} ic_j\varrho_j exp^{ac_j} = 2a$, $\sum_{j=1}^{m} c_j{}^2\varrho_j exp^{ac_j} = 2$ and $\sum_{j=1}^{m} ic_j\varrho_j exp^{-ac_j} = 2a$, $\sum_{j=1}^{m} c_j{}^2\varrho_j exp^{-ac_j} = -2$, a contradiction. Hence, $\deg_z R_i(z) \leq 1$.

If $\deg_z R_i(z) = 1$, then (6.11) and (6.12) imply that $\sum_{j=1}^{m} ic_j\varrho_j exp^{ac_j} = 2a$ and $\sum_{j=1}^{m} ic_j\varrho_j exp^{-ac_j} = 2a$, a contradiction. Hence, $R_1(z), R_2(z)$ are two nonzero constants, $R(z) = R_1(z)R_2(z)$ is a constant. By (6.5), we have

$$l(z) = \frac{t_1 exp^{az+b} + t_2 exp^{-(az+b)}}{2} + P(z),$$

where $a \neq 0$, $b \in \mathbb{C}$, $t_1, t_2 \in \mathbb{C} \setminus \{0\}$ and $P(z)$ is a polynomial of degree one. Since $\sum_{i=1}^{m} \varrho_i \neq 0$, then by (6.5), we have $P(z) \equiv 0$. So, we have

$$l(z) = \frac{t_1 exp^{az+b} + t_2 exp^{-(az+b)}}{2}, \tag{6.14}$$

where $\sum_{i=1}^{m} \varrho_i exp^{ac_i} + \sum_{i=1}^{m} \varrho_i exp^{-ac_i} = 0$, $b \in \mathbb{C}$, $R(z) = a^4 t_1 t_2$. Hence, Theorem 1.4 holds.

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Conflict of interest

The authors declare that none of the authors have any competing interests in the manuscript.

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