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STATISTICAL PROPERTIES OF EXCITED ATOMIC NUCLEI

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STATISTICAL PROPERTIES OF EXCITED ATOMIC NUCLEI

A.V. Ignatyuk

LEVEL DENSITIES

The bulk of our modern ideas on the properties of atomic nuclei are based on studies of the ground and low-lying states of nuclei. With refinement of the theory of the nucleus the observed spectroscopic data on the structure of the low-lying nuclear level density have been explained with increasing success. However, as the excitation energy increases the number of nuclear levels rises so rapidly that it becomes virtually impossible to make a detailed analysis of each level. Under these conditions it is quite natural and legitimate to resort to an averaged statistical examination of the properties of excited nuclei. The first part of this monograph is devoted to discussing the main statistical characteristic of nuclear spectra - the density of excited levels.

Chapter 1

THE INFLUENCE OF THE SHELL STRUCTURE OF A SINGLE-PARTICLE SPECTRUM ON THE STATISTICAL CHARACTERISTICS OF NUCLEI

The progress achieved by the shell model in explaining the laws of change in the quantum characteristics of the low-lying levels of nuclei is well known [1,2]. In this chapter we discuss the extension of the shell model concepts to cover the region of statistical description of the properties of excited nuclei.

1.1. The Fermi-gas model and its use in analysing experimental data

Let us consider methods of analysing the level density of excited nuclei, using the Fermi-gas model as an example. Although many of the concepts of this model are extremely simplified, it is the most convenient one for demonstrating the basic features of the theoretical methods of the nuclear level

density calculations. Moreover, because of their simplicity the relationships of this model were, and are still, widely used in the analysis and systematics of experimental data on the statistical properties of nuclei.

The density of the excited states of nuclei as a function of energy and number of particles can be defined in the general case as

$$\rho(Z, N, \mathcal{E}) = \sum_j \delta(Z - Z_j)\delta(N - N_j)\delta(\mathcal{E} - \mathcal{E}_j), \tag{1.1}$$

where $\mathcal{E}_j$ is the energy of the j-th state of a nucleus consisting of Z_j protons and N_j neutrons. Using the integral representation of δ-functions, we rewrite Eq. (1.1) in the form

$$\rho(Z, N, \mathcal{E}) = (2\pi i)^{-3} \int_{c-i\infty}^{c+i\infty} \int \exp(\beta\mathcal{E} - a_Z Z - a_N N) \times \\ \times Q(\beta, a_Z, a_N)\, d\beta da_Z da_N, \tag{1.2}$$

where

$$Q(\beta, a_Z, a_N) = \sum_j \exp(-\beta\mathcal{E}_j + a_Z Z_j + a_N N_j). \tag{1.3}$$

The latter expression, well known in statistical physics, is the relationship for the statistical sum of a large canonical ensemble, while Eq. (1.1) is a similar definition of the statistical sum of a microcanonical ensemble (MCE) [3]. Although neither definition of level density yet uses any model concepts of the nucleus, their equivalence to the general formulae of statistical physics demonstrates the statistical nature of the level density value itself, and also indicates that the methods used in the study of other physical systems can be used to consider the level density of nuclei.

Even at the early stage of development of nuclear physics Bethe suggested that the statistical properties of atomic nuclei could be described by representing the excited nucleus as a gas of non-interacting Fermi particles [4]. This suggestion was further developed in the light of the success of the shell model of the nucleus, and was widely applied in subsequent years to calculate the density of the excited levels of nuclei [5-7].

If, for the sake of simplicity, we consider only one type of particle, the statistical sum of non-interacting Fermi particles can be written in the form

$$Q(\beta, a) = \prod_\nu [1 + \exp(-\beta\epsilon_\nu + a)]^{g_\nu}, \tag{1.4}$$

where ε_ν is the energy of single-particle levels in the self-consistent single-particle potential, g_ν the degree of degeneracy of levels; $\beta = t^{-1}$ and $\alpha = \lambda/t$ are directly connected with thermodynamic temperature t and chemical potential λ of the system. The statistical sum (1.4) can be calculated in the analytical form if the continuous-spectrum approximation is used. Similar calculations are considered in many textbooks of statistical physics for the analysis of the heat capacity of the degenerate electron gas [3]. We rewrite (1.4) as

$$\ln Q(\beta,\alpha) = \int_0^\infty g(\epsilon)\ln[1+\exp(\alpha-\beta\epsilon)]\,d\epsilon. \tag{1.5}$$

Here $g(\epsilon)$ is the density of single-particle states. Using the low-temperature approximation $t \ll \lambda$ and regarding $g(\epsilon)$ as a fairly smooth function of the energy, we obtain

$$\ln Q(\beta,\alpha) = \int_0^{\alpha/\beta} g(\epsilon)(\alpha-\beta\epsilon)\,d\epsilon + \frac{\pi^2}{6\beta}g\left(\frac{\alpha}{\beta}\right) + \frac{7\pi^4}{360\beta^3}\,g''\left(\frac{\alpha}{\beta}\right), \tag{1.6}$$

where g'' is the second derivative of the density of states. With the help of (1.6) it can be easily shown that the integrand in the initial formula for level density (1.2) has a sharp extremum for certain values of the variables β_0 and α_0; because of this property, the saddle-point method can be used for calculating the corresponding integrals [3].

Within the framework of the saddle-point method the position of the extremum point is determined by the equations

$$\mathcal{E} = \partial \ln Q/\partial\beta;\quad N = \partial \ln Q/\partial\alpha. \tag{1.7}$$

Expanding in the neighbourhood of this point the exponent of the integrand in a series with an accuracy of up to second-order terms, we obtain a double Gaussian integral, the result of calculation of which we write in the form

$$\rho(N,\mathcal{E}) = \frac{Q(\beta_0,\alpha_0)\exp(\beta_0\mathcal{E}-\alpha_0 N)}{2\pi\sqrt{\det}}, \tag{1.8}$$

where

$$\det = \left|\begin{array}{cc} \dfrac{\partial^2 \ln Q}{\partial\beta^2} & \dfrac{\partial^2 \ln Q}{\partial\alpha\,\partial\beta} \\ \dfrac{\partial^2 \ln Q}{\partial\beta\,\partial\alpha} & \dfrac{\partial^2 \ln Q}{\partial\alpha^2} \end{array}\right|_{\beta=\beta_0,\,\alpha=\alpha_0} \tag{1.9}$$

If we substitute (1.6) into Eq. (1.7) determining the saddle point, by neglecting the terms containing the derivatives of g(ε) we can write them as

$$\mathcal{E} = \int_0^{\lambda_0} \epsilon g(\epsilon) d\epsilon + \frac{\pi^2}{6\beta_0^2} g(\lambda_0); \quad N = \int_0^{\lambda_0} g(\epsilon) d\epsilon. \tag{1.10}$$

These equations connect the energy and number of atoms in a system with thermodynamic temperature $t = \beta_o^{-1}$ and chemical potential λ_o, i.e. they are in no way different from the thermodynamic equations of state for the degenerate Fermi gas [3]. At zero temperature the first equation in (1.10) is used to determine the energy of the ground state of the system $\mathcal{E}_o$ and the second equation to determine the energy of the last filled level or, as it is usually called, the Fermi energy $\varepsilon_f \equiv \lambda_o$ (t = o). Substituting (1.10) into formula (1.8) and calculating the determinant of the second-order derivatives, we obtain a simple relationship for the state density

$$\rho(N, U) = \frac{1}{\sqrt{48}\, U} \exp(2\sqrt{aU}), \tag{1.11}$$

where $U = \mathcal{E} - \mathcal{E}_o$ is the excitation energy and $a = \pi^2 g(\varepsilon_f)/6$, the so-called level density parameter of the Fermi gas. The connection between the excitation energy and temperature and entropy for the given system can be expressed conveniently in terms of that parameter:

$$U = at^2; \quad S = 2at = 2\sqrt{aU}. \tag{1.12}$$

It will be seen from the above formula that it is entropy which actually determines the density of the excited states of the system.

Similarly, it is easy to consider a Fermi gas consisting of two types of particle: protons and neutrons [4-7]. For the density of excited states, in this case we obtain the relationship

$$\rho(Z, N, U) = \frac{\sqrt{\pi}}{24 a^{1/4} U^{5/4}} \exp(2\sqrt{aU}), \tag{1.13}$$

which differs from (1.11) only by the pre-exponential factor. The expressions for excitation energy and entropy retain their earlier forms (1.12), while the level density parameter is now defined simply as the sum of the respective parameters of the proton and neutron gas.

For many problems there is interest not only in the total density of the excited states of nuclei but also in the distribution of states over angular momentum J or its projection M. Introducing an additional Lagrangian multiplier, corresponding to the new characteristic of the system, into the definition of the statistical sum of the large canonical ensemble, we obtain by the above method the following Gaussian distribution for the density of states of the Fermi gas with a fixed angular-momentum projection

$$\rho(Z, N, U, M) = \frac{\rho(Z, N, U)}{\sqrt{2\pi}\,\sigma} \exp(-M^2/2\sigma^2). \tag{1.14}$$

The distribution width is determined by the spin cut-off parameter of level density

$$\sigma^2 = g(\epsilon_f)\, m_f^2\, t, \tag{1.15}$$

where m_f^2 is the average value of the square of the projection of the angular momentum of single-particle states near the Fermi energy. The quantity $\mathcal{J}_f = g(\epsilon_f)m^2_f$ is usually called the moment of inertia of the Fermi gas.

In analysing the different experiments we would be interested not only in the density of the states of the nucleus but also in the density of excited levels, each of which (2J + 1) times degenerate along the angular-momentum projection. We find the relationship for the density of levels with a specified value of angular momentum by differentiating the density of states

$$\begin{aligned} \rho(Z, N, U, J) &= \rho(Z, N, U, M = J) - \rho(Z, N, U, M = J+1) \approx \\ &\approx \frac{(2J+1)\rho(Z, N, U)}{2\sqrt{2\pi}\,\sigma^3} \exp[-(J + 1/2)^2/2\sigma^2]. \end{aligned} \tag{1.16}$$

For the total level density we accordingly obtain

$$\rho_{level}(Z, N, U) = \sum_J \rho(Z, N, U, J) = \frac{\rho(Z, N, U)}{\sqrt{2\pi}\,\sigma}. \tag{1.17}$$

In order to understand the dependence of the Fermi-gas model parameters on the number of nucleons in the nucleus, we consider a quasi-classical evaluation of $g(\epsilon_f)$ and m_f^2 for a spherical potential well. The quantization conditions which determine the spectrum of single-particle levels in such a potential take the form

$$\pi(n + 1/2) = \int_{r_{min}}^{R_0} \left[\frac{2\mu_0\epsilon}{\hbar^2} - \frac{(l + 1/2)^2}{r^2}\right]^{1/2} dr, \tag{1.18}$$

where μ_o is the nucleon mass and $R_o = r_o A^{1/3}$ the nuclear radius. The dependence of the density of single-particle states on energy and angular momentum is determined by the relation

$$g(\epsilon, l) = g_s (2l + 1) dn/d\epsilon, \tag{1.19}$$

where $g_s = 4$ is a statistical factor characterizing the spin and isospin degeneracy of single-particle states. For simplicity, we consider nuclei with Z = N = A/2. In this case,

$$g(\epsilon) = \int_0^{l_{max}} g(\epsilon, l) dl = \left(\frac{2\mu_0 R_0^2}{\hbar^2}\right)^{3/2} \frac{\epsilon^{1/2}}{3\pi}, \tag{1.20}$$

and determining the Fermi energy from the condition

$$A = \int_0^{\epsilon_f} g(\epsilon) d\epsilon = \frac{8}{9\pi}\left(\frac{2\mu_0 \epsilon_f R_0^2}{\hbar^2}\right)^{3/2}, \tag{1.21}$$

for the density of states on the Fermi surface, we obtain

$$g(\epsilon_f) = \frac{4\mu_0 r_0^2}{\hbar^2 (3\pi^2)^{1/3}} A. \tag{1.22}$$

We write the corresponding level density parameter a, MeV^{-1}, in the form

$$a = \left(\frac{\pi}{3}\right)^{4/3} \frac{2\mu_0 r_0^2}{\hbar^2} A = \frac{A}{13{,}5}, \tag{1.23}$$

where the numerical evaluation is made for $r_o = 1.2$ fm.

Similarly, we find the average value of the square of the projection of angular momentum for single-particle states on the Fermi surface

$$m_f^2 = \frac{1}{3} l_f^2 = \frac{1}{3g(\epsilon_f)} \int_0^{l_{max}} l^2 g(\epsilon_f, l) dl = \frac{(3\pi^2)^{1/3}}{10} A^{2/3}. \tag{1.24}$$

We note that in a quasi-classical approximation the moment of inertia of the Fermi gas

$$\mathcal{J}_f = g(\epsilon_f) m_f^2 = \frac{2}{5} \frac{\mu_0 r_0^2}{\hbar^2} A^{5/3} \tag{1.25}$$

coincides with the rigid-body value of the moment of inertia of the nucleus*/.

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*/ By this we mean the moment of inertia of the nucleus rotating as a rigid body.

The relationships for the Fermi-gas model given above have repeatedly been discussed by many authors [5-7] and the interested reader will find a more detailed description of the calculations in those references.

The most direct information on the level density of highly excited nuclei can be obtained from the experimental data on the density of neutron resonances. Lists of data on the observed mean distance D_o between resonances are contained in Refs [7-9]. Since for most nuclei the observed resonances correspond to neutrons with zero orbital momentum (the so-called s-neutrons), the value of D_o is linked to the density of the excited levels of the compound nucleus by the relationship

$$D_0^{-1} = \begin{cases} \frac{1}{2}\left\{\rho(B_n + \Delta E/2,\ I_0 + 1/2) + \rho(B_n + \Delta E/2,\ I_0 - 1/2)\right\} & \text{for } I_0 \neq 0; \\ \frac{1}{2}\rho(B_n + \Delta E/2,\ 1/2) & \text{for } I_0 = 0, \end{cases} \tag{1.26}$$

where B_n is the neutron binding energy, ΔE is the energy interval over which the resonances were studied, and I_o is the spin of the target nucleus. Coefficient 1/2 preceding the sum takes it into account that s-neutrons form resonances only of a particular parity. Similarly, if necessary, resonances for p-neutrons can also be included in the consideration.

The experimental values of D_o (Fig. 1) are normally used as the initial data from which the level density parameter is determined with the help of Eqs (1.16) and (1.26). An analysis of this type has repeatedly been performed [6-10]. Even in early studies on the systemization of experimental data it was noted that there were regular differences for identical excitation energies of the level density of even-even, odd and odd-odd nuclei, and that the differences were similar to those in the binding energies of such nuclei. This effect is usually taken into account by substituting into the Fermi-gas model relationships the so-called effective excitation energy defined as

$$U^* = U - \begin{cases} \delta_Z + \delta_N & \text{for even-even nuclei;} \\ \delta_Z & \text{for nuclei with even Z;} \\ \delta_N & \text{for nuclei with even N;} \\ 0 & \text{for odd-odd nuclei,} \end{cases} \tag{1.27}$$

where δ_τ is the corresponding phenomenological correction for the odd-even differences in the binding energy of nuclei [6,7]. Figure 1 shows the values of the level density parameter obtained by this analysis. The values found differ very considerably from the quasi-classical evaluation (1.23). The dependence of parameter a on mass number clearly exhibits steep dips in the region of nuclei with magic numbers of protons or neutrons, these dips directly indicating the significant role played by the shell effects in describing the statistical characteristics of the nuclei.

It should be pointed out that the values of parameter a obtained in this manner depend, in a specific manner, on the description used for the spin cut-off parameter σ^2. For the determination of σ^2 the early studies used $m_f^2 = 0.146\ A^{2/3}$ [6,7], which corresponds to mean-square averaging of the angular-momentum projections over all the occupied states of the nucleons, but not over the states in the neighbourhood of the Fermi energy. Later studies usually took a different value of $m_f^2 = 0.24\ A^{2/3}$ or directly used the rigid-body value of the moment of inertia [8-10]. The differences in the choice of the spin cut-off parameter σ^2, as well as the ambiguities in determining the corrections for odd-even differences in the binding energy of nuclei [6,7], have appreciable influence on the values of parameter a, and this should be borne in mind while comparing the results of analysis obtained by the different authors.

With accumulation of measurements of the spectra of low-lying levels in a large number of nuclei it became possible to use the data on the observed rate of growth in the total number of levels for analysis of the low-energy sector of the level density behaviour [5,6,9]. Extrapolation to this region of the Fermi-gas dependence with the parameters found from the analysis of the density of neutron resonances does not normally describe the experimental data. It has been noted that the energy dependence observed in the cumulative number of levels $\mathcal{N}(U)$ can be described much more satisfactorily by the expression

$$\mathcal{N}(U) = \exp[(U - U_0)/T], \tag{1.28}$$

where U_o and T are free parameters determined by fitting to the corresponding data of Ref. [5]. $\mathcal{N}(U)$ is connected with level density by the relation

$$\rho_{\ell ev}(U) = \frac{d\mathcal{N}}{dU} = \frac{1}{T}\exp[(U - U_0)/T]. \tag{1.29}$$

It will be seen that in its physical sense parameter T corresponds to nuclear temperature. Since the value of this parameter is assumed to be constant in the energy range considered, Eq. (1.29) is called the constant-temperature model.

In Refs [6,10] the low-energy dependence (1.29) was matched with the Fermi-gas dependence of level density (1.17) in order to have a description of level density covering the whole excitation energy region from zero to the neutron binding energy. From the condition of continuity of the function itself and the first derivatives at the matching point U_x we can in this case find the relationship between the parameters of the two models

$$U_0 = U_x - T \ln \rho_{f.g.}(U_x), \quad \frac{1}{T} = \sqrt{\frac{a}{U_x^*}} - \frac{3}{2U_x^*}, \tag{1.30}$$

where U_x^* is the effective energy of the matching point with allowance for corrections in (1.27) for even-odd effects. This phenomenological approach was used in Refs [6,10] to analyse the experimental data, and Fig. 2 shows the systematics of the nuclear temperature T and energy U_x values found. The U_x value determines the energy below which the level density cannot be satisfactorily described by means of the Fermi-gas model on the basis of definition of effective excitation energy adopted in the region of neutron resonance. It will be seen that this energy is fairly high for most nuclei.

In Ref. [11] a somewhat different approach was developed to deal with the problem of simultaneous parametrization of the density or neutron resonances and the density of the low-lying levels of nuclei. It was shown that both sets of experimental data could be described on the basis of the Fermi-gas model relationships, provided not only the level density parameter a but also the shift δ_{ef} were used as free parameters for each of the nuclei in the definition of effective excitation energy (1.27). Since for odd-odd nuclei the shift thus found was negative, this approach was called the Fermi-gas model with backshift. This model was used to analyse the whole set of experimental data on the low-lying levels of nuclei and on the density of neutron resonances, and to find the empirical values of parameters a and δ_{ef} for the entire region of mass numbers [9]. Naturally enough, as a result of redefinition of the effective excitation energy the a values obtained are somewhat lower than those given in Fig. 1. However, this does not essentially alter the general trend of the dependence of the level density parameter on mass number.

Experimental data available on the spin of the low-lying levels of some nuclei can be used to analyse the angular momentum distribution of excited levels. Examples of such an analysis are given in Fig. 3. It will be seen that the law governing distribution of excited levels (1.16) obtained within the framework of statistical consideration shows, on the whole, fair agreement with experimental data. Here, parameter σ^2 is chosen in each case from the condition of the best description of the observed distribution. Since the number of levels with identified spins is comparatively small, the errors of the spin cut-off parameter values obtained are quite large and the experimental data are still clearly insufficient to analyse the energy dependence of parameter σ^2 (U).

On the basis of the foregoing discussion it can, on the whole, be concluded that the experimental data on the level density of nuclei can be parametrized rather simply with the help of the Fermi-gas model relationships. It is not possible, however, within the framework of this model, to explain the excitation energy shift and the even-odd differences in level density nor the differences of the experimental values of the level density parameter from quasi-classical evaluations. An interpretation of all these effects can be obtained from more rigorous models, which are considered below.

1.2. Influence of shell inhomogeneities in the single-particle spectrum on the energy dependence of thermodynamic functions

To obtain a more realistic model for consideration of the statistical characteristics of nuclei, we have to discard the continuous spectrum approximation used in deriving the relations of the Fermi-gas model and calculate the level density and thermodynamic functions of nuclei taking into account the discrete shell structure of the single-particle level spectrum. We find the general form of the relationships for the density of states of a nucleus having an excitation energy U and a given angular momentum projection M, as we did above, by the saddle-point method

$$\rho(U, M) = (2\pi)^{-2} \det^{-1/2} \exp(S), \tag{1.31}$$

where S is the entropy of the excited nucleus and det = $|\partial^2 \ln Q/\partial a_i \partial a_j|$ – the determinant for the second derivatives of the logarithm of the statistical sum. For a discrete level spectrum these values are determined by:

$$\begin{aligned} S &= \sum_{\tau=Z,N} \sum_{\nu} \left\{ (\beta \epsilon_{\nu\tau} - a_\tau - a_M m_{\nu\tau}) \bar{n}_{\nu\tau} - \ln(1 - \bar{n}_{\nu\tau}) \right\}; \\ \frac{\partial^2 \ln Q}{\partial a_i \partial a_j} &= \sum_{\tau=Z,N} \sum_{\nu} C_{\nu\tau}^{(i)} C_{\nu\tau}^{(j)} \bar{n}_{\nu\tau}(1 - \bar{n}_{\nu\tau}), \end{aligned} \tag{1.32}$$

where $\bar{n}_{\nu\tau} = [1 + \exp(\beta\varepsilon_{\nu\tau} - \alpha_\tau - \alpha_M m_{\nu\tau})]^{-1}$ indicates the Fermi occupation numbers of single-particle states with energy $\varepsilon_{\nu\tau}$ and with single-particle angular-momentum projection in the given direction $m_{\nu\tau}$; sub-scripts i and j in the equation for the second derivatives take the values of Z, N, M or 0 and the coefficients: $C^{(Z)}_{\nu\tau} = \delta_{\tau Z}$, $C^{(N)}_{\nu\tau} = \delta_{\nu N}$, $C^{(M)}_{\nu\tau} = m_{\nu\tau}$ and $C^{(0)}_{\nu\tau} = \varepsilon_{\nu\tau}$ correspond to them.

The state equations in this case take form:

$$\left.\begin{aligned} \mathcal{E} &= \sum_{\tau=Z,N}\sum_{\nu} \varepsilon_{\nu\tau}\bar{n}_{\nu\tau};\quad M = \sum_{\tau=Z,N}\sum_{\nu} m_{\nu\tau}\bar{n}_{\nu\tau};\\ Z &= \sum_{\nu}\bar{n}_{\nu Z};\quad N = \sum_{\nu}\bar{n}_{\nu N}, \end{aligned}\right\} \qquad (1.33)$$

and excitation energy U is connected with total energy $\mathcal{E}$ by the relationship $U = \mathcal{E} - \mathcal{E}_0$ where $\mathcal{E}_0$ is the energy of the cold nucleus at zero temperature.

The calculations of the state density are appreciably simplified in the approximation of small momenta. In this approximation the equation for M can be solved analytically and the dependence of the state density of the nucleus on angular momentum will take the form of the Gaussian distribution (1.14), in which the spin cut-off parameter $\sigma^2_{\parallel}$ and the moment of inertia $\mathcal{J}_{\parallel}$ corresponding to it are determined by

$$\sigma^2_{\parallel} = \mathcal{J}_{\parallel} t = \sum_{\tau=Z,N}\sum_{\nu} m^2_{\nu\tau}\bar{n}_{\nu\tau}(1-n_{\nu\tau}). \qquad (1.34)$$

These equations can be used to calculate, in the case of a given single-particle level scheme, the thermodynamic functions and the density of states of excited nuclei without introducing any additional parameters. Applying this approach it is possible to investigate the interconnection between the characteristics of the highly excited nucleus and the shell effects which are reflected in the properties of the ground and low-lying states of nuclei. This method of calculating the statistical characteristics of nuclei was first introduced in Refs [12,13], and in subsequent years similar calculations came to be used widely [14-20]. In order to demonstrate the influence of the shell effects more clearly and to investigate the difference of the thermodynamic functions determined

by Eqs (1.32)-(1.34) from the equivalent functions of the Fermi-gas model, we introduce the following quantities

$$\left.\begin{aligned} &a' = S^2/4U;\ a'' = U/t^2;\ a''' = S/2t; \\ &\bar{a} = \frac{\pi^2}{6t} \sum_{\tau=Z,N} \sum_{\nu} \bar{n}_{\nu\tau}(1-\bar{n}_{\nu\tau}); \\ &\overline{m^2} = \frac{\pi^2}{6\bar{a}t} \sum_{\tau=Z,N} \sum_{\nu} m^2_{\nu\tau}\, \bar{n}_{\nu\tau}(1-\bar{n}_{\nu\tau}). \end{aligned}\right\} \qquad (1.35)$$

These relations are so chosen that in the continuous-spectrum approximation all the quantities in (1.35) are transformed into the Fermi-gas model level density parameters: $\bar{a}$ into a and $\overline{m^2}$ into m_f^2.

The results calculating the density of states $\rho(U)$, thermodynamic temperature t and nuclear temperature T, together with the quantities (1.35) for the doubly magic ^{208}Pb nucleus, are shown in Fig. 4 [14]. The calculations were carried out with the use of the level schemes for the two commonest single-particle potentials - the Nilsson potential [21] and the Saxon-Wood potential [22]. The sequences of the quantum characteristics of single-particle levels in the two potentials are similar but there are some differences in the value of the shell gap. The values obtained for a', a'', a''' and $\bar{a}$ differ noticeably from one another and depend very strongly on excitation energy. The behaviour of the thermodynamic temperature, moment of inertia $\mathcal{J}_{\parallel}$ and parameter $\overline{m^2}$ also differ substantially from that of the Fermi gas. All these results reflect a very important effect - the dependence of the average density of the single-particle states near the Fermi surface on excitation energy. This effect is absent in the traditional Fermi-gas model. The obtained dependences of the different thermodynamic characteristics of the ^{208}Pb nucleus are identical in their main features for both the level schemes although, of course, certain differences are found in the values.

The dependence of the different thermodynamic characteristics of the ^{208}Pb nucleus on angular momentum shown in Fig. 5 was obtained from a strict solution of the state equations (1.33). The figure also gives the results of calculating similar characteristics in the approximation of small angular momenta:

$$\left.\begin{aligned} &S(M) = S(M=0) - M^2/2\sigma_{\parallel}^2; \\ &t(M) = \sqrt{\frac{U - M^2/2\,\mathcal{J}_{\parallel}}{a''}}\,, \end{aligned}\right\} \qquad (1.36)$$

where parameters a'', $\mathcal{J}_{\perp}$ and σ^2 are calculated for M = 0. The results of the calculations presented show that substantial differences from the approximation of small momenta arise only at fairly high values of M, which can apparently occur only in reactions with heavy ions.

The dependence of the shell effects on mass number can be observed in Fig. 6, which presents the results of calculations of a', $\bar{a}$, $\overline{m^2}$ and $\mathcal{J}_{\parallel}$, performed for the level scheme of the Nilsson potential [14]. At a low excitation energy of U = 7 MeV the behaviour of a' and $\bar{a}$ clearly show characteristic shell dips - an effect which is well known from the analysis of experimental data (see Fig. 1). The moments of inertia $\mathcal{J}_{\parallel}$ and quantities $\overline{m^2}$ behave similarly. However, at fairly high excitation energies (U = 100 MeV) the shell inhomogeneities of the single-particle spectrum cease to have any appreciable influence on the thermodynamic characteristics of nuclei and the values obtained in the calculations can be approximated by simple quasi-classical dependences:

$$\left.\begin{aligned} \bar{a} &= 0{,}105\,A\ \text{MeV}^{-1};\\ \overline{m^2} &= 0{,}290\,A^{2/3}(1-2/3\,\xi);\\ \mathcal{J}_{\parallel} &= 1{,}85\cdot 10^{-2}A^{5/3}(1-2/3\xi)\,\hbar^2/\text{MeV} \end{aligned}\right\} \qquad (1.37)$$

Here ξ is the deformation parameter characterizing the ratio between the major semi-axis c and minor semi-axis b of an axi-symmetric spheroid $c/b = (1 + 1/3\xi)(1 - 2/3\xi)$ [21].

For the level scheme of the Saxon-Woods potential the results of the calculations of the similar quantities take the form:

$$\left.\begin{aligned} \bar{a} &= 0{,}090\,A\ \text{MeV}^{-1};\\ \overline{m^2} &= 0{,}263\,A^{2/3}(1-2/3\,\xi);\\ \mathcal{J}_{\parallel} &= 1{,}44\cdot 10^{-2}A^{5/3}(1-2/3\xi)\,\hbar^2/\text{MeV}. \end{aligned}\right\} \qquad (1.38)$$

The difference between the numerical coefficients in (1.37) and the coefficients in (1.38) is accounted for by the difference in the nucleon density distribution for the corresponding potentials.

Figure 7 shows the dependence of a' and moment of inertia $\mathcal{J}_{\parallel}$ on the deformation of the single-particle potential for the most typical ^{208}Pb and ^{176}Hf nuclei. In order to observe the changes in these values

for anomalously large deformations corresponding to the dumb-bell shape of fissioning nuclei at the point of scission, we have included in Fig. 7 the results of calculations of a' and $\mathcal{E}_{\text{II}}$ for nuclear configurations in the form of two symmetric spheroids, each of which is characterized by the deformation parameter ξ_i.

Above we have given only the most characteristic examples demonstrating the influence of the shell structure of the single-particle spectrum on the behaviour of the thermodynamic functions of nuclei. Various aspects of this method of calculating the statistical characteristics of excited nuclei have been discussed in recent years by many authors [15-20]. The studies have shown that the shell effects in the behaviour of the thermodynamic functions of nuclei are closely associated with the shell correction characterizing irregular variations in the masses and energies of deformation of nuclei. V.M. Strutinsky's studies [23] have shown the decisive role played by these corrections in the formation of many properties of atomic nuclei.

Within the framework of the initially proposed formulation the shell correction $\delta \mathcal{E}_0$ was determined for the cold nucleus as the difference between the total single-particle energy and a similar energy found for the averaged density of single-particle states:

$$\delta \mathcal{E}_0 = \mathcal{E}_0 - \tilde{\mathcal{E}}_0 = \sum_{\tau = Z, N} \left\{ \sum_{\epsilon_{\nu\tau} < \epsilon_{f\tau}} \epsilon_{\nu\tau} - \int_{-\infty}^{\tilde{\epsilon}_{f\tau}} \epsilon \tilde{g}_\tau(\epsilon) d\epsilon \right\}, \qquad (1.39)$$

where $\epsilon_{f\tau}$ is the Fermi energy for the given single-particle level scheme and $\tilde{\epsilon}_{f\tau}$ the equivalent energy determined by the conditions of conservation of particle number for the averaged spectrum.

A quantity similar to (1.39) can be likewise introduced for the heated nucleus

$$\delta \mathcal{E}(t) = \mathcal{E}(t) - \tilde{\mathcal{E}}(t) = \sum_{\tau = Z, N} \left\{ \sum_{\nu} \epsilon_{\nu\tau} \bar{n}_{\nu\tau} - \right. \qquad (1.40)$$

$$\left. - \int_{-\infty}^{\infty} \epsilon \tilde{g}_\tau(\epsilon) \bar{n}_\tau(\epsilon) d\epsilon \right\}.$$

Since at fairly high temperatures $t \gtrsim 1.5$–2.0 MeV the shell inhomogeneities of the single-particle spectrum cease to have any substantial influence on the temperature dependence of the thermodynamic functions, $\delta\mathcal{E}(t > 2 \text{ MeV}) \approx 0$. Using this result, we rewrite (1.39) in the form

$$\delta\mathcal{E}_0 = \tilde{U}(t) - U(t), \tag{1.41}$$

where $U(t) = \mathcal{E}(t) - \mathcal{E}_0$ is the excitation energy of the nucleus, and $U(t)$ is defined as

$$\tilde{U}(t) = \sum_{\tau=Z,N} \int_{-\infty}^{\infty} \epsilon \tilde{g}_\tau(\epsilon) [\bar{n}_\tau(\epsilon) - n_\tau^0(\epsilon)] d\epsilon \approx \tag{1.42}$$

$$\approx \frac{\pi^2}{6} \tilde{g}(\tilde{\epsilon}_f) t^2 + \frac{7\pi^4}{120} \tilde{g}''(\tilde{\epsilon}_f) t^4 + \ldots$$

Thus, the shell correction can be defined as the difference between the "true" excitation energy U and the excitation energy $\tilde{U}$ obtained from the asymptotic relationships of the Fermi-gas model. When the approximating polynomials needed for calculation of the averaged density of single-particle states $\tilde{g}$ (ϵ) are chosen correctly, the results of thermodynamic calculations of shell corrections (1.41) agree satisfactorily with the corrections obtained on the basis of Eq. (1.39). Their results are considered in Refs [16–20], where the details are described more fully.

The development of the shell correction method encouraged a large number of experimental and theoretical studies on the various aspects of the shell effects in nuclei. Even the earliest calculations of deformation energy by this method [23] show that the value of the shell corrections determine, to a considerable extent, the equilibrium deformations of nuclei and the form of barriers of heavy nuclei which inhibit their spontaneous decay. We shall discuss these problems below when considering the equilibrium shapes of highly excited nuclei, and analyse the statistical characteristics of nuclei during fission. Readers wishing to know more about the various aspects of the shell correction method should refer to the review paper by Brack et al. [24], and also to the original literature cited therein.

The relationship between the shell corrections and the statistical characteristics of excited nuclei stands out clearly in the experimental data for the level density parameter a obtained from the analysis of the density of neutron resonances. These data are shown at the top in Fig. 8 in the form of the ratio parameter a/mass number A. The bottom part of the figure gives the experimental values of shell corrections to the mass formula

$$\delta \mathcal{E}_0 = M_{exp}(Z, A) - M_{LDM}(Z, A, \xi_0), \tag{1.43}$$

where M_{exp} is the experimental value of the mass defect and M_{LDM} its liquid-drop component, as calculated with the Myers-Swiatecki parameters [25], for the equilibrium deformation of the nucleus ξ_0. The close interrelationship between these values is reflected by the clearly expressed correlation of the shell correction with the experimental values of parameter a. This relationship was used in Refs [26,27] to plot the phenomenological systematics of changes in the level density parameter of the Fermi-gas model.

In Ref. [26] these systematics are based on the relation

$$a(U, Z, A) = \tilde{a}(A) \left\{ 1 + \frac{f(U)\delta \mathcal{E}_0(Z, A)}{U} \right\}, \tag{1.44}$$

where $\tilde{a}(A)$ corresponds to the asymptotic value of the level density parameter at a high excitation energy, and the dimensionless function f(U) determines the energy behaviour of the parameter at lower excitation energies. The form of the function

$$f(U) = 1 - \exp(-\gamma U) \tag{1.45}$$

is found on the basis of the approximation of the above calculations of thermodynamic functions using the level spectrum of the shell model potential. For the subsequent semi-empiricial description of parameter a(U) only the form of the functional dependence f(U) was fixed, while parameter γ was chosen directly from analysis of the experimental values. To take into account the possible differences in the behaviour of the asymptotic value of the level density parameter from the quasi-classical evalution (1.23) we used two forms of the functional dependence

$$\tilde{a} = \alpha A + \beta A^2; \tag{1.46}$$

$$\tilde{a} = \alpha A + \beta A^{2/3}. \tag{1.47}$$

On the basis of Eqs (1.43)-(1.47) the experimental values of the level density parameter of the Fermi-gas model $a_{exp}(B_n^*)$ were analysed and, the sets of coefficients (in MeV^{-1} units) corresponding to different definitions of the asymptotic level density were obtained by the method of least squares:

$\tilde{a}(A)$	α	β	γ
(1.46)	0.154	-6.3×10^{-5}	0.054
(1.47)	0.114	0.162	0.054

In the case of both sets, the calculated values of parameter $a(B_n^*)$ satisfactorily reproduce all the main features of the observed changes in the experimental values of the level density parameter of the Fermi-gas model [26].

It should be pointed out that the derived relationships can be used with greater justification at excitation energies exceeding the neutron binding energy. Extrapolation of $a(B_n^*)$ to the region of energies far lower than the neutron binding energy requires a certain amount of caution. Since the given description is based on the relationships of the Fermi-gas model, in accordance with the data on the density of the low-lying levels of nuclei considered in the preceding paragraph, the Fermi-gas systematics of the level density parameter should therefore be used only at excitation energies $U > U_\chi$ (see Fig. 2). The systematics developed in Ref. [27] differ from those considered above only by having a somewhat different parametrization of the energy dependence of the level density parameter, and should have the same limits of applicability.

It should be borne in mind that the asymptotic value of the level density parameter $\tilde{a} = 0.154\ A\ MeV^{-1}$ found in the phenomenological description greatly exceeds the theoretical evaluations of this parameter (1.37) and (1.38). This difference in the parameters indicates the substantial role played by factors which are not taken into account in the theoretical calculations using the single-particle level schemes. As will be shown in Chapter 3, this difference is an empirical reflection of the influence exerted by the collective effects on the density of the excited levels of nuclei.

1.3. Changes of the single-particle spectrum with temperature

The considerable differences between the energy of the low-lying levels of nuclei and the predictions of the simplest single-particle shell model indicate that even after subtraction of the mean field the residual interactions between nucleons are sufficiently strong. Hence, nucleons moving in a self-consistent single-particle potential should, strictly speaking, be regarded as a gas of strongly interacting Fermi particles. But in his studies on the theory of the Fermi liquid Landau showed that even for strong particle interaction the behaviour of the thermodynamic characteristics of a system at low temperatures was to a large extent similar to the thermodynamic characteristics of an ideal Fermi gas [28]. The role of ideal Fermi-gas particles in a normal Fermi liquid is taken over by quasi-particles, which may be regarded as particles moving self-consistently in a cloud of other particles. The energy of the whole system in this case is by no means equal to the sum of energies of the individual quasi-particles but is a complex functional of their distribution.

It is usually not the system's energy which is of physical interest but rather changes in that energy when there is variation in the quasi-particle distribution. At low temperatures the free energy F(t) of the system of interacting Fermi particles can be written in the form [28]

$$F(t) - F_0 = \sum_{\nu} (\epsilon_\nu^0 - \lambda)\delta\bar{n}_\nu + \frac{1}{2}\sum_{\nu,\nu'} f_{\nu\nu'}\delta\bar{n}_\nu\delta\bar{n}_{\nu'} + \ldots, \tag{1.48}$$

where F_0 is the energy of the ground state, ϵ_ν^0 are energies of quasi-particles at zero temperature and $\delta\bar{n}_\nu(t)$ are changes in the quasi-particle distribution during heating. The quadratic term in Eq. (1.48) describes the interaction of quasi-particles and $f_{\nu\nu'}$ is the second derivative of the ground-state energy with respect to quasi-particle distribution. The most important feature of Eq. (1.48) is the presence of the quadratic term reflecting particle interaction. No such term is contained in the formulae for the ideal Fermi gas.

Using the simplest example, let us analyse the role of interaction in the description of the thermodynamic functions of the system. In a heated system the quasi-particle energy can be determined as a variational derivative of free energy

$$\epsilon_\nu = \epsilon_\nu^0 + \sum_{\nu'} f_{\nu\nu'}\delta\bar{n}_{\nu'} = \epsilon_\nu^0 + \delta\epsilon_\nu. \tag{1.49}$$

We write the expression for entropy in the form

$$S = -\sum_{\nu} [\bar{n}_\nu \ln \bar{n}_\nu + (1-\bar{n}_\nu)\ln(1-\bar{n}_\nu)]. \tag{1.50}$$

It is purely combinatorial in origin and does not depend on the interaction. The Fermi quasi-particle distribution function corresponds to the condition of the maximum of entropy with conservation of particle number and energy.

$$\bar{n}_\nu = \left[1+\exp\left(\frac{\epsilon_\nu - \lambda}{t}\right)\right]^{-1} \approx \bar{n}^0_\nu + \frac{\delta\epsilon_\nu}{t}\,\bar{n}^0_\nu(1-\bar{n}^0_\nu), \tag{1.51}$$

where $\bar{n}^0_\nu = \bar{n}_\nu(\epsilon_\nu = \epsilon^0_\nu)$. The changes in quasi-particle energy $\delta\epsilon$ with temperature in the approximation of a continuous homogeneous spectrum are determined by

$$\delta\epsilon = g_0 \int_{-\infty}^{\infty} \bar{f}(\epsilon,\epsilon')[\bar{n}(\epsilon',t)-\bar{n}(\epsilon',0)]\,d\epsilon' \simeq \frac{\pi^2}{6} g_0 t^2 \left.\frac{\partial \bar{f}(\epsilon,\epsilon')}{\partial \epsilon'}\right|_{\epsilon'=0}, \tag{1.52}$$

where $\bar{f}(\epsilon,\epsilon')$ is the quasi-particle internation amplitude averaged over angular variables. For entropy we obtain in this approximation the relationship

$$\begin{aligned} S &= \frac{g_0}{t}\int_{-\infty}^{\infty} \epsilon\bar{n}(\epsilon,t)\,d\epsilon - \frac{g_0}{t^2}\int_{-\infty}^{\infty}\epsilon\bar{n}(\epsilon,t)[1-\bar{n}(\epsilon,t)]\,d\epsilon \approx \\ &\approx \frac{\pi^2}{3} g_0 t\left[1-\frac{\pi^2}{6} g_0 t^2 \bar{f}''(0,0)\right]. \end{aligned} \tag{1.53}$$

At low temperatures the second term in square brackets makes a negligibly small contribution, and the entropy of the system of interacting particles will be similar to that of an ideal Fermi gas. In this case, the density of quasi-particles on the Fermi surface

$$g_0 = (2\mu_0^* \epsilon_f)^{1/2}/\pi^2\hbar^3 \tag{1.54}$$

differs from that of the single-particle states of non-interacting particles only by the effective mass of quasi-particles μ_0^*.

Thus the relationships of the Fermi-gas model can quite satisfactorily describe the temperature dependence of the thermodynamic functions of interacting Fermi particles provided the parameters g_0 of level density are determined phenomenologically from the analysis of experimental data. This

conclusion is also valid for calculations using realistic level schemes of the shell-model phenomenological potential, which is based on the systematics of the observed levels of nuclei. The above statements naturally only refer to the normal phase of the excited nucleus, i.e. to the temperature region for which there are no correlation effects of the superconducting type in the excited quasi-particle spectrum. The role of the superconducting-type effects is considered in the next chapter.

It is important, however, to bear in mind that apart from renormalization of parameters the quasi-particle interaction also leads to the appearance of a new branch of collective excitations, which has no analogue in the ideal Fermi gas [28]. In a Fermi liquid at low temperatures such excitations make a negligibly small contribution to the temperature dependence of the thermodynamic functions. The situation in nuclei is more complex since the collective excitations in nuclei, unlike those in a Fermi liquid, lie at low energies. This problem is discussed in greater detail in Chapter 3 and also during the subsequent analysis of the experimental data on nuclear fission.

The changes in quasi-particle energy with temperature, just as changes in other characteristics of the mean nuclear field, can be determined more strictly on the basis of the theory of finite Fermi systems [29]. In this theory the phenomenological amplitude $f(0,0)$ of the local interaction of quasi-particles near the Fermi surface is determined from analysis of a wide range of nuclear phenomena, and can be used to find the values of $\delta\varepsilon_\nu(t)$ for realistic single-particle level schemes. Such studies were made in Ref. [30]. Figure 9 shows the changes in the proton and neutron level schemes of the ^{214}Po nucleus obtained for temperature $t = 3.5$ MeV. Although all levels shift downwards to a slight extent, the shell structure of the spectrum changes very little. Since this shift is proportional to the square of temperature, at nuclear excitation energies of up to 100 MeV the changes of the single-particle level energy with temperature can generally be neglected when considering the statistical characteristics of nuclei. As the analysis in Ref. [30] showed, the decrease in the single-particle level energy for a heated nucleus is directly related to the increase in the mean-square radius of the nucleus with temperature. In the case of heavy nuclei this increase can be evaluated as $\delta R(t)/R_o \approx 5 \times 10^{-4} t^2$.

The changes in single-particle level energies with temperature and those in the nucleon density distribution in the heated nucleus were also studied on the basis of the Hartree-Fock approximation [31, 32]. In this approach to the problem the nucleon-nucleon interaction is parametrized in the form of effective forces dependent on nucleon density, and the equilibrium distribution of nucleon density in the nucleus and the characteristics of the single-particle levels corresponding to this distribution are determined by the subsequent self-consistent diagonalization of the single-particle Hamiltonian. All the parameters of the effective interaction are determined during the description of the ground states of nuclei and do not vary in the calculations of nucleon distribution in heated nuclei. The results of these calculations, as well as the above-described calculations in the theory of finite Fermi systems, demonstrate fairly small changes in the shell structure of the single-particle spectrum of nuclei at temperatures of up to 2 MeV [31]. Above 2 MeV the shell inhomogeneities in the single-particle spectrum no longer have any great effect on the dependence of the thermodynamic functions of nuclei on mass number, and this fact accords satisfactorily with the calculation results given in the preceding section. In this temperature region the changes with temperature in the volume and surface components of the excitation energy and free energy, MeV, which were obtained in the self-consistent approach [32], can be approximated by

$$\left.\begin{aligned} U_V(t) + U_S(t) &= (0{,}055A + 0{,}15A^{2/3})t^2 \text{ MeV}; \\ F_V(t) + F_S(t) &= -(0{,}055A + 0{,}12A^{2/3})t^2 \text{ MeV}. \end{aligned}\right\} \qquad (1.55)$$

Similar relations were found in Ref. [32] for changes in the Coulomb energy and in the energy of the symmetry of nuclei with temperature. The results of these calculations can be regarded as an evaluation of the expected changes with temperature in the different components of the liquid-drop description of the energy of nuclei. The coefficients in brackets of the temperature dependence of excitation energy and free energy (1.55) determine the asymptotic values of level density parameter $\tilde{a}$ obtained in the given approach. By comparing with (1.38) we can see that, in the case of heavy nuclei, calculations by the Hartree-Fock method give appreciably lower values of the parameters $\tilde{a}$ than those calculated with the empirical shell potential. These differences obviously stress the need for a fuller study of the dependence of the self-consistent calculation results for the thermodynamic characteristics of nuclei on the parametrization of the nucleon-nucleon

interaction. Of primary interest from the standpoint of describing heated nuclei would be the sets of strength constants which give the maximum possible effective nucleon mass inside the nucleus and consequently a higher value of the volume component of the level density parameter.

1.4. Equilibrium deformations of excited nuclei

It is well known that the nuclei in the region of rare-earth elements ($150 \leqslant A \leqslant 190$) and actinides ($A \geqslant 230$) are quite strongly deformed in the ground state, and that this property is of primary importance for a description of the low-energy nuclear spectra. Accordingly, there is naturally a problem of equilibrium deformation of highly excited nuclei. If we relate this problem to the equilibrium deformation averaged over many states of the excited nucleus, its solution can be obtained on the basis of a statistical analysis of the shape of the excited nuclei.

For fixed energies and number of particles the thermodynamic potential determining the statistical properties of a system is the entropy of the microcanonical ensemble, or which is the same thing, the logarithm of the density of excited states. Thus, if we study the dependence of entropy on the deformation of the nucleus, the maximum of entropy will correspond to the most probable equilibrium shape of the excited nucleus. In the presence of fairly complete single-particle level schemes of the deformed shell potential the entropy for a fixed excitation energy can be found from Eqs (1.31)-(1.33) considered above. It is obvious that the results of such calculations are greatly influenced by the deformation energy of cold nuclei. For a fixed total energy the surface of the deformation energies directly determines the excitation energy of the heated nucleus and, consequently, the dependance of entropy on the deformation of the nucleus.

The shell correction method has proved extremely helpful in describing deformations of cold nuclei [23, 24].

Under this method, the deformation energy of nuclei can be represented in the form of two components:

$$\mathcal{E}_{\text{def}}(\xi) = \mathcal{E}_{\text{LDM}}(\xi) + \delta\mathcal{E}_{0}(\xi), \tag{1.56}$$

where $\mathcal{E}_{\text{LDM}}(\xi)$ is the smooth energy component equivalent to the phenomenological liquid-drop model, and $\delta\mathcal{E}_{0}(\xi)$ is the irregular component due to the shell inhomogeneities of the single-particle spectrum. Written in this manner,

Eq. (1.56) implies that pair correlations have been taken into account in both its terms. In practical calculations the parameters of the liquid-drop deformation energy are determined empirically from the analysis of masses and fission barriers of nuclei [25], and the fluctuating shell correction is calculated from specific single-particle level schemes on the basis of relations of type (1.39). By this method a satisfactory quantitative description was obtained for experimental data on the equilibrium deformations of nuclei, and by including mirror-asymmetric and non-axial deformations of the shell potential in the calculations it was also possible to arrive at fairly good agreement with experiment in the analysis of the double-humped fission-barrier structure of actinide nuclei [24]. In view of all these results we can regard the deformation energy found by the shell correction method as a reliable basis for the study of entropy changes during variations in the shape of the nucleus.

A qualitative evaluation of the expected effects can be obtained even by the simplest analysis of Eqs (1.44) and (1.56). We determine the excitation energy of the nucleus for a particular deformation in accordance with (1.56) as

$$U(\xi) = \mathcal{E} - \mathcal{E}_{\mathrm{LDM}}(\xi) - \delta\mathcal{E}_0(\xi). \tag{1.57}$$

If we now use the relationships of the Fermi-gas model but take account in them the shell changes in the level density parameter (1.44), the dependence of the square of entropy on deformation of the nucleus can be represented in the form

$$\frac{S^2}{4\tilde{a}} = \mathcal{E} - \mathcal{E}_{\mathrm{LDM}}(\xi) - \delta\mathcal{E}_0(\xi)\exp(-\gamma U(\xi)). \tag{1.58}$$

It will be seen from this that at low excitation energies a decisive role is played by the last term and that entropy is maximum when the deformation corresponds to the minimum value of shell correction. Thus for low excitation the equilibrium deformation of the nucleus remains the same as in the ground state. But when we go on to the region of high excitation energies, the last term in Eq. (1.58) asymptotically tends to zero, and the dependence of entropy on deformation is then determined by the liquid-drop component of the deformation energy. The maximum of entropy in this case corresponds to the spherical shape of the nucleus. Therefore, regardless of

the equilibrium deformation of the ground state, the highly excited nuclei should be spherical.

A fuller picture of the changes in the shape of the nucleus with its excitation is given by direct microscopic calculations of entropy and other thermodynamic potentials using specific single-particle level schemes [33, 34]. As thermodynamic potentials characterizing the properties of excited nuclei, apart from entropy $S(\mathcal{E},\xi)$, we can also consider the free energy

$$F(t,\xi) = \mathcal{E}(t,\xi) - tS \tag{1.59}$$

or the energy at a fixed entropy

$$\mathcal{E}(S,\xi) = F(t,\xi) + tS. \tag{1.60}$$

In the description of the equilibrium characteristics all the potentials mentioned are equivalent, and the equilibrium shape of the nucleus can be found from the extremum condition for any of them. However, potentials (1.59) and (1.60) are more convenient for analysing the rigidity of the shape of the nucleus

$$C_{ij} = \frac{\partial^2 F(t,\xi)}{\partial\xi_i\ \partial\xi_j} = \frac{\partial^2 \mathcal{E}(S,\xi)}{\partial\xi_i\ \partial\xi_j}. \tag{1.61}$$

It should be noted that the two methods of defining rigidity will coincide only in the case of equilibrial shapes but may differ appreciably for non-equilibrial shapes. At zero temperature both thermodynamic potentials $F(t,\xi)$ and $\mathcal{E}(S,\xi)$ are transformed into the deformation energy (1.56) of the non-excited (cold) nucleus.

Figure 10 gives the calculated maps of the free energy equipotential surfaces of the ^{160}Yb nucleus for three temperatures [34]. Deformation parameter γ determines the non-axiality of the shape of the nucleus $\gamma = 0^\circ$ corresponds to the prolate ellipsoid shape and $\gamma = 60^\circ$ to the oblate ellipsoid shape. At a temperature of t = 0.2 MeV the excitation energy U < 1 MeV, and the given map of the free-energy surface virtually coincide with the deformation-energy maps discussed in Ref. [24] in the case of cold nuclei. It will be seen from the maps in Fig. 10 that at temperatures of up to 1 MeV there is only a comparatively small shift of the minimum towards smaller deformations but in the neighbourhood of the minimum the density of equi-potential lines, which characterizes the rigidity of the equilibrium shape, changes very noticeably. With a further increase in temperature to 2 MeV the

nucleus assumes a spherical shape and the map of the free-energy surface of the heated nucleus becomes similar to the deformation-energy map of the liquid-drop model.

In the case of the most typical nuclei the changes in equilibrium deformation and axial rigidity with temperature are shown in Fig. 11. In the magic spherical nucleus of ^{208}Pb the rigidity changes monotonically with temperature and exceeds approximately by a factor of two the evaluation obtained in the liquid-drop model, even at t = 2.0 MeV. Thus, in a spherical nucleus the weakening of the shell effects is comparatively slow. The situation is quite different in the originally deformed ^{126}Ba and ^{160}Er nuclei, where upon weakening of the shell effects with temperature there is destructive competition between the shell and liquid-drop deformation behaviours, as a result of which in the temperature region of t $\sim$ 1.2 MeV the rigidity decreases strongly and the shape of the nucleus may fluctuate considerably. Only with further increase in temperature is the liquid-drop rigidity of the equilibrium spherical shape restored.

The statistical approach formulated here can also be used to study the influence of angular momentum on the shape of heated nuclei. It is obvious from the above discussion that at high temperatures t $\gtrsim$ 2 MeV the changes in the shape of nuclei due to the angular momentum should correspond to the concepts of the rotating-liquid-drop model [35]. According to this model, with increase in the speed of rotation the nucleus will take the shape of an increasingly oblate ellipsoid. This deformation behaviour is maintained right up to the critical values of angular momentum J_{cr} = 70-80, at which the drop loses resistance to scission into two fragments. At the same time, studies on the shape of fast-rotating cold nuclei have shown that, owing to the influence of the shell effects, the equilibrium deformations of the states of the yrast band can differ considerably from the predictions of the liquid-drop model [36]. Thus, with excitation of the nucleus above the yrast band there should occur a transition from the shell deformations of fast-rotating nuclei to the liquid-drop deformations, which is similar to the transition considered in Figs 10 and 11 for non-rotating nuclei. The calculations - given in Ref. [34] - of the thermodynamic potentials for heated fast-rotating nuclei enable us to observe the corresponding changes in deformation for specific nuclei. The properties of nuclei with high angular momentum have in recent years been studied intensively in reactions with heavy ions. Since at the initial stage the products of these reactions generally have a

rather high excitation energy above the yrast band, it is very important for a consistent theoretical analysis of such experiments to take into consideration the changes in shape during the subsequent de-excitation of nuclei.

1.5. Parity distribution of excited states

In an analysis of experimental data on the level density of excited nuclei it is generally assumed that levels of positive and negative parity have an equiprobable distribution. Ericson's evaluation of the parity distribution of states of a non-interacting Fermi-particle system is regarded most often as justification of this assumption [5]. The evaluation is based on a simple combinatorial analysis of the probability of the occurrence of multi-particle configurations of different parity in a system of n-particles. If in the negative-parity states such a system has an even number of particles, the resultant parity of the multi-particle state of the system is positive, whereas in the case of an odd number of particles in such states the parity is negative. We denote the probability of one of the particles having the negative-parity state by p_- and one having the positive-parity state by $p_+ = 1 - p_-$. In the absence of any additional limitations the probability of finding k from n-particles in negative-parity states and the remaining n-k-particles in positive-parity states is determined by the product of the corresponding single-particle probabilities

$$\frac{n!}{k!(n-k)!} p_-^k \, p_+^{n-k}. \tag{1.62}$$

We obtain the resultant probability of the occurrence of negative parity in the system of n-particles by summing (1.62) over all values of k which lead to negative-parity states

$$P_- = \sum_{k=1,3,\ldots} \frac{n!}{k!(n-k)!} p_-^k \, p_+^{n-k} = \frac{1-(1-2p_-)^n}{2}. \tag{1.63}$$

Equation (1.63) is rigorous only for whole numbers of particles n. It can also be used, however, for probability evaluations of the parity distribution of the states of a system characterized by an average non-whole number of excited quasi-particles. For this purpose, we rewrite (1.62) in a form which is also valid for non-integer $\bar{n}$:

$$P_- = \frac{1 - m(|1-2p_-|)^{\bar{n}}}{2}; \quad P_+ = 1 - P_-, \tag{1.64}$$

where m is determined as

$$m = \left\{ \begin{array}{l} 1 \text{ for even-even and odd-odd nuclei} \\ 1 \text{ for } p_- < 1/2 \\ -1 \text{ for } p_- > 1/2 \end{array} \right\} \text{ for odd nuclei}$$

Within the framework of the Fermi-gas model the average number of excited quasi-particles is described by the expression

$$\bar{n} = 2g(\epsilon_f) \int_0^\infty \frac{dx}{1 + \exp(x/t)} = \frac{12 \ln 2}{\pi^2} \sqrt{aU^*}, \qquad (1.65)$$

and, in the case of non-magic nuclei excited by resonance neutrons, $\bar{n}$ can be evaluated as $\bar{n} \approx 0.8\sqrt{A} \approx 8\text{–}12$. For such numbers, even when p_- and p_+ differ considerably, P_- is close to 1/2 (to $p_- = 0.1$ correspond, for example, the values of $P_- = 0.42$ for $\bar{n} = 8$ and 0.46 for $\bar{n} = 12$). Hence for most nuclei the deviations from the equiprobable parity distribution of excited states should be sufficiently small even at energies corresponding to the neutron binding energy.

The above evaluation may, however, turn out to be oversimplified for near-magic nuclei*/, where the influence of the shell effects is strong and the number of excited quasi-particles comparatively small. The possibility of considerable irregularities occuring in the distribution of positive- and negative-parity levels in such nuclei is indicated in particular by the results of combinatorial calculations of level density carried out within the framework of the non-interacting particle model [37] as well as those based on the more rigorous quasi-particle-phonon model, which takes into account the collective excitations of nuclei [38, 39]. As a typical example of the latter model's results we can look at the histograms for the number of levels with angular momentum and parity $I^\pi = 1/2^+$ and $1/2^-$ in the 400 keV region in the ^{57}Fe nucleus [38]. Although the general trend of the exponential growth of level density with increase in excitation energy is in satisfactory agreement with the traditional statistical description, in the local energy sectors the level density fluctuations are fairly high and there are quite considerable differences in the number of positive- and negative-parity levels. For instance, at excitation energies close to neutron binding energy these calculations yielded a ratio of $P_+/P_- \approx 0.5$ for the ^{57}Fe nucleus and the value of $P_+/P_- = 6\text{–}8$ for the neighbouring even-even ^{58}Fe nucleus. Similar irregularities in the energy and parity distribution of levels have also been demonstrated in Refs [38, 39] for some other spherical nuclei. In the case

*/ Nuclei differing from the magic nuclei by one or two nucleons.

of deformed nuclei, the deviations from monotonic statistical dependences were substantially less and occurred only in the region of comparatively low excitation energies.

The quasi-particle-phonon model and the role of the collective excitations in the energy dependence of the level density of nuclei will be discussed in more detail in Chapter 3; at this juncture we shall try to investigate to what extent the above-mentioned deviations from the equiprobable parity distribution of excited states are due to the shell inhomogeneities of the single-particle spectrum. For this purpose, we determine the following quantities within the framework of the shell model

$$\bar{n} = \sum_j g_j \bar{n}_j; \; p_\pm = \sum_j p_j^{(\pm)} g_j \bar{n}_j / \bar{n}, \tag{1.66}$$

where g_j is the degree of degeneracy of the single-particle levels and $\bar{n}_j$ the Fermi function of the excited quasi-particle population of single-particle states; and probabilities $p_j^{(-)} = [1-(-1)^{l_j}]/2$, $p_j^{(+)} = 1 - p_j^{(-)}$ are determined by the orbital momentum of level l_j. Equations (1.66) can be used to calculate the dependence of $\bar{n}$ and p_- on temperature (or excitation energy) and Eq. (1.64) to find the contribution of the states with different parities to the total excited state density. For the most typical nuclei the calculation results for the dependence of $\bar{n}$, p_- and P_- on excitation energy are shown in Fig. 13 [40]. The calculations were based on the level scheme of the Saxon-Woods potential and took into account the influence of the pair correlations on the occupation number and the equation of state linking the temperature of the nucleus to its excitation energy (see below). It should be noted, however, that allowance for the pair correlations has little effect on the calculation results since the characteristics of the behaviour of $\bar{n}$ and p_- are determined mainly by the single-particle level scheme. In the case of ^{58}Fe and ^{124}Te nuclei, the values of P_- derived are close to the results of the above-mentioned microscopic combinatorial calculations of the corresponding values averaged over a fairly wide energy region (Fig. 13). Such averaging is necessary for eliminating energy fluctuations in the density of the positive- and negative-parity levels which occur in the combinatorial calculations.

The calculations of the quantities (1.66) performed in Ref. [40] for a wide group of nuclei show that although the differences of probability p_- from 1/2 are, as a rule, rather large for the proton and neutron single-particle level schemes, in the case of most spherical nuclei near the stability valley, these probabilities add together in a destructive sense and the parity distribution of the excited states of nuclei is very close to equiprobable.

Marked deviations from equiprobable distribution occur only in the region of nuclei with $Z \approx N = 26\text{-}30$, where $p_- \to 1$ and the number of excited quasi-particles $\bar{n}$ for $U = B_n$ is comparatively small. Thus, the combinatorial and statistical calculations of level density show that considerable deviations from the equiprobable parity distribution of the highly excited states of nuclei should evidently be expected only in nuclei of the iron group. In the case of heavier and especially deformed nuclei, these deviations at excitation energies comparable with the neutron binding energy should be small or else totally absent.

Experimental data on the parity distribution of the highly excited states of nuclei may in principle be obtained by analysing the mean distance D_1 between the neutron resonances corresponding to s- and p-neutrons. For equiprobable distribution of the excited states we should have $D_o/D_1 \approx 3$, and noticeable differences between the observed relation and this evaluation would be the most direct proof of deviations in the parity distribution. In the case of the ^{233}Th and ^{239}U isotopes the experimental values of $D_o/D_1 = 2.96 \pm 0.22$ and 2.86 ± 0.25 [41], respectively, are in satisfactory agreement with the concepts of equiprobable parity distribution of the excited states of deformed nuclei. For the ^{57}Fe and ^{59}Fe compound nuclei the observed relationships $4.1 \pm \frac{1.5}{0.9}$ and $4.0 \pm \frac{2.8}{1.8}$ [42] do not seem to confirm the considerable deviations from the equiprobable contribution of states of different parity as predicted by microscopic calculations [39]. However, the errors in experimental determination of D_o and D_1 for the above-mentioned nuclei and also for other spherical nuclei are still so high that it is evidently premature to draw any conclusions on their basis with regard to deviation from the equiprobable contribution of the positive- and negative-parity states.

1.6. Density of states for a fixed number of excited quasi-particles

In the description of level density considered above the energy and angular momentum or its projection in a given direction were chosen as the integrals of motion determining the state of the excited nucleus. There has recently been interest in the states of the nucleus which are characterized by the number of excited quasi-particles or some other equivalent quantity. Such states may occur in nuclear reaction cross-sections in the form of intermediate structures or in the form of the hard part of the evaporation spectrum during pre-equilibrium decay of the compound nucleus. These processes will be discussed below. Here we consider the results of analysing the statistical characteristics of the corresponding states within the framework of the non-interacting particle model.

Simple analytical expressions for the density of states of a nucleus with a fixed number of excited particles p and holes h can be obtained if the excited nucleus is regarded as a two-component ideal gas consisting of Boltzmann particles [43]. The statistical sum of such a gas in the continuous-spectrum approximation can be represented in the form

$$Q_{ph}(\beta) = \frac{1}{p!h!}\left(g\int_0^\infty \exp(-\beta\epsilon)\,d\epsilon\right)^{p+h} = \frac{1}{p!h!}\left(\frac{g}{\beta}\right)^{p+h} \tag{1.67}$$

and for the density of states of a system with a given excitation energy we can find a simple relationship

$$\rho_{ph}(U) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} \exp(\beta U)\, Q_{ph}(\beta)\,d\beta = \frac{g\,(gU)^{p+h-1}}{p!\,h!\,(p+h-1)!}. \tag{1.68}$$

The spin dependence of the density of the particle-hole states can be easily derived within the framework of the model of Boltzmann particles with a homogeneous spectrum of single-particle states (i.e. equidistant single-particle levels which are doubly degenerate with respect to the sign of projection of angular momentum and have an identical absolute value of projection m). The statistical sum in this case takes the form [44]:

$$Q_{ph}(\beta,\mu) = \frac{1}{p!h!}\left(\frac{g}{2\beta}\right)^{p+h} \times$$
$$\times \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty} C_p^k\, C_h^\nu \exp\left[\mu m(p+h-2k-2\nu)\right], \tag{1.69}$$

where $C_p^k = p!(p-k)!k!$. Calculating the integrals of the corresponding inverse Laplace transform for the density of states, we obtain

$$\rho_{ph}(U,M) = \frac{g\,(gU)^{p+h-1}}{2^{p+h}\,p!\,h!\,(p+h-1)!}\, C_{p+h}^{\frac{1}{2}\left(p+h-\frac{M}{m}\right)} \tag{1.70}$$

It is obvious that in the given model the angular-momentum projection M should always be a multiple of the projection of the single-particle momentum m and vary in steps of $\Delta M = 2m$. By summing (1.70) over all possible values of M, we obtain the total density of the particle-hole states (1.68).

With the help of (1.70) we find the density of levels with the given angular momentum

$$\rho_{ph}(U,J) = \frac{\left(\frac{J}{m}+1\right)\rho_{ph}(U)}{\left(p+h+2+\frac{J}{m}\right)2^{p+h-1}}\, C_{p+h}^{\frac{1}{2}\left(p+h-\frac{J}{m}\right)}. \tag{1.71}$$

Equations (1.70) and (1.71) can be easily transformed into a simpler form if we used the Stirling formula and the small-momentum approximation $M \ll m\,(p + h)$:

$$\rho_{ph}(U, M) = \sqrt{\frac{2}{\pi(p+h)}}\,\rho_{ph}(U)\exp\left[-M^2(p+h-1)/2m^2(p+h)\right]. \qquad (1.72)$$

If we go on to the density of states per unit of projection M, i.e. divide (1.72) by 2m, we get

$$\rho_{ph}(U, M) = \frac{\rho_{ph}(U)}{\sqrt{2\pi}\,\sigma_{ph}}\exp(-M^2/2\sigma_{ph}^2). \qquad (1.73)$$

This equation has a form similar to the spin dependence of the density of states of the Fermi-gas model (1.14). The spin cut-off parameter for a nucleus with a fixed number of excited particles and holes equals

$$\sigma_{ph}^2 = m^2\frac{(p+h)^2}{p+h-1} \approx m^2(p+h). \qquad (1.74)$$

Let us now go on to consider a more realistic model of non-interacting Fermi particles. We are interested in states with a specified number of excited particles and holes $n = p + h$. The statistical sum of the corresponding large canonical ensemble can be written in a form similar to (1.4) if we introduce an additional quantum number descrbing the presence of a particle or a hole in the ν-th single-particle state. We determine this number as

$$q_\nu = \begin{cases} -1, \text{ if } \epsilon_\nu < \epsilon_f; \\ +1, \text{ if } \epsilon_\nu > \epsilon_f. \end{cases} \qquad (1.75)$$

where ϵ_f is the Fermi energy. For a homogeneous single-particle spectrum in the low-temperature approximation the saddle-point equation can be written in the form

$$\left.\begin{aligned} \beta^2 U &= \frac{\pi^2}{6}g + \frac{1}{2}(\mu m+\gamma)^2 + g\int_{\mu m-\gamma}^{\mu m+\gamma}\ln\left[1+\exp(-x)\right]dx; \\ \beta n &= g\left\{\ln\left[1+\exp(\gamma+\mu m)\right] + \ln\left[1+\exp(\gamma-\mu m)\right]\right\}; \\ \beta M &= gm\left\{\ln\left[1+\exp(\gamma+\mu m)\right] - \ln\left[1+\exp(\gamma-\mu m)\right]\right\} \end{aligned}\right\} \qquad (1.76)$$

and the entropy of the nucleus can be represented as

$$S = 2\beta U - \gamma n - \mu M, \tag{1.77}$$

where γ is a Lagrange multiplier fixing the mean number of excited particles and holes. These relations can serve as the basis for the different approximations.

Let us consider the small-momentum approximation. Expanding (1.76) in a series in $\mu m \ll 1$, for the density of states $\rho_n(U,M)$ we obtain a relation of type (1.73), in which the total density of the particle-hole states is determined by the relationships

$$\left.\begin{aligned} \rho_n(U) &= \Phi_n \frac{\exp(2\beta U - \gamma n)}{g^{1/4} U^{5/4}}; \\ \Phi_n &= \left[\frac{\ln(1+e^{\gamma})}{2\pi n \sqrt{gU}}\right]^{3/2} \frac{1+e^{-\gamma}}{[1-(1+e^{-\gamma})n^2/4gU]^{1/2}}, \end{aligned}\right\} \tag{1.78}$$

and β and γ by the equations of state

$$\left.\begin{aligned} \beta^2 U &= \pi^2 g + 2g \int_0^{\gamma} \ln(1-e^{-x})dx; \\ \beta n &= 2g \ln(1-e^{\gamma}). \end{aligned}\right\} \tag{1.79}$$

The spin cut-off parameter σ_n^2 in the small-momentum approximation takes the form

$$\sigma_n^2 = M/\mu = 2gm^2/\beta(1+e^{-\gamma}). \tag{1.80}$$

The solution of Eqs (1.79) and the behaviour of $\rho_n(U)$ were studied in detail in Ref. [45] and the behaviour of σ_n^2 in Ref. [46].

For $\gamma = 0$ the expressions for entropy (1.78) and excitation energy (1.79) coincide with Eqs (1.12) of the Fermi-gas model considered earlier, and the second equation in (1.79) in this case determines the average number of particles and holes excited at the given temperature

$$\bar{n} = 2gt \ln 2 = \frac{\sqrt{24}\ln 2}{\pi}\sqrt{gU}. \tag{1.81}$$

For $M = 0$ the ratio of thermodynamic functions to those at $\gamma = 0$ depends only on the $n/\bar{n}$ ratio, i.e. the corresponding curves have a universal shape independent of the parameters of the system considered. The behaviour of

the quantities determining the density of states (1.78) in the dimensionless form is shown in Fig. 14. Comparison between the $\rho_n(U)$ values found and the much simpler calculations of the Boltzmann-gas model demonstrates very close agreement of the calculation results [44, 45]. There are differences only in the region of low-excitation energies for a fairly large number of excited particles, when the Pauli principle makes a substantial contribution. In Ref. [47] it was shown that the description of the density of the Fermi-gas particle-hole states by relationship of type (1.68) can be improved considerably if we use in them the effective excitation energy

$$U^* = U - [p(p-1) + h(h-1)]/4g. \tag{1.82}$$

The accuracy of such a description is sufficiently high over the entire energy range, excluding a small region near the threshold of the n-particle excitations.

Let us consider the problem of the maximum projection of the angular momentum in a system with a fixed number of excited quasi-particles. At the limit $\gamma \to \pm\infty$, the first equation in (1.76) determines the temperature of the system

$$t = \beta^{-1} = U/n, \tag{1.83}$$

which coincides with the temperature of the Boltzmann-particle gas. The maximum value of M in this case is determined by the number of quasi-particles

$$M'_{\max} = mn. \tag{1.84}$$

Thus, for $n \ll \bar{n}$ the Pauli principle does not impose any great limitations on the system, and the statistical characteristics of the n-quasi-particle excitations are described satisfactorily by the Boltzmann-particle gas model relationships.

For $\gamma \to \infty$ Eqs (1.76) can be transformed into

$$\begin{aligned} U &= \frac{\pi^2}{3\beta^2} g + \frac{n^2}{4g} + \frac{M^2}{4gm^2}; \\ M &= 2gm^2 \frac{\mu}{\beta}. \end{aligned} \tag{1.85}$$

For entropy and the maximum value of the angular momentum in this case we obtain the relations

$$\left.\begin{aligned} S &= 2\left[\frac{\pi^2}{3} g\left(U-\frac{n^2}{4g}-\frac{M^2}{4gm^2}\right)\right]^{1/2}; \\ M''_{max} &= 2m\left(gU-\frac{n^2}{4g}\right)^{1/2}. \end{aligned}\right\} \qquad (1.86)$$

From Eq. (1.86) we can easily find the maximum possible value of the angular-momentum projection for a given U:

$$M_{max} = m\sqrt{2gU} = \frac{\pi}{\sqrt{12}\ln 2} m\bar{n}. \qquad (1.87)$$

The solution of the system of Eqs (1.76) over the whole range of possible changes in the angular momentum was found by the authors of Refs [44], and Fig. 15 shows for different values of $n/\bar{n}$ the obtained dependence on $M/m\bar{n}$ of the ratio between the entropy $S_n(M)$ of the system with a fixed number of quasi-particles and the entropy $\bar{S} = (2\pi^2 gU/3)^{1/2}$, corresponding to the average number of quasi-particles $\bar{n}$ for M = 0. The dot-dash curves limit the region of permissible values of M in accordance with conditions (1.85) and (1.86) derived above. Maximum angular momentum (1.87) at the given excitation energy is attained for $n/\bar{n} = 1.31$. The dashed curve represents the results of calculation in the approximation of small angular momenta (Fig. 15). It will be seen that this approximation quite satisfactorily describes the entropy of the system for any M and $n/\bar{n}$, except for the region of large momenta for $n > \bar{n}$, i.e. when M is close to M''_{max}. Since the behaviour of the state of density of the system is determined mainly by the dependence of entropy on the integrals of motion, the approximation of small angular momenta quite satisfactorily describes the density of the particle-hole states virtually throughout the region of permissible values of M.

In the relationships considered above the effects due to nucleon pairing can be taken into account if it is assumed that additional energy 2Δ has to be expended in order to excite the particle-hole pair [43]. This assumption is equivalent to using in the above relations the effective excitation energy

$$U^* = \left\{\begin{array}{l} U - n\Delta \text{ for even } n; \\ U - (n-1)\,\Delta \text{ for odd } n. \end{array}\right\} \qquad (1.88)$$

The next chapter gives a more rigorous analysis of the influence of the correlation interaction of nucleons on the density of states with a fixed number of excited quasi-particles.

The influence of the shell effects on the state density of a nucleus with a fixed number of excited particles and holes can be considered if the corresponding thermodynamic functions are calculated for a realistic spectrum of single-particle states. The results of such calculations were analysed in Ref. [48]. Consideration of the shell structure of the single-particle spectrum is qualitatively equivalent to introducing the level density parameter g(U), dependent on excitation energy, into the relations considered above. However, this dependence is much less universal for states with a fixed number of excited particles than in the case of the excited nuclei in thermodynamic equilibrium considered above.

As has already been pointed out, the statistical properties of n-quasi-particle states are of interest for the study of the characteristics of excited nuclei which are in thermodynamic non-equilibrium. At present, the theory of non-equilibrium phenomena in nuclei is clearly still at an early stage of development and it is difficult to say what experimental material can be used to verify the relationshipsconsidered. These relationships are applied extensively to the model of pre-equilibrium evaporation of particles; however, as is shown below, even many of the assumptions of this model need verification as well as a sounder basis.

Chapter 2

THE ROLE OF SUPERCONDUCTING-TYPE PAIR CORRELATIONS

The strong influence exerted by the pair correlations of the interacting Fermi particles on the macroscopic properties of a system is seen most clearly in the case of superconductivity of metals at low temperatures. The microscopic theory of superconductivity developed by Bardeen, Cooper and Schriffer [49] and Bogolyubov [50] has not only explained this interesting phenomenon but also provided the basis for successfully solving a large number of general problems of the many-body theory. In atomic nuclei the theory of pair correlations of the superconducting type was developed at the end of the fifties by V.G. Solov'ev [51] and S.T. Belyaev [52]. It initiated an extensive study of the nuclear structure on the basis of the microscopic approach. Within the framework of this theory it was possible to explain such general properties of nuclei as the gap in the spectra of quasi-particle excitations of even-even nuclei and the increased density of single-particle excitations in odd nuclei, considerable decrease in the moments of inertia of deformed nuclei in relation to the rigid-body value and the numerous strengthening factors of the α-, β- and γ-transitions [51-53]. From the standpoint of a statistical description of nuclei it is quite legitimate to wonder to what extent the correlation effects occurring in the low-lying states will be reflected in the statistical properties of nuclei. The present chapter is in fact concerned with this particular question.

2.1. Description of the thermodynamic functions in the superfluid model of the nucleus

We will consider the relationships basic to the theory of superconductivity as applied to a description of the statistical characteristics of excited nuclei. Let us take the simplest form of pairing interaction between nucleons which corresponds to the Hamiltonian

$$H - \lambda N = \sum_{k,\,s} (\epsilon_k - \lambda) a^+_{ks} a_{ks} - G \sum_{k,\,k'} a^+_{k+} a^+_{k-} a_{k'-} a_{k'+}, \qquad (2.1)$$

where ϵ_k is the energy of single-particle levels in the self-consistent mean field, a^+_{ks} and a_{ks} are the particle creation and annihilation operators and G is the effective interaction constant. This relationship contains in an explicit form the sign $s = \pm$ of the projection of the single-particle angular momentum on the symmetry axis together with an implicit assumption that the

single-particle levels are doubly degenerate with respect to s. We have a level spectrum of this kind in deformed axially symmetric nuclei [53], and there is no difficulty in generalizing this consideration for the case of spherical or non-axial nuclei.

The methods of the theory of superconductivity are based on determining the transformation with the help of which calculation of the properties of a system of interacting particles with Hamiltonian (2.1) can be reduced to a model Hamiltonian for interacting particles [50]. This transformation is the transition from particle operators to quasi-particle operators

$$a_{k+} = u_k a_{k-} + v_k a^+_{k+};\ a_{k-} = u_k a_{k+} - v_k a^+_{k-}. \tag{2.2}$$

In order for the new operators to satisfy the same commutation relationships as the particle operators, coefficients u_k and v_k must satisfy the condition

$$u_k^2 + v_k^2 = 1. \tag{2.3}$$

The remaining relationships needed to determine the transformation coefficients (2.2) can be found from the conditions of the best approximation of the system's thermodynamic potentials of interest to us by the model Hamiltonian.

We determine the model Hamiltonian in the form

$$H_0 = U_0 + \sum_{k,s} E_k a^+_{ks} a_{ks} \tag{2.4}$$

and we denote the corresponding thermodynamic averages for it by

$$\langle O\rangle_0 = \mathrm{Sp}\,[O\exp(-\beta H_0)]/\mathrm{Sp}\,[\exp(-\beta H_0)]. \tag{2.5}$$

Using the canonical transformation (2.2), we can easily show that when averaging with the model Hamiltonian different from zero, there will only be the averages of the following pairs of particle operators:

$$\left.\begin{aligned} &\langle a^+_{k+} a_{k+}\rangle_0 = \langle a^+_{k-} a_{k-}\rangle_0 = u_k^2 \bar{n}_k + v_k^2 (1-\bar{n}_k);\\ &\langle a^+_{k+} a^+_{k-}\rangle_0 = \langle a_{k-} a_{k+}\rangle_0 = u_k v_k (1-2\bar{n}_k), \end{aligned}\right\} \tag{2.6}$$

where $\bar{n}_k = \langle \alpha^+_{ks}\alpha_{ks}\rangle_0 = [1 + \exp(\beta E_k)]^{-1}$ is the occupation number of the quasi-particle levels. For the average value of the Hamiltonian (2.1) we obtain the following relation as a result of transformation

$$\begin{aligned} \mathcal{E} - \lambda N = \langle H - \lambda N\rangle_0 = 2\sum_k (\epsilon_k - \lambda)[u_k^2 \bar{n}_k + v_k^2(1-\bar{n}_k)] -\\ - G[\sum_k u_k v_k (1-2\bar{n}_k)]^2. \end{aligned} \tag{2.7}$$

In this form of notation we disregarded the renormalization of the mean-field levels made necessary by the interaction, since in the self-consistent selection of the mean field the corresponding effects are included in the definition of ϵ_k.

From the variational conditions

$$\frac{\delta\langle H-\lambda N\rangle_0}{\delta v_k}=0;\quad \frac{\delta\langle H-\lambda N\rangle_0}{\delta \bar{n}_k}=\frac{\delta\langle H_0\rangle_0}{\delta \bar{n}_k} \tag{2.8}$$

we obtain the equations

$$\left.\begin{aligned} 2(\epsilon_k-\lambda)u_k v_k - G(u_k^2-v_k^2)\sum_{k'} u_{k'}v_{k'}(1-2\bar{n}_{k'})=0;\\ 2(\epsilon_k-\lambda)(u_k^2-v_k^2)+Gu_k v_k\sum_{k'} u_{k'}v_{k'}(1-2\bar{n}_{k'})=E_k,\end{aligned}\right\} \tag{2.9}$$

which, jointly with condition (2.3), determine the coefficients u_k, v_k and the quasi-particle energy E_k.

Let us derive the correlation function

$$\Delta = G\sum_k u_k v_k(1-2\bar{n}_k). \tag{2.10}$$

It will be easily seen that Eqs (2.9) have two solutions: a trivial $u_k v_k = 0$ and a non-trivial $u_k v_k \neq 0$. The trivial solution corresponds to the coefficients

$$\left.\begin{aligned} u_k=1,\ v_k=0,\ \text{if}\ \ \epsilon_k>\lambda;\\ u_k=0,\ v_k=1,\ \text{if}\ \ \epsilon_k<\lambda\end{aligned}\right\} \tag{2.11}$$

and to the quasi-particle energy $E_k = |\epsilon_k - \lambda|$, i.e. this solution is simply a change-over in the non-interacting particle model to the language of particles and holes.

For the non-trivial solution we have

$$\left.\begin{aligned} u_k^2=\frac{1}{2}\left(1+\frac{\epsilon_k-\lambda}{E_k}\right);\ v_k^2=\frac{1}{2}\left(1-\frac{\epsilon_k-\lambda}{E_k}\right);\\ E_k=\sqrt{(\epsilon_k-\lambda)^2+\Delta^2}\ .\end{aligned}\right\} \tag{2.12}$$

Substituting coefficients u_k and v_k into Eq. (2.10) and applying the condition of conservation of the number of particles, we find the system of equations to determine the chemical potential λ and correlation function Δ:

$$N = \sum_k \left[1 - \frac{\epsilon_k - \lambda}{E_k}(1 - 2\bar{n}_k)\right]; \frac{2}{G} = \sum_k \frac{1 - 2\bar{n}_k}{E_k}, \qquad (2.13)$$

Applying (2.12), we rewrite the expression for the energy of the system in the form

$$\mathcal{E} - \lambda N = \sum_k (\epsilon_k - \lambda)\left[1 - \frac{\epsilon_k - \lambda}{E_k}(1 - 2\bar{n}_k)\right] - \frac{\Delta^2}{G}, \qquad (2.14)$$

and we also obtain the entropy corresponding to the model Hamiltonian

$$S = 2\sum_k [\beta E_k \bar{n}_k - \ln(1 - \bar{n}_k)]. \qquad (2.15)$$

At zero temperature Eqs (2.13) determine the correlation function Δ_o of the ground state of the system, while Eq. (2.14) gives us the energy of the ground state.

To be used in the analysis of the thermodynamic functions of nuclei, the relations considered above need to be generalized for a two-component system. If we apply the traditional assumption of the theory of the nucleus that pair interaction occurs only between identical nucleons [53], the corresponding generalization reduces simply to independent consideration of the proton Δ_Z and neutron Δ_N correlation functions on the basis of Eqs (2.13) and to the additive summation of energy and other thermodynamic functions.

In order to make a qualitative analysis of the influence of the correlation effects on the behaviour of the thermodynamic characteristics of the nucleus, we can use the results of solving Eqs (2.13) in the continuous-spectrum approximation. Let us consider the dependence of the correlation function on temperature. For this purpose, we rewrite the second equation in (2.13) in the form

$$g\int_0^{\tilde{\epsilon}} \frac{dx}{\sqrt{x^2 + \Delta_0^2}} = g\int_0^{\tilde{\epsilon}} \frac{dx}{\sqrt{x^2 + \Delta^2}} - 2gI\left(\frac{\Delta}{t}\right), \qquad (2.16)$$

where $\tilde{\varepsilon}$ is the boundary of the single-particle spectrum, and

$$I(y) = \int_0^\infty \frac{dx}{\sqrt{x^2+y^2}\left[1+\exp\sqrt{x^2+y^2}\right]} \tag{2.17}$$

(in view of the rapid convergence of the integral the integration limit has been extended to infinity). In the region of low temperatures $t \ll \Delta$, the integral (2.17) can be calculated comparatively simply by expansion in a series with respect to $1/y$. For the correlation function in this region we obtain the temperature dependence

$$\Delta = \Delta_0 \left[1 - \sqrt{\frac{2\pi t}{\Delta_0}} \exp(-\Delta_0/t)\right] . \tag{2.18}$$

Calculations of the integral (2.17) for $\Delta \ll t$ are more cumbersome, but they are considered in detail in the literature on the theory of superconductivity [3]. Retaining the first terms of the expansion of the integral in a series for Δ/t, we rewrite the correlation function equation in the form

$$\ln\frac{\Delta_0}{\Delta} = \ln\frac{\pi t}{\gamma\Delta} + \frac{7\zeta(3)}{8\pi^2}\frac{\Delta^2}{t^2} , \tag{2.19}$$

where $\ln\gamma = C = 0.577$ is the Euler constant and $\zeta(3) = 1.202$ the Riemann zeta function. It will be seen from (2.19) that Δ vanishes at the critical temperature t_{cr} determined by

$$t_{cr} = \gamma\Delta_0/\pi = 0{,}567\,\Delta_0 . \tag{2.20}$$

Near the critical temperature, in the first order in $1-t/t_{cr}$ we obtain from (2.19)

$$\Delta = t_{cr}\left[\frac{8\pi^2}{7\zeta(3)}\left(1-\frac{t}{t_{cr}}\right)\right]^{1/2} . \tag{2.21}$$

Similarly, we can consider the basic characteristics of the behaviour of energy, entropy and other thermodynamic functions of the system. At zero temperature the inclusion of pair correlations leads to a reduction in the energy of the ground state of the system, and the gain in energy with respect to the energy of the ground state of non-interacting particles (the so-called energy of condensation) can be easily evaluated for the model of homogeneous single-particle spectrum

$$E_{cond} = \text{nonint.} - \text{inter.} = g\int_{-\tilde{\epsilon}}^{0} x\,dx -$$

$$-\left\{\frac{1}{2}g\int_{-\tilde{\epsilon}}^{\tilde{\epsilon}}x\left(1-\frac{x}{\sqrt{x^2+\Delta_0^2}}\right)dx-\frac{1}{4}g\Delta_0^2\int_{-\tilde{\epsilon}}^{\tilde{\epsilon}}\frac{dx}{\sqrt{x^2+\Delta_0^2}}\right\}=\frac{1}{4}g\Delta_0^2. \tag{2.22}$$

In this evaluation use is made of the traditional assumption $\tilde{\epsilon} \gg \Delta_o$. Above the critical temperature, Eqs (2.14) and (2.15) are equivalent to the expressions for the interacting-particle model and the temperature dependence of entropy S and excitation energy U is determined by the relationships of the Fermi-gas model

$$\left.\begin{aligned} S &= 2at, \\ U &= at^2+\frac{1}{4}g\Delta_0^2 \end{aligned}\right\} \text{ for } t \geqslant t_{cr}. \tag{2.23}$$

The shift in the excitation energy directly reflects a shift in the energy of the ground state of the system (2.22).

For $t < t_{cr}$ the calculations of the temperature dependence of the thermodynamic functions are similar to those considered above for the correlation function. Near the critical point this dependence is described by

$$\left.\begin{aligned} S &= 2at\left[1-\frac{12}{7\zeta(3)}\left(1-\frac{t}{t_{cr}}\right)\right]; \\ U &= at^2\left[1-\frac{24}{7\zeta(3)}\left(1-\frac{t}{t_{cr}}\right)\right]+\frac{1}{4}g\Delta_0^2. \end{aligned}\right\} \tag{2.24}$$

In this way the characteristics of the temperature dependence of the correlation function have a direct effect on the behaviour of the thermodynamic functions of the system. On an analogy with superconductors, the temperature t_{cr} in nuclei is called the point-of-phase transition from the superconducting (or superfluid) state to the normal Fermi-gas state. Considering the finite dimensions of nuclei, the concept of phase transition should not be understood in the literal sense of the words. This problem is discussed below in greater detail.

In the continuous-spectrum approximation the temperature dependence of the correlation function over the entire temperature range $t \leqslant t_{cr}$ can be found by expressing the integral (2.17) in terms of the Macdonald function:

$$\ln\frac{\Delta_0}{\Delta}=2\sum_{\nu=1}^{\infty}(-1)^{\nu+1}K_0\left(\frac{\nu\Delta}{t}\right). \tag{2.25}$$

Similarly, we can write the relationships for entropy and excitation energy

$$\left.\begin{aligned} S &= 2\frac{g\Delta^2}{t}\sum_{\nu=1}^{\infty}(-1)^{\nu+1}K_2\left(\frac{\nu\Delta}{t}\right); \\ U &= \frac{1}{2}St + \frac{1}{4}g(\Delta_0^2 - \Delta^2). \end{aligned}\right\} \qquad (2.26)$$

These expressions were tabulated in the studies on the theory of superconductivity [54], and the corresponding temperature dependences of the thermodynamic functions are given in Fig. 16 in comparison with the dependence of the similar functions of the Fermi-gas model.

We will now discuss some of the features involved when applying the relationships of the superconductivity theory to the problem of describing the level density of excited nuclei. The solution of this problem in the general case can be obtained by the saddle-point method, and the results thereof can be represented in a form similar to (1.8). Thus, in addition to the thermodynamic functions considered above, we further need to obtain relationships for the pre-exponential factor. In the determination of the latter relationship the problem of sequence in the behaviour of the variational transform of the superconductivity theory and calculation of contour integrals (1.2) is very important. If we perform the variational transform up to the integration point, there may be spurious terms in the model-Hamiltonian derivatives during the calculation of the pre-exponential factor through the dependence of the model Hamiltonian on the integration variables.

In order to avoid such terms, we need to formulate the variational procedure directly for the density of states. There are various alternatives for solving this problem. The result can be obtained in a fairly simple form by the saddle-point method, which should be applied, however, before using the transition to the model Hamiltonian. For a single-component system the determinant of the second derivatives in the pre-exponential factor for the density of states within the framework of this approach takes the form

$$\det = \begin{vmatrix} \langle H^2\rangle_0 - \langle H\rangle_0^2 & \langle H\rangle_0\langle N\rangle_0 - \langle HN\rangle_0 \\ \langle H\rangle_0\langle N\rangle_0 - \langle HN\rangle_0 & \langle N^2\rangle_0 - \langle N\rangle_0^2 \end{vmatrix}, \qquad (2.27)$$

where the averages for the model Hamiltonian should be calculated at temperature and chemical potential values corresponding to the saddle point. Performing

the calculations of the averages and dropping the insignificant terms proportional to G^2, we obtain

$$\det = [2\sum_k E_k^2 \bar{n}_k(1-\bar{n}_k)]\left\{2\sum_k [\bar{n}_k(1-\bar{n}_k) + \right.$$
$$\left. + \frac{\Delta^2}{2E_k^2}(1-2\bar{n}_k)^2]\right\} - [2\sum_k \bar{n}_k(1-\bar{n}_k)]^2. \qquad (2.28)$$

The temperature dependence of the determinant of the second derivatives is shown in Fig. 16, together with a similar dependence for the Fermi-gas model. There is no difficulty in generalizing Eqs (2.27) for a two-component particle system, and we will not dwell on this point.

The specific characteristics of the temperature dependence of the thermodynamic functions resulting from pair correlations of the superconducting type are most pronounced in the behaviour of the moment of inertia $\mathcal{J}_{\|}$, directly associated with the spin cut-off parameter of level density $\sigma_{\|}^2$. In the small-momentum approximation the relationship for this parameter is written usually on the analogy of the independent-particle model in the form of (1.34). For a homogeneous single-particle spectrum in this case we have

$$\mathcal{J}_{\|} = \mathcal{J}_0 \int_0^{\infty} dx \, \mathrm{ch}^{-2} \sqrt{x^2 + (\Delta/2t)^2}\,, \qquad (2.29)$$

where $\mathcal{J}_0 = gm_f^2$. We find the behaviour of the moment of inertia in the region of very low temperatures by obtaining the asymptotic dependence of the integrand for $\Delta \gg t$:

$$\mathcal{J}_{\|} = \mathcal{J}_0 \sqrt{\frac{2\pi\Delta_0}{t}} \exp(-\Delta_0/t). \qquad (2.30)$$

In the neighbourhood of the phase-transition point, using the expansion of integral (2.29) in a series in $\Delta \ll t$, we obtain

$$\mathcal{J}_{\|} = \begin{cases} \mathcal{J}_0[1-2(1-t/t_{cr})] & \text{for } t < t_{cr}, \\ \mathcal{J}_0 & \text{for } t \geqslant t_{cr}. \end{cases} \qquad (2.31)$$

The behaviour of the moment of inertia (2.29) over the whole temperature range in shown in Fig. 16.

The above relationships of the superconductivity theory began to be used in the analysis of the statistical properties of nuclei, first of all, to describe the behaviour of the angular anisotropy of the induced-fission products of transuranium nuclei [55], and then to analyse the density of excited states [56]. In all these studies the thermodynamic functions were calculated in the continuous-spectrum approximation. Although this approximation is quite convenient for qualitative consideration of the effects due to pair correlations, we cannot confine ourselves to it in many cases for quantitative analysis of the level density of excited nuclei. The shell inhomogeneities in the single-particle spectrum lead to a strong modulation of the correlation effects, and it is very important to take account of this fact in the self-consistent consideration of the characteristics of the low-lying and highly excited states of nuclei. The presence of shell inhomogeneities in the single-particle spectrum leads to the dependence of the level density parameter a and the average value of the angular momentum projection $\overline{m}^2$ on the excitation energy of the nucleus. Consideration of this dependence is equivalent to non-linear deformation of the scales on the axes in Fig. 16. An evaluation of the effects expected in this connection can be made on the basis of calculation data for shell changes in the behaviour of parameters a and $\overline{m}^2$, shown in Fig. 6. For a stricter consideration of the influence of the shell effects on the behaviour of the thermodynamic characteristics of the superfluid model it is necessary to solve Eqs (2.13)-(2.15) directly for the single-particle spectrum of levels for specific nuclei. Numerous examples of such calculations were analysed in Refs [12-18].

It should be noted that in the different studies [12, 18, 56] describing level density in the superfluid model of the nucleus the relationships for the pre-exponential factor differ markedly. The derivatives with respect to the intensive variables β and α of the correlation function of the system are, as a rule, taken into account by these relationships, and this gives rise to a discontinuity in the energy dependence of level density at the phase-transition point. Relationships (2.27) obtained by the method considered above contain no such derivatives, and the calculated level density at the critical point has only a break, as indeed there should be for phase transitions of the second kind [57].

The above relationships were obtained for the simplest form of the Hamiltonian for the superfluid model of the nucleus (2.1). If we use an interaction of a more general form

$$H_{int} = -\sum_{k,k'} G_{kk'}\, a^{+}_{k+} a^{+}_{k-} a_{k'-} a_{k'+}, \tag{2.32}$$

we get almost the same formulae (2.12) for the coefficients of variational transformation but with the correlation function Δ_k dependent on the states. The earlier form will be retained also by the expressions for the number of particles, entropy and pre-exponential factor, and it is only the relation for the energy of the system that will alter slightly

$$\mathcal{E} - \lambda N = \sum_k \left\{ (\epsilon_k - \lambda) \left[1 - \frac{\epsilon_k - \lambda}{E_k} (1 - 2\bar{n}_k) \right] - \frac{\Delta_k^2}{2E_k} (1 - 2\bar{n}_k) \right\}. \tag{2.33}$$

The correlation functions for the form of interaction used are determined by the integral equation

$$\Delta_k = \sum_{k'} G_{kk'} \frac{1 - 2\bar{n}_{k'}}{E_{k'}} \Delta_{k'}. \tag{2.34}$$

At zero temperature the solutions of this equation for the matrix elements of interaction in the form of δ-forces or Gaussian forces of a finite radius were analysed in Refs [58]. It was found that for realistic forces the diagonal matrix elements are systematically larger than the non-diagonal ones by a factor of 2–5 and, consequently, that the configuration mixing is weaker than in the approximation of G = const. These differences appear mainly in the changes in the spectroscopic factors of the states far away from the Fermi surface, and consideration of the corresponding effects can be important for describing the cross-sections of single-nucleon transfers or the factors for forbidden α-transitions.

Figure 17 shows the temperature dependence of correlation functions $\Delta_k(t)$ obtained for the proton level scheme of the Ba nucleus and the matrix elements of δ-forces [59]. The same figure gives, as a comparison, the temperature dependence of the correlation function Δ(t) corresponding to the same critical temperature in the approximation of G = const. It will be seen that, in spite of considerable differences in the values of Δ_k for different single-particle levels, their temperature dependence is similar to Δ(t) found in the continuous-spectrum approximation for the simplest form of pair interaction. In Fig. 17 we also give the results of calculation of entropy corresponding to the two models. The level density parameter a = 6.85 MeV^{-1} needed for calculations of

entropy in the continuous-spectrum approximation is determined from the condition of coincidence of entropy values at the critical point. Because of the satisfactory agreement between the derived temperature dependences of entropy below the phase-transition point we can conclude that the fluctuations of the matrix elements of interaction have little effect on the behaviour of the thermodynamic functions of the system. This result is quite natural since the thermodynamic functions reflect the integral effect of the influence of pair correlations on the properties of the excited states of nuclei.

The universal shape of the temperature dependence of the correlation functions $\Delta_k(t)$ can be used to plot approximate solutions for Eq. (2.34). In particular, the solution can be sought in a separable form

$$\Delta_k(t) = \Delta(\epsilon_k) f(t), \tag{2.35}$$

where $\Delta(\epsilon_k)$ describes the dependence of the correlation functions on the energy of the single-particle states and function f(t) determines their temperature dependence. The use of this approximation to describe the thermodynamic functions of superconductors was considered in Ref. [60] and those of the nucleus in Ref. [61].

If we substitute the separable expression for $\Delta_k(t)$ into Eq. (2.34), we get a system of equations for the determination of $\Delta(\epsilon_k)$. Since the left-hand side of such equations does not depend on temperature, the relationship $(1 - 2\bar{n}_k)/E_k$ on the right-hand side of the equations cannot be expected to depend on temperature either. By a direct comparison with the results of the above calculations of $\Delta_k(t)$ it can be demonstrated that, although this statement is not very strict, it holds good with an accuracy of a few per cent. If we take it as an additional approximation, we can then write the relationships

$$\frac{1}{[(\epsilon_k - \lambda)^2 + \Delta^2(\epsilon_k)]^{1/2}} = \frac{\mathrm{th}(E_k/2t)}{E_k} = \frac{\mathrm{th}[(\epsilon_k - \lambda)/2t]}{\epsilon_k - \lambda}, \tag{2.36}$$

with the help of which it is easy to calculate the different thermodynamic functions of the system.

Transforming (2.36), we find the energy dependence of the correlation functions

$$\Delta(\epsilon_k) = \frac{\epsilon_k - \lambda}{\mathrm{sh}[(\epsilon_k - \lambda)/2t]}. \tag{2.37}$$

Thus, in the given model the critical temperature is related to the correlation function on the Fermi surface as

$$t_{cr} = \frac{1}{2}\Delta(\epsilon_f). \tag{2.38}$$

Substituting (2.37) into (2.34), we obtain the equation determining the temperature dependence of the correlation functions

$$f(t) = \text{th}\left[\frac{t_{cr}}{t} f(t)\right]. \tag{2.39}$$

In the continuous-spectrum approximation it is easy to find an explicit form of the relations for the thermodynamic functions of the system. Thus, for the energy of condensation which determines reduction in the ground-state energy of the system as a result of the correlation effects, we have

$$E_{cond} = 2g \int_0^\infty \frac{x\,dx}{\exp(2x/t_{cr}) - 1} = \frac{\pi^2}{12} g t_{cr}^2. \tag{2.40}$$

For the excitation energy and the remaining thermodynamic functions which determine the level density of the excited nucleus in the given model below the phase transition point we get

$$\left.\begin{aligned} &U = \frac{\pi^2}{4} g t_{cr}^2 (1 - f^2),\ S = \frac{\pi^2}{3} g \frac{t_{cr}^2}{t} (1 - f^2), \\ &\sigma^2 = \mathcal{J}_\parallel t = \mathcal{J}_0 t_{cr} (1 - f^2), \\ &\det = \frac{\pi^2}{3} g^2 t_{cr}^4 (1 - f^2)(1 + f^2)^2. \end{aligned}\right\} \tag{2.41}$$

The temperature dependence f(t) of the correlation functions and of thermodynamic characteristics is shown in Fig. 18. It also gives the results of calculations of similar values obtained in the model with a constant matrix element. Since the temperature dependences of the correlation function and of the moment of inertia in these models differ very little, their differences shown in the figure have been multiplied by 100.

The models considered suggest that the principal characteristics of the behaviour of the thermodynamic functions of the superfluid model do not depend on the assumptions made regarding the form of the matrix element of the correlation interaction. The change in its functional dependence, in the

energy representation, merely leads to a renormalization of the constants determining the interrelationship between the phase-transition temperature and the correlation characteristics of the ground state of the system. Since relationships of the superfluid model for the thermodynamic functions of the system in the superconducting phase (2.41) are much simpler than in the model with G = const, their use can in many cases greatly simplify the analysis of experimental data in the region of low excitation energies. For the purpose of selection of the correlation function $\Delta(\varepsilon_f)$ it is advisable to use Eqs (2.20) and (2.38). In this case, the behaviour of the thermodynamic characteristics of the system below the phase-transition point in both models will be equivalent not only in temperature but also in energy scale.

2.2. Even-odd differences in level density

The specific feature of the pairing interaction is that it occurs only for levels filled with nucleon pairs and that the reduction in the ground-state energy caused by it will be maximal in a system with an even number of nucleons. It can be easily seen from the definition of the model Hamiltonian (2.4) that this state corresponds to a quasi-particle vacuum. When considering a system with odd Z or N we shall always have an extra particle occupying one of the levels and preventing population of this level by a nucleon pair. The simplest states of the system in this case correspond to single-quasi-particle vacuum excitations, and excitation with the minimum energy is the ground state of a system with an odd number of particles. If the above-described variational procedure is performed with allowance for blocking of the level nearest to the Fermi energy by the unpaired particle [53], we shall obtain the following equations for determining the correlation function, chemical potential and ground-state energy of the system

$$\left.\begin{aligned} \frac{2}{G} &= \sum_{k\neq s} \frac{1}{E_k}, \\ N &= 1 + \sum_{k\neq s} \left(1 - \frac{\epsilon_k - \lambda}{E_k}\right), \\ \mathcal{E}_s &= \epsilon_s + \sum_{k\neq s} \epsilon_k \left(1 - \frac{\epsilon_k - \lambda}{E_k}\right) - \frac{\Delta_s^2}{G}. \end{aligned}\right\} \qquad (2.42)$$

The same equations determine the energy and correlation function of the lowest excited states of the system which are formed during blocking by the unpaired particle of levels further away from the Fermi energy.

The role of the blocking effect can be seen by comparing the solutions of Eqs (2.42) in the continuous-spectrum approximation with the solution of similar equations for the ground state of the system with an even number of particles. Let us represent the equations for correlation functions in the form

$$g_1 \int_0^{\tilde{\epsilon}} \frac{dx}{\sqrt{x^2+\Delta_0^2}} = g_1 \int_0^{\tilde{\epsilon}} \frac{dx}{\sqrt{x^2+\Delta_s^2}} - \frac{1}{\sqrt{(\epsilon_s-\lambda)^2+\Delta_s^2}} \tag{2.43}$$

and for $\epsilon_s \approx \lambda$, we obtain

$$\Delta_s \approx \Delta_0 \exp(-1/g_1\Delta_0) \approx \Delta_0 - \frac{1}{g_1}. \tag{2.44}$$

Here g_1 determines the density of single-particle states of the component under consideration, and according to the above evaluation $g_1 = g(\epsilon_f)/2 \approx A/45\ \mathrm{MeV}^{-1}$. For nuclei in the region of the rare-earth elements this value of g_1 corresponds to a 20-25% reduction of Δ_s in comparison with Δ_0.

We find the condensation energy of the system with odd N:

$$E_{\mathrm{cond}}^{\mathrm{odd}} = g_1 \int_{-\tilde{\epsilon}}^{0'} x\,dx - \frac{1}{2} g_1 \int_{-\tilde{\epsilon}}^{\tilde{\epsilon}} x \left(1 - \frac{x}{\sqrt{x^2+\Delta_s^2}}\right) dx +$$
$$+ \frac{1}{4} g_1 \Delta_s^2 \int_{-\tilde{\epsilon}}^{\tilde{\epsilon}} \frac{dx}{\sqrt{x^2+\Delta_s^2}} + \frac{\Delta_s^2}{2E_s} \approx \frac{1}{4} g_1 \Delta_s^2 - \frac{1}{2}\Delta_s. \tag{2.45}$$

Substituting (2.44) here, we obtain

$$E_{\mathrm{cond}}^{\mathrm{even}} - E_{\mathrm{cond}}^{\mathrm{odd}} = \Delta_0 - \frac{1}{2g_1}. \tag{2.46}$$

Thus, the pair correlations lead to splitting of the ground-state energies of the system with an even and an odd number of nucleons, and this effect shows up directly in even-odd differences in the binding energies of nuclei.

Let us now consider the main characteristics of the spectra of excited states. For a system with an even N the simplest excitation is the creation

of a pair of quasi-particles from a vacuum. We will represent the energy of such excitations, neglecting corrections for the blocking effect, in the form

$$\mathcal{E}_{ss'} - \mathcal{E}_0 \approx \sqrt{(\epsilon_s - \lambda)^2 + \Delta_0^2} + \sqrt{(\epsilon_{s'} - \lambda)^2 + \Delta_0^2} \geqslant 2\Delta_0. \tag{2.47}$$

For a system with an odd N the lowest excitations correspond to displacement of the unpaired particle to levels farther away from the Fermi energy, and their energy is determined by the relationship

$$\mathcal{E}_{s'} - \mathcal{E}_s \approx \sqrt{(\epsilon_{s'} - \lambda)^2 + \Delta_0^2} - \sqrt{(\epsilon_s - \lambda)^2 + \Delta_0^2} \approx$$

$$\approx (\epsilon_{s'} - \epsilon_s) \frac{\epsilon_s + \epsilon_{s'} - 2\lambda}{\Delta_0}. \tag{2.48}$$

The expressions obtained demonstrate a very important result - the existence of an energy gap $2\Delta_0$ in the excitation spectrum of a system with an even number of particles and the absence of a gap in a system with an odd N. It will be seen from the last relationship that the pair correlations even lead to compression of the low-energy spectrum, in comparison with the non-interacting-particle model. The characteristics described here show up clearly in the level spectra of the even-even and odd nuclei, and a more detailed analysis of these data will be found in the monograph [53].

Let us go on to consider the odd-even differences in the level density of nuclei in the region of higher energies. It can be seen from the analysis of the relationships of the superfluid model in the preceding section that above the phase-transition point the influence of the pair correlations on the behaviour of the thermodynamic functions of the nucleus is reflected only by a shift in the reference point of the excitation energy (2.23). The magnitude of the shift is determined directly by the condensation energy, and the difference in the condensation energies of systems with an even and an odd number of particles (2.46) is the cause of the corresponding even-odd differences in level density and other statistical characteristics of nuclei.

As has already been pointed out, many properties of nuclei are influenced substantially by the shell structure of the spectrum of single-particle states; therefore, for a more rigorous analysis it is necessary to calculate the condensation energies for realistic proton and neutron level schemes. The results of such calculations for the single-particle level spectra of the Nilsson (a) and the Saxon-Woods (b) potentials are shown in Fig. 19 [62]. The pair interaction constants used in these calculations for the proton G_Z and

neutron G_N systems were found from the description of experimental data for the even-odd differences in the binding energies of nuclei [63]. On the whole, the derived values of the constants G, MeV, are close to the following values

$$G_Z = \frac{22{,}5 \pm 0{,}5}{A}; \quad G_N = \frac{19{,}5 \pm 0{,}5}{A}. \tag{2.49}$$

The results of these calculations show that although the condensation energy changes sharply as we approach the magic numbers, a long way from the latter the difference in the condensation energies of the even and odd systems is approximately equal to the correlation function, and this result is hardly dependent at all on the choice of the single-particle level scheme. The correlation functions vary much more smoothly than the condensation energies, and the Δ_Z and Δ_N values found are fairly close to the pairing energies used for a phenomenological description of the even-odd differences in the level density of nuclei.

Thus, the superfluid model of the nucleus gives a natural explanation of the even-odd differences observed in level density. At the same time, it should be borne in mind that a description of effective excitation energies obtained on the basis of this model differs considerably from their phenomenological description in the Fermi-gas model (1.27). The condensation energy generally exceeds the value of the phenomenologically derived correction δ, and the difference of these values can be especially large for nuclei of the transition region (Fig. 19). This should be kept in mind when comparing the experimentally derived values of the Fermi-gas level density parameter with the theoretical calculation results.

In recent years experimental data on level density have been obtained over a wide range of excitation energies for a number of nuclei near A ~ 50-70 [64], and these data can be used for direct verification of the applicability of the above relationships of the superfluid model. Figure 20 shows the experimental data and the results of calculation within the framework of the superfluid model of the level density of even-even, even-odd and odd-odd nuclei [62]. It will be seen that the theoretical curves quite satisfactorily reproduce the differences observed in the level density of nuclei with odd and even numbers of nucleons and the changes in level density for different excitation energies.

One of the most characteristic features of the superfluid model is phase transition. In nuclei this transition should not be interpreted too strictly to mean that there actually exist two clearly separate phase states: we can only say that there are two regions of different energy dependences for the thermodynamic characteristics of the nucleus. A similar effect is indicated to some extent by the results - presented in Chapter 1 - of the phenomenological systematics of the neutron resonance density and of the cumulative number of the observed low-lying levels. In particular, the data given in Fig. 2 on the position of the point of change in the Fermi-gas dependence agree, on the whole, with the evaluation of the critical energy of the phase transition

$$U_{cr} = a t_{cr}^2 + \frac{1}{4} g \Delta_0^2 = \\ = 0.778\, g \Delta_0^2 \approx 5 \div 6 \text{ MeV}, \qquad (2.50)$$

which is obtained for the quasi-classical value of the level density parameter (1.22) and the empirical value of $\Delta_0 = 12/A^{\frac{1}{2}}$ MeV. However, too much significance should not be attached to the agreement obtained since the systematics of the experimental data in Fig. 2 have been obtained on the basis of a phenomenological determination of the effective excitation energies, which only reflects the concepts of the superfluid model in a highly simplified manner.

Attempts were made in some studies to describe the observed neutron resonance density on the basis of the relationships of the superfluid model considered above [13, 65, 66]. In these calculations satisfactory agreement with experiment could generally be achieved only for near-magic nuclei. In the remaining regions theory correctly reproduced the basic trends of the dependence of neutron resonance density on the isotopic composition of nuclei although it strongly reduced the absolute resonance density value. This shows up directly in the difference between the theoretical values of the asymptotic level density parameter (1.37) and (1.38) and the similar phenomenological parameter (1.46). Since in the theoretical calculations based on realistic shell level schemes all parameters of the model are obtained from the analysis of the ground and low-lying states of nuclei, the calculations of the density of the highly excited states do not contain any free parameters by changing which one could eliminate the difference between the theoretical and experimental results. It may be assumed that the effects which are not considered in the relationships presented above make a substantial contribution to the observed neutron resonance density of nuclei far away from the magic nuclei. As we have shown in Chapter 3,

it would be quite natural to associate these effects with the collective motions of highly excited nuclei.

2.3. Moments of inertia of heated nuclei

Above we considered the relationships of the superfluid model of the nucleus for the parallel component of the moment of inertia (2.29) characterizing the distribution of the angular-momentum projections for excited states on the symmetry axis. For the analysis of deformed nuclei it is necessary also to know the distribution of the angular momentum projections on the axis perpendicular to the symmetry axis. This distribution is determined by the magnitude to the perpendicular component of the moment of inertia. At zero temperature this moment of inertia directly characterizes the level spectrum of the rotation band plotted in the ground state of the deformed nucleus, and the difference between the observed moments of inertia and the rigid-body value is a clear example of the manifestation of the superfluid properties of nuclei [67, 68].

Let us look into the behaviour of the perpendicular component of the moment of inertia in the heated nucleus. For this purpose, we derive the average value of the angular-momentum projection in the direction perpendicular to the symmetry axis and denote this direction as axis x. In accordance with the general definition of thermodynamic averages, we write

$$\langle J_x \rangle = \beta^{-1} \frac{\partial \ln Q(\omega)}{\partial \omega}, \tag{2.51}$$

where $Q(\omega) = \mathrm{Sp} \exp\left\{-\beta(H - \lambda N - \omega J_x)\right\}$ is the statistical sum of the rotating nucleus. The moment of inertia characterizes the reaction of the system to rotation, and it can be defined as

$$\mathcal{J}_x = \langle J_x \rangle / \omega. \tag{2.52}$$

Approximation of small momenta

We first calculate the statistical sum and other thermodynamic functions of the nucleus for small rotation velocities. We shall regard quantity ωJ_x in this case as a perturbation, and within the framework of the thermodynamic formulation of the perturbation theory [3] we represent $Q(\omega)$ in the form of a series in powers of ω

$$Q(\omega) = Q_0 \left\{ \sum_{\nu=0}^{\infty} \omega^{\nu} \int_0^{\beta} d\beta_1 \ldots \int_0^{\beta_{\nu-1}} d\beta_{\nu} \langle J_x(\beta_1) \ldots J_x(\beta_{\nu}) \rangle_0 \right\}. \tag{2.53}$$

Here we have introduced the notation

$$J_x(\beta) = \exp[\beta(H - \lambda N)] J_x \exp[-\beta(H - \lambda N)], \tag{2.54}$$

and statistical sum Q_0 and averaging $\langle \ldots \rangle_0$ correspond to the unperturbed Hamiltonian H - λN. If pair correlations of the superconducting type are taken into account for calculation of statistical sum Q_0 and the corresponding averages, it is necessary to use model Hamiltonian (2.4).

We write the operator of the angular momentum taking into account its symmetry properties in relation to time inversion in the form

$$J_x = \sum_{k,k'} j^{(x)}_{kk'} (a^+_{k+} a_{k'+} - a^+_{k-} a_{k'-}). \tag{2.55}$$

Let us go on to the quasi-particle operators, (2.2), and performing for the model Hamiltonian cumbersome but basically simple calculations of the terms quadratic with respect to ω in the expansion (2.53), we obtain the following relationship for the moment of inertia

$$\mathcal{F}_x = 2 \sum_{k,k'} |j^{(x)}_{kk'}|^2 \left\{ (u_k v_{k'} - v_k u_{k'})^2 \frac{1 - \bar{n}_k - \bar{n}_{k'}}{E_k + E_{k'}} + (u_k u_{k'} + v_k v_{k'})^2 \frac{\bar{n}_{k'} - \bar{n}_k}{E_k - E_{k'}} \right\}. \tag{2.56}$$

At zero temperature this relationship retains only the first term, which is a known expression for the moments of inertia of the ground states of deformed nuclei in the small-momentum approximation [53].

The influence of pair correlations on the moments of inertia of cold nuclei has been investigated by many authors [52, 67, 68]. The basic features of the temperature dependence of the perpendicular moments of inertia have been studied within the framework of quasi-classical evaluations of matrix elements in Refs [55], and on the basis of calculations with a realistic single-particle level scheme in Refs [69-71]. The results of such calculations for the single-particle level scheme corresponding to the equilibrium deformation of the ^{236}U nucleus and also to its deformation at the fission-barrier peak are shown in Fig. 21. The figure also gives the results of calculations of the parallel $\mathcal{F}_{\|}$ and effective $\mathcal{F}_{ef}$ moments of inertia, determined by the relation

$$\mathcal{F}_{ef} = \mathcal{F}_{\perp} \mathcal{F}_{\|} / (\mathcal{F}_{\perp} - \mathcal{F}_{\|}). \tag{2.57}$$

It is just this combination of moments of inertia which is part of the description of the angular-distribution anisotropy of nuclear fission fragments. It follows from the results presented here that during the break-up of pair correlations with change in temperature the value of the moments of inertia tends to the rigid-body value; however, this general tendency in the behaviour of the moments of inertia can be greatly distorted by the shell effects [69, 71].

The characteristics of the behaviour of the effective moments of inertia obtained by analysing the angular anisotropy of nuclear fission have been shown by many authors to be a direct manifestation of the phase transition predicted by the superfluid model [55, 72]. Figure 22 gives the experimental results for the polonium isotopes which have been studied in most detail at present. The figure also shows the theoretical curve. In the region of pre-actinide nuclei $\mathcal{J}_{ef}$ does not differ significantly from $\mathcal{J}_{\|}$, and the behaviour of the effective moment of inertia below the critical temperature is determined mainly by the temperature dependence of the parallel moment of inertia. Satisfactory agreement between the theoretical and experimental results confirms the validity of describing the statistical characteristics of excited nuclei by means of the superfluid model relationships and demonstrates the high sensitivity of the experimental data under consideration to the characteristics of the behaviour of the thermodynamic functions of the transition states of the fissioning nucleus. Both these factors make the study of the angular distributions of fragments a highly effective tool for studying the correlation functions of anomalously deformed transition configurations of fissioning nuclei. The results of such studies are considered in more detail at the end of Chapter 5.

In the relationships given above the moment of inertia is defined as the reaction of the system to external rotation (2.52), which is equivalent to the traditional definition of the moments of inertia of the ground states for nuclei in the cranking model [52]. The relationships for the parallel moment of inertia (2.29) are obtained similarly. It is commonly thought that the moment of inertia is associated with the corresponding spin-dependence parameters as follows

$$\sigma_i^2 = \mathcal{J}_i t. \tag{2.58}$$

This definition of the parameters σ_i is somewhat simplified since it does not take into account the microcanonical nature of the level density problem. In this problem we are interested in the dependence of the density of the

excited states of nuclei on the angular-momentum projection at a specified excitation energy, and the spin-dependence parameter should be defined as

$$\sigma_{\text{MCE}}^{-2} = -\frac{d \ln \rho (U, M)}{M d M}\Big|_{U=\text{const}}. \tag{2.59}$$

We shall show that in the presence of correlation effects the definitions (2.58) and (2.59) do not coincide. For this purpose, we will confine ourselves to an analysis only of parameter $\sigma_{\|}$, for which a considerable part of the calculations can be performed in the analytical form [57].

First of all, we find the dependence of the correlation function on the frequency of rotation. For a homogeneous single-particle level spectrum the corresponding equation can be written in the form

$$\int_0^{\tilde{\epsilon}} \frac{dx}{\sqrt{x^2+\Delta_0^2}} = \int_0^{\tilde{\epsilon}} \frac{dx}{\sqrt{x^2+\Delta^2}} [1 - \bar{n}_+(x) - \bar{n}_-(x)], \tag{2.60}$$

where $\bar{n}_\pm(x) = \left[1 + \exp\left(\frac{\sqrt{x^2+\Delta^2} \mp \omega m}{t}\right)\right]$ is the population of quasi-particle states with positive and negative angular-momentum projections on the symmetry axis. In the approximation of small angular momenta, expanding the occupation numbers in a series in ω and retaining the terms up to, and including, the second order, we rewrite Eq. (2.60):

$$\ln \frac{\Delta_0}{\Delta} = 2 \sum_{\nu=1}^{\infty} (-1)^{\nu+1} K_0\left(\frac{\nu\Delta}{t}\right)\left[1 + \left(\frac{\nu\omega m\, \Delta}{t}\right)\right]^2 \tag{2.61}$$

At temperatures close to the phase-transition temperature we can expand the Macdonald functions in a series in $(\Delta/t)^2$ and find the explicit form of the dependence of the correlation function on the rotation frequency

$$\Delta^2 = \frac{\pi^2}{\xi(3)} t^2 \Theta - 2(\omega m)^2 \left[1 - 3\frac{\xi(5)}{\xi(3)}\Theta\right]. \tag{2.62}$$

Here we have introduced the notations $\Theta = 1 - \frac{t}{t_{cr}}$; $\xi(n) = \frac{2^n - 1}{2^n}\zeta(n)$, where $\zeta(n)$ is the Riemann function. It will be easily seen that rotation leads to a reduction in the correlation function at a specified temperature and, consequently, to a fall in the critical temperature in the rotating system in comparison with critical temperature (2.20) in the absence of rotation.

Similarly, we can find the dependence of excitation energy and entropy on the rotation frequency:

$$\left.\begin{aligned} U &= \frac{\pi^2}{6} g t^2 + \frac{1}{4} g \Delta_0^2 + \frac{1}{2} g (\omega^2 m^2 - \Delta^2); \\ S &= \frac{\pi^2}{3} g t - \frac{1}{2} g \frac{\Delta^2}{t} + \frac{\omega^2}{t} (\mathcal{J}_0 - \mathcal{J}_{\parallel}), \end{aligned}\right\} \qquad (2.63)$$

where the parallel moment of inertia $\mathcal{J}_{\parallel}$ is determined by (2.29). Using the link between rotation frequency and angular-momentum projection $M = \omega \mathcal{J}_{\parallel}$ as well as between temperature and excitation energy (2.63), we rewrite the relation for entropy in the form

$$\frac{S^2}{4a} = (U - \frac{1}{4} g \Delta_0^2) \left[1 + \frac{9\Theta^2}{4\xi^2(3)}\right] - \frac{M^2 [1 - 4\Theta + \frac{3}{\xi(3)} \Theta]}{2 \mathcal{J}_0 (1 - 2\Theta)^2} . \qquad (2.64)$$

It will be seen from this formula that the dependence of entropy on angular momentum at a specified excitation energy is determined by the moment of intertia

$$\mathcal{J}_{\parallel}^{MCE} = \begin{cases} \mathcal{J}_0 [1 - \frac{3}{\xi(3)} \Theta] & \text{for } t < t_{cr}; \\ \mathcal{J}_0 & \text{for } t \geqslant t_{cr}. \end{cases} \qquad (2.65)$$

According to the definition (2.59) given above, it is this moment of inertia which is related to the spin cut-off parameter of level density (the difference of entropy from $\ln\rho(U,M)$ is associated only with the pre-exponential factor, which is of little interest for the present consideration).

The behaviour of the moment of inertia $\mathcal{J}_{\parallel}^{MCE}$ over the whole temperature range is shown in Fig. 16. The relationships needed for these calculations were examined in Ref. [57], and we shall not discuss them here. It can be seen that the temperature dependences of the moments of inertia $\mathcal{J}_{\parallel}$ and $\mathcal{J}_{\parallel}^{MCE}$ are qualitatively very similar although the differences existing between them can become important in a fairly rigorous quantitative analysis of level density parameters. It should be noted that similar differences should occur also in the case of the perpendicular moments of inertia.

The status of the above experimental data on the effective moments of inertia of fissioning nuclei does not yet permit a sufficiently reliable identification of these differences in the theoretical descriptions of moments of inertia. This is due to errors in the actual experimental data as well as

to problems entailing distortion of the obtained theoretical curves by the shell effects. The shell inhomogeneities in the single-particle spectrum lead to the dependence of the level density parameter g and the average value of the square of angular-momentum projection $\bar{m}^2$ on the excitation energy of the nucleus. Although there is not much difficulty in calculating the moments of inertia considered above for specific single-particle level schemes, the reliability of such schemes is still insufficient in the case of anomalously large nuclear deformations corresponding to the fission barriers of pre-actinide nuclei.

Arbitrary momenta

Let us consider the basic characteristics shown by the behaviour of the thermodynamic functions of the nucleus for large angular momenta, when we cannot confine ourselves to the first terms of the expansion in a series of the occupation numbers of the single-particle states. Here we shall limit ourselves to analysis of the simplest scheme of equidistant doubly-degenerate single-particle levels characterized by the identical value of the single-particle angular-momentum projections.

For such a homogeneous level scheme we can easily obtain the correlations determining the influence of angular momentum on the correlation characteristics of nuclei at zero temperature [73]. The occupation numbers for t = 0 are determined by

$$\bar{n}_{k+} = \begin{cases} 0 \text{ at } E_k > \omega m, \\ 1 \text{ at } E_k < \omega m; \end{cases} \quad \bar{n}_{k-} = 0, \tag{2.66}$$

and Eq. (2.60) for the correlation function Δ_{oM}, just as the similar equation for the average value of angular-momentum projection on the symmetry axis can be transformed into

$$\left.\begin{aligned} \Delta_0 &= \omega m + (\omega^2 m^2 - \Delta_{0M}^2)^{1/2}; \\ M &= mg(\omega^2 m^2 - \Delta_{0M}^2)^{1/2}. \end{aligned}\right\} \tag{2.67}$$

The solution of these equations

$$\Delta_{0M} = \Delta_0 \left(1 - \frac{2M}{g m \Delta_0}\right)^{1/2} \tag{2.68}$$

determines the correlation function of the lowest level with a given M. At $M \geqslant M_{cr}^{o} = gm\Delta_{o/2}$ the correlations in the system vanish, and the behaviour of the different characteristics of the system is determined by their behaviour

in the non-interacting-particle model. The relationship for the energy of this level takes the form:

$$U_M = \begin{cases} \frac{M\Delta_0}{2m} + \frac{\omega M}{2} = \frac{M^2}{2\mathcal{J}_0}\left(4\frac{M^0_{cr}}{M} - 1\right) & \text{for } M < M^0_{cr}; \\ \frac{M^2}{2\mathcal{J}_0} + \frac{1}{4}g\Delta_0^2 = \frac{M^2}{2\mathcal{J}_0}\left[1 + 2\left(\frac{M^0_{cr}}{M}\right)^2\right] & \text{for } M \geqslant M^0_{cr}. \end{cases} \tag{2.69}$$

With the help of the relationships obtained it can be shown that even in this simple case there are differences in the value of moments of inertia which are equivalent to those considered above in the approximation of small momenta. If we use definition (2.52), we find the following relationship for the moment of inertia

$$\mathcal{J}_{\parallel} = \begin{cases} \mathcal{J}_0\left(2\frac{M^0_{cr}}{M} - 1\right)^{-1} & \text{for } M < M^0_{cr}; \\ \mathcal{J}_0 & \text{for } M \geqslant M^0_{cr}. \end{cases} \tag{2.70}$$

Another method of finding the moment of inertia is based on determining the "rotation" energy proportional to M^2 in the expression for the excitation energy of the system. In this case, for the moment of inertia we obtain

$$\mathcal{J}^*_{\parallel} = \begin{cases} \mathcal{J}_0\left(4\frac{M^0_{cr}}{M} - 1\right)^{-1} & \text{for } M < M^0_{cr}; \\ \mathcal{J}_0 & \text{for } M \geqslant M^0_{cr}. \end{cases} \tag{2.70'}$$

It will be seen that in the non-interacting-particle model or above the phase-transition point the two definitions of the moment of inertia coincide, while below the critical point the behaviour of $\mathcal{J}_{\parallel}$ and $\mathcal{J}^*_{\parallel}$ differ considerably.

The break-up of the pair correlations for large angular momenta was considered in Ref. [74] for a more realistic single-particle spectrum scheme. Comparing Eqs (2.67)-(2.69) with the results of Ref. [74], we see that the simple homogeneous model gives the correct functional dependence of the correlation function Δ_{oM}, moment of inertia $\mathcal{J}_{\parallel}$ and low-lying level energy U_M on the value of the angular momentum. In this model, however, the value of the critical angular momentum is too low. This disagreement is due to a substantial difference in the distribution of the projections m_k in the single-particle level spectrum of the shell model from that of the homogeneous model. In the case of the spherical potential well, the following relations were obtained in Ref. [74] for the critical angular momentum and the critical rotation frequency:

$$\left.\begin{aligned} M_{cr}^{0} &= \frac{e^{7/3}\Delta_0}{4\sqrt{6}\,m}\quad \mathcal{F}_0 = 1{,}05\, g m \Delta_0; \\ \omega_{cr}^{0} &= \frac{e^{7/3}\Delta_0}{4\sqrt{6}\,m} = 1{,}05\,\frac{\Delta_0}{m}. \end{aligned}\right\} \qquad (2.71)$$

If these values are used in Eqs (2.69), on their basis we can obtain in spherical nuclei a simple evaluation of the characteristics of the lowest level with a given angular momentum.

The solution of the equations of state of a rotating heated nucleus at a non-zero temperature was obtained in Ref. [57] by the numerical method for a homogeneous single-particle level scheme, and the results of these calculations are shown in Figs 23-25. The curves are given in a dimensionless universal form, independent of the parameters of the system, using the critical values of the corresponding quantities.

Let us note the basic characteristics of the results obtained. Figure 23 shows, for different temperatures, the relationship between rotation frequency ω and the corresponding integral of motion - projection of angular momentum M. The ratio between these quantities, which determines the dynamic moment of inertia $\mathcal{F}_{\parallel}(t,\omega)$, and the correlation function of the system $\Delta(t,\omega)$ are also given here. It will be seen that for low temperatures there is a region of high-rotation frequencies within which the superconductivity equation has two non-zero solutions for the correlation function $\Delta(t,\omega)$. In this region, two values of the angular-momentum projection correspond to the intensive variable ω. Similar two-valued behaviour is also found in the case of the dependence of all other extensive thermodynamic functions of the system on rotation frequency. But if the state of the system is determined with the help of extensive variables, the dependence of the thermodynamic quantities of interest to us will be single-valued. This will be evident from Fig. 24, which shows, for fixed temperatures, the behaviour of excitation energy, entropy and correlation function of the system with changes in the value of the angular-momentum projection. Here the dotted line is the curve for phase transition from the superconducting to the normal state - the state to the right of the curve corresponds to $\Delta = 0$ and is determined by the equations of the non-interacting-particle model.

Since the state of the excited nucleus is described by a microcanonical ensemble, the behaviour of the thermodynamic characteristics of the system for a fixed excitation energy is of primary interest. The results of the solution of the equations of state in these variables are shown in Fig. 25.

The dotted line is the phase-transaction curve. For a fixed excitation energy the curve t = 0 determines the maximum value of the permissible angular-momentum projections. The dot-dash curve represents the behaviour of the system's entropy obtained in the small-momentum approximation

$$S(U, M) = S(U, M=0) - M^2/2\sigma^2_{MCE}, \tag{2.72}$$

where the spin cut-off parameter σ^2_{MCE} is calculated for M = 0 (see Fig. 16). It will be seen that this approximation is valid in a sufficiently wide region of angular momenta and gives strongly distorted results only for momenta close to the maximum possible momentum, where almost the whole excitation energy is expended on the rotation of the system, and the dependence of the moment of inertia on rotation frequency plays a major role in determining the position of levels with a high momentum (2.69).

It should be noted that for the normal phase (i.e. to the right of the phase-transition curve) the approximation (2.72) can be improved considerably if for large M we use a more accurate expression for the equation of state of the Fermi-gas model

$$S(U, M) = \frac{\pi^2}{3} g t = 2\left\{\frac{\pi^2}{6} g \left(U - \frac{1}{4} g \Delta_0^2 - \frac{M^2}{2\mathcal{J}_0}\right)\right\}^{1/2} \tag{2.73}$$

In the normal phase this expression is valid in the whole region of angular momenta. No such simple relation exists for the superconducting phase.

The curves derived can be used to investigate the behaviour of the thermodynamic characteristics of the nucleus throughout the angular-momentum region. For a given set of parameters g, m and Δ_o we can obtain with their help the density of excited states in the neighbourhood of the so-called yrast line, i.e. the line of the largest possible momenta. Here for calculation of M^o_{cr} and ω^o_{cr} we should use Eqs (2.71) and not the results of the homogeneous model. Calculations of the dependence of the thermodynamic functions of nuclei on angular momentum for a realistic level scheme of the deformed single-particle potential were considered in Refs [75].

The behaviour of level density for large angular momenta is of interest in the study of multi-cascade particle evaporation processes in reactions with heavy ions. The use of the small-momentum approximation in these calculations can considerably distort the results. At the same time, the relationships obtained in this approximation are sufficient in the overwhelming majority of cases for an analysis of reactions induced by lighter particles.

2.4. Pair correlations for a fixed number of quasi-particles and their relationship with the average statistical characteristics of nuclei

Let us look at the influence of the residual interactions of the correlation type on the thermodynamic characteristics of a system with a fixed number of excited quasi-particles. In order to solve this problem we need to add one more equation to Eqs (2.13) considered above

$$n = 2 \sum_k \bar{n}_k, \tag{2.74}$$

which fixes the number of excited quasi-particles n. The Lagrange multiplier γ corresponding to this equation will be part of the definition of the average occupation numbers of single-particle states $\bar{n}_k \simeq [1 + \exp(\beta E_k - \gamma)]^{-1}$ but this will not change the form of Eqs (2.13) and (2.14), which determine the correlation function and energy of the system. The relationship for entropy in the system with a fixed number of quasi-particles will take the form

$$S_n = 2 \sum_k [(\beta E_k - \gamma)\bar{n}_k - \ln(1 - \bar{n}_k)]. \tag{2.75}$$

The expressions for the pre-exponential factor are presented in Ref. [76], so we shall not discuss them here. For $\gamma = 0$ Eq. (2.74) determines the average number of excited quasi-particles in a system with a given temperature or excitation energy. If we switch off the residual interaction, Eqs (2.74) and (2.75) are transformed into similar equations for the non-interacting-particle model, and in this case n is equal to the sum of the number of excited particles and holes.

Let us consider the behaviour of the thermodynamic functions for a fixed number of quasi-particles in the case of the model of the equidistant spectrum of doubly-degenerate levels [76, 77]. At zero temperature the system should be at the lowest of the possible states with the given macroscopic characteristics. A quasi-particle vacuum n = 0 corresponds to the ground state of the system, and the correlation function Δ_0 in this state is determined by Eqs (2.13) for $\bar{n}_k = 0$. In the case of states with a non-zero number of excited quasi-particles, transition to the zero-temperature limit requires some caution and cannot be done formally. This is due to the fact that in the system under consideration the single-particle levels are degenerate with respect to the sign of angular-momentum projection. As a result of such degeneracy, at t = 0 there are two possible limiting cases of distribution of excited quasi-particles

over levels: (1) n quasi-particles occupy n levels near the Fermi energy; (2) the levels have two particles each, and n quasi-particles occupy n/2 levels near the Fermi energy.

For small n the first case corresponds to the blocking of a large number of single-particle levels; using the continuous-spectrum approximation to solve the equations of state, we obtain the dependence of the correlation function and excitation energy on the number of quasi-particles in the form

$$\left.\begin{aligned} \Delta &= \Delta_0 \left(1 - \frac{2n}{g\Delta_0}\right)^{1/2}; \\ U &= n\Delta_0 \left(1 - \frac{n}{2g\Delta_0}\right). \end{aligned}\right\} \tag{2.76}$$

As n increases the correlation function of the system decreases, and to excite a large number of quasi-particles it is advantageous for the system, from the energy standpoint, at first to destroy the condensate, i.e. to expend the energy $g\Delta_0^2/4$ on transition from the superconducting to the normal state, and then to excite n non-interacting particles. In this case, a lower energy corresponds to the second method of level distribution of excited quasi-particles and the excitation energy takes the form:

$$U = g\Delta_0^2/4 + n^2/4g. \tag{2.77}$$

We now calculate the thermodynamic functions of the system at a temperature different from zero. The solution of the equations for $\gamma = 0$ has been given above (see (2.25) and (2.26)). For the average number of excited quasi-particles for $\gamma = 0$ we rewrite relation (2.74) in the form

$$\bar{n} = 2g\Delta \sum_{\nu=1}^{\infty} (-1)^{\nu+1} K_1(\nu\Delta/t). \tag{2.78}$$

We consider the state of a system with a given temperature and number of excited particles smaller than the average. In this case ($\gamma < 0$) the average occupation numbers of levels $\bar{n}_k$ can be expanded in a series in Macdonald functions. Then the system of saddle-point equations will take the form

$$\left.\begin{aligned} \ln\frac{\Delta_0}{\Delta} &= 2\sum_{\nu=1}^{\infty} (-1)^{\nu+1} \exp(\nu\gamma) K_0\left(\frac{\nu\Delta}{t}\right); \\ n &= 2g\Delta \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \exp(\nu\gamma) K_1\left(\frac{\nu\Delta}{t}\right); \\ U &= g\Delta^2 \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \exp(\nu\gamma) K_2\left(\frac{\nu\Delta}{t}\right) + \frac{1}{4}g(\Delta_0^2 - \Delta^2). \end{aligned}\right\} \tag{2.79}$$

For entropy and the spin cut-off parameter we obtain

$$\left.\begin{aligned} S_n &= \frac{2}{t}\left[U - \frac{1}{4}g(\Delta_0^2 - \Delta^2)\right] - \gamma n; \\ \sigma_n^2 &= 2gm^2\Delta \sum_{\nu=1}^{\infty} (-1)^{\nu+1}\nu \exp(\nu\gamma) K_1\left(\frac{\nu\Delta}{t}\right). \end{aligned}\right\} \qquad (2.80)$$

If the number of excited particles n is much smaller than $\bar{n}$, then $\gamma \to -\infty$ and in the expansions we can confine ourselves to the first terms alone. In so doing we can determine the explicit form of the dependence of the system's thermodynamic functions on n:

$$\left.\begin{aligned} \ln \frac{\Delta_0}{\Delta} &= \frac{n}{g\Delta} \frac{K_0(\Delta/t)}{K_1(\Delta/t)}; \\ U^* &= U - \frac{1}{4}g(\Delta_0^2 - \Delta^2) = \frac{n\Delta}{2} \frac{K_2(\Delta/t)}{K_1(\Delta/t)}; \\ S_n &= 2\frac{U^*}{t} - n\ln \frac{n}{2g\Delta K_1(\Delta/t)}; \\ \sigma_n^2 &= m^2 n. \end{aligned}\right\} \qquad (2.81)$$

Equations (2.81) will have an especially simple form at the limit of high temperatures $\Delta/t \to 0$, when we can use the expansion of the Macdonald functions in a series:

$$\left.\begin{aligned} \ln\frac{\Delta}{\Delta_0} &= -\frac{n}{gt}\left(\ln\frac{2t}{\Delta} - C\right); \\ U^* &= nt;\ S_n = 2n - n\ln\frac{n^2}{2gU^*}, \end{aligned}\right\} \qquad (2.82)$$

where C = 0.5722 is the Euler constant.

Thus, in a system with a fixed number of excited quasi-particles, as the excitation energy increases the correlation function tends to that of the ground state while the behaviour of the thermodynamic functions at high-excitation energies is determined by the Boltzmann-particle gas equations. This result also enables us to use the Boltzmann-particle gas relationships (1.68)-(1.73) to calculate the density of the states of the system under consideration at high-excitation energies.

Let us consider the behaviour of the thermodynamic functions in the general case of an arbitrary number of excited quasi-particles. The main qualitative characteristics of the system's behaviour can now be understood on the basis of the limiting cases considered above. For a small number of excited particles $n < g\Delta_0/2$ the correlation function of the system is different from zero at any excitation energy U, and it increases with U. For large numbers

of quasi-particles $n > g\Delta_o/2$ the correlation function for the lowest states is equal to zero, and the behaviour of the system's thermodynamic functions at low-excitation energies is determined by the equations of the non-interacting Fermi-particle model with renormalized excitation energy $U^* = U - g\Delta_o^2/4$. However, with increase in excitation energy a correlation function appears in such a system, i.e. there is a change-over from the normal to the superfluid state, and for sufficiently large n the system under consideration will behave as a gas consisting of Boltzmann particles.

We have to solve a set of equations of state in order to obtain a quantitative description of the thermodynamic functions of the system. Such a solution was obtained numerically for the equidistant spectrum of doubly degenerate single-particle levels [76], and the results are given in Figs 26 and 27. The changes in the correlation function for a system with a fixed number of quasi-particles are shown in Fig. 26. The dot-dash curve represents the results of solution at zero temperature (2.76) and the hatched area is the region within which a redistribution of quasi-particles over levels takes place and solution (2.76) is transformed into (2.77). The dependence of the system's entropy and spin-cut-off parameter on excitation energy is given in Fig. 27 which also shows the energy dependence of similar values for a system with a non-fixed number of quasi-particles; the dotted line is the phase-transition curve. The calculation results given here confirm the above qualitative conclusion regarding the general trends in the behaviour of the thermodynamic functions of a system with a fixed number of quasi-particles.

Let us now consider the density of the excited states of the system. The difference of $\ln \rho_n$ from entropy is due to the presence of the pre-exponential factor, and since the latter is a weak function of the excitation energy, it needs to be taken into account only to obtain the correct absolute value of the density of states. If necessary, the pre-exponential factor can be calculated with the help of the relationships given in Ref. [76]. These calculations are very cumbersome; therefore, for practical purposes, it is of interest to obtain simpler relationships for the density of states. It can be seen from the results presented in Figs 26 and 27 that for a system with $n/n_{cr} > 0.4$ there is an energy region where the behaviour of the system's thermodynamic functions is determined by the equations of non-interacting Fermi particles. In this region the density of states will also take the form of the Fermi-gas dependence with the effective excitation energy $U^* = U - g\Delta_o^2/4$. The density

of states can be represented in the form (1.68) if a correction for the Pauli principle (1.82) is introduced into the effective excitation energy. Thus, when the latter is correctly determined, Eq. (1.68) satisfactorily describes the density of states of the system in the region of both low- and high-excitation energies. This relation can therefore be expected to be a sufficiently good approximation for calculation of the density of states of a system with a fixed number of excited quasi-particles for arbitrary values of U and n if we define the effective energy in the form

$$U^* = U - \frac{1}{4}g'(\Delta_0^2 - \Delta^2) - n(n-2)/8g \qquad (2.83)$$

and use for calculation of $\Delta(U,n)$ the results obtained above in the numerical solution of the equations (see Fig. 26). The calculations showed that the accuracy of this approximation was fairly high and that an appreciable error occurred only in the immediate neighbourhood of the excitation threshold for the given n-quasi-particle configuration. It is, therefore, better to use the accurate relationships (2.76) or (2.77) to determine the threshold.

The density of states with a fixed number of quasi-particles is used at present for the statistical analysis of intermediate structures and for describing the hard part of the spectra of particles emitted during the pre-equilibrium decay of the compound nucleus. An analysis of this kind is usually performed with the help of the Boltzmann-particle-gas model relationships without taking into account pairing corrections (1.68). The results of the studies carried out show the errors which occur in this description and give a more rigorous model for analysing the corresponding phenomena.

Using the thermodynamic functions of the system with a fixed number of excited quasi-particles to describe the averaged characteristics of nuclei discussed earlier makes for a stricter examination of the problems of the nature of phase transition in excited nuclei, the difference between the statistical characteristics of even and odd nuclei and the possibly stepped structure of the energy dependence of level density and spin cut-off parameter in the low-energy region [62].

The total density of the excited states $\rho(U)$ can be obtained by summing n-quasi-particle densities over all energetically possible configurations

$$\rho(U) = \sum_n \rho_n(U). \qquad (2.84)$$

It is obvious that for a system with an even total number of particles N the numbers of quasi-particles n in the superfluid model take even values, while for a system with an odd N only odd values are possible. The density of states $\rho_n(U)$ determines the statistical weight of configurations with a given number of quasi-particles, and can be used to obtain any average characteristic of the system under consideration

$$\bar{O}(U) = \sum_n O_n(U)\rho_n(U)/\rho(U). \qquad (2.85)$$

Figure 28 presents the results of calculation of the total density of states, the spin cut-off parameter and the average correlation function for the excited nucleus obtained with the help of relationships (2.84) and (2.85), together with the results of calculation of the same values on the basis of the ordinary thermodynamic relationships (2.25)-(2.28) which do not explicitly take account of the classification of excited states with respect to the number of quasi-particles (the values taken for parameters $g = 18.3\ \text{MeV}^{-1}$ and $\Delta_o = 0.85$ MeV correspond to the region of transuranium elements).

In the ordinary thermodynamic consideration the correlation function $\Delta(U)$ decreases with an increase in excitation energy and at an energy above the critical $U_{cr} = 0.778\ g\Delta_o^2$ it equals zero. At U_{cr} the energy dependence of the density of states, spin cut-off parameter and other thermodynamic functions of the system shows a break typical of phase transitions of the second kind. For a system with a fixed number of quasi-particles the correlation function of configurations with $n < g\Delta_o/3$ is different from zero over the whole excitation energy range, i.e. such systems only exist in the superconducting state. Systems with $n \geq g\Delta_o/3$ may exist either in the superconducting or in the normal phase but the region of normal states lies at lower excitation energies than that of the superconducting states, and the phase transition has an energetically opposite direction in relation to the phase transition of the traditional description. In the case of states with any n, the correlation functions Δ_n increase as the excitation energy rises and tend to Δ_o at high U (Fig. 26).

If we calculate the total density of states and the average statistical characteristics on the basis of Eqs (2.84) and (2.85), no breaks occur at U_{cr} in the energy dependence of the total density of states and of the spin cut-off parameter. The average correlation function decreases with an increase in

excitation energy but does not vanish at U_{cr}. These results indicate that, strictly speaking, there is no phase transition for the average statistical characteristics of the system.

The occurrence of phase transition in the traditional thermodynamic description of the characteristics of excited nuclei is the price we pay for the simplifications allowed in solving the problem. These simplifications are associated with the fact that the Hartree-Fock-Bogolyubov variational transformation is made in the simplest statistical variant, i.e. its coefficients are obtained from the condition of the best description of the most probable configuration of the system by the Hamiltonian for non-interacting quasi-particles. The phase transition in this case only reflects the fact that above the critical energy for the most probable n-quasi-particle configuration the correlation function is zero. In superconductors described by a canonical ensemble the state of the system is determined fully by the most probable configuration (the most probable and the average characteristics virtually coincide) and the temperature dependence of the correlation function is observed directly in the experiment. In the case of a nucleus which is an isolated system and is described by a microcanonical ensemble, the configurations different from the most probable make a substantial contribution to the total density of states of the system, and this manifests itself in an appreciable difference between the most probable and the average characteristics of the system (see Fig. 28). During calculation of the total density of states of the system on the basis of relationship (2.84) the Hartree-Fock-Bogolyubov transformation is carried out more rigorously, and this approach therefore corresponds to a more accurate description of the statistical properties of the given system.

However, the refinements obtained mainly concern the problem of the essential difference between the thermodynamics of the nucleus and that of superconductors, which shows up in the absence of phase transition and of any corresponding breaks in the energy dependence of the statistical characteristics of the nucleus. As for practical calculations of the density of states of the excited nucleus, above the phase-transition point the average correlation function is small ($\Delta < 0.1$ MeV) and has little influence on the behaviour of the system's thermodynamic characteristics. Therefore, the non-interacting-particle model with renormalized excitation energy $U^* = U - \frac{1}{4}g\,(\Delta_0^2 - \bar{\Delta}^2)$ in

this region quite satisfactorily describes the thermodynamic functions of the superfluid system. The differences from the ordinary thermodynamic consideration are here due to the correction $g\Delta^2/4$, which has a negligibly small influence. The traditional description gives, on the whole, the correct density of states also in the region of low energies right down to $2\Delta_o$ (see Fig. 28). Hence if we do not associate the break with any physical phenomena occurring in the nucleus, the traditional approach can be regarded as quite a satisfactory approximation for the calculation of the density of excited states in the superfluid model.

These calculations also enable us to draw some conclusions regarding the relationship between the thermodynamic and combinatorial methods of calculation. In a strictly combinatorial consideration the Hartree-Fock-Bogolyubov transformations should be carried out independently for each excited state of the nucleus. As a result, the correlation function Δ_i for any i-th state with a given number of excited quasi-particles n will differ from the thermodynamically averaged correlation function $\Delta_n(U)$ considered in the present study. However, when averaging over states the differences between the combinatorial and the thermodynamic description will be manifested only with regard to the phase transition for a configuration with fixed n, i.e. phase transitions will be absent also for the configurations determined. The refinements arising here in the density of states $\rho_n(U)$ should be negligibly small. Unfortunatley, in the combinatorial calculations which have so far been carried out [37] the variational transformations were performed only in a simplified variant without considering the question of connection with the thermodynamic description.

Using the density of states with a fixed number of excited quasi-particles, we can also more rigorously formulate the solution of the problem of even-odd differences in the level density of nuclei. From the calculation results presented in Fig. 28 for the total density of states, it will be seen that at the chosen reference point for excitation energy (the zero-excitation energy corresponds to a quasi-particle vacuum) the odd-even differences in the total density of states exist only in the region of low-excitation energies, when only one- and two-quasi-particle configurations are possible. The density of states of odd and even nuclei averaged over these structures agree quite satisfactorily with the results of the ordinary thermodynamic calculation of the density of states. Thus, over a wide energy range the even-odd differences in level density are determined only by the shift in the reference

point of excitation energy, and when the reference point is properly chosen, the thermodynamic description of the density of states is applicable to both even and odd systems. It is indeed this result that is demonstrated by the above (see Fig. 20) comparison of thermodynamic calculation with experiment.

Since configurations with a fixed number of quasi-particles occur only at an excitation energy exceeding the minimum energy (2.76), the thresholds of n-quasi-particle excitations should show up in the total density of states and in the average statistical characteristics of the system in the form of a step-like structure. At low single-particle-state density these structures are small and, consequently, insignificant for nuclei of average atomic weight; but in the region of heavy nuclei the thermodynamic functions have a fairly pronounced step-like structure at low-excitation energies. Attention was drawn a long time ago to the possibility that such structures might exist in the energy dependence of level density and also that there might be other statistical characteristics of excited nuclei [78, 79]. In experiment these effects seem to show up most clearly in the near-threshold behaviour of parameter K_o^2, which determines the angular distribution of fission fragments and is associated with the spin cut-off parameter of level density $K_o^2 = \bar{\sigma}_{\parallel}^{-2}(1 - \mathcal{J}_{\parallel}/\mathcal{J}_{\perp})^{-1}$ [72, 80].

The relationships of the Boltzmann-gas model with the phenomenologically introduced dependence of the thresholds of n-quasi-particle excitations (1.88) are generally used to describe the observed non-monotonic changes of parameter K_o^2. The results of calculation of the characteristics of n-quasi-particle states in the Boltzmann-gas model and those in the superfluid model of the nucleus are compared in Fig. 29. It will be seen that when describing the structure in the dependence of $\sigma_{\parallel}^2$ at low-excitation energies, the Boltzmann-gas model highly simplifies the picture we have of the contribution of the different configurations to the average statistical characteristics of the system and, with increase in the excitation energy, substantially reduces the value of $\sigma_{\parallel}^2$ in comparison with the superfluid model of the nucleus. It should be noted that for a fuller description of the experimental data it is also necessary to take into account the shell structure of the spectrum of single-particle states of the nucleus for deformations corresponding to the saddle point. The influence of the shell structure shows up in the dependence of parameters g and $\bar{m}^{-2}$ on excitation energy, and this dependence may weaken or strengthen the effects considered above.

The foregoing discussion shows that the thermodynamic functions of systems with a fixed number of quasi-particles are of interest not only for an analysis of the pre-equilibrium decay of nuclei but also for refining to some extent the statistical description of the average characteristics of excited nuclei.

Chapter 3

COLLECTIVE PROPERTIES OF HIGHLY EXCITED NUCLEI

Collective phenomena in nuclei are receiving considerable attention in the analysis of spectroscopic data on the characteristics of low-lying levels. Within the framework of the unified model of the nucleus developed by A. Bohr and B. Mottelson [82] it has been possible to explain and systematize the extensive experimental material on the structure of these levels. Various microscopic methods of describing the structure of collective levels [29,53,83] are also widely used at present to consider the interrelationship between the collective excitations and the single-particle motion of nucleons in a self-consistent nuclear potential. In view of the success of the unified model it is natural to raise the question of how collective effects influence the statistical properties of highly excited nuclei. Below we discuss the existing theoretical approaches to the analysis of these effects, together with the experimental data which can be used to answer this question.

3.1. Adiabatic evaluation of the contribution of the rotational and vibrational excitations to level density

The methods of level density calculation considered above are based on representing the energy of the nucleus in the form of the sum of all possible combinations of the energy of excited non-interacting quasi-particles. A long time ago Ericson [84] drew attention to the substantial difference between this approach and the phenomenological methods of constructing the spectrum of low-lying levels in the unified model of the nucleus, according to which the quasi-particle excitations of the nucleus are supplemented adiabatically by collective excitations. However, as long as the methods of level density analysis were purely empirical in nature, practically no questions arose concerning the role of the collective effects in highly excited nuclei. But after it was found that the microscopic calculations performed for realistic single-particle level schemes were not in a position to explain the observed density of neutron resonances, the matter of taking the collective effects into account became very urgent. The possibilities for solving this problem within the framework of the concepts of the unified model of the nucleus were analysed in Ref. [85].

If the nucleus has a deformation or any other characteristic which enables us to speak of its spatial orientation, the spectrum of its energy states will be determined not only by the internal excitations but also by the rotation of the nucleus as a whole. This rotation may lead to a considerable increase in the density of nuclear levels. If the rotation energy is regarded as independant of internal excitations, the level density in the case of a deformed axially symmetric nucleus is determined by the relationship

$$\rho(U, J) = \frac{1}{2} \sum_{K=-J}^{J} \rho_{\text{int}} (U - E_{\text{rot}}^{JK}, K), \tag{3.1}$$

where K is the projection of the angular momentum of the internal excitations on the symmetry axis and $E_{\text{rot}}^{JK} = [J(J+1) - K^2]/2\mathcal{J}_{\perp}$ the rotation energy. The factor ½ in Eq. (3.1) reflects the assumption regarding additional mirror symmetry of the shape of the excited nucleus similar to the symmetry of the equilibrium deformations of the ground states of nuclei. Using the above Gaussian distribution for the density of the internal excitations of the nucleus (1.14), we rewrite Eq. (3.1) in the form

$$\begin{aligned} \rho(U, J) &= \frac{\rho_{\text{int}}(U)}{2\sqrt{2\pi}\sigma_{\parallel}} \sum_{K=-J}^{J} \exp\left[-\frac{J(J+1)}{2\sigma_{\perp}^2} - \frac{K^2}{2\sigma_{\text{ef}}^2}\right] \approx \\ &\approx \frac{(2J+1)\rho_{\text{int}}(U)}{2\sqrt{2\pi}\sigma_{\parallel}} \exp[-J(J+1)/2\sigma_{\perp}^2], \end{aligned} \tag{3.2}$$

where $\sigma_i^2 = \mathcal{J}_i t$ indicates the corresponding spin cut-off parameters of level density. The summing over K in the last formula is evaluated on the assumption that $\overline{K}^2 \ll 2\sigma_{\text{ef}}^2$. Comparing (3.2) with the relationships of type (1.16) obtained in the non-interacting quasi-particle model, we can easily see that allowance for the rotation of the nucleus gives us an increase in the excited level density by a factor of $\sigma_{\perp}^2$.

The increase in level density due to rotation of the nucleus can be still higher if the symmetry of shape in the excited nucleus is lower than in the ground states of deformed nuclei. In the case of loss of mirror symmetry, the factor ½ should be dropped from Eq. (3.1), i.e. we shall have an additional twofold increase in level density with respect to the definition considered above. The increase will be still higher with loss of axial symmetry. For

level density in this case we shall obtain the relation

$$\rho(U,J) = \sum_{J_x, J_y, J_z} \rho_{\text{int}}\left(U - \frac{J_x^2}{2\mathcal{J}_x} - \frac{J_y^2}{2\mathcal{J}_y} - \frac{J_z^2}{2\mathcal{J}_z}\right) \delta(J^2 - J_x^2 - J_y^2 - J_z^2) \approx \tag{3.3}$$

$$\approx (2J+1)\rho_{\text{int}}(U)\exp(-J(J+1)/2\bar{\sigma}^2),$$

where $\bar{\sigma}^2 = (\sigma_x^2 + \sigma_y^2 + \sigma_z^2)/3$ is the averaged value of the spin cut-off parameter. By comparison with the mirror and axially symmetric shape of the nucleus (3.2) we obtain an additional increase in level density by a factor of $2\sqrt{2\pi\sigma_{\parallel}}$.

The above analysis of the influence of the rotation of the nucleus on excited level density is to a large extent similar to the adiabatic separation - well known in statistical physics - of the statistical sum of a gas of polyatomic molecules into independent factors corresponding to the statistical sum of the internal excitations of molecules and to the statistical sums of all possible types of collective motion of molecules [3]. The rotational increase in level density obtained above coincides directly with the adiabatic evaluations of the rotational statistical sum for a given symmetry of the shape of the nucleus:

$$K_{\text{rot}}(t) = \begin{cases} \mathcal{J}_\perp t & \text{for mirror and axially symmetric shapes;} \\ 2\,\mathcal{J}_\perp t & \text{for axially symmetric shapes;} \\ 2t^{3/2}(2\pi\,\mathcal{J}_x\,\mathcal{J}_y\,\mathcal{J}_z)^{1/2} & \text{for shapes having no symmetry.} \end{cases} \tag{3.4}$$

In addition to the rotational structure of levels, the nuclear spectra also clearly show excitations of a vibrational nature reflecting collective oscillations of the shape of the nucleus [86]. In the adiabatic approximation the increase in excited level density due to these oscillations is determined by the magnitude of the vibrational statistical sum

$$K_{\text{vibr}}(t) = \prod_{\lambda} [1 - \exp(-\omega_\lambda/t)]^{-g_\lambda}, \tag{3.5}$$

where ω_λ is the energy and g_λ the degree of degeneracy of the corresponding vibrational excitations. In accordance with this evaluation the increase in level density will be considerable only if low-energy vibrational excitations with $\omega_\lambda < t$ exist in the nucleus. In particular, an increase in level density in the neutron binding energy region of $K_{\text{vibr}}^{(2+)} \approx 4$-$6$ corresponds to the observed energies of quadrupole excitations of nuclei in the rare-earth region and $K_{\text{vibr}}^{(3-)} \approx 8$-$10$ to the similar octupole excitations. These figures will be higher

in the case of transuranium nuclei, where the octupole states drop noticeably lower [86]. Such evaluations are, of course, much on the high side since they do not take into account, first, the well-known non-adiabatic nature of the separation of vibrational modes in the cold nucleus [53] and, second, the possible changes in the vibrational excitation energy in heated nuclei.

Since the properties of the observed low-energy vibrational excitations are influenced considerably by the shell inhomogeneities of the single-particle spectrum, the value of the statistical sum K_{vibr} obtained on the basis of the liquid-drop model can be taken for an evaluation reflecting the weakening of the shell effects in the heated nucleus. In this model the surface oscillation energy of the nucleus is determined by the relation

$$\omega_\lambda^2 = \frac{\hbar^2 \alpha}{\rho_0 R_0^3} \lambda(\lambda-1)(\lambda+2), \tag{3.6}$$

where α is the coefficient of surface tension, ρ_0 the density of the nuclear matter and λ the multipolarity of the oscillations. To simplify the analysis, we have omitted in Eq. (3.6) the terms associated with the Coulomb energy of the charged drop. From (3.6) we can find the surface oscillation density of the drop

$$g_s(\omega) = (2\lambda+1)\frac{d\lambda}{d\omega} \approx \frac{4}{3}\left(\frac{\rho_0 R_0^3}{\hbar^2 \alpha}\right)^{2/3} \omega^{1/3} \tag{3.7}$$

and determine with its help the energy and entropy of the surface oscillations of the heated nucleus

$$\left.\begin{aligned} U_s &= \int_0^\infty g(\omega)\frac{\omega d\omega}{\exp(\omega/t)-1} = \frac{4}{3} C_{4/3}\left(\frac{\rho_0}{\hbar^2\alpha}\right)^{2/3} R_0^3 t^{2/3}; \\ S_s &= \int_0^t \frac{dt'}{t'}\frac{dU_s}{dt'} = \frac{7}{3} C_{4/3}\left(\frac{\rho_0}{\hbar^2\alpha}\right)^{2/3} R_0^3 t^{4/3}. \end{aligned}\right\} \tag{3.8}$$

The integral contained in these expressions

$$C_{4/3} = \int_0^\infty \frac{x^{4/3}\,dx}{\exp(x)-1} = 1{,}694$$

is well known from analysis of the thermodynamic functions of a degenerate Bose gas [3]. For the statistical sum of the surface oscillations we find

$$K_s = \exp(S_s - U_s/t) = \exp\left\{ C_{4/3}\left(\frac{\rho_0}{\hbar^2\alpha}\right)^{2/3} R_0^2 t^{4/3} \right\}. \tag{3.9}$$

The equations of the liquid-drop model considered here were used to analyse the level density of nuclei even in the early studies on the statistical theory [4]. However, initially they were discussed as an alternative to the Fermi-gas model, whereas in the concepts of the unified model they should be treated as a contribution by the surface oscillations of the nucleus to the internal quasi-particle oscillations. If, in (3.9), we substitute the value of the surface tension coefficient $\alpha \approx 1.2$ MeV/fm^2 corresponding to the phenomenological parameters of the mass formula, we obtain an increase of $K_s = 2.4$ for the level density in the neutron resonance region. This evaluation may be somewhat on the low side since consideration of the Coulomb energy of the drop reduces the surface oscillation energy (3.6) and consequently increases K_s.

Strictly speaking, any separation of collective variables should be accompanied by a corresponding decrease in the number of internal degrees of freedom. But since collective motions are formed owing to deep-lying nucleons, while internal excitations are determined basically by the single-particle levels adjacent to the Fermi surface, exclusion of the extra degrees of freedom in the low-temperature region should not strongly affect the density of the internal excited states. Under these conditions, adiabatic consideration can be fully justified, at least as a first step in the analysis of the rotational and vibrational increases, in the level density of nuclei.

Inclusion of the rotational increase of level density in the calculations based on the above thermodynamic relationships of the superfluid model greatly improves the agreement between the theoretical values and the observed density of neutron resonances [65,66]. In particular, by including this increase in the calculations with the single-particle spectrum of the Nilsson oscillator potential, Huizenga obtained a satisfactory description of the experimental data for both spherical and deformed nuclei [66]. The use of more realistic level schemes of the Saxon-Woods potential, however, gave a less satisfactory accordance with experiment, and the theoretical values of the density of the resonances in the case of deformed nuclei were systematically lower than those observed by a factor of 2-5 [65]. The deviation from experiment was still higher in the region of nuclei $100 \leq A \leq 150$, which have no equilibrium deformation in the ground state and for which there are consequently no grounds for introducing a rotational increase of level density. It may be assumed that all these differences reflect, to some extent, the role of the collective vibrational excitations not included in the calculations under consideration. This

assumption tallies with the above adiabatic evaluations of the vibrational increase of level density. A more convincing quantitative analysis of the influence of vibrational effects is considered in the following sections.

On the basis of the adiabatic evaluation of the coefficients of vibrational and rotational increase in level density, an attempt can be made to obtain the phenomenological systematics of experimental data which would not only reproduce the observed neutron resonance density but also incorporate the main results of the theoretical concepts about the shell and correlation effects. Systematics of this type were considered by us in Ref. [87]. In order to describe the level density above the point of phase transition from the superconducting to the normal state, we used the Fermi-gas model relationships with the excitation energy shifted by the value of condensation energy (2.22), and the influence of the shell effects on the value of the level density parameter a(U) was taken into account on the basis of relationships equivalent to (1.44). The equations of the superfluid model (2.41) were used to describe the energy dependence of level density below the phase-transition point, and the ground-state correlation functions $\Delta_o = 12/\sqrt{A}$ MeV were taken in accordance with the systematics for even-odd differences in the masses of nuclei. When calculating the rotational increase in the level density of deformed nuclei, we took the rigid-body value of the perpendicular moment of inertia in the normal phase and took into account the corresponding changes in the moment of inertia with temperature in the superfluid phase. The vibrational increase in level density was considered on the basis of the liquid-drop evaluation (3.9). The relations of this approach are, of course, more cumbersome than the simple equations of the Fermi-gas model. But we cannot avoid making the model more complicated if we want to gain a uniform and self-consistent description of level density and also of other statistical characteristics of nuclei over a wide range of excitation energies.

Within the framework of the model considered, from the description of experimental data on the density of neutron resonances of nuclei with $A \geq 150$ we found the values of the level density parameter $a(B_n)$ at an excitation energy equal to the neutron binding energy and the asymptotic values of parameter $\tilde{a}$ corresponding to the region of high excitation energies. The results

obtained are presented in Fig. 30, together with data from a similar analysis within the framework of the Fermi-gas model. From the systematics of the parameters obtained with allowance for the collective increase in the level density of nuclei we also determined the coefficients

$$\tilde{a}/A = 0{,}0931\ \text{MeV}^{-1};\ \gamma = 0{,}064\ \text{MeV}^{-1}, \tag{3.10}$$

which give the best approximation for correlation of the energy changes in the level density parameter with shell correction (1.44). The calculated values of the parameters $a(B_n)$ corresponding to the given set of coefficients are also presented in Fig. 30.

It may seem at first sight that both the systematics of resonance data shown in Fig. 30 are equally valid since they give approximately the same description of level density at excitation energies close to the neutron binding energy. However, these descriptions were obtained at different absolute values of the level density parameter. Consideration of the collective effects appreciably reduces the value of $a(B_n)$ and the asymptotic values of parameters $\tilde{a}$, which consequently show satisfactory agreement with the results of theoretical calculations of the level density parameters using the realistic level schemes of the Saxon-Woods potential (1.38). As we have already pointed out in Chapter 1, the Fermi-gas systematics of the parameters give no such agreement. But this means that the predictions of the two models will differ markedly as we move away from the binding energy, and the analysis of the differences can serve as a method of studying the collective increase in the excited nuclei level density. There is obviously a dire need to take the collective effects into account when describing the observed energy dependence of the fissionability of heavy nuclei. This problem will be discussed in more detail in the final chapter.

3.2. Phonon excitations of nuclei and combinatorial analysis of the density of multiphonon excitations

Although the phenomenological collective models are very useful for describing many properties of nuclei, it is natural to wish for an explanation of these properties at a deeper microscopic level. The microscopic consideration generally enables us not only to establish the limits of applicability of

the phenomenological models but also to gain a better understanding of the nature of the phenomena under study.

A common property of the interacting Fermi-particle systems is that they have a branch of collective excitations caused by the coherent motion of a large number of particles. We come across this property not only in nuclei but also in quantum fluids, plasmas and so on. The method of linearizing equations of motion, with all sorts of modifications worked out by many authors [88,89], is widely used in branches of physics to analyse these excitations. In the theory of the nucleus this approach is successfully employed for the microscopic description of the low-lying states of nuclei - collective as well as single-particle states [83,86]. For the purpose of the problems discussed here, it would be of great interest to obtain an equivalent description of the statistical characteristics of nuclei.

Before considering the possible ways of solving this problem, let us take a simple example and discuss the basic features of the microscopic methods of analysis of the coherent effects. The general idea of the method of linearization of equations can be formulated as the self-consistent separation, within a system of interacting particles, of elementary excitations which seem to be more or less independent. The corresponding equations determining the creation Q_λ^+ and destruction Q_λ operators of such excitations are usually written in the form

$$[H, Q_\lambda^+] = \omega_\lambda Q_\lambda^+; \quad [H, Q_\lambda] = -\omega_\lambda Q_\lambda, \tag{3.11}$$

where H is the Hamiltonian of the given system and the square brackets denote a commutator. The elementary excitations may correspond to excitations of individual particles but they may also be collective fluctuations of density, shape or another equilibrium characteristic of the system. Such collective fluctuations are called phonons in solid-state physics, and this name is generally used also when considering collective excitations of nuclei. According to the concepts of elementary excitations, the wave function for the ground states Ψ_0 of even-even nuclei should, in this case, be defined as a phonon vacuum, i.e. $Q_\lambda \Psi_0 = 0$, and the excited states as one-phonon states $Q_\lambda^+ \Psi_0$, two-phonon states $Q_\lambda^+ Q_\lambda^{+\prime} \Psi_0$ and so on. To orthonormalize the wave functions of the ground and excited states, the phonon operators should satisfy the commutation relations:

$$\left.\begin{aligned} &[Q_\lambda, Q_{\lambda'}] = [Q_\lambda^+, Q_{\lambda'}^+] = 0; \\ &[Q_\lambda, Q_{\lambda'}^+] = \delta_{\lambda\lambda'}. \end{aligned}\right\} \tag{3.12}$$

Let us consider, as an example, the model nuclear Hamiltonian, in which the effective interaction of particles moving in a self-consistent mean field is represented in a separable form

$$H = \sum_\nu \epsilon_\nu a_\nu^+ a_\nu - \frac{\chi}{2}\Big(\sum_{\nu\nu'} f_{\nu\nu'} a_\nu^+ a_{\nu'}\Big)^2. \tag{3.13}$$

Here f(r) indicates form factors and χ the strength constant of the effective forces, and for the sake of simplicity we shall assume that all the matrix elements of the form factors $f_{\nu\nu'}$ are real. It is also implicit that the effective interaction no longer contains components which could be included in the levels of the mean field ϵ_ν.

Let us seek the phonon operators in the form of a linear superposition of particle-hole excitations of the nucleus:

$$\left.\begin{aligned} Q_\lambda^+ &= \sum_{ph} (Y_{ph}^\lambda a_p^+ a_h - Z_{ph}^\lambda a_h^+ a_p); \\ Q_\lambda &= \sum_{ph} (Y_{ph}^\lambda a_h^+ a_p - Z_{ph}^\lambda a_p^+ a_h). \end{aligned}\right\} \tag{3.14}$$

Here, the subscript h denotes the states below the Fermi level corresponding in the non-interacting quasi-particle model to hole excitations, and subscript p the states above the Fermi level corresponding to particle excitations. When calculating commutators $[H, Q_\lambda^+]$ we shall regard operators $a_p^+ a_h$ as boson operators, i.e. we shall assume that the following commutation relationships are valid for them

$$[a_h^+ a_p, a_{p'}^+ a_{h'}] \approx \delta_{pp'}\delta_{hh'}. \tag{3.15}$$

This approach is usually called the quasi-boson approximation or random-phase approximation. Within its framework, the equations of motion (3.11) give us a system of linear algebraic equations for the transformation coefficients (3.14):

$$\left.\begin{aligned} (\epsilon_p - \epsilon_h - \omega_\lambda) Y_{ph}^\lambda &= \chi f_{ph} \sum_{p'h'} f_{p'h'} (Y_{p'h'}^\lambda + Z_{p'h'}^\lambda); \\ (\epsilon_p - \epsilon_h + \omega_\lambda) Z_{ph}^\lambda &= \chi f_{ph} \sum_{p'h'} f_{p'h'} (Y_{p'h'}^\lambda + Z_{p'h'}^\lambda). \end{aligned}\right\} \tag{3.16}$$

From these equations we find the secular equation for phonon self-energies

$$1 = 2\chi \sum_{ph} f_{ph}^2 \frac{\omega_{ph}}{\omega_{ph}^2 - \omega_\lambda^2}, \tag{3.17}$$

where $\omega_{ph} = \varepsilon_p - \varepsilon_h$. Using commutation relations (3.12) we can also easily obtain the explicit form of the expressions for coefficients Y_{ph}^λ and Z_{ph}^λ.

The solution of Eq. (3.17) for the schematic model of a degenerate single-particle spectrum, when all differences ω_{ph} are equal to ω_o, is well known [83]. In this case, we have

$$\omega = [\omega_0^2 - 2\chi\omega_0 \sum_{ph} f_{ph}^2]^{1/2} \tag{3.18}$$

for the energy of the lowest phonon excitation ($\chi > 0$) and $\omega = \omega_o$ for the others. It will be seen from (3.18) that the interaction of all particle-hole pairs is reflected coherently in the energy of the lowest excitation alone. With increase in the value of the strength constant this coherent excitation moves increasingly further away from the particle-hole excitation energy, i.e. becomes increasingly more collective. We will not involve ourselves in a discussion of the solutions of secular equation (3.17) since the relevant problems have been treated fairly exhaustively in the literature [83,86]. The monograph [53] also gives numerous examples of the practical application of the quasi-boson approximation to describe the low-lying collective states of spherical and deformed nuclei.

Let us now return to the statistical description of the properties of excited levels and consider how the approach outlined above can be applied to the analysis of these properties. In Ref. [90] direct combinatorial calculations of the density of possible multiphonon excitations of nuclei were suggested for this purpose. Here, of course, it is not assumed that the structure of the highly excited states of the nucleus corresponds to the picture of ideal phonons. Interaction of the various types of nuclear excitations makes the structure of nuclear levels much more complex even at small excitation energies, and the complexity increases sharply as one moves up the energy scale. However, studies [90] on the dispersion equations which take into account the interaction of quasi-particles with phonons have shown that the number of roots of these equations over a not too small energy interval is equal, with good accuracy, to the number of multiphonon base states in the same interval. So it can be assumed that an integral characteristic such as level density is not affected

significantly by the complication in the structure of multiphonon excitations. Thus, the problem of calculating the level density of realistic nuclei can be reduced to combinatorial recombination of all multiphonon excitations lying within the energy interval considered. Although such combinatorial calculations are extremely laborious, they are perfectly feasible with the use of modern computer techniques.

The approach formulated above was developed in Refs [38,39] and has been applied to a wide range of problems associated with nuclear level density. These calculations took into account, first of all, that in spherical nuclei considerable fluctuations existed in the energy dependence of level density and that there were specific differences in the density of levels with positive and negative parity (see Fig. 12). Moreover, it was evidently for the first time that within the framework of this approach a realistic evaluation of the vibrational increase in level density had been obtained. For this purpose, the results of level density calculations performed for empirical values of the strength constants of the effective multipole-multipole and spin-multipole interactions generating collective phonons were compared with similar calculations for zero constants, where the collectivity effect totally vanishes. The derived coefficients for vibrational increase in level density are shown in Fig. 31, together with the liquid-drop evaluation given above and the statistical calculation results, which will be discussed in more detail in the next section. Unfortunately, in the combinatorial calculations the analysis of the coefficients of increase in level density was made for a comparatively small number of nuclei. It is therefore difficult at this stage to draw any conclusion on their basis about the laws of change in these values with the excitation energy and shell structure of nuclei.

Since the phonon excitations determine only the density of the internal non-rotational states, in the level density calculations for deformed nuclei the phonon states obtained in the combinatorial approach with a fixed value of the angular-momentum projection on the symmetry axis were supplemented in the adiabatic approximation of (3.1) with a rotational band. This approach was adopted in Refs [38,39] to calculate the level densities of all nuclei for which experimental data on the density of neutron resonances were available. It was found that the results of this approach, without any special adjustment of parameters, agreed with experiment to within a factor of 1.5-2.0 in deformed

nuclei and a factor of 2-4 in spherical nuclei. This agreement is undoubtedly satisfactory, especially if we take into account that in many spherical nuclei the disagreements between the different experiments reach a factor of 2-3.

Together with the many advantages which have been mentioned, this approach has a number of disadvantages, the influence of which increases with excitation energy. They include not only the laboriousness of the combinatorial calculations, the elimination of which is basically a technical problem - more important are the limitations imposed on the multiphonon states by the Pauli principle. Because of violation of the Pauli principle, spurious states occur among the multiphonon excitations, and the number of them rises rapidly with increase in the number of excited phonons. Moreover, in complex multiphonon states it is important to take into account the changes in phonon structure due to weakening of the pair correlations and to the possible changes with excitation energy in the characteristics of the mean field. Rigorous consideration of all these effects is an extremely complex problem, to which no satisfactory solution has yet been found. It is, therefore, advisable to use the combinatorial description of multiphonon excitations only at energies of the order of, or lower than, the neutron binding energy, where the errors of the method are still small. At higher energies we have to take into account, at least approximately, all the above effects and this can be done at present only within the framework of a thermodynamic description of excited nuclei.

3.3. Thermodynamic description of the collective motions of heated nuclei

The above method of linearization of the equations of motion in the many-body theory is widely used, not only for analysis of the elementary excitations of the ground state of the system but also when considering the coherent effects at a non-zero temperature [88,89]. The coherent phenomena in highly excited nuclei can be studied in a similar manner [91]. In this approach, temperature is a statistical characteristic of the excitation (heated state) of the nucleus. The problem reduces to a self-consistent determination of phonons, whose thermodynamic properties reflect the role of the coherent effects.

To solve this problem we can use, as before, the equations of motion (3.11) and the commutation properties of phonons (3.12); but we have to calculate the phonon states taking into account the changes with temperature in the population of quasi-particle levels in the heated nucleus. More particularly, for the model Hamiltonian (3.13) considered in Section 3.2 we shall seek the phonon operators in the form

$$\left.\begin{aligned} Q^+_\lambda &= \sum_{\nu > \nu'} (Y^\lambda_{\nu\nu'} a^+_\nu a_{\nu'} - Z^\lambda_{\nu\nu'} a^+_{\nu'} a_\nu); \\ Q_\lambda &= \sum_{\nu > \nu'} (Y^\lambda_{\nu\nu'} a^+_{\nu'} a_\nu - Z^\lambda_{\nu\nu'} a^+_\nu a_{\nu'}). \end{aligned}\right\} \tag{3.19}$$

This definition differs from (3.14) only by the absence of limitations on the population of single-particle states. We will also take into account similar changes in the commutation relationships of the random-phase approximation, which we can represent in the form

$$[a^+_{\nu_1} a_{\nu_2}, a^+_{\nu'_1} a^+_{\nu'_2}] \approx \langle |a^+_{\nu_1} a_{\nu_2}, a^+_{\nu'_1} a_{\nu'_2}| \rangle \approx \delta_{\nu_1\nu'_2} \delta_{\nu_2\nu'_1} (\bar{n}_{\nu_1} - \bar{n}_{\nu_2}), \tag{3.20}$$

where the angular brackets denote statistical averaging and $\bar{n}_\nu$ the occupation numbers of single-particle states corresponding to this averaging. Using (3.20) for calculation of the commutators (3.11), we obtain the following equations for the coefficients of phonon transformation (3.19)

$$\left.\begin{aligned} (\omega_{\nu_1\nu_2} - \omega_\lambda)\, Y^\lambda_{\nu_1\nu_2} &= \chi f_{\nu_1\nu_2} \sum_{\nu'_1 > \nu'_2} f_{\nu'_1\nu'_2} (\bar{n}_{\nu'_2} - \bar{n}_{\nu'_1}) (Y^\lambda_{\nu'_1\nu'_2} + Z^\lambda_{\nu'_1\nu'_2}); \\ (\omega_{\nu_1\nu_2} + \omega_\lambda)\, Y^\lambda_{\nu_1\nu_2} &= \chi f_{\nu_1\nu_2} \sum_{\nu'_1 > \nu'_2} f_{\nu'_1\nu'_2} (\bar{n}_{\nu'_2} - \bar{n}_{\nu'_1}) (Y^\lambda_{\nu'_1\nu'_2} + Z^\lambda_{\nu'_1\nu'_2}). \end{aligned}\right\} \tag{3.21}$$

The secular equation for the self-energies of phonons in the heated nucleus takes the form:

$$1 = 2\chi \sum_{\nu_1 > \nu_2} f^2_{\nu_1\nu_2} \frac{\omega_{\nu_1\nu_2} (\bar{n}_{\nu_2} - \bar{n}_{\nu_1})}{\omega^2_{\nu_1\nu_2} - \omega^2_\lambda}. \tag{3.22}$$

It will be seen that at zero temperature relationships (3.20)-(3.22) coincide with the similar formulae in Section 3.2.

On the basis of commutation relationships (3.12) and (3.20) we can find the transformationsinverse to (3.19), and write the initial Hamiltonian (3.13) in the phonon representation

$$H_{\text{vibr}} = U_0 + \frac{1}{2} \sum_\lambda \omega_\lambda (Q^+_\lambda Q_\lambda + Q_\lambda Q^+_\lambda). \tag{3.23}$$

These transformations determine the phonon Hamiltonian with an accuracy up to the constant term U_o. However, the value of U_o can be fixed by requiring that when the interaction is switched off, the influence of the formally introduced phonons should not in any way be reflected in the thermodynamic functions of the system. This condition can be written as

$$\langle H_{\text{vibr}}\rangle_{\chi=0} = U_0 + \frac{1}{2}\sum_{\lambda} g_\lambda \omega_\lambda^0 \operatorname{cth}\frac{\omega_\lambda^0}{2t} = 0,$$

where ω_λ^0 denote the poles of the secular equation (3.22), which coincide with the solutions of the equation for zero-strength constant, and g_λ the degree of degeneracy of the solutions.

Since the phonon representation of the Hamiltonian (3.23) corresponds to an ideal gas made of Bose particles, the changes associated with phonons in the entropy and excitation energy of the nucleus are determined by

$$\left.\begin{aligned} \delta S &= \sum_{\lambda} g_\lambda \left\{ \frac{\omega_\lambda}{2t} \operatorname{cth}\frac{\omega_\lambda}{2t} - \ln\left(2\operatorname{sh}\frac{\omega_\lambda}{2t}\right)\right\} - \\ &- \sum_{\lambda} g_\lambda \left\{ \frac{\omega_\lambda^0}{2t} \operatorname{cth}\frac{\omega_\lambda^0}{2t} - \ln\left(2\operatorname{sh}\frac{\omega_\lambda^0}{2t}\right)\right\}; \\ \delta U &= \frac{1}{2}\sum_{\lambda} g_\lambda \left\{ \omega_\lambda\left(\operatorname{cth}\frac{\omega_\lambda}{2t} - 1\right) - \omega_\lambda^0\left(\operatorname{cth}\frac{\omega_\lambda^0}{2t} - 1\right)\right\}. \end{aligned}\right\} \qquad (3.24)$$

If we disregard similar changes in the pre-exponential factor, the influence of the coherent effects of a vibrational nature associated with phonons on nuclear level density can be represented in the form

$$K_{\text{vibr}} \approx \exp\left(\delta S - \frac{\delta U}{t}\right) = \prod_{\lambda}\left[\frac{1-\exp(-\omega_\lambda^0/t)}{1-\exp(-\omega_\lambda/t)}\right]^{g_\lambda}. \qquad (3.25)$$

When the difference between ω_λ and ω_λ^0 is small, the ratio of the corresponding components of (3.25) tends to unity; hence, a substantial contribution to the coefficient of vibrational increase in level density is made only by the collectivized excitations, for which the roots of the secular equation are quite strongly shifted relative to the poles. The occurrence of the statistical sum of poles in (3.25) reflects the non-adiabatic character of the effects under consideration. It should be noted, however, that if the roots and poles of the secular equation have a dense spectrum, a situation may possibly occur where none of the solutions of the equation is sufficiently collectivized but, as a whole, the increase in level density is appreciable. A study of such situations could be of considerable interest for the analysis of coherent effects which are not associated with a particular collective mode.

In Ref. [91] it is shown that this consideration is equally valid for a more general Hamiltonian than (3.13). Inclusion in the Hamiltonian of correlation interactions of the superconducting type, just as the use of effective interactions of a non-separable form, will make the secular equations more complex but the relationships for the thermodynamic functions (3.24) will remain the same. There is no need here to write out the corresponding secular equations since their explicit form is not really important for the problems under discussion.

Study of the structure of the low-lying levels of nuclei within the framework of the microscopic approach has shown that in the formation of the collective properties of levels the greatest role is played by the quadrupole-quadrupole and octupole-octupole components of the effective residual quasi-particle interaction [53,86]. For analysis of the statistical characteristics of excited nuclei it is therefore important, first of all, to study the influence of precisely these components of interaction.

The calculation results for the coefficients of increase in level density (3.25) due to the quadrupole-quadrupole quasi-particle interaction are presented for the most characteristic spherical nuclei in Fig. 32 [92]. To demonstrate the differences between the coherent effects in cold and hot nuclei, the figure also shows the results of calculations of similar coefficients for the spectrum of the roots and poles obtained at zero temperature. The value of coefficients K_{vibr} in the latter case is determined almost entirely by the energy of the first vibrational level and for practical purposes coincides with the adiabatic evaluation of its value (3.5). The temperature dependence of the spectrum for the roots of the secular equation is very important for the analysis of the coherent effects. The non-monotonic dependence of the coefficients K_{vibr} for the ^{58}Fe and ^{120}Sn nuclei (see Fig. 32, lower part) is associated with the decrease of the correlation functions of the nucleus in the temperature region from 0.4 to 0.6 MeV and with the corresponding rearrangement of the elementary excitation spectrum. The breakup of the correlation effects of the superconducting type in the highly excited (heated) nucleus leads to considerable weakening of the coherent effects in the quadrupole excitation spectrum of the nucleus, and this is reflected directly by the value of the coefficient of increase in level density. Figure 33 presents the calculation results for coefficients K_{vibr} due to octupole-octupole effective quasi-particle interaction. The pair correlations have little influence on the octupole excitation spectrum and their breakup in heated nuclei does not substantially affect the temperature dependence of the corresponding coefficients.

It should be pointed out that at temperatures exceeding the critical temperature of breakup of pair correlations the spectrum of low-energy solutions of the secular equations is very dense and may contain many roots with $\omega_\lambda < t$. As a rule, none of these roots is sufficiently collectivized but as a result they can make an appreciable contribution to the coefficient of vibrational increase in level density. To some extent, this result may be due to errors in the approximation used for linearization of the equations of motion for $\omega_\lambda \approx \omega_\lambda^0$. For a rigorous solution of the problem it is necessary in this case to go beyond the framework of linear approximation and to take into account the damping of the temperature phonons under consideration. An approximate evaluation of the role of damping can be made on the basis of the similarity of the coherent excitations of the nucleus with zero sound in a Fermi liquid. The region where the zero sound occurs is determined by the inequality $\omega > \hbar/\tau$, where τ is the relaxation time or the mean lifetime of the excited quasi-particles [93]. The relaxation time is inversely proportional to the square of the temperature, and for the nucleus the proportionality factor can be evaluated from the imaginary part of the optical potential. On the basis of such an analysis we should limit the energies of the coherent phonons in the nucleus by the inequality $\omega_\lambda > (0.1 \text{ to } 0.5)t^2$ MeV. If the coefficient of vibrational increase in level density (3.25) is determined by the higher-energy phonons, the results obtained can be regarded as sufficiently reliable. If not, the results of calculations in the linear approximation are hardly trustworthy. As the analysis showed, in calculations of the coefficients this limitation is not important in practice for the bulk of the spherical nuclei at temperatures below 1.0-1.5 MeV. In the case of the transitional and deformed nuclei the situation is less favourable but depends on the consistency of the choice of single-particle level schemes and strength constants of effective interaction.

The results of thermodynamic description of the vibrational increase in level density can be compared with those of combinatorial analysis of the density of multiphonon excitations of nuclei considered in the preceding section. Conceptually, both approaches are fairly close to each other and are based on an identical effective Hamiltonian. However, the relationships utilized in the level density calculations differ appreciably, and this definitely affects the results. In the combinatorial analysis of the multiphonon excitations the phonon spectrum is determined by the solutions of secular equations similar

to (3.17), and the values of K_{vibr} given at the top in Figs 32 and 33 should correspond to this approach. Consideration of the rearrangement of the quasi-particle excitation spectrum in the thermodynamic description leads to a change in these values (see the lower parts of Figs 32 and 33). However, when calculating the increase due to the combined influence of all vibrational modes the differences referred to are partially compensated for, and the integral K_{vibr} obtained in the thermodynamic calculations of level density agree, on the whole, satisfactorily with the combinatorial calculation results (see Fig. 31). The quantitative differences in these results may be due to some differences in the single-particle level schemes and to a difference in the modelling of the effective forces. Unfortunately, the two approaches were not compared under identical conditions.

Approximation of residual two-particle interactions of nucleons by multipole-multipole forces is used widely in the microscopic description of the collective excitations of atomic nuclei [53,86]. The choice of these forces is justified mainly by the mathematical simplicity of the solution of the equations which determine the energy and intensity of electromagnetic transitions in nuclei in the random-phase approximation or in other equivalent modifications of harmonic approximation. A more realistic form of the effective forces can be obtained if we use the invariance principles or other self-consistency conditions to choose them. Various formulations of these conditions have been considered by many authors for cold nuclei [86,94-98]. Similiar conditions can be obtained also for the effective forces which determine the coherent excitations of heated nuclei [99-102].

As the simplest self-consistency condition, we require that the changes in time of the mean field $\delta V(\tau)$ determining the single-particle motion of nucleons in the nucleus should occur self-consistently with changes in the density matrix $\delta\rho(\tau)$ [86,95]. In the co-ordinate representation this condition can be written in the form

$$\delta V(\mathbf{r},\tau) = \int V(\mathbf{r},\mathbf{r}')\delta\rho(\mathbf{r}',\tau)d\mathbf{r}', \qquad (3.26)$$

where V(r,r') is the effective residual interaction. To simplify further consideration, we shall assume that the residual interaction does not depend on particle type (on particle isospin) and that the mean field can be approximated by the local single-particle potential. In the nucleus we do not know the

true residual interactions resulting from subtraction of the self-consistent field and we can only model them within the framework of the corresponding approximation when describing different experimental data. In practical applications it is very convenient to model the effective forces in the separable form

$$V(\mathbf{r},\mathbf{r}') = \sum_{\lambda,\mu} \varkappa_\lambda f_\lambda(r) f_\lambda(r') Y^*_{\lambda\mu}(\Omega) Y_{\lambda\mu}(\Omega'). \tag{3.27}$$

Applying condition (3.26), we can require that the effective forces (3.27) should generate the same changes in the self-consistent field as the true residual interactions. If this requirement is strictly met, then, within the framework of harmonic approximation, the calculations of the characteristics of collective modes (nuclear density oscillations in the nucleus) with effective forces (3.27) should give the same results as the calculations with the realistic two-particle interaction.

When using the effective forces (3.27) to model the residual interactions, we should not, of course, require detailed coincidence of all fluctuations in the field and average density of nucleons in the nucleus. It is quite sufficient if the basic components of their spatial oscillations are self-consistent. If we assume that these oscillations can be described unambiguously by the set of parameters $\beta_{\lambda\mu}(\tau)$ characterizing a family of equipotential surfaces with radius

$$r(\Omega) = \bar{r}\,[1 + \sum_{\lambda \geqslant 2} \sum_{\mu \leqslant \lambda} \beta_{\lambda\mu}(\tau)\, Y^*_{\lambda\mu}(\Omega)], \tag{3.28}$$

then the variations in the mean field and density of nucleons can be represented in the form

$$\left.\begin{aligned} \delta V(\mathbf{r},\tau) &= -r \frac{d\bar{V}(r)}{dr} \sum_{\lambda\mu} \beta_{\lambda\mu}(\tau) Y^*_{\lambda\mu}(\Omega); \\ \delta\rho(\mathbf{r},\tau) &= -r \frac{d\bar{\rho}(r)}{dr} \sum_{\lambda\mu} \beta_{\lambda\mu}(\tau) Y^*_{\lambda\mu}(\Omega), \end{aligned}\right\} \tag{3.29}$$

where $\bar{\rho}(r)$ corresponds to the steady-state nucleon density distribution in the local mean field $\bar{V}(r)$. Using the self-consistency condition (3.26) and relationships (3.29) we can easily show that the radial dependence of the form factors of effective forces (3.27) should take the form

$$f(r) = r \frac{d\bar{V}(r)}{dr}. \tag{3.30}$$

For the interaction constant we obtain the following relationship from the self-consistency condition

$$\chi = -\left[\int_0^\infty r^4 \frac{d\bar{V}}{dr} \frac{d\bar{\rho}}{dr} dr \right]^{-1} . \tag{3.31}$$

Unlike the ordinary multipole-multipole forces [86], this constant does not depend on λ.

For the density matrix of the spherical single-particle potential of the shell model we rewrite relationship (3.31) in the form

$$\chi^{-1} = \sum_j \frac{2j+1}{4\pi} \left\langle j \left| \frac{1}{r^2} \frac{d}{dr} r^4 \frac{d\bar{V}}{dr} \right| j \right\rangle [u_j^2 \bar{n}_j + v_j^2 (1 - \bar{n}_j)], \tag{3.32}$$

where $\bar{n}_j$ is the population of single-particle levels at a given temperature t and u_j and v_j are the coefficients of the variational transformation (2.2). At zero temperature the relationships (3.30)-(3.32) coincide with the relationships of a similar nature considered in Refs [86,95]. A simple evaluation of the integral (3.31) can be obtained for the Saxon-Woods potential

$$\chi = \left[\frac{3V_0}{4\pi r_0^3} \int_0^\infty \left(r^2 \frac{dg(r)}{dr} \right)^2 dr \right]^{-1} = \frac{32\sqrt{2\pi}}{3r_0 \gamma V_0} A^{-4/3}, \tag{3.33}$$

where V_o is the potential well depth and $g(r) = [1 + \exp\gamma(r - R_o)]^{-1}$ its corresponding form factor (R_o is the well radius and γ the diffusivity parameter).

In order to find the temperature dependence of the interaction constant, we obtain its evaluation for the harmonic-oscillator potential. Disregarding the correlation interaction of nucleons, for the oscillator potential we find

$$\chi^{-1} = \frac{5\omega_0}{2\pi} \sum_{N=0}^{\infty} \left(N + \frac{3}{2} \right) (N+1)(N+2) \bar{n}_N \approx \frac{5\epsilon_f^4}{8\pi\omega_0^3} \left[1 + 2\left(\frac{\pi t}{\epsilon_f} \right) \right]^2 . \tag{3.34}$$

Here, N is the principal quantum number and $\epsilon_N = (N + 3/2)\omega_o$ its corresponding single-particle level energy; $\epsilon_f = (3A/2)^{\frac{1}{3}}\omega_o$ is the Fermi energy and $\omega_o = 41.5\ A^{-1/3}$ MeV. At zero temperature the constant (3.34) coincides with the constant of the traditional quadrupole-quadrupole forces [86].

In the case of a number of nuclei, we carried out [99] numerical calculations of the interaction constant (3.32) for the spectrum of single-particle states and the corresponding single-particle wave functions of the shell model. The values obtained for the interaction constant in the cold nucleus are shown in the lower part of Fig. 34 and the corresponding temperature dependence at the top of the same figure. The absolute values of the interaction constant show a fairly good agreement with evaluation (3.33), while the temperature dependence of the interaction constant for the Saxon-Woods potential is even weaker than for the harmonic oscillator. The characteristics of the behaviour of the strength constants $\chi(t)$ in the low-temperature region (see top of Fig. 34) for nuclei with substantial pairing are due to correlation effects of the superconducting type. On the whole, the changes in the constants with temperature are sufficiently small, and consideration of these changes is not reflected noticeably in the results of calculation of vibrational increase in the level density of near-magic nuclei. However, if the shape of the nucleus changes during heating, the self-consistent determination of the changes in the interaction constants with temperature is very important under these conditions.

The interaction constant of the traditional multipole-multipole forces is usually adjusted with respect to the position of the first vibrational level in the nucleus under consideration [53]. For modified effective forces the values of the constants thus obtained are shown in the lower part of Fig. 34. The difference between these values and those obtained on the basis of relationship (3.32) is due, firstly, to some inaccuracy of the harmonic approximation and, secondly, to the insufficient completeness of the base used in similar calculations for the associated single-particle levels. To ensure completeness of the base we should also include in the calculations the single-particle states in the continuous-spectrum region. From the practical point of view, this problem is extremely laborious. Therefore, although the basic qualitative features of the change in the characteristics of vibrational excitations in different nuclei represented by the calculations with theoretical values of the constants are sufficiently accurate, for the purpose of a quantitative description of the coefficients of vibrational increase in level density it is better to use the values of strength constants corrected by the available experimental data on the energy of collective levels. Such a correction, to some extent,

compensates for the drawbacks of the simple model Hamiltonians used in the practical calculations. As the analysis performed in Ref. [99] shows, when the strength constants are adjusted with respect to the position of the vibrational collective level in the cold nucleus, the form of the radial dependence of the effective forces does not strongly influence the calculation results, at least for the vibrational excitations of low multipolarity.

For more complete modelling of the various components of the effective forces we can also use the invariance requirements imposed on the nuclear Hamiltonian by the laws of conservation of momentum, angular momentum and so on. Use of the invariance principles was found to be very rewarding in the study of the collective levels of cold nuclei [96-98], and the possibility of utilizing similar methods for heated nuclei has been demonstrated in Refs [100-102]. These methods enable us to introduce self-consistently isoscalar and isovector components of the effective forces and also, if necessary, to study the effective interaction of other types of symmetry. The problems of self-consistent determination of interactions are important not only for the analysis of vibrational increase in level density but also for study of the spectral intensity of radiative transitions in highly excited nuclei [101]. At present, we are still at a comparatively early stage in the study of the collective properties of highly excited nuclei and possess very scant experimental data on the possible coherent effects.

The above analysis of the density of neutron resonances shows that a consistent explanation of the observed values cannot be found without using the rotational and vibrational increases in level density. Unfortunately, the lack of sufficiently reliable experimental data on the absolute value of level density at excitation energies exceeding the neutron binding energy substantially restricts the possibilities of critically verifying the theoretical description of the energy dependence of the vibrational increase in level density.

It should be noted that during transition to the region of excitation energies $U > 50$ MeV, the increase in the number of excited quasi-particles and decrease in their mean lifetime causes the coherent effects in the motion of particles to be weakened, and consequently the vibrational increase in level density should vanish. As a result of transition to the spherical

equilibrium shape in highly excited nuclei, the effects of rotational increase in level scheme should also disappear. Thus, in the high-energy region the level density of nuclei can again be described by the relationships of the Fermi-gas model; however, the values of level density parameter a in this case should correspond to the results of the shell model definition (1.38) or to the systematics (3.10) considered above and not to the results of the Fermi-gas analysis of the density of neutron resonances (1.46).

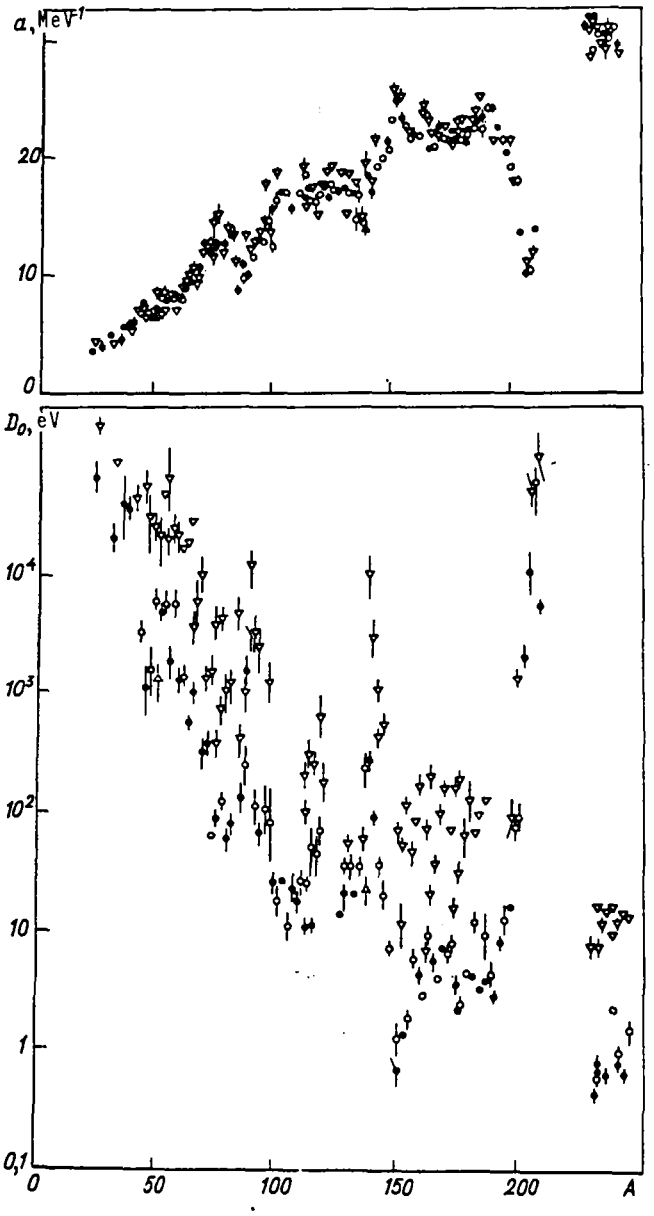

Fig. 1. Experimental data on the mean distance between neutron resonances (below) and the values of the Fermi-gas level density parameter a obtained from their analysis (above):

o - even-even nuclei; ∇ - even-odd nuclei; Δ - odd-even nuclei;
● - odd-odd nuclei.

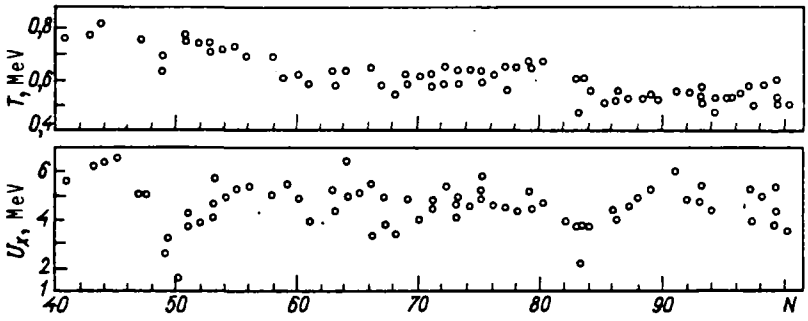

Fig. 2. The dependence of nuclear temperature T and excitation energy U_x, below which the level density behaviour is approximated by the constant-temperature model, on the number of neutrons [10].

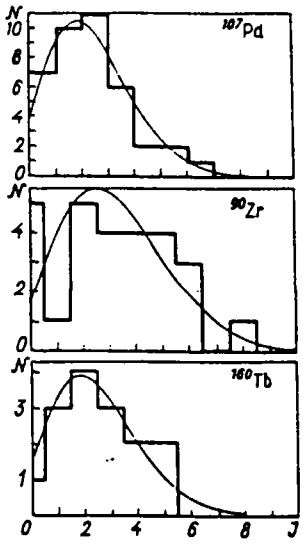

Fig. 3. Spin distributions of the identified low-lying levels of nuclei (histograms) and their approximation by the statistical distribution (1.6) (curves) [10].

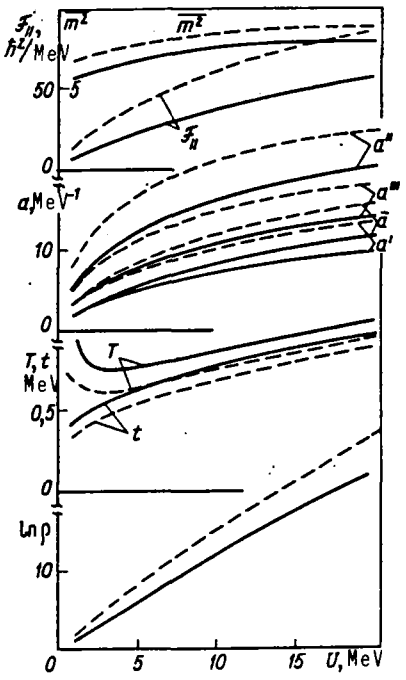

Fig. 4. The energy dependence of the thermodynamic characteristics of the ^{208}Pb nucleus as calculated for single-particle levels of the Saxon-Woods potential (solid curves) and those of the Nilsson oscillator potential (dashed curves).

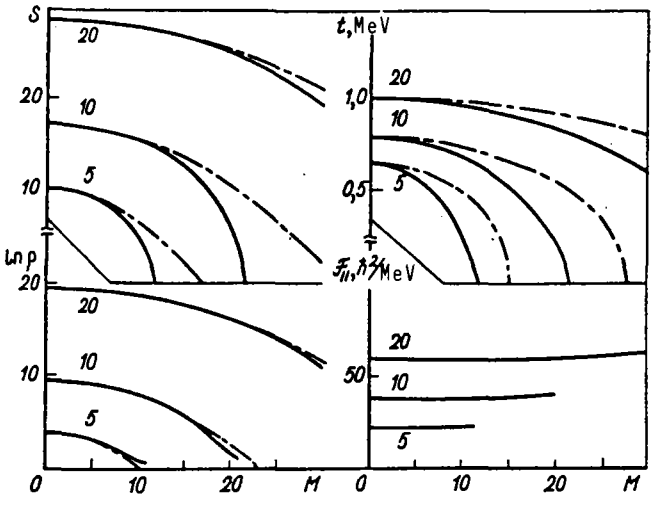

Fig. 5. The dependence of the thermodynamic characteristics of the ^{208}Pb nucleus on the angular-momentum projection as obtained from a rigorous solution of the equations of state (sold curves) and in the approximation of small angular momenta (dot-dash curves). The numbers on the curves denote excitation energy in MeV.

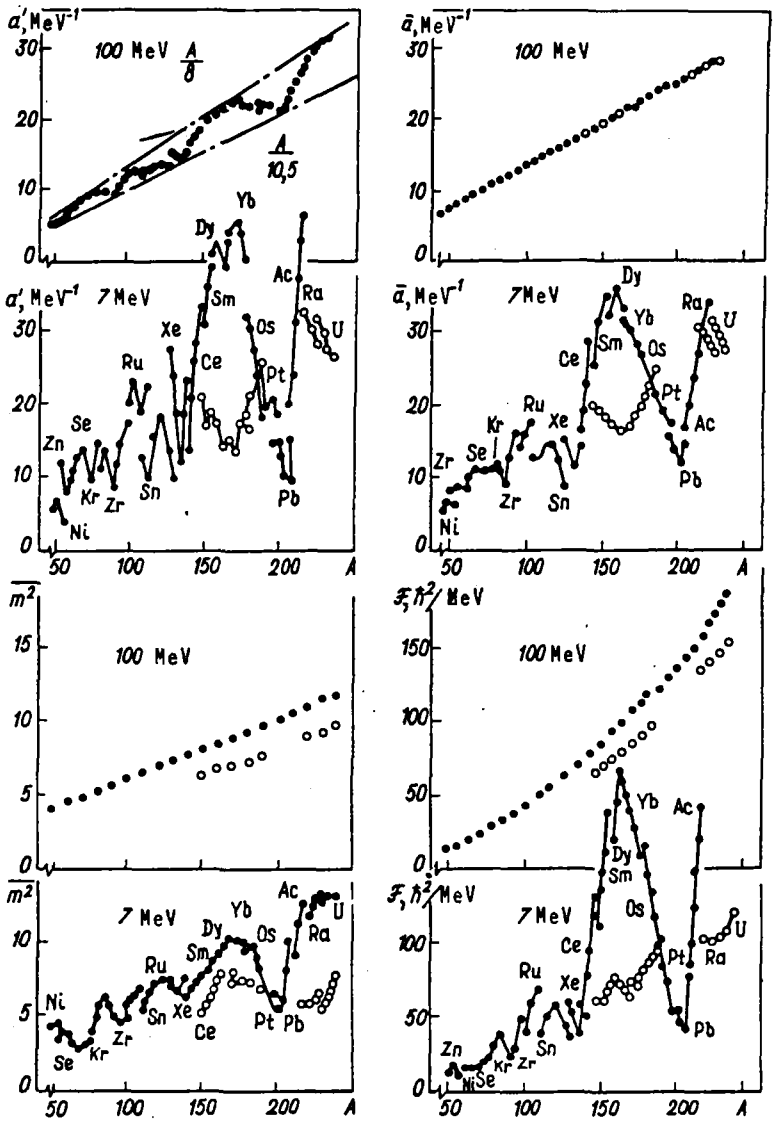

Fig. 6. Parameters a', $\bar{a}$, $\bar{m}^2$ and $\mathcal{F}_\hbar$ in the non-interacting-particle model for two excitation energies: ● - results of calculation with the single-particle spectrum of spherical nuclei; o - same for deformed nuclei.

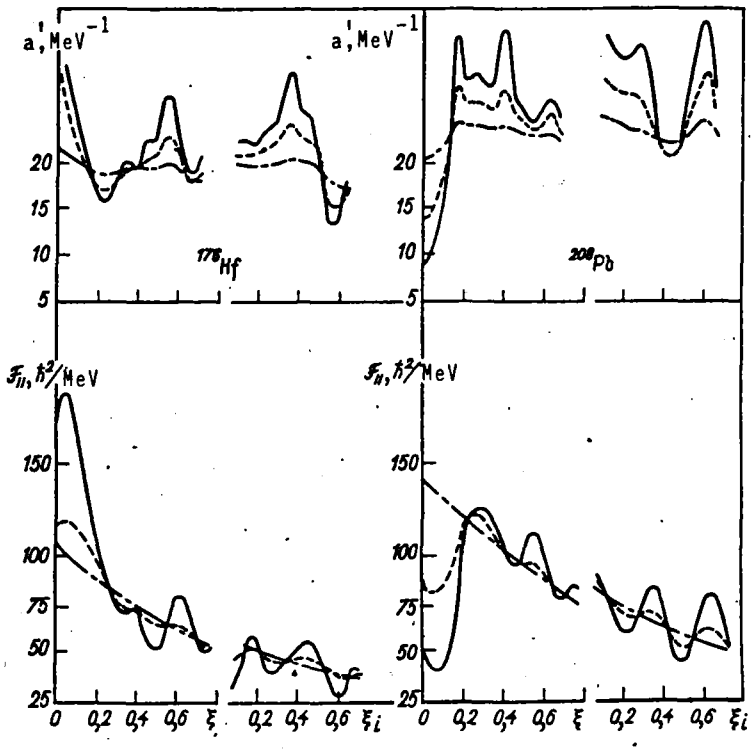

Fig. 7. The dependence of the level density parameters a' and moments of inertia $\mathcal{F}_\hbar$ at different excitation energies on deformation ξ. The solid curves denote 7 MeV, the dashed curves 20 MeV and the dot-dash curves 100 MeV.

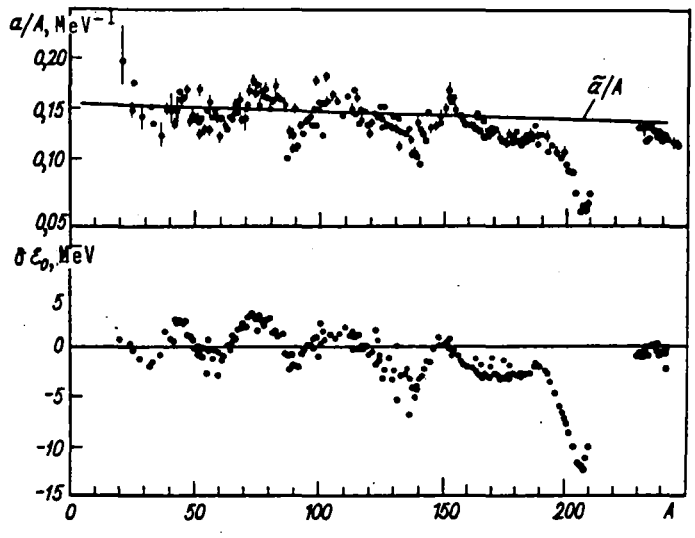

Fig. 8. The ratio of level density parameter a to mass number (above), and the shell correction in the mass formula [25] (below).

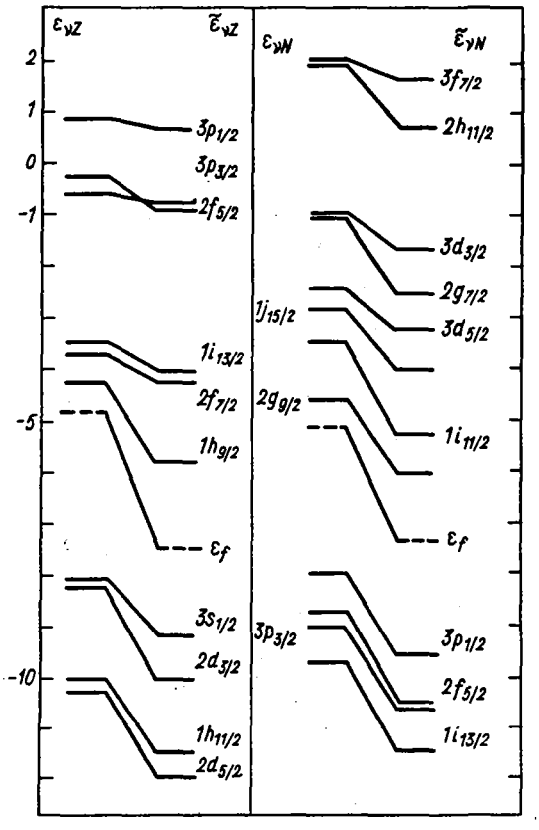

Fig. 9. Proton and neutron level schemes of the ^{214}Po nucleus at zero temperature (left) and at a temperature of 3.5 MeV (right).

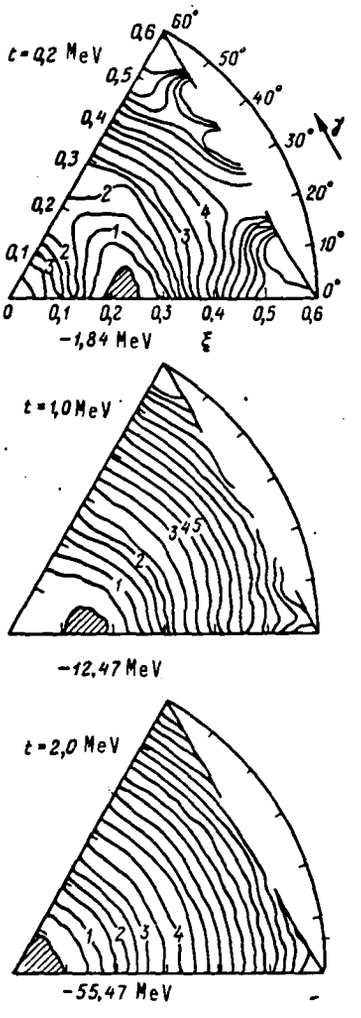

Fig. 10. Free-energy equipotential-surface maps for the ^{160}Yb nucleus at different temperatures [34]. The free energy at the point of the minimum is indicated below each map. The numbers on the equipotential lines give the changes in free energy.

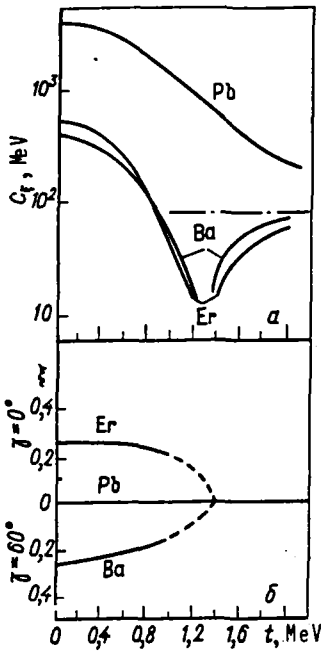

Fig. 11. The temperature dependence of the axial rigidity (a) and equilibrium deformation (b) of the ^{126}Ba, ^{160}Er and ^{208}Pb nuclei. The value of rigidity for the liquid-drop model is shown by the dot-dash curve.

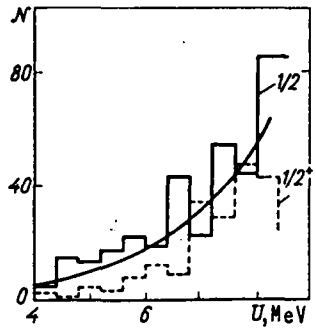

Fig. 12. The histogram of the number of levels with $J^{\pi} = 1/2^{+}$ and $1/2^{-}$ in the 400 keV region obtained in the combinatorial calculations for the ^{57}Fe nucleus [38]. The solid curve corresponds to the Fermi-gas description of level density with parameter a found from the density of neutron resonances.

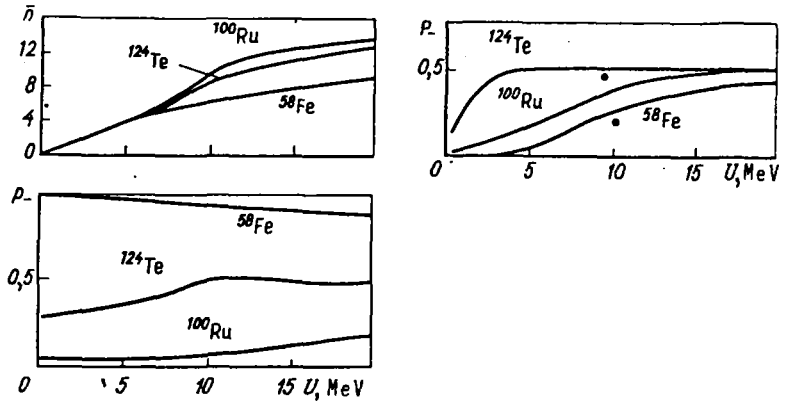

Fig. 13. The dependence of the average number of excited quasi-particles $\bar{n}$, probability p_- and relative contribution P_- of the negative-parity states to the level density of the nucleus. The black dots denote the results of the combinatorial calculations for the ^{58}Fe and ^{124}Te nuclei.

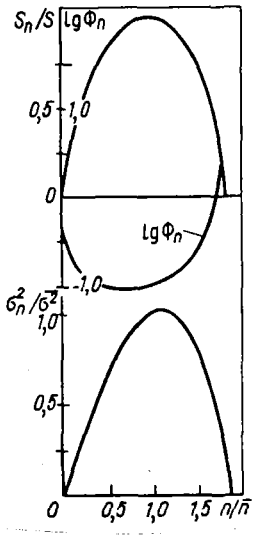

Fig. 14. The dependence of the thermodynamic characteristics of the excited particle-hole states in the Fermi-gas model on the $n/\bar{n}$ ratio.

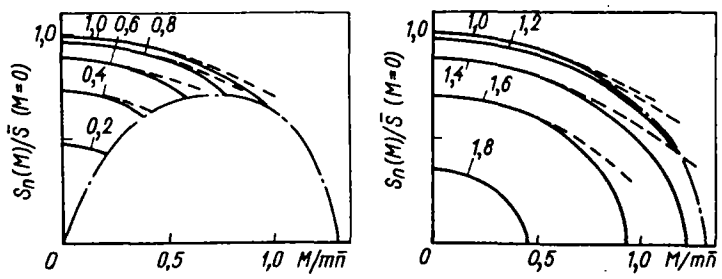

Fig. 15. The dependence of the $S_n/\bar{S}$ ratio on angular-momentum projection $M/m\bar{n}$. The number on the curves denotes the $n/\bar{n}$ ratio. The dashed curve represents the dependence of entropy in the approximation of small momenta and the dot-dash curve the maximum attainable values of angular momenta.

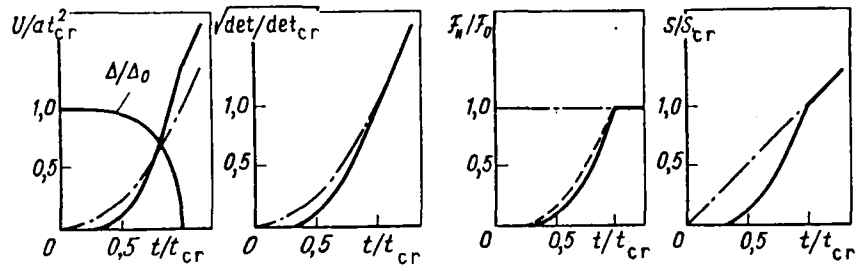

Fig. 16. The temperature dependence of the thermodynamic characteristics of the superfluid model of the nucleus (solid curves) and of the Fermi-gas model (dot-dash curves). For moments of inertia the solid curve denotes $\mathcal{I}_{||}^{MCE}$ (2.59) and the dashed curve $\mathcal{I}_{||}$ (2.29).

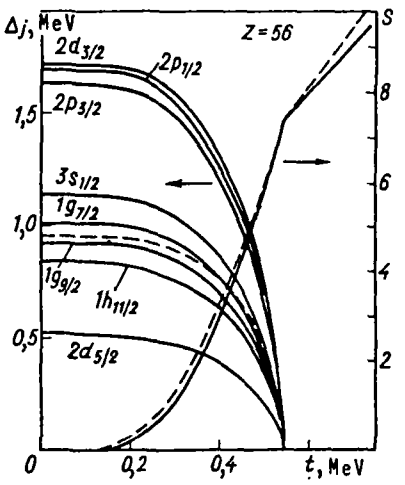

Fig. 17. The temperature dependence of the correlation functions and entropy of proton levels for ^{138}Ba for pairing interaction in the form of δ-forces (solid curves) and in the approximation of G = const (dashed curves).

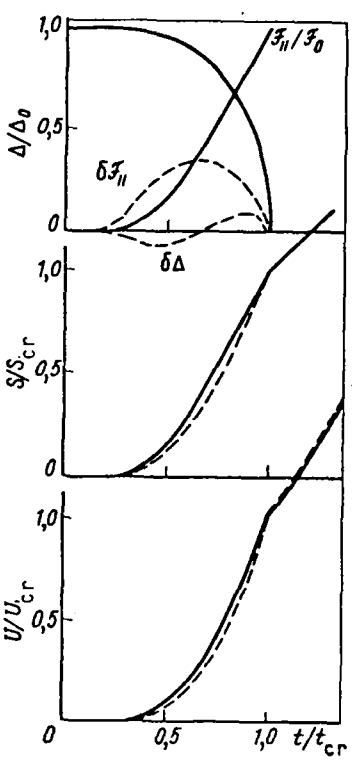

Fig. 18. The temperature dependence of the thermodynamic functions of the modified superfluid model (solid curves) and of the traditional model with G = const (dashed curves). In the case of the correlation function and the moment of inertia, the dashed line shows the difference between the respective values, multiplied by 100.

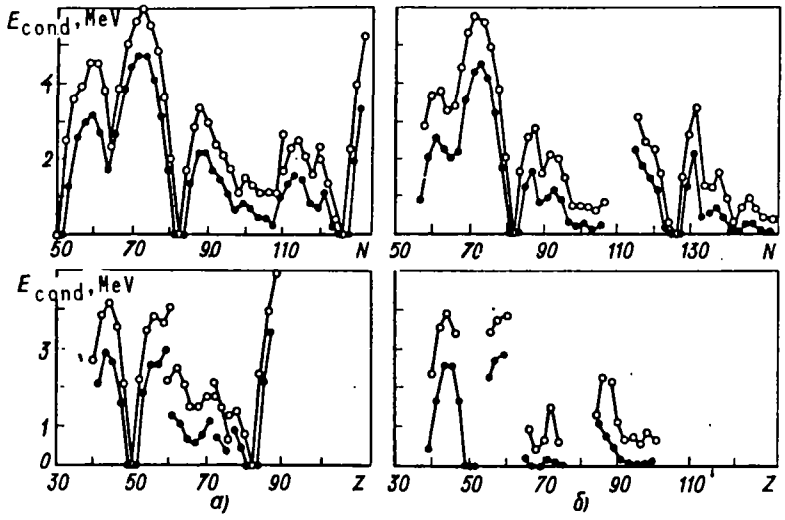

Fig. 19. Condensation energies for systems with an even (o) and an odd (•) number of protons and neutrons.

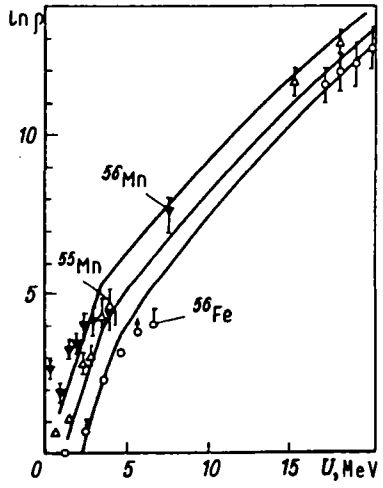

Fig. 20. The energy dependence of the level density of nuclei with even and odd numbers of neutrons and protons.

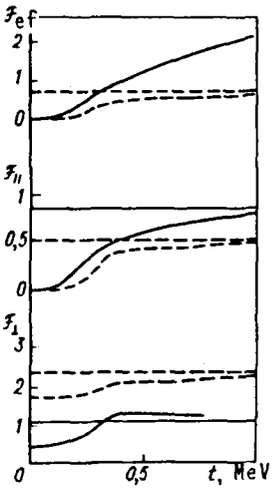

Fig. 21. The temperature dependence of the perpendicular $\mathcal{J}_{\perp}$, parallel $\mathcal{J}_{\parallel}$ and effective $\mathcal{J}_{ef}$ moments of inertia of the ^{236}U nucleus for equilibrium deformation (solid curves) and for transitional configurations (dashed curves) corresponding to the fission barrier peak [69]. The moments of inertia are given in units of $\mathcal{J} = \frac{2}{5}\mu_o AR_o^2$, and the straight lines parallel to the abscissa axis determine the rigid-body values of the moments of inertia for corresponding deformations.

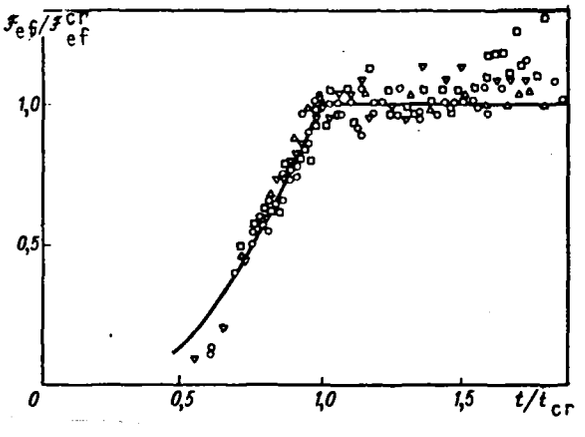

Fig 22. The temperature dependence of the effective moment of inertia of the transitional configurations of polonium isotopes: ▽ - ^{208}Po; o - ^{210}Po; □ - ^{211}Po; and Δ - ^{212}Po. The solid curve corresponds to the predictions of the superfluid model in the continuous-spectrum approximation.

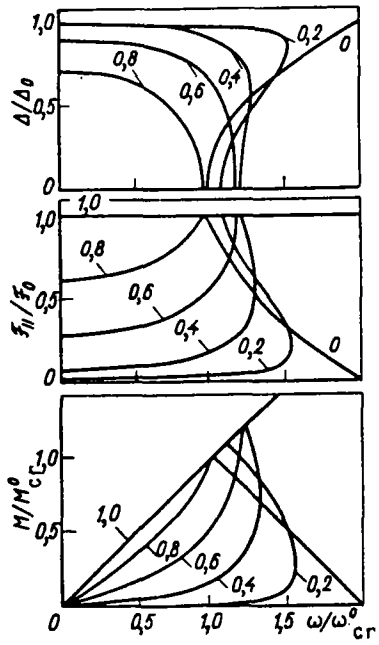

Fig. 23. The dependence of correlation functions, moments of inertia and angular-momentum projections on rotation frequency at a fixed temperature. The numbers on the curves denote temperature in units of t/t_{cr}.

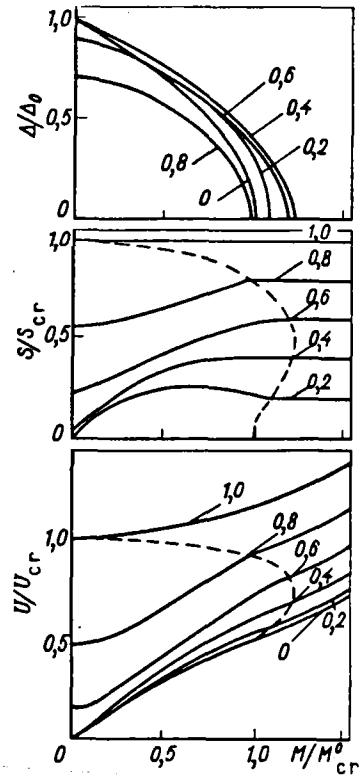

Fig. 24. The dependence of thermodynamic functions on angular-momentum projection at a fixed temperature. The numbers on the curves denote temperature.

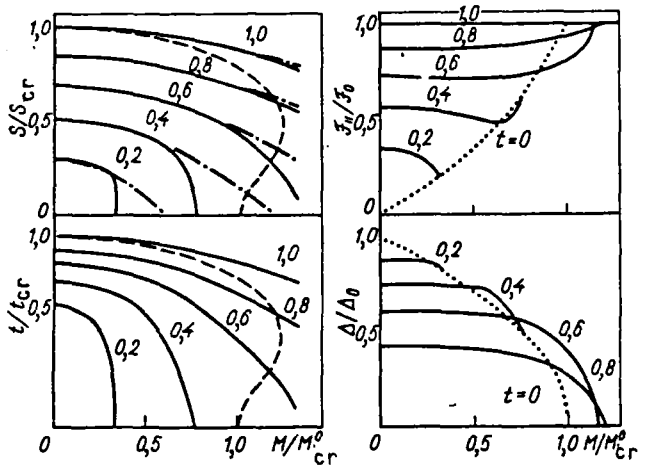

Fig. 25. The dependence of thermodynamic functions on angular-momentum projection for a fixed excitation energy. The dot-dash curve shows the dependence of entropy in the approximation of small momenta and the dashed line the phase-transition curve. The numbers on the curves indicate excitation energy in units of U/U_{cr}.

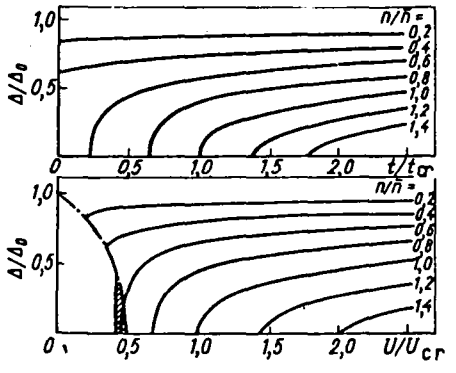

Fig. 26. The dependence of correlation functions Δ_n on temperature and excitation energy.

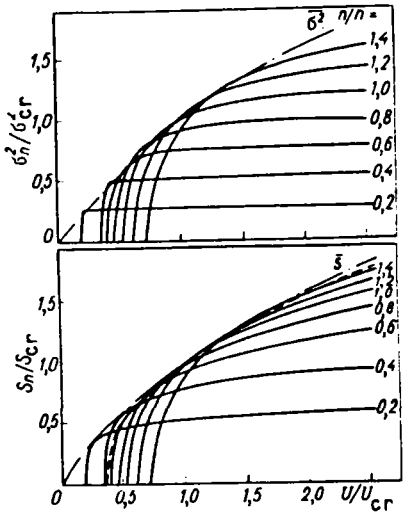

Fig. 27. The dependence of entropy and spin cut-off parameter on excitation energy. The dot-dash curve represents the behaviour of the average values of $\bar{S}(\gamma = 0)$ and $\bar{\sigma}^2(\gamma = 0)$.

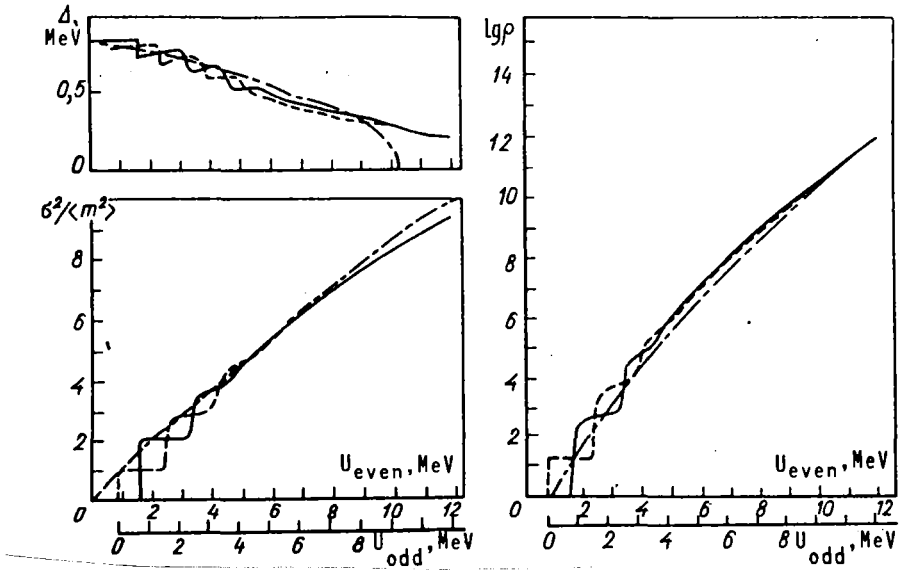

Fig. 28. The energy dependence of the correlation functions, spin cut-off parameters and density of excited states for a system with an even (solid curves) and an odd (dashed curves) number of particles. The dot-dash lines indicate the results of the thermodynamic description which takes no account of the discrete nature of the quasi-particle excitations.

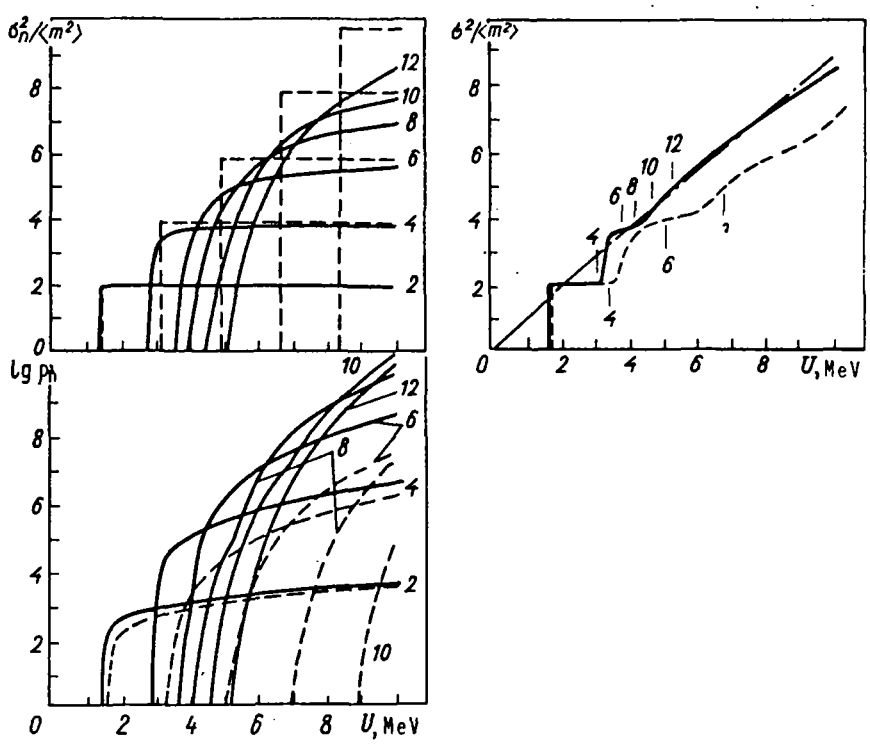

Fig. 29. The energy dependence of the characteristics of n-quasi-particle excitations in the superfluid model of the nucleus (solid curves) and in the Boltzmann-gas model (dashed curves). The dot-dash curve represents the thermodynamic description of the spin cut-off parameter without a fixed number of quasi-particles; the thresholds of n-quasi-particle excitations are also shown here.

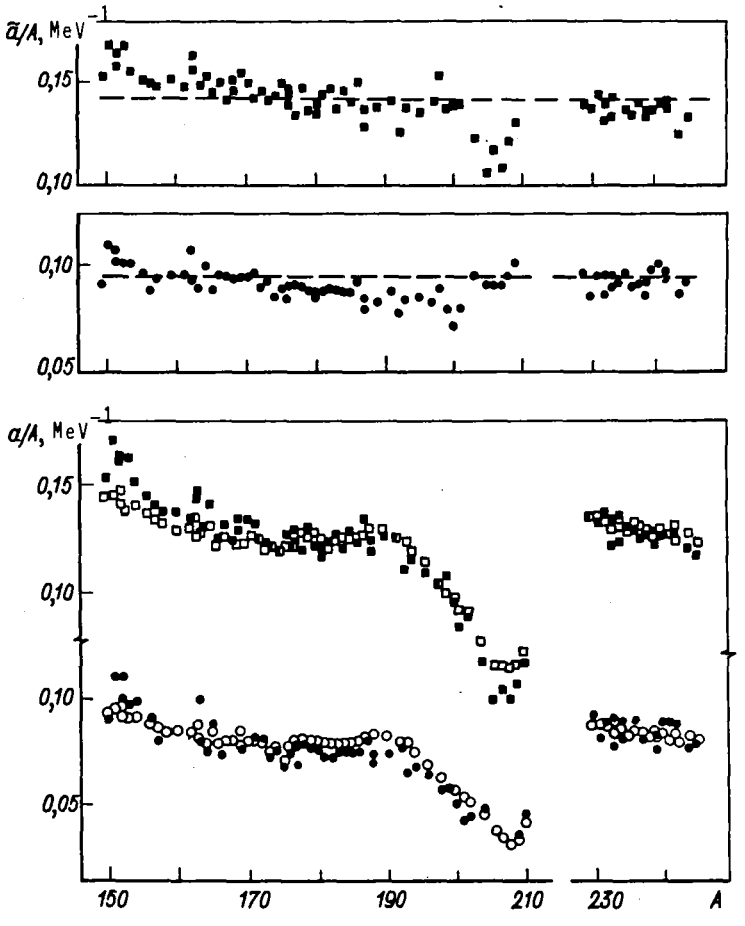

Fig. 30. Level density parameters $a(B_n)$ and their asymptotic values ã obtained from the analysis of the density of neutron resonances in the Fermi-gas model (■) and in the superfluid model of the nucleus with allowance for collective effects (●). The open symbols indicate the results of the phenomenological description (1.44) for sets of parameters (1.46) and (3.10).

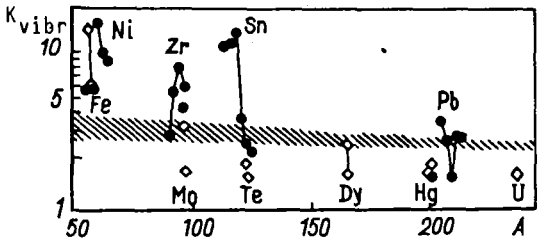

Fig. 31. The coefficients of vibrational increase in level density for different nuclei at excitation energy $U = B_n$. The hatched region corresponds to the evalution (3.9); ◇ indicates the results of combinatorial calculations [39] and ● the results of thermodynamic calculations [103].

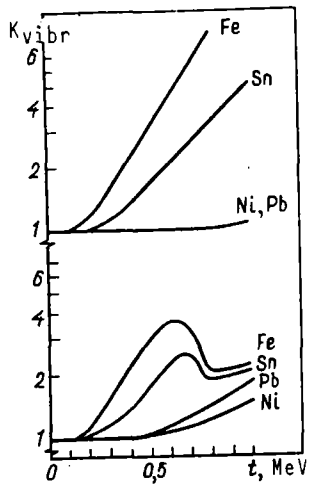

Fig. 32. The temperature dependence of the coefficients of increase in level density due to quadrupole coherent modes. The calculation results for the spectrum of roots and poles of the cold nucleus are shown above and those with allowance for the temperature dependence of the roots of secular equation (3.22) are given below.

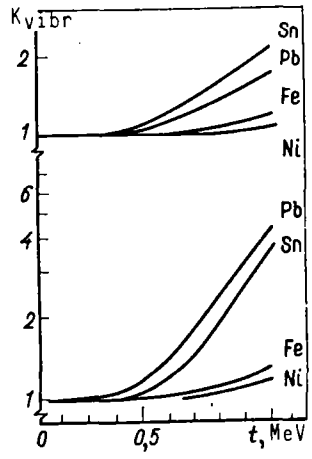

Fig. 33. The same as in Fig. 32 for octupole modes.

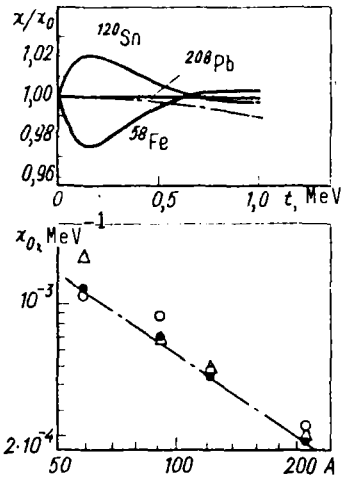

Fig. 34. The temperature dependence of the strength constant of effective multipole interaction (above) and the dependence of the strength constant on mass number for cold nuclei (below). The black dots represent the results of calculation using relationship (3.32) and the open symbols the same with the fitting of the solutions of the secular equation to experimental data on the position of the $2^+(0)$ and 3^- (Δ) levels. The dot-dash curves correspond to quasi-classical evaluations of χ_o and χ/χ_o on the basis of relationships (3.33) and (3.34).

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