Num. PDE

Recall:

Forward dif $\quad \dfrac{f(x+h)-f(x)}{h} \sim f'(x)$

Backward dif $\quad \dfrac{f(x)-f(x-h)}{h} \sim f'(x)$

(best) central dif
$$\left\{ \begin{array}{l} \dfrac{f(x+h)-f(x-h)}{2h} \sim f'(x) \\ \dfrac{f(x+h)-2f(x)+f(x-h)}{h^2} \sim f''(x) \end{array} \right.$$

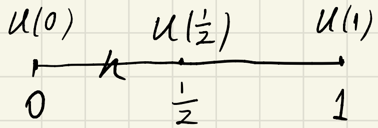
Will do two applications
$$\left\{ \begin{array}{ll} \text{Tue}: & \text{Boundary value (ODE)} \\ \text{Wed}: & \text{Heat equ} \end{array} \right.$$

Next week (old exams)

Ex:
$$\left\{ \begin{array}{l} u''(x) + x\,u'(x) + x\,u(x) = x^2 + x \qquad (*) \\ u(0)=0 \quad , \quad u(1)=1 \end{array} \right.$$

Exact solution : $u(x) = x$ $\quad (\Rightarrow u(\tfrac{1}{2}) = \tfrac{1}{2})$

Num. $u(\frac{1}{2})$?



$h = \frac{1}{2}$

Idea: Num $(*) \rightarrow$ gives a linear equ for $u(\frac{1}{2})$

$(*)$ at $(\frac{1}{2})$ $\sim$ $\underbrace{\dfrac{u(0)-2u(\frac{1}{2})+u(1)}{\frac{1}{4}}}_{u''(\frac{1}{2})} + \frac{1}{2}\underbrace{\dfrac{u(1)-u(0)}{1}}_{u'(\frac{1}{2})} + \frac{1}{2}u(\frac{1}{2})$

$= (\frac{1}{2})^2 + \frac{1}{2}$

$$\Big\Downarrow \; \begin{aligned} u(0)&=0 \\ u(1)&=1 \end{aligned}$$

$$4\left(0-2u(\tfrac{1}{2})+1\right) + \tfrac{1}{2}(1) + \tfrac{1}{2}u(\tfrac{1}{2}) = \tfrac{3}{4}$$

$$\Downarrow$$

$u(\frac{1}{2}) = \frac{1}{2}$ (Lucky!)

$\downarrow$

(we get the exact solution)

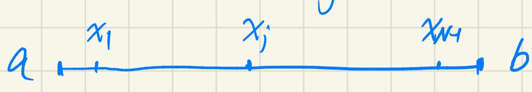
General theory

$$\begin{cases} u'' + p(x)u' + q(x)u = r(x) & x \in [a,b] \\ u(a) = u_a, \quad u(b) = u_b \end{cases}$$

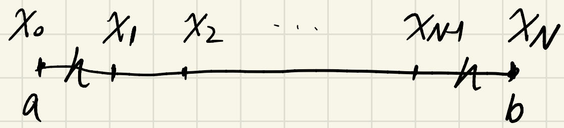
Idea:
$$\begin{cases} u'(x) \sim \dfrac{u(x+h)-u(x-h)}{2h} \\ u''(x) \sim \dfrac{u(x-h)-2u(x)+u(x+h)}{h^2} \end{cases}$$

What we know: $u(a)$, $u(b)$ the boundary values.

Aim: know $u(x)$ for $a<x<b$



Step 1



$$\left\{\begin{array}{l} x_0 = a \quad x_N = b \\ h = \dfrac{b-a}{N} \\ x_j = a + jh \\ \qquad 0 \le j \le N \end{array}\right.$$

$(N \geq 2)$

(Aim: Find Num. value for $\underline{u(x_1), \cdots, u(x_{N-1})}$)

Step 2: Num. equ at $x_1, \cdots, x_{N-1}$.

*1 Num equ $(x_1) \Rightarrow$

$$\overset{u''(x_1)}{\overset{\sim}{\frac{u(x_0)-2u(x_1)+u(x_2)}{h^2}}} + p(x_1)\,\overset{u'(x_1)}{\overset{\sim}{\frac{u(x_2)-u(x_0)}{2h}}} + q(x_1)u(x_1) \sim r(x_1)$$

$\cdots\ \text{---} = (x_2)$

$\vdots$

*N-1 Num $\cdots$ (x_{N-1})

$$\frac{u(x_{N-2})-2u(x_{N-1})+u(x_N)}{h^2} + p(x_{N-1})\,\frac{u(x_N)-u(x_{N-2})}{2h} + q(x_{N-1})u(x_{N-1}) \sim r(x_{N-1})$$

Step 3: Solve the associated linear equ that is very close to our ODE.

Write $u(x_j)$ as U_j

Unknown: $U_0, \cdots, U_N$

$\underline{U_0,\ U_N}$ ✓

*0 $\quad U_0 = u_a$

*1 $\quad \dfrac{U_0 - 2U_1 + U_2}{h^2} + p(x_1)\dfrac{U_2 - U_0}{2h} + q(x_1)U_1 = \underline{r(x_1)}$

$\vdots$

x_{N-1} $\quad \dfrac{U_{N-2} - 2U_{N-1} + U_N}{h^2} + p(x_{N-1})\dfrac{U_N - U_{N-2}}{2h} + q(x_{N-1})U_{N-1} = r(x_{N-1})$

*N $\quad U_N = u_b$

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Remark:

$$*_j \iff \frac{U_{j+1}-2U_j+U_{j-1}}{h^2} + p(x_j)\,\frac{U_{j+1}-U_{j-1}}{2h} + q(x_j)\,U_j = r(x_j)$$

$$\Big\Updownarrow \cdot h^2$$

$$U_{j+1} - 2U_j + U_{j-1} + \frac{p(x_j)}{2}h\,(U_{j+1}-U_{j-1}) + q(x_j)h^2 U_j = r(x_j)\cdot h^2$$

$$\Big\Updownarrow$$

$$\underbrace{\left(1-\frac{p(x_j)}{2}h\right)}_{\overset{\text{def}}{=}\, v_j} U_{j-1} + \underbrace{\left(q(x_j)h^2-2\right)}_{=:\, d_j} U_j + \overbrace{\left(1+\frac{p(x_j)}{2}\cdot h\right)}^{w_j \,=:} U_{j+1} = r(x_j)\,h^2$$

Write linear equ as a matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ v_1 & d_1 & w_1 & 0 & \cdots & 0 \\ 0 & v_2 & d_2 & w_2 & \cdots & \\ & & \ddots & \ddots & \ddots & \\ & & & v_{N-1} & d_{N-1} & w_{N-1} \\ 0 & \cdots & & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_N \end{pmatrix} = \begin{pmatrix} u_a \\ r(x_1)h^2 \\ \vdots \\ r(x_{N-1})h^2 \\ u_b \end{pmatrix}$$

(Python code) $AU = b$ 9-15

Ex:

$$\begin{cases} u'' + 2u' - 3u = 9x \\ u(0) = 1 \\ u(1) = e^{-3} + 2e - 5 \end{cases}$$

Exact solution
$u(x) = e^{-3x} + 2e^{x} - 3x - 2$

$p = 2 \Downarrow q = -3$

$$\begin{cases} v_j = 1 - \frac{p(x_j)}{2} h = 1 - h \\ d_j = q(x_j) h^2 - 2 = -3h^2 - 2 \\ w_j = 1 + \frac{p(x_j)}{2} h = 1 + h \end{cases}$$

$h = \frac{1}{N}$

$\Downarrow$

$$A = \begin{pmatrix} 1 & & & & \\ 1-h & -3h^2-2 & 1+h & & \\ & & \ddots & & \\ & & 1-h & -3h^2-2 & 1+h \\ & & & & 1 \end{pmatrix}$$

Computer:

$N = 10$ Max error 8.499×10^{-4}

$N = 20$ — — 2.133×10^{-4}

$N = 40$ — — 5.348×10^{-5}

$N \to 2N$ error $\to$ error/4 (order 2)

$$\text{error} \sim \frac{C}{N^2}$$

$$\frac{C}{(2N)^2} \sim \frac{1}{4} \cdot \frac{C}{N^2}$$

Ex: (2019 resit P6)

$$\begin{cases} u'' + 4xu = r(x) & x \in [2,5] \\ u(2) = 3 \\ u(5) = 4 \end{cases}$$

$$h = \frac{5-2}{N} = \frac{3}{N}$$

step1 $x_j = 2 + \frac{3j}{N}$ $\quad 0 \le j \le N$

step2: Num. equ at $x_1, \cdots, x_{N-2}, x_{N-1}$.

step3: matrix form $\begin{cases} p = 0 \Rightarrow v_j = w_j = 1 \\ q = 4x \Rightarrow d_j = 4x_j h^2 - 2 \end{cases}$

$$\begin{pmatrix} 1 & & & & & \\ 1 & 4x_1h^2-2 & 1 & & & \\ & 1 & 4x_2h^2-2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4x_{N-1}^2h^2-2 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} U_0 \\ \vdots \\ \vdots \\ U_N \end{pmatrix} = \begin{pmatrix} 3 \\ r(x_1)h^2 \\ \vdots \\ r(x_{N-1})h^2 \\ 4 \end{pmatrix}$$

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Other boundary conditions

Ex:

$$\begin{cases} u''+2u'-3u=9x \\ u'(0)=-4 \\ u(1)=e^{-3}+2e-5 \end{cases}$$

Exact solution

$$u(x)=e^{-3x}+2e^{x}-3x-2.$$

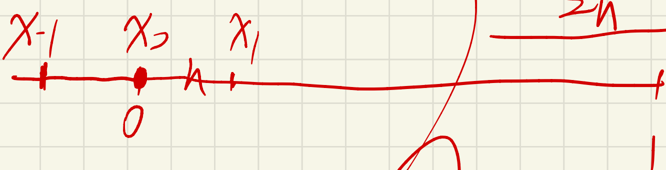
Problem: how to use $u'(0)=-4$?

Method (1) $u'(0)\sim \frac{U_1-U_0}{h}$ | $\frac{U_1-U_0}{h}=-4$

order 1 method

Method (2): add x_{-1}, Use central diff

$$\frac{U_1-U_{-1}}{2h}=-4 \quad (u'(0)=-4)$$



before $x_1,\cdots,x_{N-1}$

Num equ at $x_0,\cdots x_{N-1}$

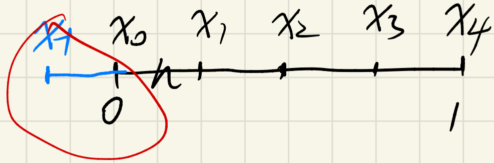
$$U_N=e^{-3}+2e-5$$

N+2 equ. N+2 unknowns $U_{-1}, U_0, \cdots U_N$

look at

$$\begin{cases} u'' + 2u' - 3u = 9x \\ u'(0) = -4 \\ u(1) = e^{-3} + 2e - 5 \end{cases}$$

Step 1: $N = 4$ $\Downarrow$ $h = \frac{1}{4}$



Step 2: Write $u(x_j)$ as U_j $-1 \le j \le 4$

$$\begin{cases} \dfrac{U_1 - U_{-1}}{\frac{1}{2}} = -4 \\ \text{Num equ } (x_0) \quad \dfrac{U_{-1} - 2U_0 + U_1}{(\frac{1}{4})^2} + 2\dfrac{U_1 - U_{-1}}{\frac{1}{2}} - 3U_0 = 0 \\ \vdots \\ \text{Num equ } (x_3) \text{ ———— ————} \\ U_4 = e^{-3} + 2e - 5 \end{cases}$$

Step 3: Write it as a matrix form, solve it!