AMCS/MATH 608
Problem set 9 due November 25, 2014
Dr. Epstein

Reading: There are many excellent references for this material; I especially like *Real Analysis* by Elias Stein and Rami Shakarchi.

Standard problems: The solutions to the following problems do not need to be handed in.

1. Let $\delta = (\delta_1, \ldots, \delta_d)$ be a d-tuple of positive numbers; for a subsest $E \subset \mathbb{R}^d$ define

$$\delta E = \{(\delta_1 x_1, \ldots, \delta_d x_x) : (x_1, \ldots, x_d) \in E\}. \tag{1}$$

   If E is measurable, then show that δE is as well and

$$m(\delta E) = \delta_1 \cdots \delta_d m(E). \tag{2}$$

2. Suppose that $A \subset E \subset B$, with A and B measurable sets. Show that if $m(A) = m(B)$, then E is measurable with $m(E) = m(A)$.

3. Let a and b be positive numbers. Show that $(a + b)^\gamma \geq a^\gamma + b^\gamma$ whenever $\gamma \geq 1$, and that the opposite inequality holds if $0 \leq \gamma \leq 1$.

4. An alternate way to define measurable sets is to say that a set E is measurable, if for every $\epsilon > 0$ there is a closed set $F \subset E$, such that $m_*(E \setminus F) < \epsilon$. Show that this definition gives the same collection of measurable sets as the definition used in class.

Homework assignment: The solutions to the following problems should be carefully written up and handed in.

1. In class we constructed a Cantor set by successively removing the middle thirds of the remaining intervals. For any number $0 < \xi < 1$ a similar construction can be performed by successively removing the middle ξ-part of the remaining intervals. Prove that the result of this is a closed, totally disconnected set of measure zero.

We can also construct a set where we remove a different fraction from the centers of the remaining interval at each step.We let $\{\ell_k : k = 1, 2, \ldots\}$ be a sequence choosen so that for each k

$$\ell_1 + 2\ell_2 + \cdots + 2^{k-1}\ell_k < 1. \tag{3}$$

At the k^{th}-stage of our construction we remove the 2^{k-1} centrally located portions of length ℓ_k of the remaining intervals. Call this set C_k, and let $C = \cap_{k=1}^{\infty} C_k$.

(a) If the $\{\ell_j\}$ are choosen small enough so that

$$\sum_{k=1}^{\infty} 2^{k-1}\ell_k < 1, \tag{4}$$

then show that $m(C) > 0$, and in fact

$$m(C) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k \tag{5}$$

(b) Show that C is totally disconnected.

(c) Show that C is uncountable.

2. Suppose that E is a given set, and define the open set:

$$O_n = \{x : d(x, E) < 1/n\}. \tag{6}$$

If E is compact, then show that $m(E) = \lim_{n\to\infty} m(O_n)$. Show that there are both closed and open sets for which this is false.

3. In this problem we prove the Borel-Cantelli Lemma. Suppose that $\{E_k : k = 1, 2, \ldots\}$ are measurable sets for which

$$\sum_{k=1}^{\infty} m(E_k) < \infty. \tag{7}$$

We define the set

$$E = \{x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k\}. \tag{8}$$

Show that E is measurable and that $m(E) = 0$. Hint: $E = \cap_{k=1}^{\infty} \cup_{l=k}^{\infty} E_l$.

4. Let $A = C$ the middle thirds Cantor set, and $B = C/2$. Show that the set of sums, $A + B$, satisfies $[0, 1] \subset A + B$. This shows that its possible for two closed sets of measure zero to have a sum with $m(A + B) > 0$.

5. Let x be an irrational number. Show that there exist infinitely many fractions p/q of relatively prime integers so that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}. \tag{9}$$

Hint: First show (using the pigeon hole principle for example) that for every integer n at least one element of the set $\{x, 2x, \ldots, (n-1)x\}$ differs from an integer by less than $1/n$.

Use the Borel-Cantelli lemma to show that the set of $x \in \mathbb{R}$ for which there are infinitely many p/q with

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3}. \tag{10}$$

is a set of measure zero.

6. Let E be subset of $\mathbb{R}$ with $m_*(E) > 0$. Prove that for every $0 < \alpha < 1$, there exists an open interval I such that

$$m_*(E \cap I) \geq \alpha m_*(I). \tag{11}$$

Hint: Choose an open set $O \supset E$, such that $m_*(E) \geq \alpha m_*(O)$. Write O as a disjoint union of open intervals, and show that at least one of these intervals must satisfy the desired estimate.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let

$$\Gamma = \{(x, f(x)) : x \in \mathbb{R}\}. \tag{12}$$

Show that Γ is measurable and that $m(\Gamma) = 0$. That is: the 2-dimensional measure of a graph is always zero.