A Note on the Risch Differential Equation*

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Abstract

This paper relates to the technique of integrating a function in a purely transcendental regular elementary Liouville extension by prescribing degree bounds for the transcendentals and then solving linear systems over the constants. The problem of finding such bounds explicitly remains yet to be solved due to the so-called third possibilities in the estimates for the degrees given in R. Risch's original algorithm.

We prove that in the basis case in which we have only exponentials of rational functions, the bounds arising from the third possibilities are again degree bounds of the inputs. This result provides an algorithm for solving the differential equation $y' + f'y = g$ in y where f, g and y are rational functions over an arbitrary constant field. This new algorithm can be regarded as a direct generalization of the algorithm by E. Horowitz for computing the rational part of the integral of a rational function (i.e. $f' = 0$), though its correctness proof is quite different.

1. Introduction

The problem of finding an elementary integral of a function in a regular elementary Liouville extension, that is a function composed recursively by the four basic arithmetic operations and applications of logarithms and exponentials, but not algebraics, ultimately leads to solving the differential equation $y' + f'y = g$ in y, where f, g and y are elements in the field of the integrand itself. (cf. Risch [9]; for further motivation, see also Rosenlicht [5, p. 160]). It is at this stage of the decision procedure at which the size of the answer can become unproportionally large compared to the integrand. An example, stated by several authors, is $\int \exp(x + 100 \log x)\, dx$, the closed form of which is a polynomial with 101 terms in x times $\exp(x)$. This blow-up is accounted for in Risch's original proof by the so-called third possibilities in the estimates for the degrees of the answer. However, these bounds can only be calculated if one has, in addition to just performing arithmetic in the constant field C of the integrand including testing for equality to 0, some specialized routines for C.

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* This research was partially supported by the National Science and Engineering Council of Canada under grant 3-643-126-90. This is a corrected version of a paper published in the Procceedings EUROSAM '84, Springer Lecture Notes in Computer Science **174**, 359-366 (1984).

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Risch's original algorithm requires polynomial factorization over the constant field as well as testing for integrality, but it has been shown by M. Rothstein [7, 8] that the computation of integral roots of polynomials over C is sufficient.

It is the subject of the so-called parallel Risch algorithm to compute these bounds without eliminating the exponentials or logarithms in succession (cf. Norman and Moore [4]). Unfortunately, no complete algorithm is known though progress has been made in special cases, e.g. by Davenport [1] who focuses on logarithmic extensions. In this note we will prove that in the basis case the third possibilities for the degree estimates of y in $y' + f'y = g$, $f, g, y \in C(x)$, if they arise, have also bounds expressible in the degrees in f and g. An immediate consequence of this result is, that for size blow-up to occur the nested logarithm in the previous example is essential. It also eliminates the need for the special purpose routines for C. The result may explain why the heuristic bounds (cf. Fitch [2]) appear so robust, in practice.

Another interpretation of our result is possible. E. Horowitz [3] has given an algorithm for finding the rational part of a rational function integral by solving a linear system without performing the squarefree factorization of the integrands denominator. Our result leads to an equivalent algorithm which by setting $f' = 0$, has Horowitz' approach as a special case.

Our paper is arranged in the following way. Section 2 introduces some notation and establishes preliminary facts needed for the proof. Section 3 contains the formulation and proof of the main result as well as links this result to Horowitz' algorithm.

2. Notation and Preliminary Results

The ring $C[x]$ of polynomials with coefficients in the field C becomes a differential ring with derivation $'$ if we prescribe that $c' = 0$ for any $c \in C$ and $x' = 1$. Its field of quotients, $C(x)$, is of course the field of rational functions over C. We always assume that C is a field of characteristic 0. By $\operatorname{ldcf}(f)$ we denote the leading coefficient of a polynomial $f \in C[x]$ and we call f monic if $\operatorname{ldcf}(f) = 1$. As is well known, every polynomial $q \in C[x]$ can be decomposed by GCD computations into a product $q_1 q_2^2 \cdots q_r^r$ of squarefree polynomials q_i with $\operatorname{GCD}(q_i, q_j) = 1$ for $i \neq j$. This squarefree decomposition is also unique up to a scalar factor in C. Furthermore, every rational function p/q, $p, q \in C[x]$, q monic, can be uniquely expanded into partial fractions with respect to the squarefree decomposition of q, viz.

$$\frac{p}{q} = p_0 + \sum_{i=1}^{r} \sum_{j=1}^{i} \frac{p_{ij}}{q_i^j}\,, \quad p_0\,,\ p_{ij} \in C[x]\,, \quad \deg(p_{ij}) < \deg(q_i)\,.$$

The following lemma will be applied in various places of our argument.

Lemma 1: Let $u, v \in C[x]$, $\operatorname{GCD}(u, v) = 1$ and let $(u/v)' = p/q$, $p, q \in C[x]$, $\operatorname{GCD}(p, q) = 1$. Assume that $w \in C[x]$ is squarefree such that w divides q. Then w divides v and if r is the multiplicity of w in v, w^{r+1} must divide q.

Proof: Since

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = \frac{p}{q}$$

it is clear that w must divide v. Assume now that $v = w^r\hat{w}$ with $\mathrm{GCD}\,(w,\hat{w}) = 1$. We show that w^r does not divide $u'v - uv'$. Suppose the contrary. Since w^r divides $u'v$ and $\mathrm{GCD}\,(w,u) = 1$, w^r then would have to divide $v' = rw^{r-1}w'\hat{w} + w^r\hat{w}'$, hence w needed to divide $w'\hat{w}$. But this is impossible since w is squarefree. Therefore w^{r+1} must remain in the reduced denominator of $(u/v)'$. $\diamondsuit$

This very elementary lemma provides us with quite a powerful tool. To demonstrate this, consider the rational integral

$$\int \frac{p(x)}{q(x)}\,dx\,, \quad p,q \in C[x]\,, \quad \deg(p) < \deg(q)\,.$$

Set $\bar{q} = \mathrm{GCD}\,(q,q')$ and $q^* = q/\bar{q}$, i.e. q^* is the largest squarefree factor of q. We can show that there exist unique polynomials $g, h \in C[x]$ such that

$$\int \frac{p(x)}{q(x)}\,dx = \frac{g(x)}{\bar{q}(x)} + \int \frac{h(x)}{q^*(x)}\,dx\,, \quad \deg(g) < \deg(\bar{q})\,, \quad \deg(h) < \deg(q^*)\,. \tag{2.1}$$

For, differentiating (2.1) and multiplying by the common denominator q gives

$$p = q^*g' - \frac{q^*\bar{q}'}{\bar{q}}g + \bar{q}h \tag{2.2}$$

where

$$\frac{q^*\bar{q}'}{\bar{q}} = \frac{q^*}{\bar{q}}\left(\frac{q}{q^*}\right)' = \frac{q'}{\bar{q}} - q^{*\prime} \in C[x]\,,$$

since $\bar{q}$ divides q'. Plugging unknown coefficients for g and h into (3.2) and equating the coefficients of equal powers of x^i, $0 \le i \le \deg(q) - 1$, we get a linear system in $\deg(q)$ equations and $\deg(q)$ unknowns. This system has a unique solution if the only solution for $p = 0$ is $g = h = 0$. This follows from lemma 1 since $(g/\bar{q})' = -h/q^*$ has no non-trivial solution because q^* is squarefree. We have incidentally also shown that $\int h/q^*$ cannot be a rational function.

Setting up and solving the linear system resulting from (2.2) is Horowitz' algorithm. It should be noted that neither the full squarefree factorization of q nor Hermite's reduction is needed in this method, the latter not even in the correctness proof.

3. Main Result

We now show how to solve the differential equation $y' + f'y = g$ in y, where $f, y, g \in C(x)$. We first repeat Risch's original argument, enhanced by Rothstein's observations.

Theorem 3.3: Let $C(x)$ be the transcendental extension of the constant field C with $x' = 1$. Assume that f and $g \in C(x)$ are given. Then we can solve

$$y' + fy = g\,, \quad y \in C(x) \tag{3.1}$$

in a finite number of arithmetic operations in C, including computing integer roots of polynomials over C.

Proof: We represent, by GCD computations,

$$f(x) = \frac{F(x)}{q_1(x)^{k_1} \cdots q_n(x)^{k_n}}\,, \qquad g(x) = \frac{G(x)}{q_1(x)^{l_1} \cdots q_n(x)^{l_n}}$$

where F, G, $q_1, \ldots, q_n \in C[x]$, $q_1, \ldots, q_n$ monic, squarefree and pairwise relatively prime, $k_i \geq 0$, $l_i \geq 0$ for $1 \leq i \leq n$. From lemma 1 we conclude that if $y(x)$ solves (3.1) then

$$y(x) = \frac{Y(x)}{q_1(x)^{j_1} \cdots q_n(x)^{j_n}}$$

with $Y(x) \in C[x]$, $j_i \geq 0$ for $1 \leq i \leq n$. We first compute a bound $\bar{\jmath}_i$ for j_i, $1 \leq i \leq n$, and then a bound $\bar{\alpha}$ for $\deg_x(Y)$. Let

$$y(x) = \frac{A_{i,j_i}(x)}{q_i(x)^{j_i}} + \ldots\,, \quad f(x) = \frac{B_{i,k_i}(x)}{q_i(x)^{k_i}} + \ldots\,, \quad g(x) = \frac{D_{i,l_i}(x)}{q_i(x)^{l_i}} + \ldots\,.$$

be the partial fraction expansion of y, f, and g with $A_{i,j_i}, B_{i,l_i}, D_{i,l_i} \in C[x]$ non-zero and $\deg_x(A_{i,j_i}) < \deg(q_i)$ unless $j_i = 0$, $\deg(B_{i,k_i}) < \deg(q_i)$ unless $k_i = 0$. Substituting these expansions into (3.1) we get

$$-\frac{j_i q_i' A_{i,j_i}}{q_i^{j_i+1}} + \ldots + \frac{B_{i,k_i} A_{i,j_i}}{q_i^{j_i+k_i}} + \ldots = \frac{D_{i,l_i}}{q_i^{l_i}} + \ldots \quad .$$

We first observe that $j_i + 1 \leq l_i$ is equivalent to $j_i + k_i \leq l_i$ since otherwise one of the leading terms could not cancel on the left-hand side. The third possibility is that $j_i + 1 = j_i + k_i > l_i$. In this case,

$$k_i = 1 \quad \text{and} \quad q_i \quad \text{divides} \quad -j_i q_i' A_{i,j_i} + B_{i,k_i} A_{i,j_i}\,,$$

which implies that $\text{GCD}\,(-j_i q_i' + B_{i,k_i}, q_i) \neq 1$. Therefore, j_i must be a root of the resultant

$$R(z) = \text{resultant}_x(B_{i,k_i}(x) - z q_i'(x), q_i(x)) \in C[z]\,.$$

First of all, $R(z) \neq 0$ because otherwise for some root β of $q_i(x)$, $B_{i,k_i}(\beta) - z q_i'(\beta) = 0$ meaning $q_i'(\beta) = 0$ which contradicts the squarefreeness of q_i. Let m_i be the largest positive integral root of $R(z)$, if any, otherwise let $m_i = 0$. Then

$$j_i \leq \bar{\jmath}_i = \max(\min(l_i - 1, l_i - k_i)\,,\ m_i)\,.$$

We now set

$$y(x) = \frac{Y(x)}{q_1(x)^{\bar{\jmath}_1} \cdots q_n(x)^{\bar{\jmath}_n}} = \frac{Y(x)}{\bar{q}(x)}$$

and substitute into (3.1). Multiplying out with a common denominator we get

$$uY' + vY = t \tag{3.2}$$

with

$$Y(x) = y_\alpha x^\alpha + \ldots + y_0\,, \qquad\qquad u(x) = a_\beta x^\beta + \ldots + a_0 \in C[x]\,,$$
$$v(x) = b_\gamma x^\gamma + \ldots + b_0 \qquad \text{and} \qquad t(x) = d_\delta x^\delta + \ldots + d_0 \ \in C[x]\,.$$

Again it behooves us to determine a bound for α. Substitution in (3.2) gives

$$(a_\beta x^\beta + \ldots)(\alpha y_\alpha x^{\alpha-1} + \ldots) + (b_\gamma x^\gamma + \ldots)(y_\alpha x^\alpha + \ldots) = d_\delta x^\delta + \ldots\,. \tag{3.3}$$

Thus $\alpha + \beta - 1 \le \delta$ if and only if $\alpha + \gamma \le \delta$ or the third case $\alpha + \beta - 1 = \alpha + \gamma > \delta$ which implies that $\alpha a_\beta + b_\gamma = 0$. Let ρ be $-b_\gamma/a_\beta$ if this is a positive integer, otherwise let $\rho = 0$. Then

$$\alpha \le \bar{\alpha} = \max(\min(\delta - \beta - 1, \delta - \gamma)\,,\ \rho)\,.$$

Multiplying (3.3) out and equating powers of x^i we obtain a linear system in the y_i's with coefficients in C. $\diamondsuit$

We now further inspect the third possibilities in the case that f is the derivative of a rational function.

Theorem 2: If one applies the algorithm given in the proof of theorem 1 to the differential equation $y' + f'y = g$, the third possibility for the bound $\bar{\jmath}_i$ can never occur and the only time the third possibility for the bound $\bar{\alpha}$ can happen is when $\rho = \deg(\bar{q})$.

Proof: Assume that $j_i + 1 > l_i$ which implies that $k_i = 1$. Thus the partial fraction expansion

$$f'(x) = \frac{b_{i,k_i}(x)}{q_i(x)} + \ldots$$

which is impossible as shown in lemma 1. Now let

$$y(x) = \frac{Y(x)}{\bar{q}(x)}\,, \quad f'(x) = \frac{p(x)}{\hat{q}(x)}\,, \quad g(x) = \frac{s(x)}{q(x)}\,.$$

Notice that $\bar{q}(x)$ divides $q(x)$. Substituting into our differential equation $y' + f'y = g$ we get

$$\frac{Y'(x)}{\bar{q}(x)} + \left(\frac{p(x)}{\hat{q}(x)} - \frac{\bar{q}'(x)}{\bar{q}(x)}\right)\frac{Y(x)}{\bar{q}(x)} = \frac{s(x)}{q(x)} \tag{3.4}$$

Since the bound $\bar{\alpha}$ depends only on the difference $\delta - \beta$ and $\delta - \gamma$ as well as the quotient b_γ/a_β it does not matter for the determination of $\bar{\alpha}$ if we multiply (3.4) with a larger than the least common denominator. We get

$$(\bar{q}\hat{q})Y' + q(p\bar{q} - \hat{q}\bar{q}')Y = \bar{q}^2\hat{q}s\,.$$

The third possibility implies that

$$\beta = \deg(\bar{q}\hat{q}q) = \gamma + 1 = \deg(q(p\bar{q} - \hat{q}\bar{q}')) + 1\,.$$

If $\deg(p) \ge \deg(\hat{q})$, this is clearly impossible. Thus $\deg(p) < \deg(\hat{q})$ which, since

$$\frac{p}{\hat{q}} = f' = \left(\frac{d}{e}\right)' = \frac{d'e - de'}{e^2}\,, \quad d, e \in C[x]\,,$$

implies that we may choose $\deg(d) < \deg(e)$ and thus get $\deg(p) \leq \deg(\hat{q}) - 2$. Therefore, $a_\beta = \text{ldcf}\,(\bar{q}\hat{q}q) = 1$, $b_\gamma = \text{ldcf}\,(q(p\bar{q} - \hat{q}\bar{q}')) = \text{ldcf}\,(-\bar{q}')$ and thus $-b_\gamma/a_\beta = \rho = \deg(\bar{q})$. $\diamondsuit$

We now present an example showing that the case $\deg(Y) = \deg(\bar{q}) > \max(0, \min(\delta - \beta - 1, \delta - \gamma))$ can occur.

Example: Let $f' = -1/x^2$, $g = -(x+1)/x^4$. Then $q_1 = x$, $k_1 = 2$, $l_1 = 4$, $\jmath_1 = \min(l_1 - 1, l_1 - k_1) = 2$ and

$$\left(\frac{Y}{x^2}\right)' - \frac{1}{x^2}\frac{Y}{x^2} = \frac{-x+1}{x^4} \quad \text{with} \bar{q}(x) = x^2 \,.$$

This leads to

$$x^2 Y' - (2x+1)Y = -x - 1 \,.$$

Thus, $\beta = 2$, $\gamma = 1$, $\delta = 1$ and

$$\bar{\alpha} = \max(\min(\delta - \beta - 1, \delta - \gamma)\,,\ \deg(\bar{q})) = \max(\min(-2, -1), 2) = 2 \,.$$

Solving for $Y = y_2 x^2 + y_1 x + y_0$ we get $y_2 = 1$, $y_1 = -1$, $y_0 = 1$. Hence

$$\int \frac{x+1}{x^4} \exp\left(\frac{1}{x}\right) = -\frac{x^2 - x + 1}{x^2} \exp\left(\frac{1}{x}\right) \,. \quad \diamondsuit$$

It is surprisingly easy to show that the solution to $y' + f'y = g$, $f' \neq 0$, is unique. Suppose the contrary that is y_1 and $y_2 \in C[x]$ solve the differential equation. Then with $\bar{y} = y_1 - y_2 \neq 0$ we must have $\bar{y}'/\bar{y} = -f'$. It is easy to see from $\int \bar{y}'/\bar{y} = \log(\bar{y})$ that $\bar{y}'/\bar{y}$ can be written as h/q^* with $h, q^* \in C[x]$ and q^* squarefree. But as mentioned in section 1, $\int h/q^*$ cannot be the rational function $-f$.

We now derive from theorem 2 an algorithm for solving

$$y' + f'y = g\,, \quad f' = \frac{p}{\hat{q}} \in C[x]\,, \quad g = \frac{s}{q} \in C(x)$$

equivalent to Horowitz' rational function integration algorithm. We choose $\bar{q} = \text{GCD}\,(q, q')$ as the denominator of y. This choice is equivalent to setting $\jmath_i = l_i - 1$, which is not necessarily the sharpest bound but which avoids computing the full squarefree factorization of $\hat{q}$ and q. Then we calculate the bound $\bar{\alpha}$ for degree of the numerator of y and solve the resulting linear system as discussed in the proof of theorem 1. One upper bound for $\bar{\alpha}$ is

$$\bar{\alpha} \leq \max(\deg(\bar{q}) - \deg(q) + \deg(s) - 1, \deg(\bar{q}))$$

which is slightly pessimistic but which, we hope, exhibits the similarity to Horowitz' algorithm.

Due to M. Rothstein [7, 8], the degree bound $\bar{\alpha}$ for Y in $uY' + vY = t$ can also be reduced in the following way. If $\text{GCD}\,(u, v) \neq 1$ then we divide u, v and t by this GCD. Obviously, if the division of t leaves a remainder then the differential equation has no solution. Thus we may assume that $\text{GCD}\,(u, v) = 1$ and we can find unique polynomials $d, e \in C[x]$ with

$$ud + ve = t\,, \quad \deg(e) < \deg(u)\,.$$

Now $Y = \bar{Y}u + r$, $\deg(r) < deg(u)$, if and only if

$$r = e \quad \text{and} \quad u\bar{Y}' + (u' + v)\bar{Y} = d - e' .$$

Thus solving for $\bar{Y}$ with $\deg(\bar{Y}) = \bar{\alpha} - \beta$ is sufficient. Of course, we can repeat this process until either $\deg(\bar{Y}) < \beta$ or $\deg(u) = 0$. In the first case $u\bar{Y}' + v\bar{Y} = t$ implies $\bar{Y} \equiv tv^{-1}$ (mod u). Thus we only need to invert v modulo u. The second case must be handled by solving linear systems as discussed above.

One could argue that so far we have only shown that no exponent blow-up due to "third possibilities" occurs when integrating elements in $C(x, \exp f(x))$, $f(x) \in C(x)$. It is relatively easy to show that this remains true when integrating elements in

$$C(x)[\exp f_1(x), \ldots, \exp f_n(x)], \quad f_i(x) \in C(x). \tag{4.1}$$

We assume that the $\exp f_i$ are algebraically independent over $C(x)$ and introduce no new constants. An element in (4.1) can be written as

$$\sum g_{e_1,\ldots,e_n}(x) \exp(e_1 f_1(x) + \cdots + e_n f_n(x)),\ g_{e_1,\ldots,e_n}(x) \in C(x),\ e_i \in \mathbf{Z}. \tag{4.2}$$

By our assumption, the arguments to the exponentials in (4.2) cannot differ by just a constant. It is now an old result by Liouville (cf. Rosenlicht [6, p.295]) that the integral of (4.2) is elementary if and only if each

$$\int g_{e_1,\ldots,e_n}(x) \exp(e_1 f_1(x) + \cdots + e_n f_n(x))$$

is elementary. This integral leads, of course, to solving a differential equation of type (3.1). Therefore, theorem 2 implies that the exponents in the integral of an integrand in (4.1) depend only on the exponents in the integrand.

Our conclusions partially generalize in the case in which C is replaced by a regular elementary purely transcendental Liouville extension of $C(x)$ and x is replaced by a logarithm or exponential. However, as we have already exemplified in the introduction, bounds for the third possibilities cannot be derived from the input degrees alone, in general.

4. Conclusion

The complexity of R. Risch's algorithm for deciding whether a function in an elementary purely transcendental Liouville extension field possesses an elementary integral is little understood. The fact that one seems to need the closed form solution for just recognizing elementary integrals indicates that the given decision procedure might not even be elementary recursive. Here we have settled two questions. Firstly, we have shown that even in the basis case third possibilities can arise. Secondly, however, we have put this basis case into the class of polynomial-time problems.

Acknowledgement

I like to express my gratitude to the students and colleagues in my integration course at University of Toronto for their many thoughts on the subject. James Davenport pointed out an improvement in the bound computation what I wish to acknowledge. Thanks go also to Guy Cherry whose trick of summing up arguments to exponentials I have used in section 3.

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