Siegel Discs in Complex Dynamics

MATH 31400 (Spring 13) Analysis - 3: Final project

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Abstract

Siegel disc is a connected component in the Fatou set where the dynamics is analytically conjugated to an irrational rotation. In what follows, we try to give a brief overview of some of the key questions regarding the study of Siegel discs; namely, conditions for its existence, its geometry and the topology of its boundary.

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1 | Introduction

1.1 | Setting and Terminology

A dynamical system is a physical setting together with rules for how the setting changes or evolves from one moment of time to the next or from one stage to the next. The simplest model of a dynamical system supposes that the $(n+1)-$th state x_{n+1} can be determined solely from a knowledge of the previous stage x_n, i.e. $x_{n+1} = f(x_n)$. Such a system is called a **discrete dynamical system**. Note that for a well-chosen change of variable $y = \varphi(x)$, the new map $y_{n+1} = h(y_n)$ where $h := \varphi \circ f \circ \varphi^{-1}$ may be easier to iterate.

In the study of **complex dynamical systems** of one variable, the evolution of the system is realized by iteration of entire or meromorphic complex functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$. We will mostly consider rational functions of degree at least two or transcendental functions in this article. Also by the notation f^n we will mean $\underbrace{f \circ \ldots \circ f}_{n\text{times}}$.

Definition 1.1. Fatou set and Julia set: The set where $\{f^n\}$ is a normal family is known as the Fatou set of f, denoted $\mathcal{F}(f)$. The complement of $\mathcal{F}(f)$ in $\widehat{\mathbb{C}}$ is known as the Julia set of f, denoted $\mathcal{J}(f)$.

Fatou Components: A Fatou component is a maximal connected open subset of $\mathcal{F}(f)$.

A component U of $\mathcal{F}(f)$ is n-**periodic** if n is the smallest natural number to satisfy $f^n(U) \subseteq U$. Then $\{U, f(U), f^2(U), \ldots, f^{n-1}(U)\}$ is called a $n-$periodic cycle. If $n = 1$, U is called **an invariant component**. A Fatou component U is said to be **completely invariant** if it is invariant and satisfies $f^{-1}(U) \subseteq U$. A component U of $F(f)$ is said to be **pre-periodic** if there exists a natural number $k > 1$ such that $f^k(U)$ is periodic.

Definition 1.2. Singular Values: A point z is a critical point of f if $f'(z) = 0$. The value of f at a critical point is called a critical value of f. A point a is called an asymptotic value of f if there exists a path $\gamma(t)$ in $\widehat{\mathbb{C}}$ satisfying $\lim_{t\to\infty} f(\gamma(t)) = a$.

All the critical and asymptotic values of a function is called the singular values of f, denoted by S_f.

Definition 1.3. Periodic Point: A point $z \in \mathbb{C}$ is a $p-$periodic point if p is the smallest natural number such that $f^p(z) = z$. If $p = 1$, z is called the fixed point of f.

1.2 | Linearization

Let z be a fixed point of f. If f is differentiable, it is well approximated by its **linear part**, i.e. its differential at z. Linear maps are easier to iterate and so we ask the question whether or not a change of variable can be found for which f is conjugate to its linear part, in other words, whether or not f is linearizable.

The Multiplier: If f is holomorphic and a is a $p-$periodic point of f, then **the multiplier** is the complex number $\lambda = (f^p)'(a)$.

Note that the multiplier is invariant under holomorphic conjugacy. The linear part of f at the periodic point is the map $z \mapsto \lambda z$. a is said to be **attracting, indifferent or repelling** if $|\lambda| < 1, = 1$ or > 1 respectively. a is called **superattracting** if $\lambda = 0$. Further, an indifferent point is called **irrationally indifferent** if $\lambda = e^{2i\pi t}$ for some irrational t. It is called **rationally indifferent**, if t is rational.

We will talk more about Linearization problem later in regard to existence of Siegel discs.

1.3 Classification of Periodic Fatou component

Suppose U is an $n-$periodic Fatou component. Then exactly one of the following possibilities occur:

- *Attracting Basin:* If for all points $z \in U$, $\lim_{k\to\infty} f^{kn}(z) = p$ where p is an attracting $n-$periodic point lying in U, the component U is said to be an attracting basin.

  Example 1.4. An example of this type is the components of the map

  $$f(z) = z - \frac{z^3 - 1}{3z^2}$$

  containing attracting points that are solutions to $z^3 = 1$.

- *Parabolic Basin:* In this case ∂U contains a rationally indifferent $n-$periodic point p and $\lim_{k\to\infty} f^{kn}(z) = p$ for all $z \in U$.

- *Baker Domains:* If for $z \in U$, $\lim_{k\to\infty} f^{kn}(z) = \infty$, then the Fatou component is called a Baker domain. Note that this comes up only in case of transcendental functions and not for polynomials and rational functions.

  Example 1.5. An example of this type is:

  $$f(z) = z - 1 + (1 - 2z)e^z$$

- *Herman Rings:* Suppose there exists an analytic homeomorphism $\varphi : U \to A$ where $A = \{z : 1 < |z| < r\}$ for some $r > 1$, such that

  $$\varphi(f^k(\varphi^{-1}(z))) = e^{2i\pi\alpha}z$$

  for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then U is called a Herman ring.

  Example 1.6. An example of a rational function which has a Herman ring is given by

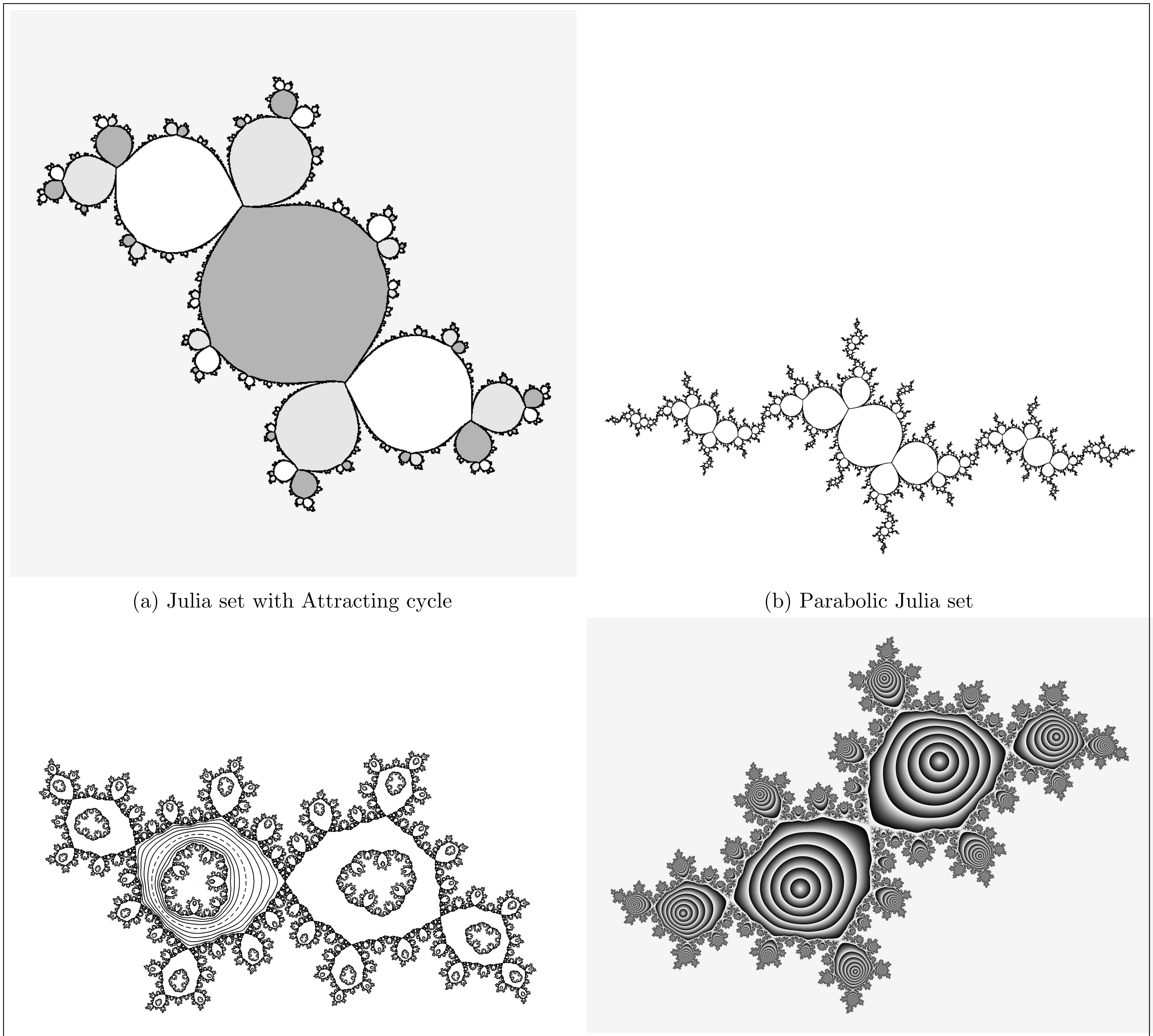
  $$f(z) = e^{2i\pi\tau}\frac{z^2(z - 4)}{1 - 4z}$$

  where $\tau = 0.6151732\ldots$ such that the rotation number of f on the unit circle is $\frac{\sqrt{5}-1}{2}$. See figure 1c.

- *Siegel Discs:* A Fatou component U is said to be a Siegel disc if there exists an analytic homeomorphism $\varphi : U \to \mathbb{D}$ such that

  $$\varphi(f^k(\varphi^{-1}(z))) = e^{2i\pi\alpha}z$$

  for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Note that by definition, Siegel discs are simply connected.



(a) Julia set with Attracting cycle

(b) Parabolic Julia set

(c) Julia set with Herman ring

(d) Julia set with Siegel disc

Figure 1: Examples of Periodic Fatou Components

Remark 1.7. In the case of a degree one rational map with an irrationally indifferent fixed point, there is a Möbius map conjugating it to a rotation on the whole Riemann sphere; therefore it has exactly two fixed points, both are linearizable, the Fatou set is the whole sphere and the Siegel disc of each fixed point is the sphere minus the other fixed point.

2 | Existence of Siegel Disc

2.1 | Linearization Problem

The question of the existence of Siegel discs is a subset of the linearization problem. In this section we first survey what is known about local linearization problem and prove some of the easier results.

Lemma 2.1. *Let f be a rational function of degree 2 or more with an irrationally indifferent fixed point z_0. Then the following conditions are equivalent:*

- *f is locally linearizable around z_0.*
- *$z_0 \in \mathcal{F}(f)$.*
- *The connected component U of $\mathcal{F}(f)$ containing z_0 is a Siegel disc.*

Proof. If f is locally linearizable around z_0, then the iterates of f in a suitable neighborhood of z_0 correspond to iterated rotations of a small disc, and hence form a normal family. Thus $z_0 \in \mathcal{F}(f)$. Now since z_0 is irrationally indifferent, we know that U is wither Herman ring or Siegel disc. But Herman rings do not have fixed points. Thus U must be a Siegel disc. □

Proposition 2.2. *An invariant Siegel disc S contains an irrationally indifferent fixed point of f.*

Proof. Consider the point $\varphi^{-1}(0)$ where $f(\varphi^{-1}(z)) = \varphi^{-1}(e^{2\pi i\alpha}z)$ with an irrational α. □

We observe that the linearizability of a function f is a topological and not an analytical property. Indeed,

Lemma 2.3. *For holomorphic maps of one variable, existence of a homeomorphic linearizing map implies the existence of a holomorphic one.*

Proof. Let h be a local homeomorphism satisfying $f \circ f = R_\theta \circ f$ near 0, where $R_\theta(z) = e^{2i\pi\theta}z$. Then for small $\epsilon > 0$, $U = h^{-1}(\mathbb{D}(0,\epsilon))$ is a topological disc containing the fixed point 0 and is invariant under the action of f. Let $\varphi : U \to \mathbb{D}$ denote a conformal isomorphism given by the Riemann mapping theorem which satisfies $\varphi(0) = 0$. Then Schwarz lemma tells us that $\varphi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ is the rigid rotation R_θ, meaning φ is a holomorphic linearizing map for f. □

Remark 2.4. By above proof it is clear that the linearizing coordinates of the Siegel disc is given by the Riemann mapping.

The correct answer to whether a local linearization is possible or not, depends on a careful study of the extent to which the rotation number α can be very closely approximated by rational numbers.

Theorem 2.5. ***(Siegel[1942]):*** *If $\frac{1}{|\lambda^q - 1|}$ is less than some polynomial function of q, then every germ of a holomorphic function with fixed point of multiplier λ is locally linearizable.*

We prove later that Theorem 2.5 implies the following corollary:

Corollary 2.6. *For every ξ outside of a set of Lebesgue measure zero, we can conclude that every holomorphic germ with a fixed point of multiplier $e^{2i\pi\xi}$ is locally linearizable.*

In other words, if the angle $\xi \in \mathbb{R}/\mathbb{Z}$ is "randomly chosen" with respect to Lebesgue measure, then with probability 1 every rational function with a fixed point of multiplier $e^{2i\pi\xi}$ will have a corresponding Siegel disc.

Definition 2.7. Diophantine Numbers: Let κ be a positive real number. By definition, an irrational number α is said to be **Diophantine of order** $\leq \kappa$ if there exists $\epsilon > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{\epsilon}{q^\kappa}$$

for every rational number p/q.

The class of all such number will be denoted by $\mathcal{D}(\kappa)$. Clearly,

$$\mathcal{D}(\kappa) \subset \mathcal{D}(\eta) \text{ whenever } \kappa < \eta.$$

Setting $\lambda = e^{2i\pi\xi}$ as above, and choosing p to be the closest integer to $q\xi$ so that $|q\xi - p| \leq \frac{1}{2}$, we can prove that

$$4|q\xi - p| \leq |\lambda^q - 1| \leq 2\pi|q\xi - p|.$$

It follows that being Diophantine is equivalent to the requirement that

$$|\lambda^q - 1| > \frac{\epsilon'}{q^{k-1}} \iff \frac{1}{|\lambda^q - 1|} < cq^{k-1}$$

for some $\epsilon' > 0$, with some value of κ and with $c = \frac{1}{\epsilon'}$. Thus Siegel's theorem 2.5 can be restated as follows:

Theorem 2.8. *If the angle $\xi \in \mathbb{R}/\mathbb{Z}$ is Diophantine of any order, then any holomorphic germ with multiplier $\lambda = e^{2i\pi\xi}$ is locally linearizable.*

It turns out that $\mathcal{D}(\kappa)$ is empty for $\kappa < 2$. However the following theorem makes it easier to find Diophantine numbers of higher integer order as it proves every algebraic number is Diophantine.

Theorem 2.9. ***(Liouville).*** *If the irrational number α satisfies a polynomial equation $f(\alpha) = 0$ of degree d with integer coefficients, then $\xi \in \mathcal{D}(d)$.*

Proof. We may assume that $f(p/q) = 0$. Clearing denominators, it follows that $|f(p/q)| > \frac{1}{q^d}$. On the other hand, if M is an upper bound for $|f'(x)|$ in the interval of length 1 centered at α, then

$$f\left(\frac{p}{q}\right) \leq M\left|\alpha - \frac{p}{q}\right|$$

Choosing $\epsilon < \frac{1}{M}$, we then obtain $|\epsilon - p/q| < \epsilon/q^d$, as required. $\square$

It follows that any irrationally indifferent fixed point with algebraic rotation number is locally linearizable.

Theorem 2.10. *Denote by $\mathcal{D}(2+)$ the set*

$$\mathcal{D}(2+) = \bigcap_{\kappa > 2} \mathcal{D}(\kappa)$$

Then this set $\mathcal{D}(2+)$ has full measure in the circle $\mathbb{R}/\mathbb{Z}$.

Proof. Let $U(\kappa, \epsilon)$ be the open set consisting of all $\xi \in [0, 1]$ such that $|\xi - p/q| \leq \epsilon/q^\kappa$ for some p/q. This set has measure at most

$$\sum_{q=1}^{\infty} q.\frac{2\epsilon}{q^\kappa}$$

since for each q there are q possibilities for $\frac{p}{q}$ mod 1. If $\kappa > 2$, then this sum converges, and hence $\to 0$ as $\epsilon \to 0$. Therefore the intersection $\bigcap_{\epsilon > 0} U(\kappa, \epsilon)$ has measure zero, and its complement $\mathcal{D}(\kappa)$ has full measure. Taking intersection as $\kappa \searrow 2$, we see that $\mathcal{D}(2+)$ also has full measure. $\square$

Thus Theorem 2.5 and Theorem 2.10 together imply Corollary 2.6. We finish this section with the following result, which give a much sharper picture of the local linearization problem.

2.2 Bryuno's Arithmetic Condition

For an irrational $\alpha \in (0,1)$, we consider the continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

where a_i are uniquely defined strictly positive integers. The rational number

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \frac{1}{a_2 + \dots \frac{}{a_{n-1} + \frac{1}{a_n}}}}$$

is called the nth convergent to α. Denote $\lambda = e^{2i\pi\alpha}$ as usual.

Theorem 2.11. ***(Bryuno [1965],Rüssman [1967])*** *With λ and $\{q_n\}$ as above, if*

$$\sum_n \frac{\log(q_{n+1})}{q_n} < \infty \tag{$\star$}$$

then any holomorphic germ with a fixed point of multiplier λ is locally linearizable.

The number α is said to satisfy Bryuno's condition whenever $(\star)$ holds. Clearly the Diophantine numbers satisfy Bryuno's condition. Yoccoz showed that theorem 2.11 is a best possible result.

Theorem 2.12. ***(Yoccoz[1988])*** *Conversely, if the sum $(\star)$ diverges, then the quadratic map $f(z) = z^2 + \lambda z$ has a fixed point at the origin which is not locally linearizable. Furthermore, this fixed point has the **small cycles property**: Every neighborhood of the origin contains infinitely many periodic orbits.*

Evidently such small cycles provide an obstruction to linearizability.

Remark 2.13. In general, a fixed point with any non-Bryuno rotation number α may be linearizable. Indeed $z \mapsto \lambda z$ is linearizable by definition.

We try to give some intuitive ideas as to what above theorems mean in the polynomial case. Whenever the summand $(\log q_{n+1})/q_n$ is large the rotation number will be extremely close to p_n/q_n, so that f will be extremely close to a parabolic map with a period q_n cycle of repelling directions. It follows that the basin of infinity for f will have a period q_n cycle of deep fjords which penetrate towards zero, squeezing the size of a possible Siegel disc. When the sum is infinite, such a Siegel disc can no longer exist.

3 | Siegel Disk boundaries

Although the behavior of irrational rotations, and hence of the dynamics inside a Siegel disc, is completely understood; things are different with Siegel disc boundaries. Even today only partial answers are known in regard to what they look like, or where it is.

Are Siegel disc boundaries analytic curves? Boundary of a Siegel disc U cannot be analytic cosed curve contained in the domain of definition of f since otherwise, the linearizing coordinates can be extended beyond U by **Schwarz's reflection principle**. So there are two possibilities:

- Either the Siegel disc goes to infinity or touches the boundary of the domain of definition of f. In this case, the boundary may or may not be analytic.
- The boundary is contained in the domain of definition of f and nowhere analytic.

3.1 | Topology of Siegel disc boundaries

Definition 3.1. A Siegel disc U of f is called **bounded** if it is contained in a compact subset of the domain of definition of f. Otherwise, the Siegel disc is called **unbounded**.

3.1.1 | Bounded Siegel discs

Evidently, since the domain of definition of a rational or polynomial function is $\widehat{\mathbb{C}}$, which is compact; every Siegel discs of such maps are bounded. A complete characterization is known for the boundary of a bounded Siegel disc.

Theorem 3.2. ***(J.T.Rogers[1992])*** *Let B denote the boundary of a bounded Siegel disc U. Then B satisfies one of the two possibilities:*

- *The conformal map φ from U to $\mathbb{D}$ extends continuously to $\psi : B \to \partial\mathbb{D} = S^1$.*
- *B is an **indecomposable continuum**: a compact connected metric space that cannot be written as the union of two closed connected proper subspaces.*

In the first case, we say that B is **tame** and in the second case, we say that B is **wild**. A typical example of a tame boundary is any **Jordan curve**. In fact we can say more:

Lemma 3.3. *If the boundary B of a bounded Siegel disc is locally connected, then it is a Jordan curve. Moreover, the conjugacy to a rotation extends to the boundary to a homeomorphism that still conjugates the map to the rotation.*

We also have the corresponding theorem for wild B, which can be stated as follows:

Lemma 3.4. *If the boundary B of a Siegel disc of a polynomial of degree ≥ 2 contains a periodic point, then B is an indecomposable continuum.*

3.1.2 | Unbounded Siegel Discs

An unbounded Siegel disc may be created from any function with a linearizable fixed point by arbitrarily restricting the domain of definition, but more interesting examples arise from the study of transcendental entire or meromorphic functions.

Example 3.5. We look at a specific example

$$f(z) = e^{2i\pi\theta}(e^z - 1)$$

where $\theta = \frac{\sqrt{5}-1}{2}$ is the golden ratio. Its Siegel disc Δ is shown in 2.

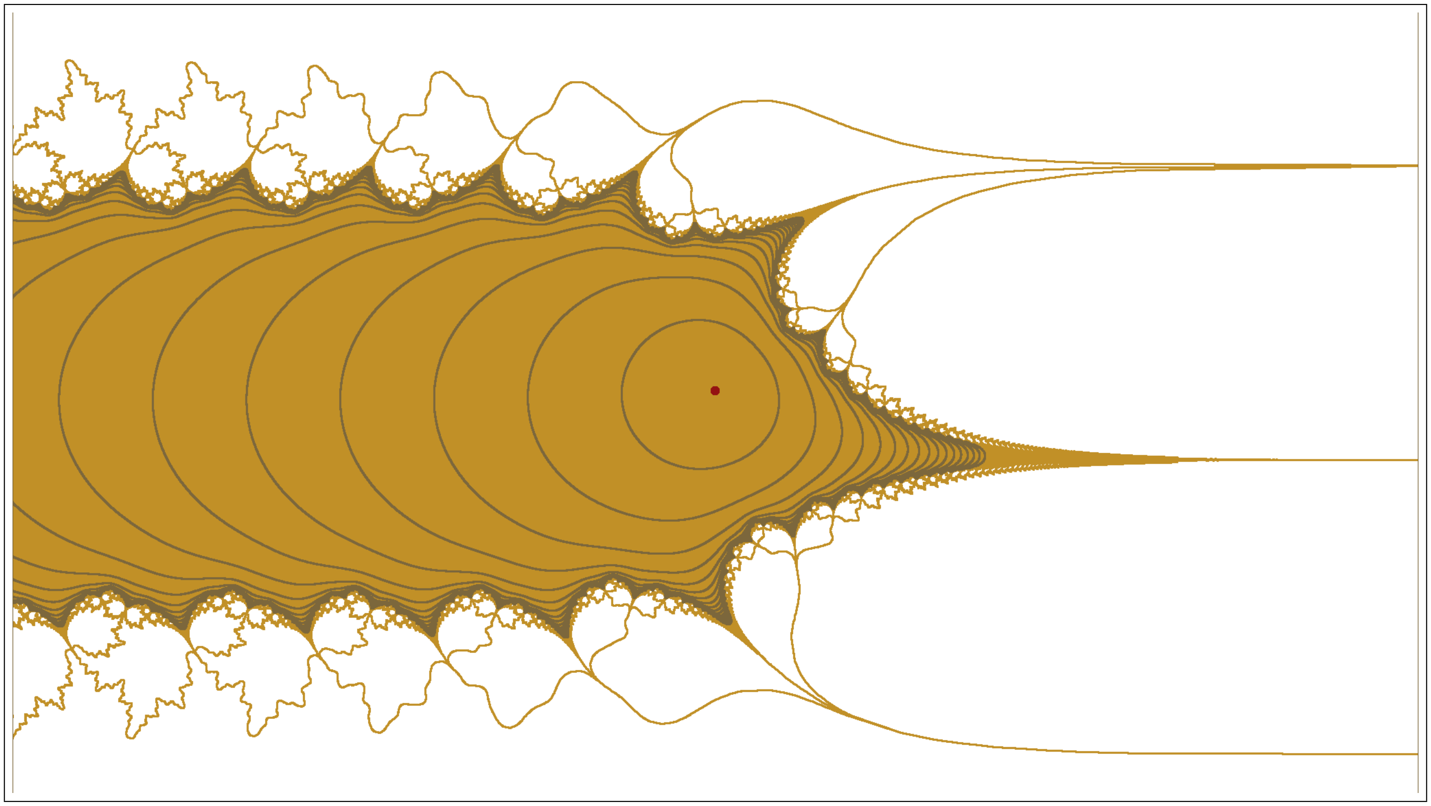


Figure 2: Exponential golden mean Siegel disc, with "strands" to infinity

Computer experiments suggest that there exists a curve γ in Δ that tends to infinity in one direction, but it is still an open question as to whether infinity is accessible from Δ **(Baker's Conjecture)**. But assuming the conjecture is true; we observe that any iterated preimage of the curve γ will also be a curve to infinity, so Δ is in fact unbounded in infinitely many directions.

It follows that (assuming Baker's conjecture) the boundary B of Δ satisfies neither of the two alternatives from Roger's theorem 3.2. If in addition to Baker's conjecture, we also assume that $-\infty$ is accessible from the Julia set, then we can make the following statement:

Lemma 3.6. ***(Rempe[2007]):*** *B is not itself indecomposable, but contains uncountably many indecomposable continua, all pairwise disjoint except for the point at infinity.*

3.2 Obstructions

We now focus on the following question: what happens at the boundary of a Siegel disc U that prevents it from extending further?

Notation 3.7. B will denote the boundary of U. The domain of definition of f will be denoted by $\mathrm{Def}(f)$.

Note that there are three obstructions:

- U must be contained in $\mathrm{Def}(f)$. So the most obvious obstruction is that its boundary meets the boundary of $\mathrm{Def}(f)$. A more general and intrinsically defined obstruction is that U be unbounded.

- Since f is conjugate to a rotation, it is injective on U. Thus U cannot have a critical point.
- Since f is conjugate an *irrational* rotation, U cannot have a periodic point.

Iterating f , it follows that U cannot contain any preimage of a critical point, of a periodic point, or any point that eventually gets mapped out of $\mathrm{Def}(f)$. So we look for whether the same are possible on B.

3.2.1 Critical Points on the boundary

We consider the case of quadratic polynomial with a Siegel disc. Let

$$P_\lambda(z) = \lambda z + z^2$$

Then the P_λ's corresponding to Siegel discs are obtained as limits of P_λ's with $|\lambda| < 1$. Note that P_λ has a critical point at $z = -\lambda/2$. Naturally, we can hope that the critical point stays on the boundary of the Siegel disc, as it is the largest disc on which the conjugation holds.

Due to the following theorem by Peterson and Zakeri , we can say that for almost all quadratic polynomial with a Siegel disc, there is a critical point on the boundary.

Let $\mathcal{E}$ be the set of all irrational numbers $\alpha = [a_1, a_2, a_3, \ldots]$ in $\mathbb{R}/\mathbb{Z}$ for which

$$\log(a_n) = O(\sqrt{n}) \text{ as } n \to \infty$$

It is not hard to check that $\mathcal{D}(2) \subsetneq \mathcal{E} \subsetneq \mathcal{D}(\kappa)$ for every $\kappa > 2$. Moreover, $\mathcal{E}$ has full measure in $\mathbb{R}/\mathbb{Z}$.

Theorem 3.8. ***(Peterson-Zakeri)*** *If $\alpha \in \mathcal{E}$, then the Julia set of $P_\alpha(z) = e^{2i\pi\alpha} z + z^2$ is locally connected and has measure zero. In particular, B is a Jordan curve passing though the critical point.*

Herman [1985] also proved that for all Diophantine numbers, P_λ has the critical point on its boundary. He also showed the following:

Theorem 3.9. *Let θ be Bryuno number. Let $P(z) = e^{2\pi i\theta} z + z^2$ and U be its Siegel disc. Then there exists θ such that U has no critical point on the boundary.*

3.2.2 Can infinity be on the boundary

Consider the exponential map

$$E_\theta(z) = e^{2i\pi\theta}(e^z - 1)$$

Clearly 0 is a fixed point of this map with multiplier $\lambda = e^{2i\pi\theta}$. Herman proved that for a.e. rotation number, the Siegel disc of E_θ is unbounded. Thus infinity belongs to its boundary. There are paths going to infinity whose image under E_θ tends to the omitted value $s = -\lambda$. Thus s is an asymptotic value and Geyer,Buff and Fagella proved that if the Siegel disc of E_θ is unbounded, then s belongs to its boundary, even though it is not known yet if there exists or not a path going to infinity within the Siegel disc.

3.2.3 Periodic points on the boundary

Periodic points on the boundary seem less likely to exist. Indeed, Lemma 3.4 points in that direction. However we note that for rational or transcendental f, $B \subseteq \mathcal{J}(f)$, which equals the closure of the set of repelling periodic points of f.

3.3 Boundaries of Siegel discs and the postsingular set.

Definition 3.10. Postsingular Set: The Postsingular set of f is defined to be the closure of the union of the orbits of the singular values, denoted by

$$CL = \overline{\bigcup_{n\geq 0} \{f^n(S_f)\}}$$

Note that CL is important since on its complement all branches of f^{-n} are locally defined and analytic.

Theorem 3.11. ***(Fatou)*** *If U is a Siegel Disc or Herman ring, then the boundary B of U is contained in CL.*

Proof. Let U be a rotation domain that is invariant under f, and suppose CL does not contain B. Let D be an open disc disjoint from CL which meets B. We assume also that D is disjoint from some open invariant subset $V \neq \emptyset$ of U. Define f_n to be any branch of f^{-n} on D. since the f_n's omit V, they form a normal family on D. Now f is one-to-one on U, so there are other components of $f^{-1}(U)$. Since inverse iterates of any fixed point of $\mathcal{J}(f)$ are dense in $\mathcal{J}(f)$, there is for suitable $m \geq 1$, a component W of $f^{-m}(U)$ distinct from U that meets D. If $z \in D \cap W$, then $f_j(z)$ and $f_k(z)$ belong to different components of $\mathcal{F}(f)$ for $j \neq k$, or else they would belong to a periodic component, which could not be iterated eventually to W then U. Hence $f^k(z)$ tends to $\mathcal{J}(f)$ for $z \in D \cap W$, and since $\mathcal{J}(f)$ has no interior, any normal limit of the f^k's is constant on $D \cap W$. On the other hand, since the f^k's are rotations of U, any normal limit is non constant on $D \cap U$. This contradiction establishes the theorem. □

We finish this section with the following generalization due to Fatou:

Theorem 3.12. *Let f be a meromorphic function, and let $\mathcal{C} = \{U_0, U_1, \ldots, U_{p-1}\}$ be a periodic cycle of Siegel discs (or Herman rings). Then $\partial U_j \subset \overline{\bigcup_{n\geq 0} \left\{f^n(S_{f^{-1}})\right\}}$ for all $i = 0, \ldots, p-1$.*

4 | Further Results

Throughout this section S stands for an invariant Siegel disc.

Proposition 4.1. *Suppose the Fatou set of f, $\mathcal{F}(f)$ contains a Siegel disc. then there are pre-periodic components in $\mathcal{F}(f)$.*

Proof. By Picard's theorem for transcendental functions and quite obviously for rational functions of degree more than one, given any point $z \in S$, there are more than one point whose f-image is z. As f is one-one on S, any point $z \in S$ has only one preimage in S. The other preimages must lie in Fatou components other than S. If U is such a component, then $f(U) = S$ which is invariant (one-periodic). So U is pre-periodic. □

Remark 4.2. A Siegel disc S does not satisfy $f^{-1}(S) \subseteq S$. So these are not completely invariant.

Remark 4.3. **Picard's theorem** states that every point in $\widehat{\mathbb{C}}$, except at most two has infinitely many pre-images under f. Hence for transcendental functions, $\mathcal{F}(f)$ may contain infinitely many pre-periodic components.

Recall that we proved existence of irrationally indifferent fixed point in S earlier. There are also invariant subsets of S. Suppose C_s is a circle of radius s centered at the origin where $0 \leq s \leq 1$. Denote by C_s^*, the set $\varphi^{-1}(C_s^*)$ where φ is the analytic homeomorphism that exists from S onto $\mathbb{D}$ by definition of Siegel discs. Here C_0 and C_0^* are assumed to be 0 and the irrationally indifferent fixed point in S respectively.

Theorem 4.4. *If S is an invariant Siegel disc of f, then*

$$S = \bigcup_{0 \leq s < 1} C_s^*$$

where each C_s^* is invariant and $C_s^* \cap C_t^* = \emptyset$ for $s, t \in [0, 1)$ and $s \neq t$.

Proof. For any $s \in [0, 1)$, $C_s^* \subset S$. So, $\bigcup\limits_{0 \leq s < 1} C_s^* \subseteq S$. Let $w \in S$. Then $|\varphi(w)| < 1$ by definition of φ. Denote the circle having radius $|\varphi(w)|$ and centered at origin by $C_{|\varphi(w)|}$. Now $\varphi^{-1}\left(C_{|\varphi(w)|}\right)$ is nothing but $C^*_{|\varphi(w)|}$ which contains w. This implies $S \subseteq \bigcup\limits_{0 \leq s < 1} C_s^*$.

To show that each C_s^* is invariant, let $z \in C_s^* = \varphi^{-1}(C_s)$. From the definition of Siegel disc, it follows that $f = \varphi^{-1}\rho\varphi$ on S. Now $\varphi(z) \in C_s$ and C_s is preserved by ρ. So $f(z) \in C_s^*$. Thus C_s^* is invariant.

Lastly, $C_s \cap C_t = \emptyset$ for $s \neq t$ and the injectivity of φ implies $C_s^* \cap C_t^* = \emptyset$ for $s \neq t$. □

Corollary 4.5. *All the limit functions of $\{f^n\}$ on S are non constant.*

Proof. Suppose there is a subsequence $\{f_{n_k}\}$ of $\{f^n\}$ that converges uniformly on each compact subset of S to a constant c. Let C_s^* be an invariant curve in S (as in the previous theorem) such that $c \notin C_s^*$. Now for all n_k and $z \in C_s^*, f_{n_k}(z) \in C_s^*$. Thus a neighborhood around c can be found which does not contain any $f_{n_k}(z)$. So $f_{n_k}(z)$ can not converge to c on C_s^*, contradiction!! Therefore, any limit function of $\{f^n\}$ is non constant. □

Note 4.6. All limit functions of $\{f^n\}$ on an attracting or parabolic Fatou component are constants.

We finish this section with characterization of certain functions which do not have Siegel discs in their Fatou set.

Theorem 4.7. *Suppose for a function $f : \mathbb{C} \to \widehat{\mathbb{C}}$, $\bigcup\limits_{n \geq 0} f^n(S_f) \subseteq \gamma$ where γ is bounded but not a closed curve in $\mathbb{C}$. Then $\mathcal{F}(f)$ does not contain any Siegel disc.*

Proof. Suppose S is an invariant Siegel disc of $\mathcal{F}(f)$. By the Theorem 3.11, $\partial S \subseteq CL$. The boundary of S must be a simple closed curve in $\widehat{\mathbb{C}}$. But $\bigcup\limits_{n \geq 0} f^n(S_f)$ is given to be a subset of γ which is bounded. So ∂S must be a simple closed curve in $\mathbb{C}$ which is no longer true as $\partial S \subseteq CL \subseteq \gamma$ and γ is not closed. Contradiction!! Thus no Siegel disc can exist. □

5 References

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