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# ANALYSIS AND CONTROL OF DISPLACED PERIODIC ORBITS IN THE EARTH-MOON SYSTEM

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## ABSTRACT

We consider displaced periodic orbits at linear order in the circular restricted Earth-Moon system, where the third massless body is a solar sail. These highly non-Keplerian orbits are achieved using an extremely small sail acceleration. In this paper we will use solar sail propulsion to provide station-keeping at periodic orbits above the $L_2$ point. We start by generating a reference trajectory about the libration points. By introducing a first-order approximation, periodic orbits are derived analytically at linear order. These approximate analytical solutions are utilized in a numerical search to determine displaced periodic orbits in the full nonlinear model. Because of the instability of the collinear libration points, orbit control is needed for a spacecraft to remain in the vicinity of these points. The reference trajectory is then tracked using a Linear Quadratic Regulator (LQR). Finally, simulations are given to validate the control strategy. The importance of finding such displaced orbits is to obtain continuous communications between the equatorial regions of the Earth and the polar regions of the Moon.

## 1. INTRODUCTION

Solar sailing technology has been widely investigated over the past decade. It appears as a promising form of advanced spacecraft propulsion, which can enable exciting new space-science mission concepts such as solar system exploration and deep space observation. A solar sail is propelled by reflecting solar photons and therefore can transform the momentum of the photons into a propulsive force. Although solar sailing has been considered as a practical means of spacecraft propulsion only relatively recently, the fundamental ideas are by no means new (see McInnes [1] for a detailed description).

Solar sails can also be utilised for highly non-Keplerian orbits, such as closed orbits displaced high above the ecliptic plane (see Waters and McInnes [2]). Solar sails are especially suited for such non-Keplerian orbits, since they can apply a propulsive force continuously. This allows some exciting and unique trajectories. In such trajectories, a sail can be used as a communication satellite for high latitudes. For example, the orbital plane of the sail can be displaced above the orbital plane of the Earth, so that the sail can stay fixed above the Earth at some distance, if the orbital periods are equal. Orbits around the collinear points of the Earth-Moon system are also of great interest because their unique positions are advantageous for several important applications in space mission design (see e.g. Szebehely [3], Roy [4], Vonbun [5], Gómez et al. [6]).

In the recent years several authors have tried to determine more accurate approximations (quasi-Halo orbits) of such equilibrium orbits [7]. The orbits were first studied by Farquhar [8], Farquhar and Kamel [7], Breakwell and Brown [9], Richardson [10], Howell [11]. Halo orbits near the collinear libration points in the Earth-Moon system are of great interest, particulary around the $L_1$ and $L_2$ points because their unique positions. However, a linear analysis shows that the collinear libration points $L_1$, $L_2$, and $L_3$ are of the type $saddle \times center \times center$, leading to the instability in their vicinity, whereas the equilateral equilibrium points $L_4$, and $L_5$ are stable ($center \times center \times center$). Although the libration points $L_4$, and $L_5$ are naturally stable and require a small acceleration, the disadvantage is the longer communication path length from the lunar pole to the sail. It is essential to note that the equilateral libration points $L_4$ and $L_5$ of the Earth-Moon system can be found to be unstable if the gravitational effect of the sun is included (see Szebehely [3]). If the orbit maintains visibility from Earth, a spacecraft on it (near the $L_2$ point) can be used to provide communications between the equatorial regions of the Earth and the lunar poles. The establishment of a bridge for radio communications is crucial for forthcoming space missions, which plan to use the lunar poles. McInnes [12]and Simo and McInnes [13, 14] investigated a new family of displaced solar sail orbits near the Earth-Moon libration points. In Baoyin and McInnes [15] and McInnes [16], the authors describe the new orbits which are associated with artificial lagrange points in the Earth-Sun system. These artificial equilibria have potential applications for future space physics and Earth observation missions. In McInnes and Simmons [17], the authors investigate large new families of solar sail orbits, such as Sun-centered halo-type trajectories, with the sail executing a circular orbit of a chosen period above the ecliptic plane. In Ozimek et al. [18] and Wawrzyniak and Howell [19], the authors used collocation approach to the problem of computing solar sail lunar pole-sitter orbits. We have recently investigated displaced periodic orbits at linear order in the Earth-Moon restricted three-body system, where the third massless body utilizes a hybrid of solar sail and a solar electric propulsion system (see Simo and McInnes [20]). Then, a feedback linearization control scheme was proposed and implemented. The main idea of this approach is to cancel the nonlinearities and to impose desired linear dynamics satisfied by the solar sail.

In the present study, we will focuss on linear control technique to the problem of tracking and maintaining the solar sail on prescribed orbits. The first-order approximation is introduced for the linearized system of equations. The Laplace transform is used to produce the first-order analytic solution of the out-of plane motion. It will be shown for example that, with a suitable sail attitude control program, a 1750 $km$ displaced, out-of-plane trajectory around the $L_2$ point may be executed with a small sail acceleration.This unstable orbit will therefore be used as a reference trajectory for the solar sail and tracked using the Linear Quadratic Regulator (LQR) control method (see Cielaszyk and Wie [21]). This paper is organized as follow: Section 2 provides the mathematical expressions describing the motion of the sail in the circular restricted three-body problem. Section 3 is devoted to the study of the periodic orbits around the Lagrange points in the Earth-Moon system. The periodic solutions to the linearized equations of motion are derived analytically. In section 4 a LQR control is developed and implemented. Section 5 is concerned with the numerical computation around the libration point $L_2$ in the Earth-Moon system. Finally some numerical results are presented to illustrate our approach, and compared to the feedback linearization control technique, which has been successfully applied in [20] to reduce the fuel expenditure.

## 2. SYSTEM MODEL

In this work $m_1$ represents the larger primary (Earth), $m_2$ the smaller primary (Moon) and we will be concerned with the motion of a hybrid sail that has negligible mass. It is always assumed that the two more massive bodies are moving in circular orbits with constant angular velocity $\omega$ about their common center of mass, and the mass of the third body is too small to affect the motion of the two more massive bodies. The unit mass is taken to be the total mass of the system $(m_1 + m_2)$ and the unit of length is chosen to be the constant separation $R^\star$ between $m_1$ and $m_2$. The time unit is defined such that $m_2$ orbits around $m_1$ in time $2\pi$. Under these considerations the masses of the primaries in the normalized system of units are $m_1 = 1 - \mu$ and $m_2 = \mu$, with $\mu = m_2/(m_1 + m_2)$ (see Figure 1). Thus, in the Earth-Moon system, the nondimensional unit acceleration is $a_{ref} = \omega^2 R^\star = 2.7307\ mm/s^2$ where the Earth-Moon distance $R^\star = 384400\ km$. The dashed line in Figure 1 is a line parallel to the Sun-line direction.

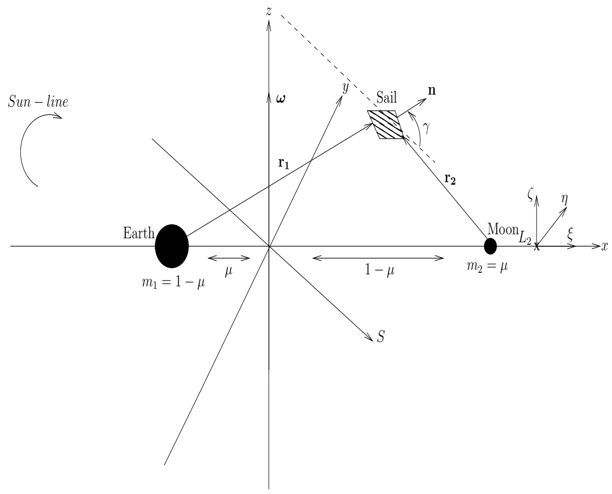



Figure 1. Schematic geometry of the Earth-Moon restricted three-body problem.

### 2.1. Equations of Motion

The vector dynamical equation for the solar sail in a rotating frame of reference is described by

$$\frac{d^2\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt} + \nabla U(\boldsymbol{r}) = \boldsymbol{a}, \tag{1}$$

where $\boldsymbol{\omega} = \omega\hat{z}$ ($\hat{z}$ is a unit vector pointing in the direction of $z$) is the angular velocity vector of the rotating frame and $\boldsymbol{r}$ is the position vector of the solar sail relative to the center of mass of the two primaries. The three-body gravitational potential $U(\boldsymbol{r})$ and the solar radiation pressure acceleration $\boldsymbol{a}$ are defined by

$$\begin{aligned} U(\boldsymbol{r}) &= -\left[\frac{1}{2}|\boldsymbol{\omega} \times \boldsymbol{r}|^2 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right], \\ \boldsymbol{a} &= a_0(\boldsymbol{S} \cdot \boldsymbol{n})^2\boldsymbol{n}, \end{aligned}$$

where $\mu = 0.1215$ is the mass ratio for the Earth-Moon system. The sail position vectors w.r.t. $m_1$ and $m_2$ respectively, are defined as $\boldsymbol{r}_1 = [x + \mu, y, z\ ]^T$ and $\boldsymbol{r}_2 = [x - (1 - \mu), y, z]^T$, and $a_0$ is the magnitude of the solar radition pressure force exerted on the sail. The unit normal to the sail $\boldsymbol{n}$ and the Sun-line direction are given by

$$\boldsymbol{n} = \left[\ \cos(\gamma)\cos(\omega_\star t) \quad -\cos(\gamma)\sin(\omega_\star t) \quad \sin(\gamma)\ \right]^T,$$

$$\boldsymbol{S} = \left[\ \cos(\omega_\star t) \qquad -\sin(\omega_\star t) \qquad 0\ \right]^T,$$

where $\omega_\star = 0.923$ is the angular rate of the Sun line in the corotating frame in a dimensionless synodic coordinate system.

### 2.2. Linearized system

We now want to investigate the dynamics of the sail in the neighborhood of the libration points. The libration points are the equilibrium solutions of the restricted three-body problem, which describes the motion of a particle (very small mass) under the gravitational attraction of two massive bodies.

We denote the coordinates of the equilibrium point as $\boldsymbol{r}_L = (x_{L_i}, y_{L_i}, z_{L_i})$ with $i = 1, \cdots, 5$.

Let a small displacement in $\boldsymbol{r}_L$ be $\delta\boldsymbol{r}$ such that $\boldsymbol{r} \to \boldsymbol{r}_L + \delta\boldsymbol{r}$. We will not consider the small annual changes in the inclination of the Sun line with respect to the plane of the system.

Therefore, the linear equations for the solar sail are

$$\frac{d^2\delta\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\delta\boldsymbol{r}}{dt} + \nabla U(\boldsymbol{r}_L + \delta\boldsymbol{r}) = \boldsymbol{a}(\boldsymbol{r}_L + \delta\boldsymbol{r}), \quad (2)$$

and retaining only the first-order term in $\delta\boldsymbol{r} = [\delta x, \delta y, \delta y]^T$ in a Taylor-series expansion, the gradient of the potential and the acceleration can be expressed as

$$\begin{aligned} \nabla U(\boldsymbol{r}_L + \delta\boldsymbol{r}) &= \nabla U(\boldsymbol{r}_L) + \left.\frac{\partial \nabla U(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L} \delta\boldsymbol{r} \quad (3) \\ &\quad + O(\delta\boldsymbol{r}^2), \\ \boldsymbol{a}(\boldsymbol{r}_L + \delta\boldsymbol{r}) &= \boldsymbol{a}(\boldsymbol{r}_L) + \left.\frac{\partial \boldsymbol{a}(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L} \delta\boldsymbol{r} \quad (4) \\ &\quad + O(\delta\boldsymbol{r}^2). \end{aligned}$$

It is assumed that $\nabla U(\boldsymbol{r}_L) = 0$, and the acceleration is constant with respect to the small displacement $\delta\boldsymbol{r}$, so that

$$\left.\frac{\partial \boldsymbol{a}(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L} = 0.$$

The linear variational system associated with the libration points at $\boldsymbol{r}_L$ can be determined through a Taylor series expansion by substituting Eqs. (4) and (5) into (2)

$$\frac{d^2\delta\boldsymbol{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\delta\boldsymbol{r}}{dt} - K\delta\boldsymbol{r} = \boldsymbol{a}(\boldsymbol{r}_L), \quad (5)$$

where the matrix $K$ is defined as

$$K = -\left[\left.\frac{\partial \nabla U(\boldsymbol{r})}{\partial \boldsymbol{r}}\right|_{\boldsymbol{r}=\boldsymbol{r}_L}\right].$$

Using the matrix notation the linearized equation about the libration point (Equation (5)) can be represented by the inhomogeneous linear system $\dot{\mathbf{X}} = A\mathbf{X} + \mathbf{b}(t)$, where the state vector $\mathbf{X} = (\delta\boldsymbol{r}, \delta\dot{\boldsymbol{r}})^T$, and $\mathbf{b}(t)$ is a $6 \times 1$ vector, which represents the solar sail acceleration.

The Jacobian matrix $A$ has the general form

$$A = \begin{pmatrix} 0_3 & I_3 \\ K & \Omega \end{pmatrix}, \quad (6)$$

where $I_3$ is a identity matrix, and

$$\Omega = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For convenience the sail attitude is fixed such that the sail normal vector $\boldsymbol{n}$, which is a unit vector that is perpendicular to the sail surface, points always along the direction of the Sun line with the following constraint $\boldsymbol{S}\cdot\boldsymbol{n} \geq 0$. Its direction is described by the pitch angle $\gamma$ relative to the Sun-line, which represents the sail attitude.

By making the transformation $\boldsymbol{r} \to \boldsymbol{r}_L + \delta\boldsymbol{r}$ and retaining only the first-order term in $\delta\boldsymbol{r} = (\xi, \eta, \zeta)^T$ in a Taylor-series expansion, the linearized nondimensional equations of motion relative to the collinear libration points can be written as

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} - U^o_{xx}\xi &= a_\xi, \quad (7) \\ \ddot{\eta} + 2\dot{\xi} - U^o_{yy}\eta &= a_\eta, \quad (8) \\ \ddot{\zeta} - U^o_{zz}\zeta &= a_\zeta, \quad (9) \end{aligned}$$

where $U^o_{xx}$, $U^o_{yy}$, and $U^o_{zz}$ are the partial derivatives of the gravitational potential evaluated at the collinear libration points, and the solar sail acceleration is defined in terms of three auxiliary variables $a_\xi$, $a_\eta$, and $a_\zeta$.

The acceleration components are given by

$$\begin{aligned} a_\xi &= a_0\cos(\omega_\star t)\cos^3(\gamma), \\ a_\eta &= -a_0\sin(\omega_\star t)\cos^3(\gamma), \\ a_\zeta &= a_0\cos^2(\gamma)\sin(\gamma). \end{aligned}$$

## 3. ANALYTICAL APPROACH

Considering the dynamics of motion near the collinear libration points. We may choose a particular periodic solution in the plane of the form (see Farquhar [**?**])

$$\begin{aligned} \xi(t) &= \xi_0\cos(\omega_\star t), \quad (10) \\ \eta(t) &= \eta_0\sin(\omega_\star t). \quad (11) \end{aligned}$$

By inserting Equations (10) and (11) in the differential equations, we obtain the linear system in $\xi_0$ and $\eta_0$,

$$\begin{cases} \left(U^o_{xx} - \omega_\star^2\right)\xi_0 - 2\omega_\star\eta_0 = a_0\cos^3(\gamma), \\ -2\omega_\star\xi_0 + \left(U^o_{yy} - \omega_\star^2\right)\eta_0 = -a_0\cos^3(\gamma). \end{cases} \quad (12)$$

Then the amplitudes $\xi_0$ and $\eta_0$ are given by

$$\begin{aligned} \xi_0 &= a_0\frac{\left(U^o_{yy} - \omega_\star^2 - 2\omega_\star\right)\cos^3(\gamma)}{\left(U^o_{xx} - \omega_\star^2\right)\left(U^o_{yy} - \omega_\star^2\right) - 4\omega_\star^2}, \quad (13) \\ \eta_0 &= a_0\frac{\left(-U^o_{xx} + \omega_\star^2 + 2\omega_\star\right)\cos^3(\gamma)}{\left(U^o_{xx} - \omega_\star^2\right)\left(U^o_{yy} - \omega_\star^2\right) - 4\omega_\star^2}, \quad (14) \end{aligned}$$

and we have the equality

$$\frac{\xi_0}{\eta_0} = \frac{\omega_\star^2 + 2\omega_\star - U^o_{yy}}{-\omega_\star^2 - 2\omega_\star + U^o_{xx}}. \quad (15)$$

Then the trajectory will be an ellipse centered on the collinear libration points. We can find the required radiation pressure be solving the equation (13)

$$a_0 = \cos^{-3}(\gamma)\left[\frac{\omega_\star^4 - \omega_\star^2(U^o_{xx} + U^o_{yy} + 4) + U^o_{xx}U^o_{yy}}{U^o_{yy} - 2\omega_\star - \omega_\star^2}\right]\xi_0.$$

By applying the Laplace transform, the uncoupled out-of-plane $\zeta$-motion defined by the equation (9) can be solved. The transform version is obtained as

$$\begin{aligned} s^2 Z - s\xi_0 - \dot{\xi}_0 - U^o_{zz} Z &= \frac{a_0\cos^2(\gamma)\sin(\gamma)}{s}, \quad (16) \\ (s^2 - U^o_{zz})Z &= \dot{\xi}_0 + s\xi_0 \quad (17) \\ &\quad + \frac{a_0\cos^2(\gamma)\sin(\gamma)}{s}, \end{aligned}$$

also

$$Z(s) = \frac{1}{s^2 - U^o_{zz}}\left(\dot{\xi}_0 + s\xi_0 + \frac{a_0\cos^2(\gamma)\sin(\gamma)}{s}\right). \quad (18)$$

The frequency of the out-of-plane motion is given by solving the equation

$$s^2 - U^o_{zz} = 0,$$

where $s_{1,2} = \pm i\sqrt{|U^o_{zz}|} = \pm i\omega_\zeta$.

Using Mathematica, we can find the inverse Laplace transform, which will be the general solution of the out-of-plane component

$$\begin{aligned} \zeta(t) &= \zeta_0\cos(\omega_\zeta t) + \dot{\zeta}_0|U^o_{zz}|^{-1/2}\sin(\omega_\zeta t) \quad (19) \\ &\quad + a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1}[U(t) - \cos(\omega_\zeta t)], \\ &= U(t)a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1} \quad (20) \\ &\quad + \dot{\zeta}_0|U^o_{zz}|^{-1/2}\sin(\omega_\zeta t) \\ &\quad + \cos(\omega_\zeta t)[\zeta_0 - a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1}], \end{aligned}$$

where the nondimensional frequency is defined as

$$\omega_\zeta = |U^o_{zz}|^{1/2}$$

and $U(t)$ is the unit step function.
Specifically for the choice of the initial data $\dot{\zeta}_0 = 0$, equation (32) can be more conveniently expressed as

$$\begin{aligned} \zeta(t) &= U(t)a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1} \quad (21) \\ &\quad + \cos(\omega_\zeta t)[\zeta_0 - a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1}]. \end{aligned}$$

The solution can be made to contain only the periodic oscillatory modes at an out-of-plane distance

$$\zeta_0 = a_0\cos^2(\gamma)\sin(\gamma)|U^o_{zz}|^{-1}. \quad (22)$$

Furthermore, the out-of-plane distance can be maximized by an optimal choice of the sail pitch angle determined by

$$\begin{aligned} \frac{d}{d\gamma^\star}\cos^2(\gamma^\star)\sin(\gamma^\star) &= 0, \\ \gamma^\star &= \tan^{-1}(2^{-1/2}), \\ \gamma^\star &= 35.264^\circ. \quad (23) \end{aligned}$$

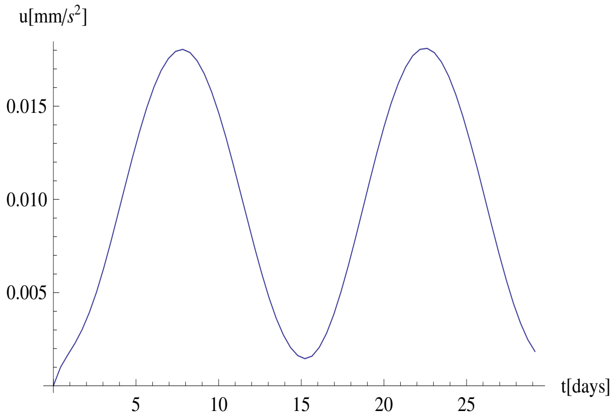
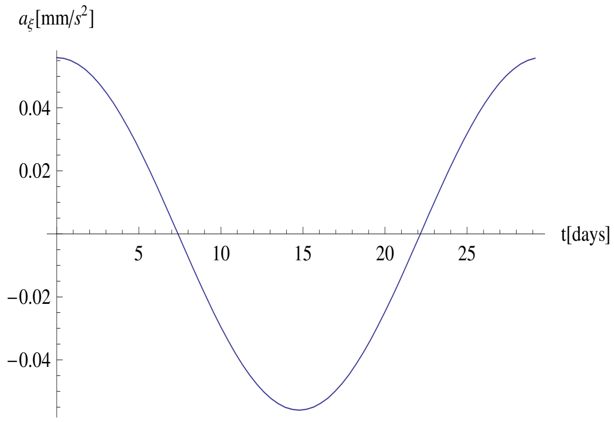
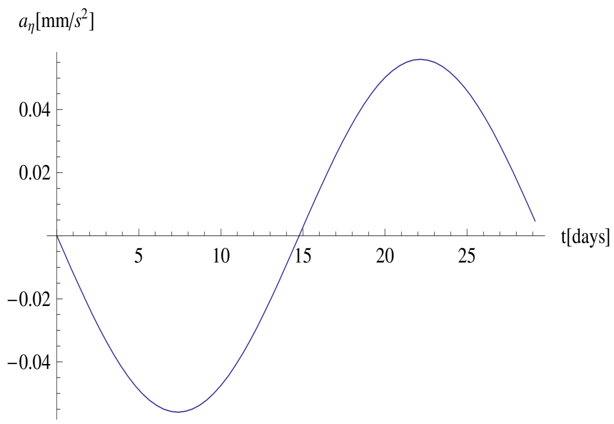



Figure 2. Magnitude of the total control effort about the $L_2$ point.



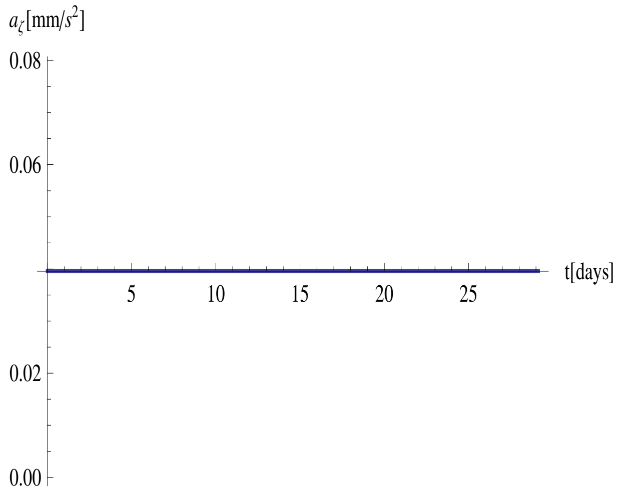
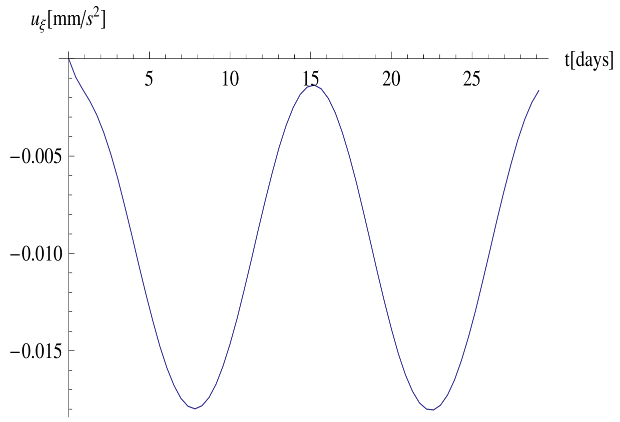
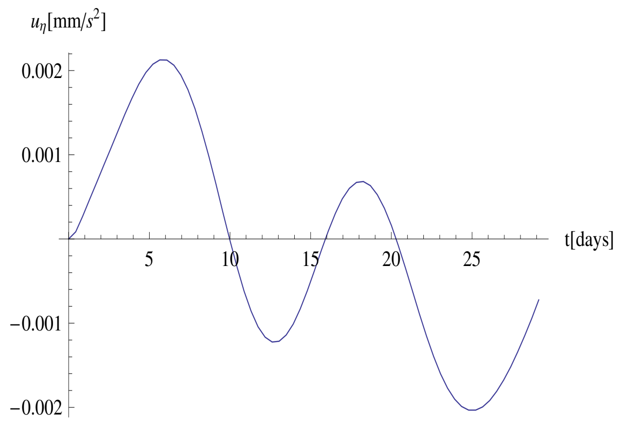



Figure 3. Accelearation derived from the solar sail about the $L_2$ point.



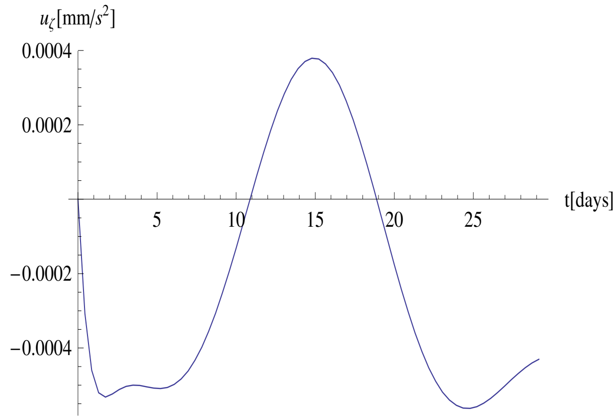




Figure 4. Control acceleration inputs needed for maintaining a $1750-km$ reference trajectory.

## 4. TRAJECTORY TRACKING

The nonlinear equations of motion of the solar sail can be obtained by simply adding the control acceleration vector $\mathbf{u}(t)=[u_\xi \quad u_\eta \quad u_\zeta]^T$ as

$$\begin{aligned}
\ddot{\xi} &= 2\dot{\eta}+(x_{L_2}+\xi)-(1-\mu)\frac{(x_{L_2}+\xi)+\mu}{r_1^3} \qquad (24)\\
&\quad -\mu\frac{(x_{L_2}+\xi)-1+\mu}{r_2^3}+a_\xi+u_\xi,\\
\ddot{\eta} &= -2\dot{\xi}+\eta-\left(\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}\right)\eta \qquad (25)\\
&\quad +a_\eta+u_\eta,\\
\ddot{\zeta} &= -\left(\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}\right)\zeta+a_\zeta+u_\zeta, \qquad (26)
\end{aligned}$$

where

$$\begin{aligned}
r_1 &= \sqrt{(x+\mu)^2+y^2+z^2},\\
r_2 &= \sqrt{(x-1+\mu)^2+y^2+z^2}.
\end{aligned}$$

Similary, the nondimensional linearized equations of motion can be derived in state space form as

$$\dot{\mathbf{X}}=A\mathbf{X}+\mathbf{b}(t)+B\mathbf{u}(t), \qquad (27)$$

where the state vector $\mathbf{X}=(\delta\boldsymbol{r},\delta\dot{\boldsymbol{r}})^T$, $\mathbf{b}(t)$ is a $6\times 1$ vector, which represents the solar sail acceleration, $\mathbf{u}(t)=[u_x \quad u_y \quad u_z]^T$, A is given by Eq. (6), and the matrix $B=[O_{3\times 3} \quad I_{3\times 3}]^T$.

Thus, we can consider a linear state feedback controller of the form

$$\mathbf{u}(t)=-\mathbf{K}(\mathbf{x}(t)-\mathbf{x}_{ref}(t)), \qquad (28)$$

where $\mathbf{K}$ is the constant gain matrix to be determined, and the reference trajectory $\mathbf{x}_{ref}(t)=[\xi_{ref}, \quad \eta_{ref}, \quad \zeta_{ref}]^T$ is given by the analytical solution

$$
\begin{aligned}
\xi_{ref}(t) &= \xi_0 \cos(\omega_\star t), \\
\eta_{ref}(t) &= \eta_0 \sin(\omega_\star t), \\
\zeta_{ref}(t) &= z_0.
\end{aligned}
$$

A feedback controller that minimizes the cost function

$$J = \frac{1}{2} \int_0^\infty \left[ \mathbf{x}(t)^T Q \mathbf{x}(t) + \mathbf{u}(t)^T R \mathbf{u}(t) \right] dt, \tag{29}$$

while tracking the reference trajectory has the form

$$\mathbf{u}(t) = -R^{-1} B P \mathbf{x}(t), \tag{30}$$

where P is unique, positive semidefinite solution to the algebraic Riccati equation

$$-A^T P - PA + PBR^{-1} B^T P - Q = 0. \tag{31}$$

The matrices Q and R represent the weight of the state error and the control input.

## 5. CONTROL ANALYSIS RESULTS

Prior results using the feedback linearization control strategy have been successfully developed to track the linear displaced trajectory about the $L_2$ point (see [20]). One alternative controller used in this paper is the LQR method. The simulations show that the LQR as well the feedback linearization control results are similar.

The simulation was performed around the collinear libration point $L_2$ for a period of one month. The magnitude of the total control effort appears in Figure 2. Thus, the control acceleration effort $\boldsymbol{u}(t)$ required to track the reference orbit varies up to $0.015\ mm/s^2$ about the $L_2$ point.The acceleration derived from the solar sail (denoted by $a_\xi, a_\eta, a_\zeta$) is plotted in terms of components for one-month orbits in Figure 3 about $L_2$, and the control acceleration inputs needed for maintaining a $1750 - km$ reference trajectory appears in Figure 4 about $L_2$. The control acceleration inputs (denoted by $u_\xi, u_\eta, u_\zeta$) are order of $10^{-3} - 10^{-4}\ mm/s^2$, while the acceleration derived from the solar sail is approximately $10^{-2}\ mm/s^2$. The small control acceleration input is then applied to ensure that displacement of the periodic orbit is constant. The solar sail provides a constant out-of-plane force. The average $\Delta V$ per orbit found by integrating the control acceleration inputs is about 28 $m/s$. It was shown in [20] that an average $\Delta V$ per orbit of approximately 23 $m/s$ is needed to maintain a $1750 - km$ linear displaced orbit about the $L_2$ point. This observation explain why it is useful to develop more accurate, nonlinear reference orbits that are closer to an exact periodic solution of the nonlinear equations of motion. For this reason, the more accurate reference orbits are utilized to reduce the fuel expenditure.

## 6. CONCLUSIONS

We consider displaced periodic orbits at linear order in the Earth-Moon system. Using the linearized equations of motion around the collinear libration points, periodic orbits that are displaced can be derived, which will be interesting for future mission design. Due to the instability of these points, active control is required to maintain the sail on prescribed orbits. An approximate solution is introduced to design a reference trajectory. An LQR technique is then implemented to track this orbit. The Linear Quadratic Regulator control approach is shown to be able to maintain a linear displaced orbit despite the nonlinear dynamical effects.

## 7. ACKNOWLEDGMENTS

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## 8. REFERENCES

[1] Colin R. McInnes. *Solar sailing: technology, dynamics and mission applications*. Springer Praxis, London, 1999.

[2] T. Waters and C. McInnes. Periodic Orbits Above the Ecliptic in the Solar-Sail Restricted Three-Body Problem. *Journal of Guidance, Control and Dynamics*, 30(3):687–693, 2007.

[3] Victor Szebehely. *Therory of Orbits: the restricted problem of three bodies*. Academic Press, New York and London, 1967.

[4] A E Roy. *Orbital Motion*. Institute of Phisics Publishing, Bristol and Philadelphia, 2005.

[5] F.O. Vonbun. "A Humminbird for the L$_2$ Lunar Libration Point". *Nasa TN-D-4468*, April 1968.

[6] G. Gómez, Á. Jorba., J.Masdemont, and C. Simó. *Dynamics and Mission Design Near Libration Points*, volume III, IV. World Scientific Publishing Co.Pte.Ltd, Singapore.New Jersey.London.Hong Kong, 2001.

[7] R.W. Farquhar and A.A. Kamel. Quasi-periodic orbits about the trans-lunar libration point. *Celestial Mechanics*, 7:458–473, 1973.

[8] R. Farquhar. The utilization of halo orbits in advanced lunar operations. *Nasa technical report*, 1971.

[9] J.V. Breakwell and J.V. Brown. The 'halo' family of 3-dimensional periodic orbits in the earth-moon restricted 3-body problem. *Celestial Mechanics*, 20:389–404, 1979.

[10] D. L. Richardson. Halo orbit formulation for the ISEE-3 mission. *Journal Guidance and Control*, 3(6):543–548, 1980.

[11] K.C. Howell. Three-dimensional, periodic, 'halo' orbits. *Celestial Mechanics*, 32:53–71, 1984.

[12] C. McInnes. Solar sail Trajectories at the Lunar $L_2$ Lagrange Point. *Journal of Spacecraft and Rocket*, 30(6):782–784, 1993.

[13] Jules Simo and Colin R. McInnes. Solar sail orbits at the Earth-Moon Libration points. *Communications in Nonlinear Science and Numerical Simulation*, 14(12):4191–4196, December 2009.

[14] Jules Simo and Colin R. McInnes. Asymptotic Analysis of Displaced Lunar Orbits. *Journal of Guidance, Control and Dynamics*, 32(5):1666–1671, September-October 2009.

[15] H. Baoyin and C. McInnes. Solar sail halo orbits at the Sun-Earth artificial $L_1$ point. *Celestial Mechanics and Dynamical Astronomy*, 94(2):155–171, 2006.

[16] C. R. McInnes. Artificial Lagrange points for a non-perfect solar sail. *Journal of Guidance, Control and Dynamics*, 22(1):185–187, 1999.

[17] C.R. McInnes, A.J.C. McDonald, J.F.C. Simmons, and E.W. McDonald. Solar sail parking in restricted three-body systems. *Journal of Guidance, Control and Dynamics*, 17(2):399–406, 1994.

[18] Martin Ozimek, Daniel Grebow, and Kathleen Howell. A Collocation Approach for Computing Solar Sail Lunar Pole-Sitter Orbits. *In AIAA/AAS Astrodynamics Specialist Conference*, Pittsburgh, Pennsylvania, August 2009. Paper AAS 09-378.

[19] Geoffrey G. Wawrzyniak and Kathleen Howell. Accessing the Design Space for Solar Sails in the Earth-Moon system. *In AIAA/AAS Astrodynamics Specialist Conference*, Pittsburgh, Pennsylvania, August 2009. Paper AAS 09-348.

[20] Jules Simo and Colin R. McInnes. Designing displaced lunar orbits using low-thrust propulsion. *Journal of Guidance, Control and Dynamics*, to appear, 2009.

[21] David Cielaszyk and Bong Wie. New Approach to Halo Orbit Determination and Control. *Journal of Guidance, Control and Dynamics*, 19(2):266–273, March-April 2009.