Last time: (Hensley's) Zaremba's Conj: $\exists A < \infty : \forall d \geq 1$ $(\gg 1)$, $(A=5)$. $(A=2?)$.

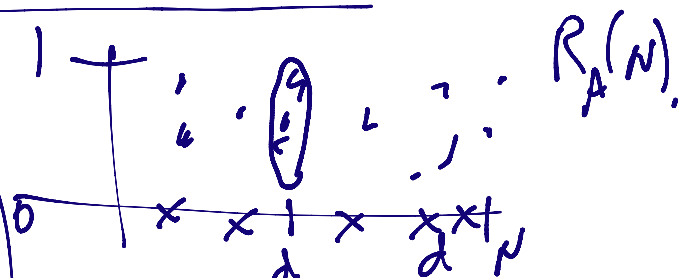
$\exists (b,d)=1$, $\frac{b}{d} = [0, a_1, \ldots, a_\ell]$, all $a_j \leq A$.

Let $R_A = \{ \frac{b}{d} : \quad \}$, $D_A = \{ d \geq 1 : \exists (b,d)=1, \frac{b}{d} \in R_A \}$.

Thm: (Hensley)
$$\# R_A(N) = \# \{ \tfrac{b}{d} \in R_A : d < N \} = N^{2\delta_A + o(1)}$$

Thm Hensley '96: $\delta_A = 1 - \frac{6}{\pi^2} \cdot \frac{1}{A} + o\left(\frac{1}{A}\right)$. $\left(\frac{1}{2}, 1\right) \ni \delta_A = H.\dim \underbrace{C_A}_{} = \Lambda(R_A)$

$$\# D_A \cap [1,N] \gg N^{2\delta_A - 1 - o(1)}.$$



$$m_A(d) = \# \{ b < d, (b,d)=1, \tfrac{b}{d} \in R_A \} = \iota^{-1}(d), \quad \iota : R_A \to D_A$$
$$\tfrac{b}{d} \mapsto d.$$
$$\leq d.$$

$$\sum_{d<N} m_A(d) = N^{2\delta_A + o(1)}.$$

So by making A large $\Rightarrow \delta_A \to 1$, $\Rightarrow 2\delta - 1 \to 1$,

$\# D_A \cap [1,N] \gg N^{1-\varepsilon}$. Can't get $\# D_A \cap [1,N] \gg N$.

Thm (B-K): $\exists A \left(\begin{matrix} =50 \\ =5 \end{matrix} \right)$ Frolenkov-Kan, Huang s.t. $\frac{1}{N} \# D_A \cap [1,N] \to 1$.

Key (trivial) observation: $\frac{b}{d} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \frac{1}{a_\ell}}} \iff \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & * \\ b & * \end{pmatrix}$

Exercise: $\boxed{\text{unique if } \ell \text{ even}}$

Main object: $\Gamma_A^{(0)} := \left\langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} : a \in A \right\rangle^{+}$ ← semigp. $\subset GL_2(\mathbb{Z})$.

Let $\Gamma_A = \Gamma_A^{(0)} \cap SL_2$ = even-length words in $\Gamma_A^{(0)}$,

$\Gamma_A^{(0)} = \left\{ I, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \right\} \cdot \Gamma_A$.

Exercise: 1-1 corr $R_A \leftrightsquigarrow \Gamma_A$.

$\frac{b}{d} = [0; a_1, \ldots, a_\ell]$, $\ell \equiv 0 \ (2)$ $\longmapsto \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix} \in \Gamma_A$

Really care about:

$\frac{b}{d} \longleftarrow \begin{pmatrix} d & * \\ b & * \end{pmatrix}$

↑ unique! ~~det!~~

$\Gamma_A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathcal{O}$ orbit.

$\boxed{\text{Exercise, } \forall \gamma \in \Gamma_A, \ \gamma_{1,1} \text{ is largest.}}$

Exercise: If $\gamma_1 = \begin{pmatrix} d & \alpha \\ b & \beta \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} d & \alpha' \\ b & \beta' \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\gamma_1 = \gamma_2 \cdot \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some n.

Pf 1: Chinese Rem Thm.

Pf 2: $\gamma_2^{-1} \cdot \gamma_1 = \begin{pmatrix} \beta' & -\alpha' \\ -\delta & d \end{pmatrix} \begin{pmatrix} d & \alpha \\ b & \beta \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Claim: $\Gamma_A \subset SL_2(\mathbb{Z})$ is thin.

Def (K '14): Let $S \subset \mathbb{Z}^N$. Let

$V = Zcl(S) := \{z : f(z) = 0, \forall f \in \{f \in \mathbb{C}[x_1, \dots, x_N] : f(s) = 0, \forall s \in S\}\}$

S is thin iff: $\boxed{\frac{\# S \cap B_X}{\# V(\mathbb{Z}) \cap B_X} \to 0}$

E.g.: $Zcl(\Gamma_A) = SL_2$, i.e. if $f(\Gamma_A) = 0$,

then $f(\gamma) = (\det(\gamma) - 1) \cdot f_1(\gamma)$, and

$\# \Gamma_A \cap B_X \asymp \# R_A(X) = X^{2\delta_A + o(1)}$ $\boxed{\Rightarrow \frac{\# \Gamma_A \cap B_X}{\# SL_2(\mathbb{Z}) \cap B_X} \to 0}$,

$\# SL_2(\mathbb{Z}) \cap B_X = \# \{ \underbrace{|a|, |b|, |c|, |d| < X}_{X^4 \text{ pos}} : \underbrace{ad - bc = 1}_{P = \frac{1}{X^2}} \} = X^{2 + o(1)}$.

Slight detour: Hensley conjecture & more:

$R_A = \left\{ \frac{b}{d} = [0, a_1, \ldots, a_\ell] \;\middle|\; a_j \leq A \right\}$, Let $\mathcal{A} \subseteq \mathbb{N}$ "alphabet".

Let $R_{\mathcal{A}} = \{ \text{- - - -} \mid a_j \in \mathcal{A} \}$.

$\delta_{\mathcal{A}} = C_{\mathcal{A}} = \Lambda(R_{\mathcal{A}})$

$\#R_{\mathcal{A}}(N) = N^{2\delta_{\mathcal{A}} + o(1)}$

$> 1/2$.

Hensley Conj ('96): If $\delta_{\mathcal{A}} > 1/2 \Rightarrow D_{\mathcal{A}} \supseteq \mathbb{N}_{\geq 1}$.

Lemma: Let $\mathcal{A} = \{2, 4, 6, 8, 10\}$. Then $\delta_{\mathcal{A}} = 0.517\ldots > 1/2$,

But $D_{\mathcal{A}} \bmod 4 = \{0, 1, 2\}$ ← "local" obstruction

Given $\mathcal{A}$, call d admissible if $\forall q \geq 1$, $d \in D_{\mathcal{A}} \bmod q$,

Conj (B-K): If $\mathcal{A}$ has $\delta_{\mathcal{A}} > 1/2$, then every suff large, admissible $d \in D_{\mathcal{A}}$.

Took difficult, but is not. Determining local conditions is easy. $\Gamma_A \subset SL_2(\mathbb{Z})$. Z-dense ($|A| \geq 2$).

<u>Strong Approx of Z-dense Semisimple groups:</u>

If $\Gamma < SL_2(\mathbb{Z})$, $Zcl(\Gamma) = SL_2$. $\Longrightarrow$ $\Gamma \bmod p = SL_2(\mathbb{Z}/p)$, for $p \gg_\Gamma 1$!

"SL_2" is semisimple "GL_2" only <u>reductive</u>.

Strong Approx is <u>false</u> for reductive groups:

<u>E.g.:</u> $GL_2(\mathbb{Z}) < GL_2(\mathbb{Z})$. $Zcl(GL_2(\mathbb{Z})) = GL_2$.

$GL_2(\mathbb{Z}) \bmod 5$ = all matrices in $\mathbb{Z}/5\mathbb{Z}$ with $\det = \pm 1$.

$\subsetneq$

Compare to: $GL_2(\mathbb{Z}/5)$ = all matrices in $\mathbb{Z}/5\mathbb{Z}$ with $\det \in (\mathbb{Z}/5)^\times$.

$\boxed{\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \text{ vs } \begin{pmatrix} \sqrt{4} & 0 \\ 0 & 1/\sqrt{4} \end{pmatrix}.}$

But $\Gamma_A \subset SL_2(\mathbb{Z})$. Can SA apply?

Zcl thin group

$$T_A \bmod q \underset{\text{sub semigp}}{\subset} SL_2(\mathbb{Z}/q\mathbb{Z}).$$

$// \ \langle T_A \rangle \bmod q.$

Pick up inverses for free (Fermat/Euler).

So to understand $T_A \bmod q$, only need to know $\langle T_A \rangle < SL_2(\mathbb{Z})$.

Exercise

But already for $A = \{1,2\}$, $= SL_2(\mathbb{Z})$.

Hensley: Maximal alphabets having local obstruction

$$\delta\{\underline{2,4,6,8,10,}\dots\} = 0.70\dots$$

$$\delta\{\underline{1,9,17,25},\overset{???}{\dots}\} = 0.56\dots$$

Modified Hensley/BK Conj: If $\delta > \max \Rightarrow \mathcal{D}_A \supset \mathbb{N}_{\gg 1}$.

Thm (B-K): If $\delta_A > .99$ $(\delta_A > .80)$, $\frac{\#\mathcal{D}_A \cap [1,N]}{N} \to 1$.

"Circle method"