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Mechanism of folding a strip into isotetrahedra or rectangle dihedra

Kiyoko Matsunaga

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Abstract. Any net of an isotetrahedron I (a tetrahedron with all congruent four faces) and a rectangle dihedra RD satisfies the Conway criterion. Does the converse proposition hold? If so, are there practical algorithms for it? The difficult part of the proof of it comes down to the following problem [5], [6]: Find the practical algorithm for folding a parallelogramic strip into I or RD. The process of how to cover a thin rectangular board by a long tape without making gaps or overlaps has been known as a folklore among natives in various places globally [1]. By generalizing the known folklore foldings, this paper gives the practical algorithms to obtain all isotetrahedra and rectangle dihedra into which a strip can be folded. As a result of it, it is also proved that there is no other way to fold a given strip into rectangle dihedra other than the known two types of folklore foldings.

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§1. Introduction

A polyhedron or a dihedron (or a doubly covered polygon) is called a tile maker if every unfolding of P tiles the plane. [2] proved that tile makers belong to the following five families: (a) isotetrahedra (b) rectangle dihedra (c) equilateral triangle dihedra (d) isosceles right triangle dihedra (e) half equilateral triangle dihedra. Especially for isotetrahedra and rectangle dihedra, the following theorem was obtained in [3]:

Theorem 1 ([3]). *Every unfolding of an isotetrahedron or a rectangle dihedron satisfies the Conway criterion.*

The Conway criterion which is used throughout the paper is the following.

Definition 1 (Conway criterion [12])**.** A given region (figure) can tile the plane using only translations and 180° rotations if its perimeter can be divided into six parts by six consecutive points A, B, C, D, E and F, all located on its perimeter, such that:

(a) The perimeter part AB is congruent to the perimeter part ED by translation τ in which

$$\tau(A) = E, \quad \tau(B) = D.$$

(b) Each of the perimeter parts BC, CD, EF and FA is centrosymmetric, i.e., each part coincides with itself when the region (figure) is rotated by 180° around its midpoint.

(c) Some of the six points may coincide but at least three of them must be distinct.

A region satisfying the Conway criterion is called a *Conway tile* (see Fig.1 for an example). It is natural to consider the converse of Theorem 1, which is

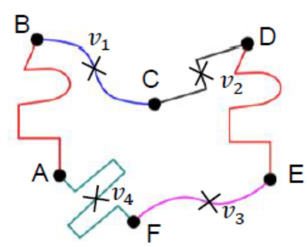

Fig. 1: A Conway tile. X stands for a midpoint.

the following problem:

Problem 1. *Can every Conway tile be folded into either an isotetrahedron or a rectangle dihedron?*

This problem belongs to the theory of foldings. Since the theory of foldings has a wide range of applications in other fields, this field is actively studied [8]. In mathematics, engineering, the arts and everyday life, a variety of folding problems has appeared, e.g. mathematics of origami, computational origami, the fold and one cut problem, the Miura Map Fold, folding a given polygon into polyhedra, flattening polyhedra, designing pleated origami, et. al. The theory of folding has been receiving a great deal of interest, since it was introduced in the books "*Geometric Folding Algorithms*" by E. D. Demaine and J. O'Rourke [10], and "*How to fold it*" by J. O'Rourke [11]. The topic *Folding*

Polygons to Polyhedra is taken up in a few sections of both [10] and [11]. They mention that

1. A polygon can be folded into a convex polyhedron if it has an Alexandrov gluing.

2. While Alexandrov's theorem guarantees unique existence of a polyhedron into which a polygon with Alexandrov gluing can be folded, there is no known practical algorithm for reconstructing the 3D shape of the polyhedron. In general, it is not easy to characterize the 3D shape of the polyhedron into which a polygon can be folded.

A Conway tile is a considerably extensive set of plane regions which includes arbitrary triangles, arbitrary quadrangles, pentagons with at least a pair of parallel parts, hexagons with at least a pair of parallel sides, some non-convex polygons and some plane regions with curved lines. Therefore, finding a practical algorithm for folding Conway tiles into a polyhedron or a dihedron makes sense.

For a Conway tile N, a **4-base of** N is defined as a set of four midpoints of centrosymmetric parts of N under the assumption that the midpoint of a centrosymmetric part XY is X $(= Y)$ if X coincides with Y. Thus, there exists a 4-base for any Conway tile N. Notice that a Conway tile may have many different 4-bases. Fig.2 shows three different 4-bases of a Latin cross which is a Conway tile.

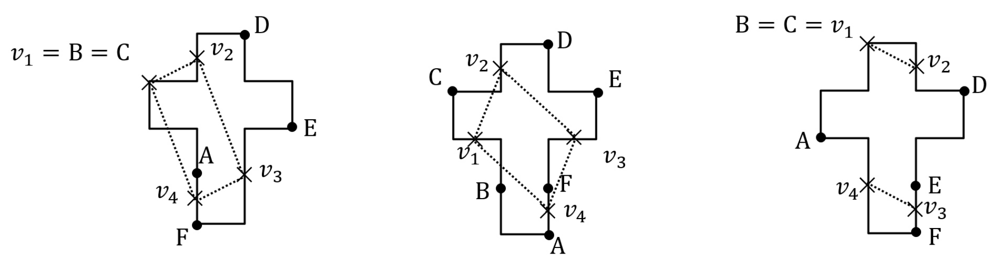

Fig. 2: Three different 4-bases $\{ v_1, v_2, v_3, v_4\}$ of a Latin cross.

Theorem 2 ([4]). *Let N be a Conway tile with its 4-base v_1, v_2, v_3 and v_4. Then, these four points form a parallelogram.*

The four points in the 4-base of a Conway tile N play an important role when N is folded into an isotetrahedron or a rectangle dihedron.

By Alexandrov's theorem and the characteristic of Conway tiles, every Conway tile is guaranteed to be folded into an isotetrahedron or a rectangle dihedron as shown in the following theorem.

Theorem 3 ([6]). *Every Conway tile is foldable into either an isotetrahedron or a rectangle dihedron whose vertices are four points of its 4-base.*

Proof. Let N be an arbitrary Conway tile. Let A, B, C, D, E and F be the six consecutive points on the perimeter of N, which satisfy the conditions of the Conway criterion. A perimeter of N consists of at most four centrosymmetric pairs BC, CD, EF and FA of the perimeter parts with their midpoints v_1, v_2, v_3 and v_4, respectively, and at most one pair of parallel-congruent perimeter parts AB and ED. Glue together parts of the perimeter of a Conway tile N such that the two halves of each centrosymmetric edge are glued to each other, and the two congruent edges related by a translation are glued to each other. The gluing result is a topological sphere. Furthermore, the gluing result has just four points, v_1, v_2, v_3 and v_4, where the sum of the face angles is 180°. The sum of the face angles of the other remaining points is 360°. By Alexandrov's theorem (see details at [10] or [11]), N is folded into either a polyhedron or a dihedron whose vertices are v_1, v_2, v_3 and v_4. That is, N is folded into either an isotetrahedron or a rectangle dihedron whose four vertices are four points of the 4-base of a Conway tile N. □

Let the gluing of the perimeter of a Conway tile N be called *Conway gluing* when parts of perimeter of N are glued such that the two halves of each centrosymmetric edge are glued to each other, and the two congruent edges related by a translation are glued to each other. Conway gluing is a special case of Alexandrov gluing.

In [9], [10], and [11], all convex polyhedra and dihedra are determined into which a square is folded. Fig.3 illustrates four cases of them where a square is folded into an isotetrahedron or a rectangle dihedron with vertices v_1, v_2, v_3 and v_4. In all these cases, the perimeter of a square is glued by Conway gluing, where six points A, B, C, D, E and F on the perimeter of a square satisfies the Conway criterion. It is easy to fold a Conway tile into an isotetrahedron I or a rectangle dihedron RD if its 4-base parallelogram can be divided into two identical acute or right triangles by its diagonal as shown in the following Lemma 1 [4], [5], [6].

Lemma 1. *Suppose that a 4-base parallelogram of a Conway tile N is divided into two acute triangles T_As or two right triangles T_Rs by its diagonal. Then, N can be folded into an istotetrahedron whose faces are congruent to T_A* (Fig.4 (a)) *or a rectangle dihedron whose faces are $T_R \cup T_R$* (Fig.4 (b)).

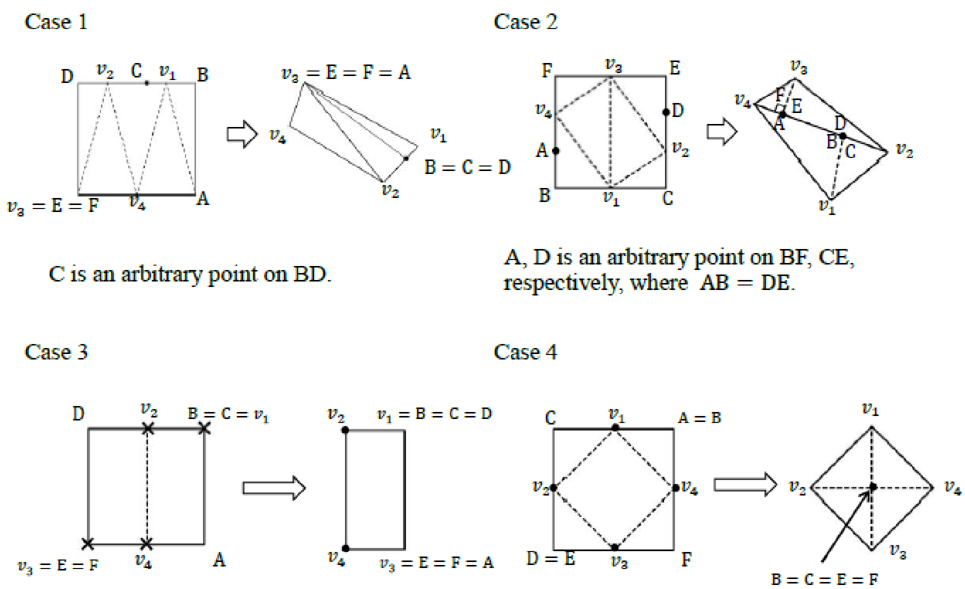

Fig. 3: Four different Conway gluing results of a square.

However, it becomes difficult to fold a Conway tile N into I or RD whose 4-base parallelogram is thin and long, and can not be divided by its diagonal into two identical acute or right triangles. Folding such a Conway tile N comes down to thinking about how to fold a parallelogram strip S, which is reversible to N around the 4-base of N [4], [5], [6], [7]. A strip S is a Conway tile with many different 4-bases. Therefore the following problem arises: Find practical algorithms for folding a strip into an isotetrahedron or a rectangle dihedron whose vertices are arbitrary 4-base of the strip.

§2. Known folding a long strip into a rectangle dihedron.

Folk arts in various countries have presented rectangle dihedra (we denote a rectangle dihedron by RD throughout the paper) by folding a parallelogramic strip. Two ways of such foldings are known; Fig.5 and Fig.6 illustrate the procedure of how a long strip is folded into a rectangle dihedron [1]. These methods of folding are called a *folklore folding 1(FF1)*, a *folklore folding 2 (FF2)*, respectively [5], [6], [7]. One of differences between them is the positions of the left and right sides (i.e., AB and CD) of the strip in rectangle dihedra. In *FF1*, CD (blue) is attached to AB (red), but not in $FF2$ as shown in Fig.5 and Fig.6.

In [5], decomposing a parallelogramic strip S into right triangles brings the

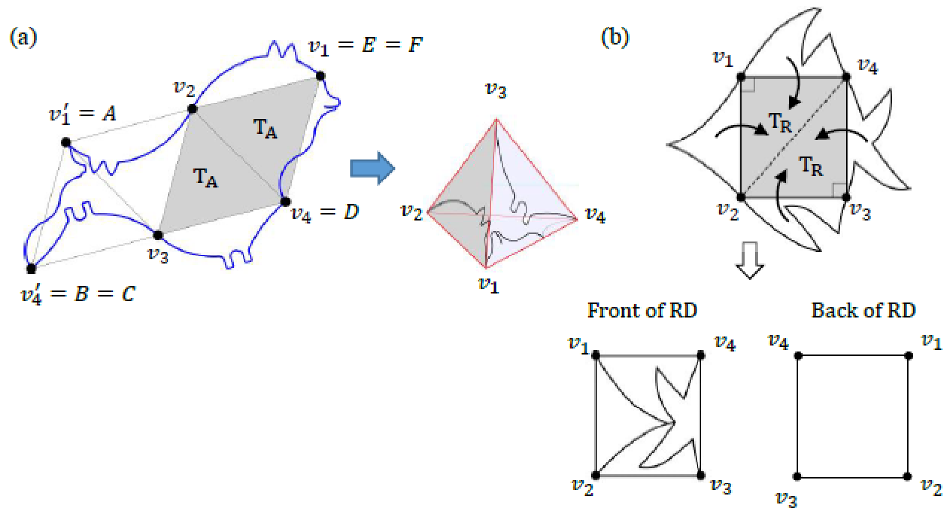

Fig. 4: Conway tiles whose 4-base parallelogram is divided into two non-obtuse triangles.

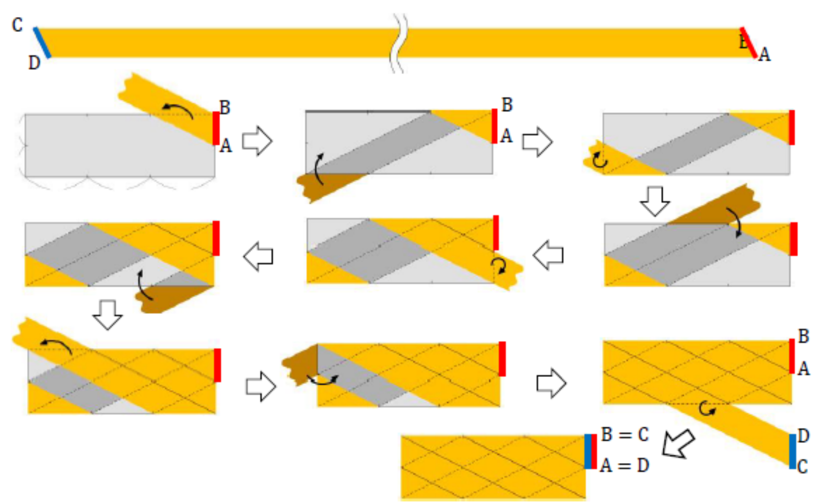

Fig. 5: Example of the procedure of the folklore folding 1 (*FF1*).

mechanism of *FF1* and *FF2* into sharp relief.

Lemma 2 ([5]). *A parallelogramic strip S can be folded into RD by FF1 if S can be decomposed into $4n\,(n+1)$ identical right triangle Ts, as shown in Fig.7. Four corners $v_1,\ v_2, v_3$ and v_4 of RD are vertices with the 90° angle in the triangle labeled $1,\ 2\,(n+1)\,,\ 2n\,(n+1)+1$ and $2(n+1)^2$, where all right*

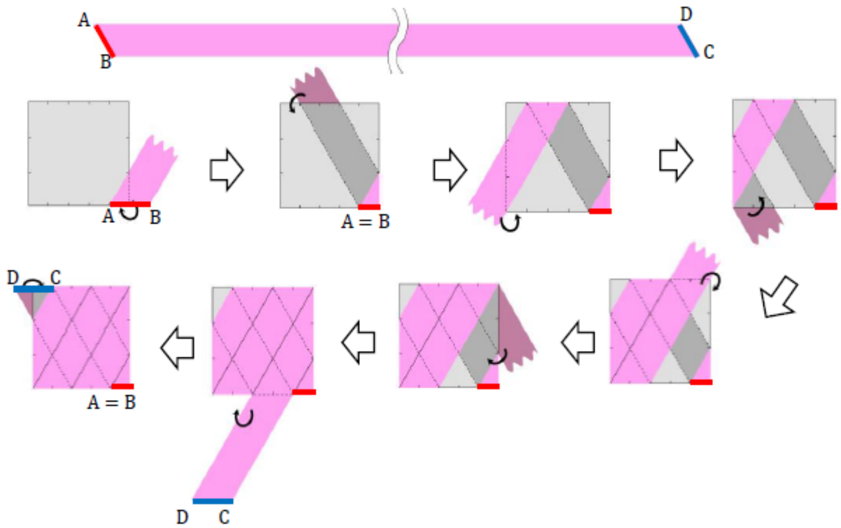

Fig. 6: Example of the procedure of the folklore folding 2 (*FF2*).

triangles in S are labeled with 1 *to* $4n\,(n+1)$ *from right to left.*

Proof. Following the procedure illustrated in Fig.8, S can be folded into RD with corners v_1, v_2, v_3 and v_4. Fig.8 shows the case when $n = 3$, but the same procedure works for every integer n. Note that in the case of $n = 3, v_1, v_2, v_3, v_4$ are the vertices with the $90°$ angle in the triangles labeled 1, $2 \times (3+1) = 8$, $2 \times 3 \times (3+1) + 1 = 25$, $2(3+1)^2 = 32$, respectively.

Fold lines of S to make RD by *FF1* are determined as shown below. Fig.9 shows the case for $n = 3$. First, let us direct our attention to the right triangle T with label $2\,(n+1)$ with the right angle v_2. Let s_a, s_b be the sides of T incident to v_2 with the length a, b, respectively. Let l_a, l_b be the lines containing s_a, s_b, respectively. On the side of S containing v_2, take points a_k, b_m whose distances from v_2 are $k \cdot (n+1)d$ $(k = 1, 2, \cdots)$, $m \cdot nd$ $(m = 1, 2, \cdots)$, respectively, where $d = l/2n(n+1)$ (in Fig.9, $d = l/24$). Draw lines through a_k, b_m parallel to l_a, l_b, respectively. These lines are fold lines (drawn in blue lines in Fig.9). □

Remark 1. The folding in Fig.5 is the case of *FF1* for $n = 2$ as shown in Fig.10. In *FF1*, we observe that the first triangle with label 1 is attached to the last triangle with label $4n(n+1)$ (= 24 in Fig.10 (a), (b), (c)); therefore, it makes a ring. Consequently, any parallelogram shape with the same length is folded into the identical RD by *FF1* procedure. Thus, in the case of *FF1*, it is sufficient to consider only a rectangle strip $ABCD$ instead of other parallelogramic strips.

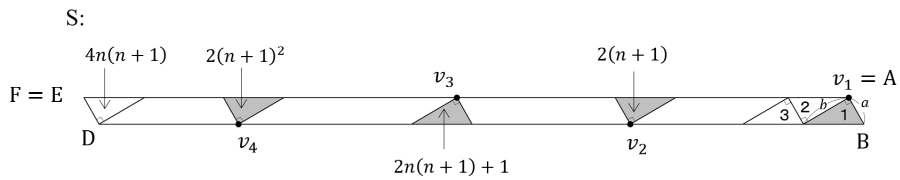

Fig. 7: The strip S, which consists of $4n(n+1)$ right triangles, is folded into RD with the size $na \times (n+1)b$.

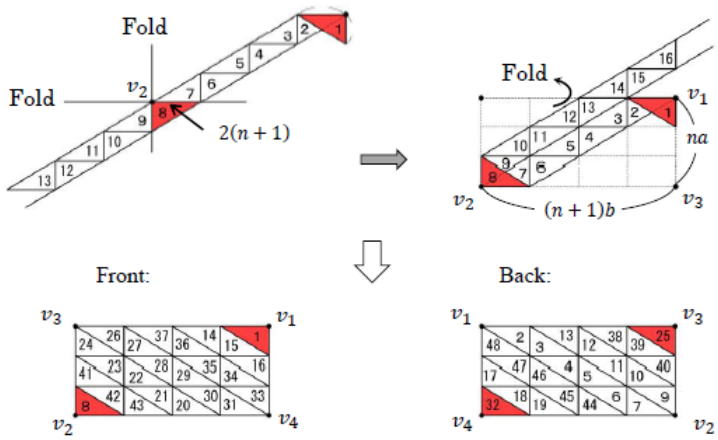

Fig. 8: How to fold S into RD by *FF1* ($n = 3$)

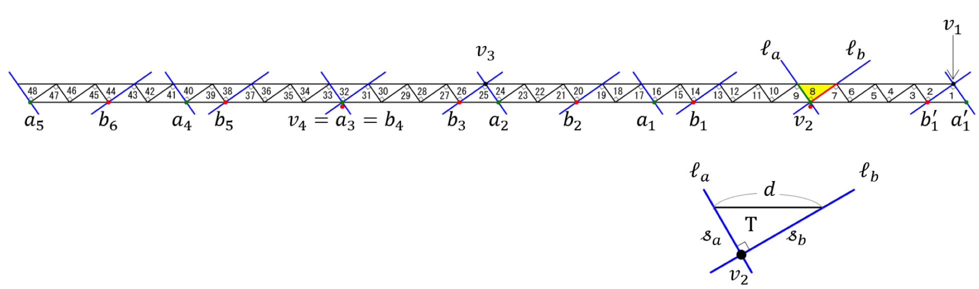

Fig. 9: Fold lines of *FF1* for $\boldsymbol{n = 3}$

Lemma 3 ([5])**.** *A parallelogramic strip S can be folded into RD by FF2 if S can be decomposed into* $4(2n-1)\ (2n+1)$ *identical right triangles* ($n = 2,\ 3,\ \ldots$)*, as shown in Fig.11. Four corners of RD are two midpoints of sides of S* (v_1 *and* v_3) *and the intersection points of pairs of triangles labeled "*$4n\ (2n-1)+1$ *and* $4n\ (2n-1)-3$*" and "*$4n(2n+1)$ *and* $4n\ (2n+1)-4$*".*

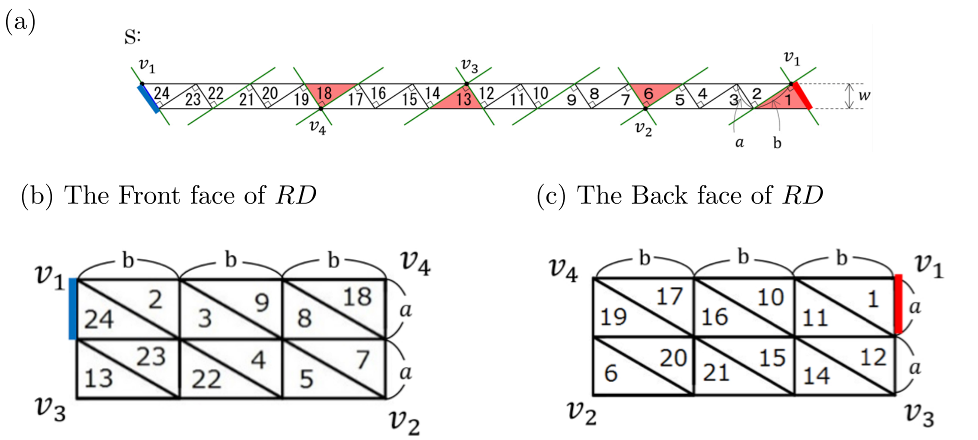

Fig. 10: The strip S, which consists of 24 right triangles, is folded into RD with the size $2a \times 3b$ by *FF1*.

Proof. Following the procedure illustrated in Fig.12, S can be folded into an RD with corners v_1, v_2, v_3 and v_4. Fig.12 shows the case when $n = 3$, but the same procedure works for any integer $n \geq 2$. Note that $4n\ (2n-1)+1$ or$-3 =$ 57, 61 and $4n\ (2n+1)+0$ or $-\ 4 = 80,\ 84$, when $n = 3$.

Fold lines to make S into RD by *FF2* are determined as follows (see Fig.13 which shows the case for $n{=}\ 3$): Draw lines v_1Q_1 and $v_4P_{(2n-1)(2n+1)-1}$. Draw lines $v_2P_{n(2n-1)-2}$ (say ℓ_a) and $v_2P_{n(2n-1)}$ (say ℓ_b) where $Q_{n(2n-1)} = v_2$. Take points A_i, B$_j$ with the distance $i{\cdot}(2n+1)\,d$, $j{\cdot}(2n-1)\ d\ (i,\ j = 1,\ 2,\ 3,\ \cdots)$ from $v_2\left(= Q_{n(2n-1)}\right)$ on ML, respectively. Draw lines a_i and $a_i^{'}, b_j$ and $b_j^{'}$ through A_i, B_j parallel to $\ell_a,\ \ell_b$, respectively. These lines are fold lines (drawn in orange lines). □

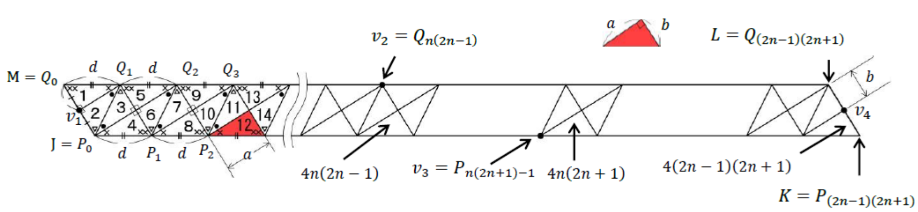

Fig. 11: S is decomposed into $4(2n-1)(2n+1)$ right triangles

Remark 2. The strip S in Fig.6 is the case of *FF2* for $n = 2$ as shown in Fig.14.

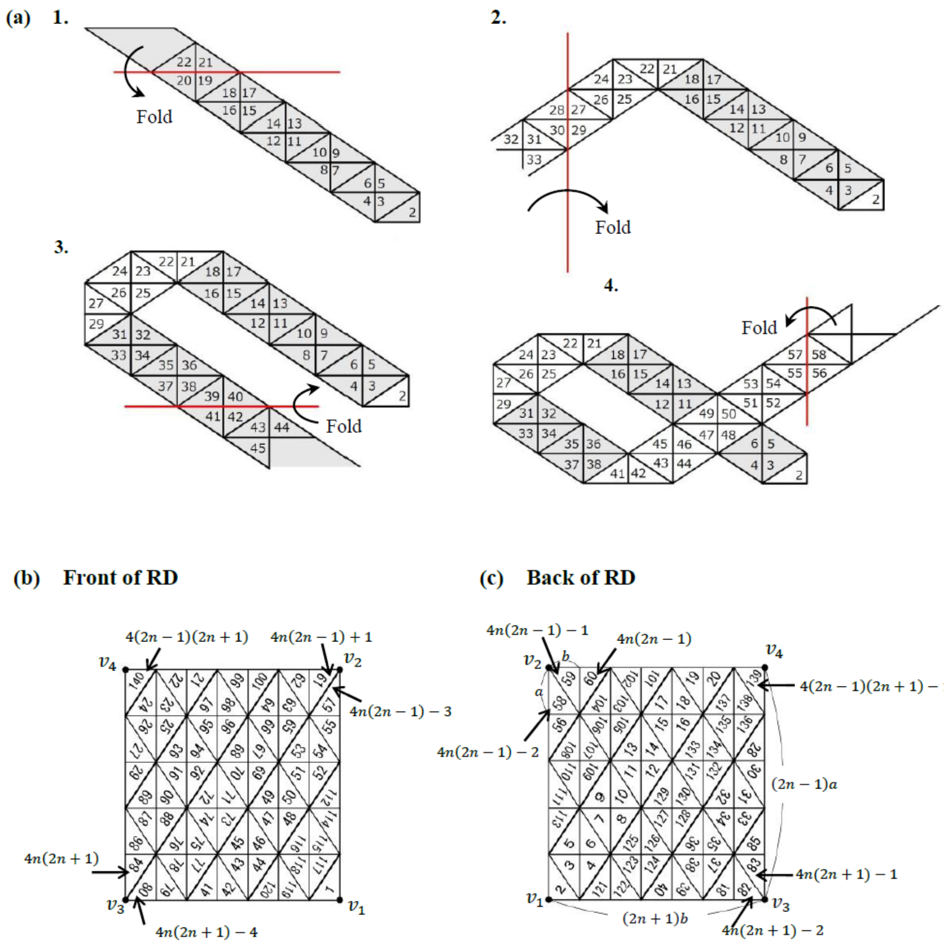

Fig. 12: How to fold S into RD. One side of S is shaded.

§3. Folding a rectangle strip into isotetrahedra or RD.

In section 3, we determine all isotetrahedra and RD obtained from one rectangle strip. Although we deal with only rectangle strips in the following theorems, similar results hold for any other parallelogramic strips.

Lemma 4. *For a given rectangle strip S with the size $l \times w$, there are six essentially different locations of consecutive six points A, B, C, D, E and F on the perimeter of S satisfying the Conway criterion.*

Proof. Depending on whether a parallel-congruent pair exists or not and where such a pair locates, S can be divided into the following six cases:

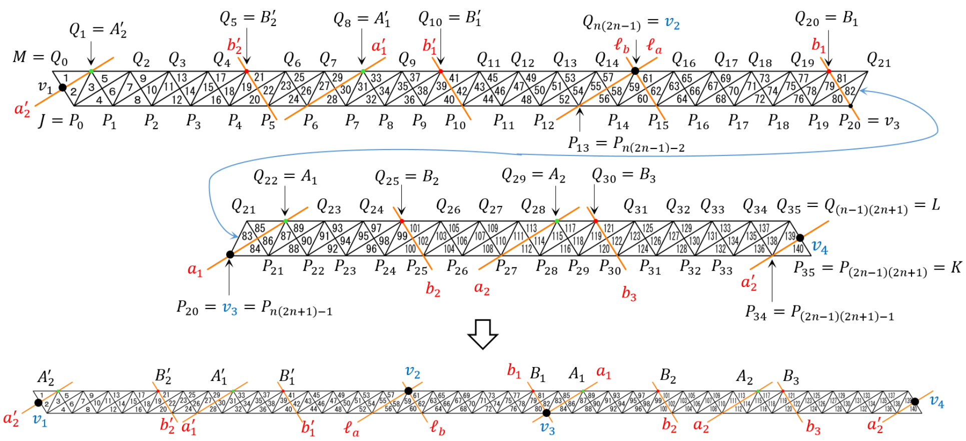

Fig. 13: Fold lines of *FF2* for $n = 3$.

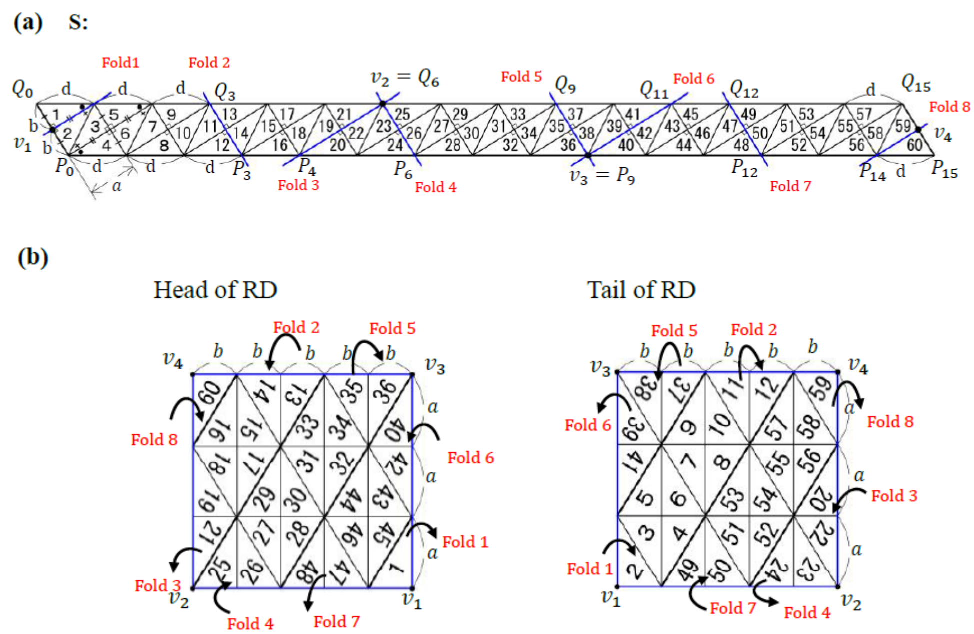

Fig. 14: The strip S, which consists of 60 right triangles, is folded into RD with size $3a \times 5b$ by *FF2*.

Case I. S has no pair of parallel-congruent parts (i.e., the case where $A = B$ and $D = E$ in Conway criterion) as shown in Fig.15 (a).

Case II. Vertical sides of S are a pair of parallel-congruent parts and each of horizontal sides is a centrosymmetric part (i.e. $B = C$ and $E = F$ in the

Conway criterion) as shown in Fig.15 (b).
Case III. Vertical sides of S are a pair of parallel-congruent parts and at least one of horizontal sides has two centrosymmetric parts as shown in Fig.15 (c) and (d). By gluing a pair of parallel-congruent parts (i.e., AB and ED), S becomes a ring. Therefore, the case of Fig.15 (d) comes down to the case in Fig.15 (c) when we consider the Conway gluing for S.
Case IV. Parts AB and ED of vertical sides of S are a parallel-congruent pair as shown in Fig.15 (e).
Case V. Horizontal sides AB and ED of S are a parallel-congruent pair as shown in Fig.15 (f) .
Case VI. Parts AB and ED of horizontal sides of S are a parallel-congruent pair as shown in Fig.15 (g).

□

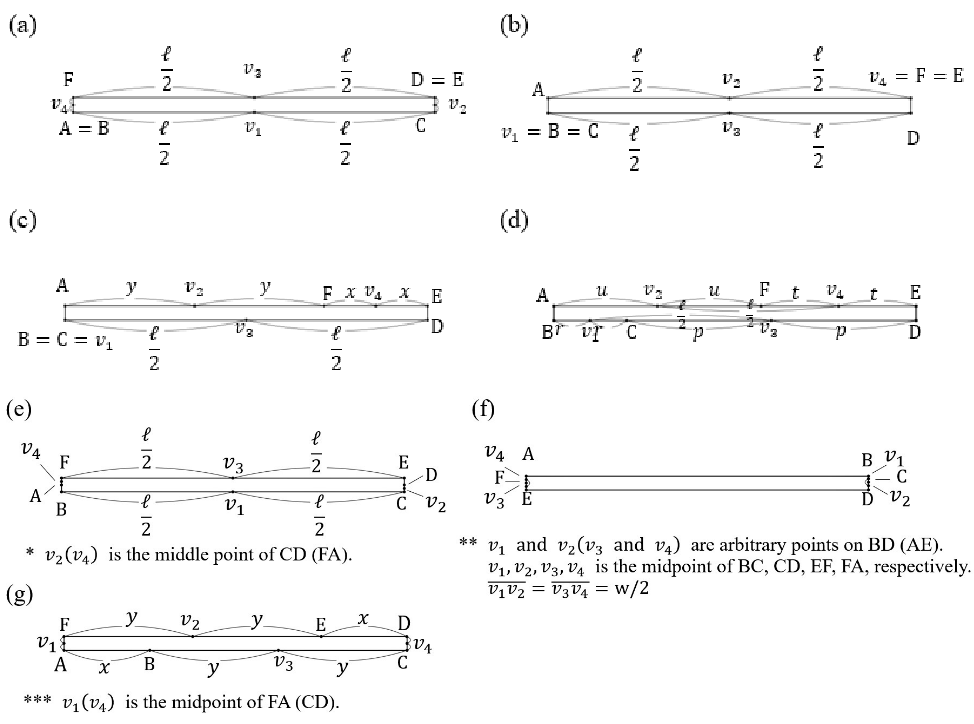

Fig. 15: Six essentially different locations of consecutive six points A, B, C, D, E and F.

Next we determine the accurate shapes of resultant isotetrahedra I or rectangle dihedra RD which is foldable from a given rectangle strip S. From Theorem 3, four vertices of I or RD must be four points of a 4-base of S.

That is, all we need to do is to find I or RD for all cases (Case I∼Case VI in Lemma 4) concerning the location of 4-base on the perimeter of S. Resultant I and RD can be divided into two types; one of which is easy to fold by Lemma 1 (let them be called *single type*), and the other is hard to fold. First, we determine all single type I or RD in Theorem 4. The rest of them (i.e., difficult cases) is solved in Theorem 5, 6, 7 and 8.

Theorem 4. *Let S be a rectangle strip with size $l \times w$. There are four different single types isotetrahedra and four single types of rectangle dihedra into which S is fodable into.*

Proof. In the following six cases of Lemma 4 (Fig.16∼19), all single types of isotetrahedra and rectangle dihedra are obtained from a rectangle strip.

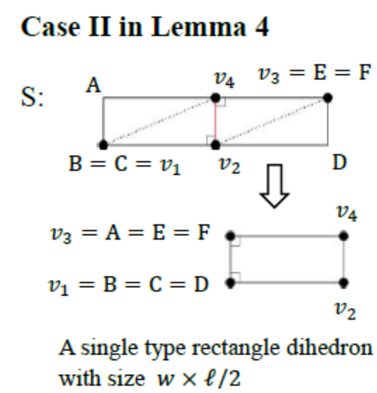

Fig. 16: Case II

CaseV-1 in Lemma 4 (Case V when at least one of $v_i v_j$ is parallel to AB)

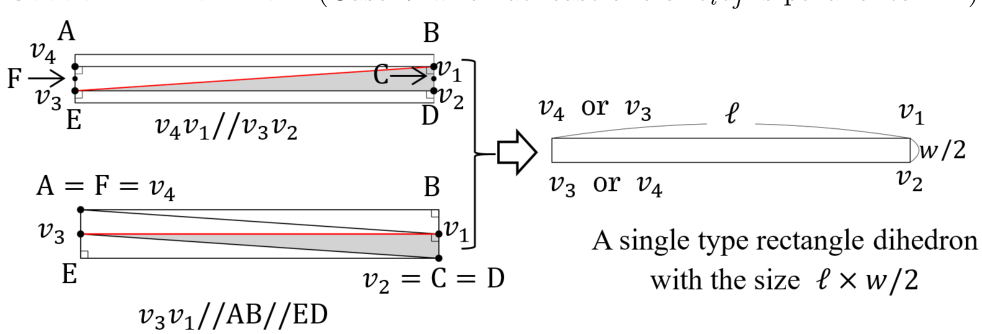

Fig. 17: Case V-1

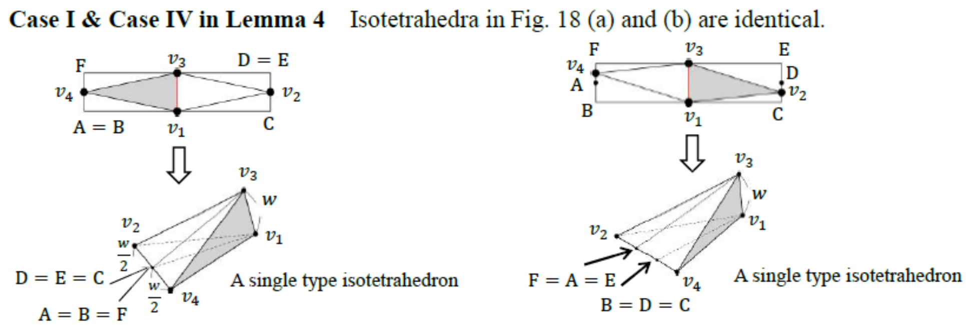

Fig. 18: Case I and Case IV

CaseV-2 (Case V when none of v_iv_j is parallel to AB)

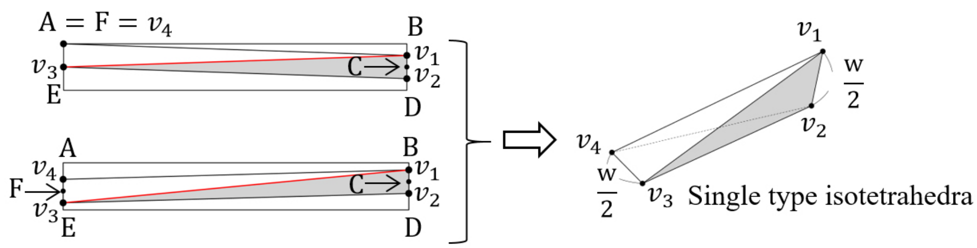

Fig. 19: Case V-2

Case III in Lemma 4 We divide this case into two subcases when $\ell < 4w$ or not.

Case III-1 **($l < 4w$)** For any location of F on AE, the 4-base-parallelogram $v_2v_1v_3v_4$ can be divided into two identical congruent acute triangles by its diagonal v_2v_3(Fig.20 (a)). Then, a strip S can be folded into various non-similar single type isotetrahedra each of whose faces is congruent to T_i, where $T_i(i = 1, 2, 3$ and $4)$ changes its shape depending on the location of F on AE (Fig.20 (b)).

Case III-2 **($\ell \geq 4w$)** Draw the semicircle $\mathfrak{C}$ with diameter v_1v_3 (Fig.21 (a)). We denote the intersection of AE and $\mathfrak{C}$ by G and H, respectively as shown in Fig.21(a). If $G = v_2$, a single type rectangle dihedron is obtained (Fig.21 (c)). If $Av_2 < AG$ or $AH < Av_2 < \ell/2$, then a strip S is folded into a single type isotetrahedron (Fig.21 (b)).

Case VI in Lemma 4 We divide this case into two subcases when $\ell < 4w$ or not.

Case VI-1 ($\ell \leq 4w$) For any location of E on FD (and B on AC), a 4-base parallelogram $v_1v_2v_4v_3$ is divided into two acute triangles by its diagonal (v_2v_3

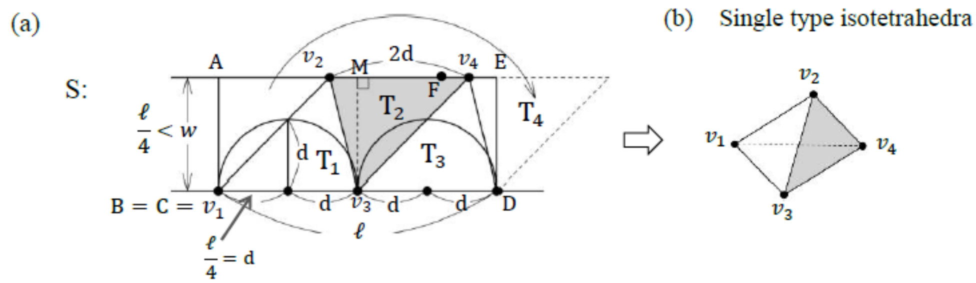

Fig. 20: Case III-1

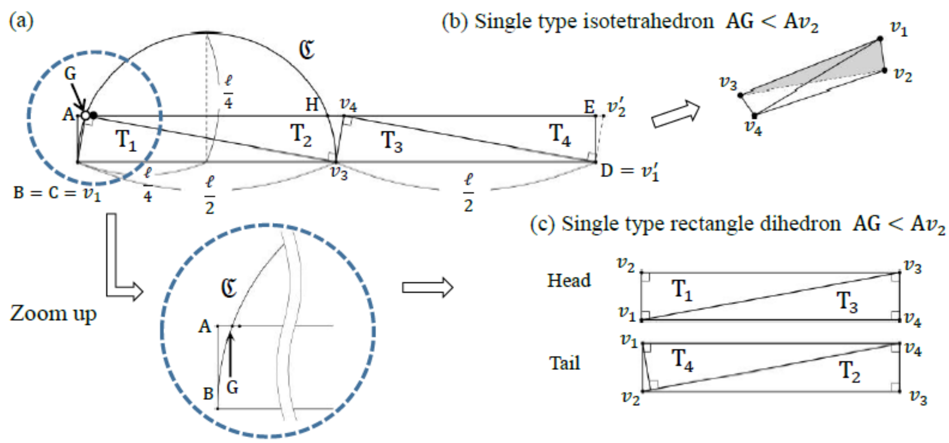

Fig. 21: Case III-2

or v_1v_4, Fig.22 (a)). Then, S is folded into a single type isotetrahedron (Fig.22 (b)).

Case VI-2 ($\ell > 4w$) Let M be the center of S (Fig.23 (a)). If v_2 is the intersection G or H of AE and the semicircle $\mathfrak{C}$ with the diameter v_1M, then S is folded into a single type RD (Fig.23 (c)). If v_2 is the intersection I or J of FD and the circle $\mathfrak{C}'$ with the diameter Mv_4, S is also folded into a single type RD.

If E is chosen on FD such that $Fv_2 < FG$ or $FH < Fv_2 < \ell/2$, a 4-base parallelogram is divided into two identical acute triangles by its diagonal. Thus, S is folded into a single type isotetrahedron (Fig.23 (b)). □

We have already dealt with rectangle strips in Case I, II, IV, V, III-1, III-2 and VI-1, VI-2 in Theorem 4. Henceforth, we consider two cases which are not dealt with:

1. Let III-3 be a subcase of the Case III with the additional condition

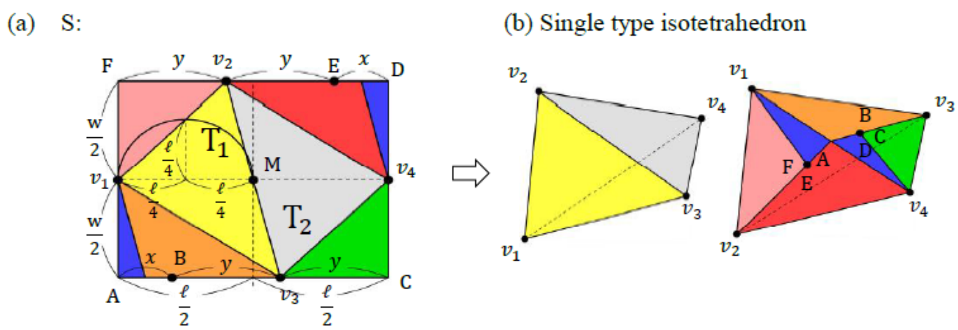

Fig. 22: Case VI-1

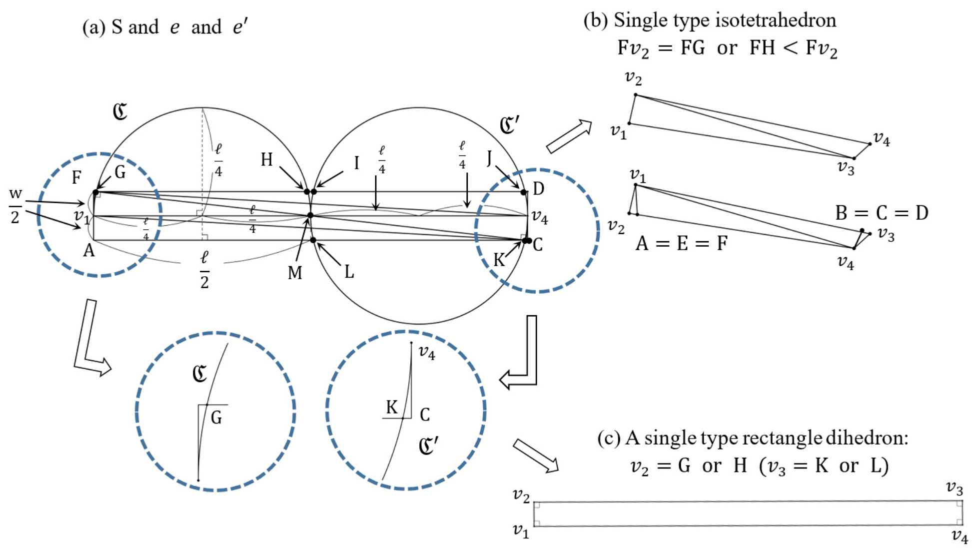

Fig. 23: Case VI-2

$AG < Av_2 < AH$, where G and H are the intersections of AE and $\mathfrak{C}$ as shown in Fig.21(a).

2. Let VI-3 be a subcase of Case VI with the additional conditions $4w \leqq \ell$ and $FG < Fv_2 < FH$ or $FI < Fv_2 < FJ$, where G and H (I and J) are the intersections of AE and $\mathfrak{C}$ ($\mathfrak{C}'$) as shown in Fig.23.

In Case III-3, it is sufficient to consider the case $AG < Av_2 < \ell/4$ because of the symmetry of v_2 and v_4 (Fig.24). Let a rectangle strip S with points $A, B, C, \cdots, F, v_1, v_2, v_3$ and v_4 as shown in Fig.24 be a *III-3 strip S*.

Observation 1. On applying Conway gluing to a III-3 strip, it automatically coils proper n times to make a topological sphere TS with four vertices

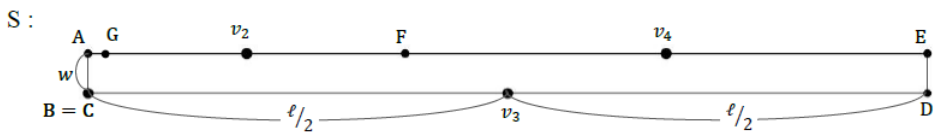

Fig. 24: A III-3 strip S ($AG < v_2 < \ell/4$).

v_1, v_2, v_3 and v_4 on the condition that the length of S is fixed and the location of v_2 and the width of S vary as shown in the last but one of Fig.25 (a), (b) and (c), respectively. Recrease TS into the other one (the last one of Fig.25 (a), (b),(c)). S coils S coils like $FF1$ for $3, 2, 5$ as shown in the last one of Fig. 25 (a), (b), (c), respectively. These topological sphere TS with proper $n + (n + 1)$-coils compose of two parallelograms each of which can be divided into two acute or right triangles. Proper n ($\in \mathbb{N}$) is uniquely determined for each of III-3 strips.

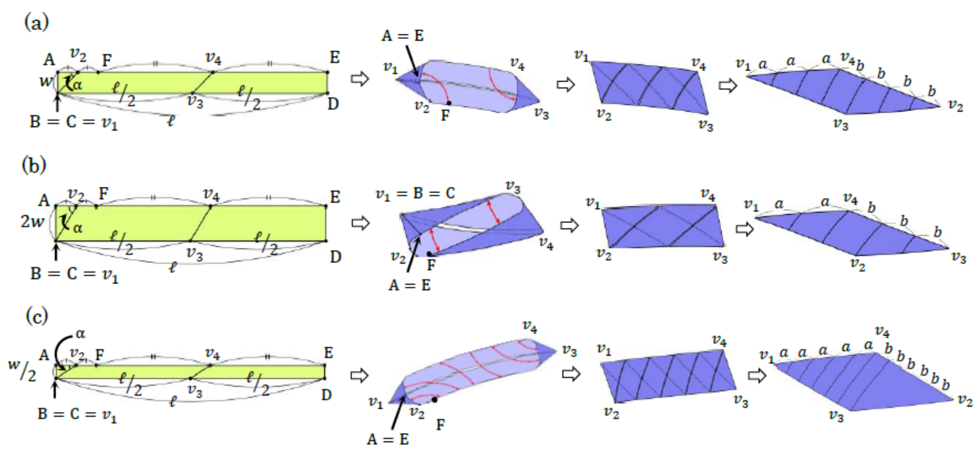

Fig. 25: On applying Conway-gluing to III-3 strips

Proper n for a given III-3 strip S can be calculated, taking into account the length of v_1v_2 and the length v_2v_4. Therefore, the following Theorem 5 holds.

Theorem 5. *A III-3 strip S coils $n = \lceil a\ell/2\left(a^2 + w^2\right)\rceil$ times to make a topological sphere on applying Conway-gluing to it, where a, ℓ and w are the length of Av_2 , the length of S and the width of S, respectively.*

By using the value of proper n for a given III-3 strip, we determine the practical algorithm for folding a III-3 strip S with arbitrarily chosen 4-base into an isotetrahedron or a rectangle dihedron in the following Theorems 6.

Theorem 6. *A III-3 strip S is foldable into isotetrahedra I or rectangle dihedra RD accoding to the practical algorithm which is generated by generalizing the procedure of FF1.*

Proof. First, determine the value of proper n for a given III-3 strip S using Theorem 5.
Second, decompose a III-3 strip S into $4n(n+1)$ triangles according to the following procedure:

1. Decompose the side BD into $2n(n+1)$ segments $P_0P_1, \cdots, P_{2n(n+1)-1}$ $P_{2n(n+1)}$ with the same length $d{=}\ell/2n\,(n+1)$, where $v_1 = B = C = P_0$, $D = P_{2n(n+1)}$ and $v_3 = P_{n(n+1)}$.

2. Decompose the side $v_2E \cup Av_2$ into $2n(n+1)$ segments, consecutive $Q_{n+1}Q_{n+2}, Q_{n+2}Q_{n+3}, \cdots, Q_{2n(n+1)}Q_1, Q_1Q_2, \cdots, Q_nQ_{n+1}$ where $v_2 = Q_{n+1}$ and $Q_{2n(n+1)}Q_1$ is the union of $Q_{2n(n+1)}E$ and AQ_1 (Fig.26). Notice that v_4 is automatically assigned to $Q_{(n+1)^2}$. Draw $P_{i-1}Q_i$ and P_iQ_i $(i = 1, \cdots, 2n(n+1))$, and then the strip is decomposed into $4n(n+1)$ identical triangles.

3. By $2i$ and $2i{+}1$, denote the triangles $P_{i-1}Q_iP_i$, and $Q_iP_iQ_{i+1}$ $(i = 1, 2, \cdots, 2n(n+1))$ where $P_{2n(n+1)} = P_0, Q_{2n(n+1)+1} = Q_1$, and $Q_{2n(n+1)}P_{2n(n+1)}$ $Q_{2n(n+1)+1}$ is the triangles with label 1, respectively, as shown in Fig.26 which is the case of $n = 2$.

Let a decomposed III-3 strip S above be called an *n-FF1 decomposed S*.

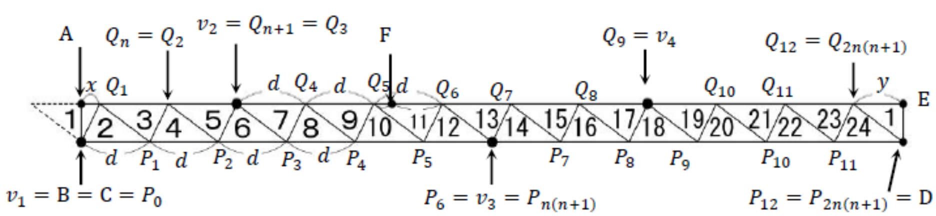

Fig. 26: An *n-FF1* decomposed S (In this figure the case of $n = 2$).

Third, fold an *n-FF1* decomposed S along the same fold lines of S to make RD by *FF1* which is shown in the proof of Lemma 2 and then TS like a doubly covered parallelogram (denotet it by DCP) is obtained (Fig.27). Notice that this way of decomposing and folding a III-3 strip is nothing else but applying Conway gluing to a III-3 strip (independent of values of n).
Recrease a parallelograms of the front (and back) of the DCP and each of them can be divided into two acute or right triangles by its diagonal (red lines

in Fig.27 (a)). Thus, it is folded into an isotetrahedron or a rectangle dihedron (Fig.27 (b). In this case, an isotetrahedron is obtained).
Infinitely many non-similar isotetrahedra I and rectangle dihedra RD are obtained out of a given rectangle strip, since shapes of a cell (triangle) and front and back of DCP vary as the location of v_2 moves on AE under the condition $AG < Av_2 \leq \ell/4$.
Therefore, the strip S in this Case III-3 is foldable into infinitely many isotetrahedra or rectangle dihedra by the above generalized *FF1* procedure. □

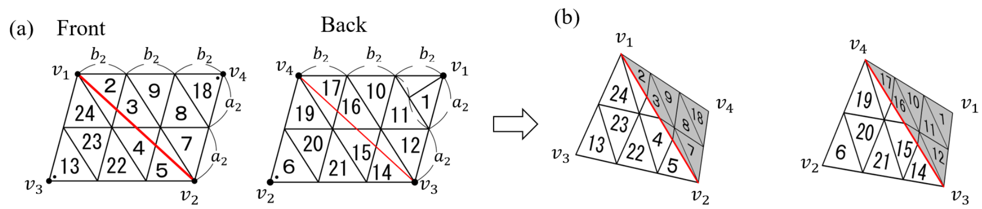

Fig. 27: A III-3 strip S in Fig.24 is folded into an isotetrahedron.

Last we consider the case VI-3. Let a rectangle strip S with points $A, B, C, \cdots, F, v_1, v_2, v_3$ and v_4 as shown in Fig.28 be a *VI-3 strip S*. That is, on the perimeter of a VI-3 strip S, consecutive six points $A, B, C, D, E, F,$ $v_1,\ v_2, v_3$ and v_4 locate such that $AB = ED, Fv_1 = Av_1 = Dv_4 = Cv_4, Bv_3 = Cv_3,\ Fv_2 = Ev_2$ and the angle $v_1v_2v_3 > 90°$.

Fig. 28: A VI-3 Strip S

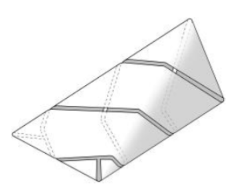

Fig. 29: On applying Conway gluing to a VI-3 Strip ($n = 3$ in this case)

Observation 2.
On applying the Conway gluing to a VI-3 strip, it automatically coils proper

n times to make a topological sphere with four vertices $v_1,\ v_2, v_3$ and v_4 as shown in Fig.29. In the same manner as a III-3 strip, a proper $n\,(\in\mathbb{N})$ is uniquely determined for each of VI-3 strips.

Proper n for a given VI-3 strip S can be calculated, taking into account the length and the width of S in the following Theorem 7.

Theorem 7. *A VI-3 strip S coils $n{=}\left\lfloor\left(x\ell+x^2+w^2\right)/2\left(x^2+w^2\right)\right\rfloor$ times to make a topological sphere on applying Conway gluing to it, where $x,\ \ \ell$ and w are the length of AB, the length and the width of S, respectively.*

By using the value of the proper n for a given VI-3 strip, we determine the practical algorithm for folding a VI-3 strip S with arbitrarily chosen 4-base into an isotetrahedron in the following Theorem 8.

Theorem 8. *A VI-3 strip S is foldable into isotetrahedra I according to the practical algorithm which is generated by generalizing the procedure of FF2.*

Proof. First, we determine the value of the proper n for a given VI-3 strip S using Theorem 7. Second, we define an *n-FF2 decomposed* S as a VI-3 strip S decomposed into $4\,(2n-1)(2n+1)$ triangles as follows:
We modify the process in which the perimeter S is divided by the *FF2* procedure in order to glue the perimeter parts AB and ED regardless of the location of B on AC (and E on DF). Divide $AB,\ ED$ into $2n-1$ segments with the same length d_1, respectively. Next, divide BC, EF into $2n(2n-1)$ segments with the same length d_2, respectively. (Note that v_2 is an arbitrary point such that $FG < Fv_2 < FH$, while $Fv_2 : ED = n : 1$ in the *FF2* procedure as shown in Fig.11 and Fig.14. Thus, in the *FF2* procedure, $d_1 = d_2$. But d_1 may not be equal to $d_2$in this case).
Lastly, draw $v_1Q_1, P_{i-1}Q_i, P_iQ_i, P_{i-1}Q_{i+1}, P_{(2n-1)(2n+1)-1}Q_{(2n-1)(2n+1)}$and $P_{(2n-1)(2n+1)-1}v_4$ $(i = 1, 2, \cdots, (2n{-}1)(2n{+}1){-}1)$. Fig.30 illustrates a 3-$FF2$ decomposed S for a VI-3 strip S of Fig.28.
Fold an n-$FF2$ decomposed strip S along the same fold lines of S to make *RD* by *FF2* which is shown in the proof of Lemma 3 (i.e., Fig.13) and then TS like a doubly covered parallel-octagons (parallelograms on rare occasions) are obtained (denote it by DCP). Note that this way of decomposing and folding a VI-3 strip is nothing but applying Conway gluing to a VI-3 strip. Shapes of front and back of DCP are not always parallelograms, but four points $v_1,\ v_2,\ v_3$ and v_4 on them form parallelograms (Fig.31). By recreasing the front and back of DCP along the lines v_iv_j, the recreased DCP are found to be composed of two identical parallelograms $v_1v_2v_4v_3$ (Fig.32).
The parallelograms of front and back of the recreased DCP can be divided into two identical acute triangles by its diagonal (Fig.32). Thus, a VI-3 strip S

is folded into an isotetrahedron I with vertices $v_1,\ v_2,\ v_3$ and v_4 each of whose faces is congruent to a half of the parallelogram of the recreased DCP (Fig.32, 33). Infinitely many non-similar I are obtained out of a given rectangle strip since shapes of both $4(2n-1)(2n+1)$ triangles and front and back of DCP vary as the location of $B(E)$ moves on $AC(FD)$.
Therefore, the strip S in this Case VI-3 is foldable into infinitely many isotetrahedra by the above generalized *FF2* procedure. □

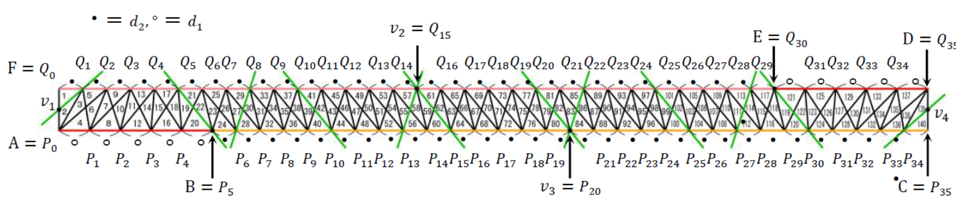

Fig. 30: $3 - FF2$ decomposed S of Fig.2828. Fold lines are green.

Front: $v_2Bv_4P_6v_3Av_1Q_1$ Back: $v_2Bv_4P_6v_3Av_1Q_1$

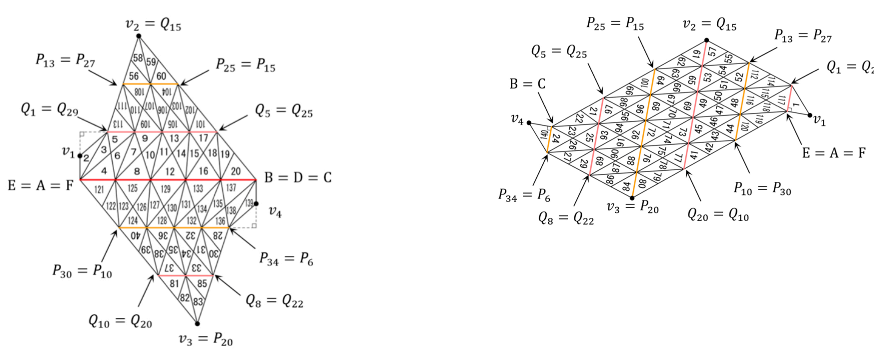

Fig. 31: DCP composed of these two parallel-octagons ($n = 3$).

§4. No other than known two ways of folding a strip into RD

The following Theorem 9 are obtained as a by-product of Theorem 6 and Theorem 8.

Theorem 9. *In folding a parallelogram strip into a rectangle dihedron, there is no other way of folding it into coiled RD other than well-known methods FF1 and FF2.*

Proof. Any net of a rectangle dihedron is a Conway tile. Then, the strip must be Conway glued when the strip is folded into a rectangle dihedron. The coiled

Front: $v_1v_2v_4v_3$ Back: $v_1v_2v_4v_3$

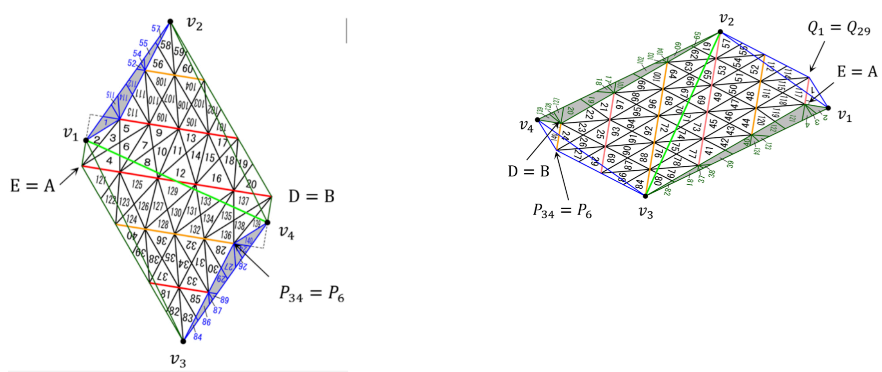

Fig. 32: Recreased $DCP(n = 3)$. The shape of front and back of it is parallelogram.

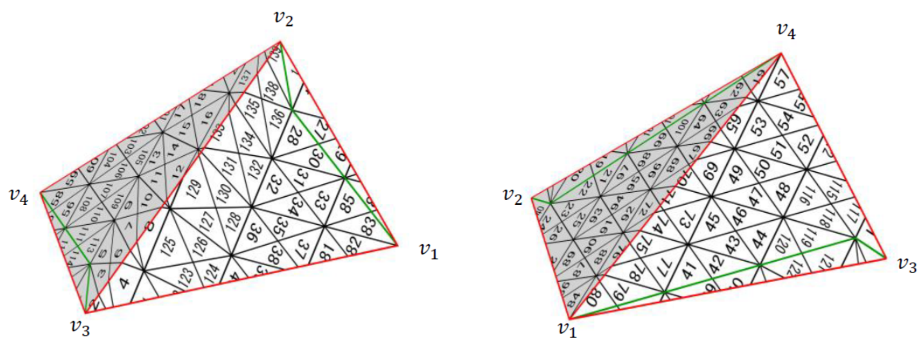

Fig. 33: An isotetrahedron is obtained from a rectangle strip S of Fig.28, seen from two different viewpoints

Conway gluing result of a strip appears if and only if consecutive six points are chosen on the perimeter of the strip in the manner of Case III-3 and Case VI-3. In particular, a rectangle dihedron is obtained in Case III-3 in Theorem 6 only when the strip can be divided into $4n(n+1)$ identical right triangles. This case is nothing but *FF1* as shown in Fig.7.

And a rectangle dihedron is obtained in Case VI-3 only when the shape of strip is a special parallelogram which can be divided into $4(2n-1)(2n+1)$ identical right triangles (i.e., S is the union of $(2n-1)\,(2n+1)-1$ rhombi and two halves of this rhombus). This case is nothing but *FF2* as shown in Fig.11. $\square$

§5. Acknowledgement

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Kiyoko Matsunaga
Tokyo University of Science
Research Center for Math & Sci Education
1-3 Kagurazaka, Shinjuku, Tokyo 162-8601, JAPAN
E-mail: `matsunaga@mathlab-jp.com`