THE BASEL PROBLEM

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The Basel Problem is that of evaluating the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots,$$

which was first raised in 1650 by Pietro Mengoli. It was first solved by Euler in 1734 and is named for his city of origin, although by then he was already in St Petersburg.

There are now many proofs that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1}$$

and all require some non-trivial limiting operation. The most generic are connected with the theory of Fourier series or some other aspect of harmonic analysis, or with the properties of periodic functions such as $\sin x$. My favourite proof is the following, which although motivated by the theory of Fourier series avoids any sophisticated limiting theorems such as the Riemann-Lebesgue Lemma and should be comprehendible to anyone who has taken a basic course of calculus and has met complex numbers.

We use the notation

$$e(\alpha) = e^{2\pi i \alpha}.$$

Then we have

$$\int_0^1 e(h\alpha) d\alpha = \begin{cases} 1 & (h = 0), \\ 0 & (h \in \mathbb{Z} \setminus \{0\}). \end{cases} \tag{2}$$

For a positive integer H we introduce the *Fejér Kernel*

$$K_H(\alpha) = H^{-1} \left| \sum_{n=1}^{H} e(n\alpha) \right|^2 = \sum_{h=-H}^{H} \left(1 - \frac{|h|}{H} \right) e(h\alpha). \tag{3}$$

To see that the second formula follows from the first write the first as

$$H^{-1} \sum_{m=1}^{H} e(m\alpha) \sum_{n=1}^{H} e(-n\alpha)$$

and collect together those terms for which $m-n=h$. There can only be a non-zero contribution when $-H\le h\le H$. Moreover when $0\le m-n=h\le H$ we have $m=n+h$ and there are only such pairs m,n when $h+1\le n+h\le H$ and then m is uniquely determined by n. Thus the number of such pairs is $H-h=H-|h|$. By symmetry we have the same conclusion when $-H\le h<0$.

Now consider for an arbitrary integer h

$$I(h)=\int_0^1\left(\tfrac12\alpha^2-\tfrac12\alpha+\tfrac1{12}\right)e(h\alpha)d\alpha.$$

At once

$$I(0)=0.$$

When $h\ne 0$, by integration by parts twice we have,

$$\begin{aligned}I(h)&=\left[\left(\tfrac12\alpha^2-\tfrac12\alpha+\tfrac1{12}\right)\frac{e(h\alpha)}{2\pi ih}\right]_0^1-\int_0^1\left(\alpha-\tfrac12\right)\frac{e(h\alpha)}{2\pi ih}d\alpha\\&=0-\left[\left(\alpha-\tfrac12\right)\frac{e(h\alpha)}{(2\pi ih)^2}\right]_0^1+\int_0^1\frac{e(h\alpha)}{(2\pi ih)^2}d\alpha\\&=-\frac{1}{(2\pi ih)^2}\\&=\frac{1}{4\pi^2h^2}.\end{aligned}$$

Now we combine this with the formula for the Fejér kernel (3),

$$\begin{aligned}\sum_{h=1}^{H}\frac{1}{2\pi^2h^2}\left(1-\frac{h}{H}\right)&=\sum_{\substack{h=-H\\h\ne0}}^{H}\frac{1}{4\pi^2h^2}\left(1-\frac{|h|}{H}\right)\\&=\int_0^1\left(\tfrac12\alpha^2-\tfrac12\alpha+\tfrac1{12}\right)K_H(\alpha)d\alpha.\end{aligned}$$

By (2) we also have

$$\int_0^1\tfrac1{12}K_H(\alpha)d\alpha=\sum_{h=-H}^{H}\left(1-\frac{|h|}{H}\right)\frac{1}{12}\int_0^1e(h\alpha)d\alpha=\frac{1}{12}.$$

Thus

$$\sum_{h=1}^{H}\frac{1}{2\pi^2h^2}\left(1-\frac{h}{H}\right)=\frac{1}{12}+\int_0^1\left(\tfrac12\alpha^2-\tfrac12\alpha\right)K_H(\alpha)d\alpha. \tag{4}$$

The sum

$$\sum_{h=1}^{H} e(h\alpha)$$

is the sum of the terms of a geometric progression with common ratio $e(\alpha)$ and so when $\alpha \notin \mathbb{Z}$ we have

$$\sum_{h=1}^{H} e(h\alpha) = \frac{e\big((H+1)\alpha\big) - e(\alpha)}{e(\alpha)-1} = e\big((H+1)\alpha/2\big)\frac{\sin(\pi H\alpha)}{\sin \pi\alpha}.$$

Hence, by (3),

$$K_H(\alpha) = H^{-1}\left(\frac{\sin(\pi H\alpha)}{\sin \pi\alpha}\right)^2 \tag{5}$$

and we see that this also holds when $\alpha = k \in \mathbb{Z}$ provided we interpret the right hand side as H, which would follow from *l'Hôpital's rule* on taking the limit as $\alpha \to k$. We also have

$$\big(\tfrac{1}{2}(1-\alpha)^2 - \tfrac{1}{2}(1-\alpha)\big)K_H(1-\alpha) = \big(\tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\alpha\big)K_H(\alpha).$$

Thus in the integral in (4), when we replace the variable α by $1-\alpha$ on the interval $[1/2, 1]$ we obtain

$$\sum_{h=1}^{H} \frac{1}{2\pi^2h^2}\left(1 - \frac{h}{H}\right) = \frac{1}{12} + \int_0^{1/2} (\alpha^2 - \alpha)K_H(\alpha)d\alpha.$$

By (3), $0 \le K_H(\alpha) \le H$ and by (5) and the well known inequality $|\sin \pi\beta| \ge 2|\beta|$ for $|\beta| \le 1/2$ we have

$$K_H(\alpha) \le H^{-1}\alpha^{-2}$$

when $0 < \alpha \le \frac{1}{2}$.

Let δ be a real number with $0 < \delta < \frac{1}{2}$. Then

$$\begin{aligned}\left|\int_0^{1/2} (\alpha^2-\alpha)K_H(\alpha)d\alpha\right| &\le \int_0^{\delta} \alpha H d\alpha + \int_{\delta}^{1/2} H^{-1}\alpha^{-1}d\alpha \\ &< \delta^2 H + H^{-1}\log\frac{1}{2\delta}.\end{aligned}$$

If we take $\delta = H^{-1}$ then the above $\to 0$ as $H \to \infty$. We also have

$$\sum_{h=1}^{H} \frac{1}{2\pi^2h^2}\left(1 - \frac{h}{H}\right) = \sum_{h=1}^{H} \frac{1}{2\pi^2h^2} - \sum_{h=1}^{H} \frac{1}{2\pi^2hH}$$

and the second sum here is as most

$$\frac{1}{2\pi^2 H}\left(1+\int_1^H \frac{dx}{x}\right) = \frac{1+\log H}{2\pi^2 H}$$

and this also $\to 0$ as $H \to \infty$. Hence

$$\sum_{h=1}^{\infty} \frac{1}{2\pi^2 h^2} = \frac{1}{12}.$$

By the way, the polynomial

$$\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha + \frac{1}{12}$$

at the core of the proof is the second, $b_2(\alpha)$, of a family of polynomials named for Jacob Bernoulli, who also came from Basel. The first is

$$b_1(\alpha) = \alpha - \frac{1}{2}$$

and one can define them iteratively by

$$b_{k+1}(\alpha) = \int_0^\alpha b_k(\beta)d\beta - \int_0^1 b_k(\beta)(1-\beta)d\beta. \tag{6}$$

Some authors define them so that the leading coefficient is 1 by replacing the right hand side of (6) by

$$b_{k+1}(\alpha)/(k+1).$$

They have many interesting properties and applications.