# Notation

| | |
|---|---|
| $\mathcal{L}$ | Laplace transform, §2.6 |
| $\mathcal{L}^{-1}$ | inverse Laplace transform, §2.14 |
| $\mathcal{B}$ | Borel transform, §4.4a |
| $\mathcal{LB}$ | Borel / $\mathsf{BE}$ summation operator, §4.4, §4.4f |
| $p$ | usually, Borel plane variable |
| $\tilde{f}$ | formal expansion |
| $H(p)$ | Borel transform of $h(x)$ |
| $\sim$ | asymptotic to, §1.1a |
| $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ | the positive integers, integers, rationals, real numbers, complex numbers respectively. |
| $\mathbb{H}$ | open right half complex-plane. |
| $C_a$ | absolutely continuous functions, [47] |

# *Asymptotics and Borel summability*

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# Contents

# Foreword

The field of asymptotics has evolved substantially in the last thirty years or so; for instance relatively general tools have been developed to recover exact solutions from formal expansions in relatively general settings. Many of these new developments are still scattered in relatively specialized articles. The present book is intended to provide a self-contained introduction to asymptotic analysis (with some emphasis on methods needed in exponential asymptotics, and on applications which are not part of usual asymptotics books, such as asymptotics of Taylor coefficients), and to explain basic ideas, concepts and methods of generalized Borel summability, transseries and exponential asymptotics.

To provide a sense on how these latter methods are used in a systematic way, general nonlinear ODEs near a generic irregular singular point are analyzed in detail. The analysis of difference equations, PDEs and other types of problems, only superficially touched upon in this book, while certainly providing a number of new challenges, is not radically different in spirit. Mastering the basic techniques in the context of ODEs should provide most of the necessary background to give a smoother access to the many existing articles on other types of problems that are not covered at this introductory level.

The level of difficulty is uneven; sections marked with $*$ are more difficult and not crucial for following the rest of the material.

The book assumes standard knowledge of Real and Complex Analysis. Chapters one through four, and parts of chapter five a and six are suitable for use in a graduate or advanced undergraduate course. Most exercises are meant to be relatively straightforward; more challenging exercises are marked with $(*)$.

The book provides complete mathematical rigor, but it is written so that many proofs can be omitted at a first reading.

# Chapter 1

# Introduction

## 1.1 Expansions and approximations

Classical asymptotic analysis studies the limiting behavior of functions when singular points are approached. It shares with analytic function theory the goal of providing a detailed description of functions, and it is distinguished from it by the fact that the main focus is on singular behavior. Asymptotic expansions provide increasingly better approximations as the special points are approached yet they rarely converge to a function.

The asymptotic expansion of an analytic function at a regular point is the same as its convergent Taylor series there. The local theory of analytic functions at regular points is largely a theory of convergent power series.

We have $-\ln(1-x) = \sum_{k=0}^{\infty} x^k/k$; the behavior of the log near one is transparent from the series, which also provides a practical way to calculate $\ln(1-x)$ for small $x$. Likewise, to calculate $z! := \Gamma(1+z) = \int_0^{\infty} e^{-t}t^z dt$ for small $z$ we can use

$$\ln\Gamma(1+z) = -\gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k\zeta(k)z^k}{k}, \ (|z|<1), \text{ where } \zeta(k) := \sum_{j=1}^{\infty} j^{-k} \quad (1.1)$$

and $\gamma = 0.5772..$ is the Euler constant (see Exercise 4.62 on p. 106). Thus, for small $z$ we have

$$\begin{aligned}\Gamma(1+z) &= \exp\left(-\gamma z + \pi^2 z^2/12 \cdots\right) \\ &= \exp\left(-\gamma z + \sum_{k=2}^{M} (-1)^k \zeta(k) k^{-1} z^k\right)\left(1 + O(z^{M+1})\right) \quad (1.2)\end{aligned}$$

where, as usual, $f = O(z^j)$ means that $|z^{-j}f| \le C$, for some $C > 0$, when $z$ is small.

$\Gamma(z)$ has a pole at $z=0$; $z\Gamma(z) = \Gamma(1+z)$ is described by the convergent power series

$$z\Gamma(z) = \exp(-\gamma z + \sum_{k=2}^{M} k^{-1}(-1)^k\zeta(k)z^k k(1 + O(z^{M+1})) \quad (1.3)$$

This is a perfectly useful way of calculating $\Gamma(z)$ for small $z$.

Now let us look at a function near an essential singularity, *e.g.* $e^{-1/z}$ near $z=0$. Of course, multiplication by a power of $z$ does not remove the singularity, and the Laurent series contains all negative powers of $z$:

$$e^{-1/z} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!z^j} \tag{1.4}$$

Eq. (1.4) is fundamentally distinct from the first examples. This can be seen by trying to calculate the function from its expansion for say, $z = 10^{-10}$: (1.1) provides the desired value very quickly, while (1.4), also a convergent series, is virtually unusable. Mathematically, we see that error bounds as in (1.2) and (1.3) do not hold for (1.4). On the contrary, we have

$$e^{-1/z} - \sum_{j=0}^{M} \frac{1}{j!z^j} \gg z^{-M}, \text{ as } z \to 0 \tag{1.5}$$

where $\gg$ means much larger than. The series (1.4) is convergent, but **antiasymptotic**: the terms of the expansion get larger and larger as $z \to 0$. The function needs to be calculated there in a different way, and there are certainly many good ways. Surprisingly perhaps, the exponential, together with related functions such as $\log, \sin x$ (and powers, since we prefer the notation $x$ to $e^{\ln x}$) are the only ones that we need in order to represent many complicated functions, asymptotically. This fact has been noted already by Hardy, who wrote [35] "No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms". This reflects some important fact about the relation between asymptotic expansions and functions which will be clarified in §4.9.

If we need to calculate $\Gamma(x)$ for very large $x$, the Taylor about one given point would not work, since the radius of convergence is finite (due to poles on $\mathbb{R}^-$). Instead we have Stirling's series,

$$\ln(\Gamma(x)) \sim (x-1/2)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{\infty} c_j x^{-2j+1}, \; x \to +\infty \tag{1.6}$$

where $2j(2j-1)c_j = B_{2j}$ and $\{B_{2j}\}_{j\in\mathbb{N}} = \{1/6, -1/30, 1/42...\}$ are Bernoulli numbers. This expansion is *asymptotic* as $x \to \infty$: successive terms get smaller and smaller. Stopping at $j=3$ we get $\Gamma(6) \approx 120.00000086$ while $\Gamma(6) = 120$. Yet, the expansion in (1.6) cannot converge to $\ln(\Gamma(x))$, and in fact, it has to have zero radius of convergence, since $\ln(\Gamma(x))$ is singular at all $x \in -\mathbb{N}$ (why is this an argument?).

Unlike asymptotic expansions, convergent but antiasymptotic expansions do not contain manifest, detailed information. Of course, this is not meant to understate the value of convergent representations, nor to advocate for uncontrolled approximations.

### 1.1a Asymptotic expansions

An asymptotic expansion of a function $f$ at a point $t_0$, usually dependent on the direction along which $t_0$ is approached, is a formal series[1] of simpler functions $f_k$,

$$\tilde{f} = \sum_{k=0}^{\infty} f_k(t) \tag{1.7}$$

in which each successive term is much smaller than its predecessors. For instance if the limiting point is $t_0$, approached from above along the real line, this requirement is written

$$f_{k+1}(t) = o(f_k(t)) \quad (\text{or} \quad f_{k+1}(t) \ll f_k(t)) \text{ as } t \to t_0^+ \tag{1.8}$$

meaning

$$\lim_{t \to t_0^+} f_{k+1}(t)/f_k(t) = 0 \tag{1.9}$$

We will often use the variable $x$ when the limiting point is $+\infty$ and $z$ when the limiting point is zero.

### 1.1b Functions asymptotic to an expansion, in the sense of Poincaré

The relation $f \sim \tilde{f}$ between an actual function and a formal expansion is defined as a sequence of limits:

**Definition 1.10** *A function $f$ is asymptotic to the formal series $\tilde{f}$ as $t \to t_0^+$ if*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = f(t) - \tilde{f}^{[N]}(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{1.11}$$

Condition (1.11) can be written in a number of equivalent ways, useful in applications.

**Proposition 1.12** *If $\tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k(t)$ is an asymptotic series as $t \to t_0^+$ and $f$ is a function asymptotic to it, then the following characterizations are equivalent to each other and to (1.9).*

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[1] That is, there are no convergence requirements. More precisely, formal series are sequences of functions $\{f_k\}_{k \in \mathbb{N} \cup \{0\}}$, written as infinite sums, with the operations defined as for convergent series; see also §1.1c .

*(i)*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.13}$$

*where* $g(t) = O(h(t))$ *means* $\limsup_{t\to t_0^+} |g(t)/h(t)| < \infty$.

*(ii)*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = f_{N+1}(1 + o(1)) \quad (\forall N \in \mathbb{N}) \tag{1.14}$$

*(iii) There is a function* $a : \mathbb{N} \mapsto \mathbb{N}$ *such that* $a(N) \geq N$ *and*

$$f(t) - \sum_{k=0}^{a(N)} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.15}$$

*This condition seems strictly weaker, but it is not. It allows us to use less accurate estimates of remainders, provided we can do so to all orders.*

**PROOF** We only show (iii), the others being immediate. Let $N \in \mathbb{N}$. We have

$$\frac{1}{f_{N+1}(t)}\left(f(t) - \sum_{k=0}^{N} \tilde{f}_k(t)\right)$$
$$= \frac{1}{f_{N+1}(t)}\left(f(t) - \sum_{k=0}^{a(N)} \tilde{f}_k(t)\right) + \sum_{j=N+1}^{a(N)} \frac{f_j(t)}{f_{N+1}} = O(1) \tag{1.16}$$

since in the last sum in (1.16) $N$, and thus the number of terms, is fixed, and thus the sum converges to 1 as $t \to t_0^+$. ☐

Simple examples of asymptotic expansions are

$$\sin z \sim z - \frac{z^3}{6} + ... + \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!} + \cdots \quad (|z| \to 0) \tag{1.17}$$

$$f(z) = \sin z + e^{-\frac{1}{z}} \sim z - \frac{z^3}{6} + ... + \frac{(-1)^{n+1} z^{2n+1}}{(2n+1)!} + \cdots (z \to 0^+) \tag{1.18}$$

$$e^{-1/z} \int_1^{1/z} \frac{e^t}{t} dt \sim \sum_{k=0}^{\infty} k! z^{k+1} \quad (z \to 0^+) \tag{1.19}$$

The series on the right side of (1.17) converges to $\sin z$ for any $z \in \mathbb{C}$ and it is asymptotic to it for small $|z|$. The series in the second example converges for any $z \in \mathbb{C}$ but not to $f$. In the third example the series is nowhere convergent, in short it is a *divergent* series. It can be obtained by repeated integration by parts:

$$f_1(x) := \int_1^x \frac{e^t}{t}dt = \frac{e^x}{x} - e + \int_1^x \frac{e^t}{t^2}dt$$
$$= \cdots = \frac{e^x}{x} + \frac{e^x}{x^2} + \frac{2e^x}{x^3} + \cdots + \frac{(n-1)!e^x}{x^n} + C_n + n!\int_1^x \frac{e^t}{t^{n+1}}dt \tag{1.20}$$

with $C_n = -e\sum_{j=0}^n j!$. For the last term we have

$$\lim_{x\to\infty} \frac{\displaystyle\int_1^x \frac{e^t}{t^{n+1}}dt}{\displaystyle\frac{e^x}{x^{n+1}}} = 1 \tag{1.21}$$

(by L'Hospital) and (1.19) follows.

**Note 1.22** *The constant $C_n$ cannot be included in (1.19) using the definition (1.11), since its contribution vanishes in any of the limits implicit in (1.11).*

By a similar calculation,

$$f_2 = \int_2^x \frac{e^t}{t}dt \sim e^x\tilde{f}_0 = \frac{e^x}{x} + \frac{e^x}{x^2} + \frac{2e^x}{x^3} + \cdots + \frac{n!e^x}{x^{n+1}} + \cdots \text{ as } x \to +\infty \tag{1.23}$$

and now, unlike the case of (1.17) versus (1.18) there is no obvious function to prefer, insofar as asymptoticity goes, on the left side of the expansion.

Stirling's formula (1.6) is another example of a divergent asymptotic expansion.

**Remark 1.24** *Asymptotic expansions cannot be added, in general. Otherwise, since on the one hand $f_1 - f_2 = \int_1^2 dse^s/s = 3.0591...$, and on the other hand both $f_1$ and $f_2$ are asymptotic to the same expansion, it would follow that* $3.0591... \sim 0$. *This is one reason for considering, for restricted expansions, a weaker asymptoticity condition, see §1.1c .*

Examples of expansions that are *not asymptotic* are (1.4) for small $z$, or

$$\sum_{k=0}^{\infty} \frac{x^{-k}}{k!} + e^{-x} \quad (x \to +\infty) \tag{1.25}$$

(because of the exponential term, this is not an ordered *simple series* satisfying (1.8)). Note however expansion (1.25), *does* satisfy all requirements in the *left* half plane, if we write $e^{-x}$ in the first position.

**Remark 1.26** *Sometimes we encounter expansions for large $x$ of the form* $\sin x(1 + a_1x^{-1} + a_2x^{-2} + \cdots)$ *which, while very useful, have to be understood differently and we will discuss this question in §3.5c . They are not asymptotic expansions in the sense above, since* $\sin x$ *can vanish. Usually the approximation itself fails near zeros of* sin.

### 1.1c Asymptotic power series

A special role is played by power series which are series of the form

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k, \;\; z \to 0^+ \tag{1.27}$$

With the transformation $z = t - t_0$ (or $z = x^{-1}$) the series can be centered at $t_0$ (or $+\infty$, respectively).

**Definition 1.28 (Asymptotic power series)** *A function possesses an asymptotic* power *series as $z \to 0$ if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \;\; (\forall N \in \mathbb{N}) \;\;\; \text{as } z \to 0 \tag{1.29}$$

**Remark 1.30** An asymptotic series is *not* an asymptotic expansion in the sense of Definition 1.10 and (1.29) is not a special case of (1.13) unless all $c_k$ are nonzero.

The asymptotic **power series** at zero in $\mathbb{R}$ of $e^{-1/z^2}$ is the zero series. However, the asymptotic *expansion*, or behavior, of $e^{-1/z^2}$ cannot be just zero.

### 1.1d Operations with asymptotic power series

Addition and multiplication of asymptotic power series are defined as in the convergent case:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} d_k z^k = \sum_{k=0}^{\infty}(Ac_k + Bd_k) z^k$$

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} d_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j d_{k-j}\right) z^k$$

**Remark 1.31** *If the series $\tilde{f}$ is convergent and $f$ is its sum, $f = \sum_{k=0}^{\infty} c_k z^k$, (note the ambiguity of the sum notation), then $f \sim \tilde{f}$.*

The proof follows directly from the definition of convergence.

The proof of the following lemma is immediate:

**Lemma 1.32 (Algebraic properties of asymptoticity to a power series)** *If $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ and $g \sim \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$ then*

*(i) $Af + Bg \sim A\tilde{f} + B\tilde{g}$*

*(ii) $fg \sim \tilde{f}\tilde{g}$*

**Corollary 1.33 (Uniqueness of the asymptotic series to a function)** *If* $f(z) \sim \sum_{k=0}^{\infty} c_k z^k$ *as* $z \to 0$ *then the* $c_k$ *are unique.*

**PROOF** Indeed, if $f \sim \sum_{k=0}^{\infty} c_k z^k$ and $f \sim \sum_{k=0}^{\infty} d_k z^k$, then, by Lemma 1.32 we have $0 \sim \sum_{k=0}^{\infty} (c_k - d_k) z^k$ which implies, inductively, that $c_k = d_k$ for all $k$. ◻

Algebraic operations with asymptotic series are limited too. Division of asymptotic series is not always possible. $e^{-1/z^2} \sim 0$ in $\mathbb{R}$ while $1/\exp(-1/z^2)$ has no asymptotic series at zero.

### 1.1d .1 Integration and differentiation of asymptotic power series.

Asymptotic relations can be integrated termwise as Proposition 1.34 below shows.

**Proposition 1.34** *Assume* $f$ *is integrable near* $z = 0$ *and that*

$$f(z) \sim \tilde{f}(z) = \sum_{k=0}^{\infty} c_k z^k$$

*Then*

$$\int_0^z f(s)ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{c_k z^{k+1}}{k+1}$$

**PROOF** This follows from the fact that $\int_0^z o(s^n)ds = o(z^{n+1})$ as it can be seen by straightforward inequalities. ◻

Differentiation is a different issue. Many simple examples show that asymptotic series cannot be freely differentiated. For instance $e^{-1/x^2} \sin e^{1/x^4} \sim 0$ as $x \to 0$ on $\mathbb{R}$, but the derivative is unbounded.

Asymptotic power series of analytic functions can be differentiated if they hold in a region which is not too rapidly shrinking. Such a region is often a sector or strip in $\mathbb{C}$, but can be allowed to be thinner.

### 1.1d .2 Asymptotics in regions in $\mathbb{C}$

**Proposition 1.35** *Let* $M > 0$ *and assume* $f(x)$ *is analytic in the region* $S_a = \{x : |x| > R, |\mathrm{Im}\,(x)| < a|\mathrm{Re}\,(x)|^{-M}\}$, *and*

$$f(x) \sim \sum_{k=0}^{\infty} c_k x^{-k} \quad \text{as } |x| \to \infty$$

*in any subregion of the form* $S_{a'}$ *with* $a' < a$.

*Then*

$$f'(x) \sim \sum_{k=0}^{\infty}(-kc_k)x^{-k-1}$$

*as* $|x| \to \infty$ *in any subregion of the form* $S_{a'}$ *with* $a' < a$.

**PROOF** Here, Proposition 1.12 (iii) will come in handy. Let $N > 2M+2$. By the analyticity and asymptoticity assumptions, there is some constant $C$ such that $|f(x) - \sum_{k=0}^{N} c_k x^{-k}| < C|x|^{-N}$ in $S_{a'}$ $(a' < a)$. Let $a'' < a'$; then the points $x$ such that $|\mathrm{Im}\,(x)| \le a''$ are contained in $S_{a'}$, together with a disk around them of radius $(a' - a'')|x|^{-M}/2$ if $x$ is large enough. We have

$$\left| f'(x) - \sum_{k=0}^{N}(-kc_k)x^{-k-1} \right| = \left| \frac{1}{2\pi i} \oint_{\mathcal{C}} (s-x)^{-2} \left( f(s) - \sum_{k=0}^{N} c_k s^{-k} \right) ds \right|$$
$$\le \frac{4C}{(a'-a'')^2}|x|^{2M}|x|^{-N} \tag{1.36}$$

and the result follows. ◻

**Exercise 1.37** Consider the following integral related to the error function

$$F(z) = e^{z^{-2}} \int_0^z s^{-2} e^{-s^{-2}} ds$$

It is clear that the integral converges at the origin, if the origin is approached through real values (see also the change of variable below); thus we *define* the integral to $z \in \mathbb{C}$ as being taken on a curve $\gamma$ with $\gamma'(0) > 0$, and extend $F$ by $F(0) = 0$. The resulting function is analytic in $\mathbb{C} \setminus 0$, see Exercise 3.8 below.

What about the behavior at $z = 0$? It depends on the direction in which 0 is approached! Substituting $z = 1/x$ and $s = 1/t$ we get

$$E(x) = e^{x^2} \int_x^{\infty} e^{-t^2} dt =: \frac{\sqrt{\pi}}{2} e^{x^2} \mathrm{erfc}(x) \tag{1.38}$$

Check that if $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ and $f'(x) \to L$ as $x \downarrow 0$, then $f$ is differentiable to the right at zero and this derivative equals $L$. Use this fact, Proposition 1.35 and induction to show that the Taylor series at $0^+$ of $F(z)$ is indeed given by (3.7).

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## Formal and actual solutions.

Few calculational methods have longer history than successive approximation. Suppose $\epsilon$ is small and we want to solve the equation $y - y^5 = \epsilon$. Looking first for a small solution, we see that $y^5 \ll y$ and then, writing

$$y = \epsilon + y^5 \tag{1.39}$$

as a first approximation, we have

$$y \approx y_1 = \epsilon \tag{1.40}$$

We can use $y \approx y_1$ in (1.39) to improve in accuracy over (1.40):

$$y_2 = \epsilon + \epsilon^5$$

and further

$$y_3 = \epsilon + \left(\epsilon + \epsilon^5\right)^5$$

Repeating this procedure indefinitely, the right side becomes

$$\epsilon + \epsilon^5 + 5\epsilon^9 + 35\epsilon^{13} + 285\epsilon^{17} + 2530\epsilon^{21} + \cdots \tag{1.41}$$

**Exercise 1.42** *Show that this series converges for* $|\epsilon| < 4 \cdot 5^{-5/4}$. *(Hint: one way is to use implicit function theorem.)*

Regular differential equations can be locally solved much in the same way. Consider the Painlevé equation

$$y'' = y^2 + z$$

near $z = 0$ with $y(0) = y_0$ and $y'(0) = y_1$ small. If $y$ is small like some power of $z$, then $y''$ is far larger than $y^2$ and then, to leading approximation,

$$y'' = z$$

and

$$y = y_0 + y_1 z + \frac{z^2}{2}$$

We can substitute this back into the equation and get a better approximation of the solution, and if we repeat the procedure indefinitely, we get the actual solution of the problem (since, as it follows from the general theory of differential equations, the solution is analytic).

Let us look at the equation

$$f' - f = x^{-1}, \quad x \to +\infty \tag{1.43}$$

If $f$ is small like an inverse power of $x$, then $f'$ should be even smaller, and we can apply again successive approximations to the equation written in the form

$$f = x^{-1} - f' \tag{1.44}$$

To leading order $f \approx f_1 = 1/x$, we then have $f \approx f_2 = 1/x - 1/x^2$ and now if we repeat the procedure indefinitely we get

$$f \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots - \frac{(-1)^n n!}{x^{n+1}} + \cdots \tag{1.45}$$

Something must have gone wrong here. We do not get a solution (in any obvious meaning) to the problem: for no value of $x$ is this series convergent. What distinguishes the first two examples from the last one? In the first two, the next approximation was obtained from the previous one by algebraic operations and integration. These processes are regular, and they produce, at least under some restrictions on the variables, convergent expansions. We have, e.g., $\int \cdots \int x = x^n/n!$. But in the last example, we iterated upon differentiation a regularity-reducing operation. We have $(1/x)^{(n)} = n!/x^{n+1}$.

### 1.1e Limitations of representation of functions by expansions

Prompted by the need to eliminate apparent paradoxes, mathematics has been formulated in a precise language with a well defined set of axioms [55], [52] within set theory. In this language, a function is defined as *a set of ordered pairs* $(x, y)$ [2] such that *for every $x$ there is only one pair with $x$ as the first element.* All this can be written precisely and it is certainly foundationally satisfactory, since it uses arguably more primitive objects: sets.

A tiny subset of these general functions can arise as unique solutions to well defined problems, however. Indeed, on the one hand it is known that there is no specific way to distinguish two arbitrary functions based on their intrinsic properties alone[3]. On the other hand, a function which is known to be the unique solution to a specific problem can a fortiori be distinguished from any other function. By the same argument, clearly it cannot be possible to represent general functions by constructive expansions.

In some sense, most functions just exist in an unknowable realm, and only their collective presence has mathematical consequences. We can usefully restrict the study of functions to those which do arise in specific problems, and hope that they have, in general, better properties than arbitrary ones. For instance, solutions of specific equations, such as systems of linear or nonlinear ODEs or difference equations with meromorphic coefficients, near a regular or singular point, can be described completely in terms of their expansion at such a point (more precisely, they are completely described by their *transseries*, a generalization of series described later).

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[2] Here $x, y$ are themselves sets, and $(x, y) := \{x, \{x, y\}\}$; $x$ is in the domain of the function and $y$ is in its range.

[3] More precisely, in order to select one function out of an arbitrary, *unordered* pair of functions, some form of the *axiom of choice* [52] is needed.

Conversely, we can write formal expansions without a natural function counterpart. The formal expression

$$\sum_{q\in\mathbb{Q}} \frac{1}{x+q}; \quad x \notin \mathbb{Q} \tag{1.46}$$

(true, this is not an asymptotic series whatever $x$ is) cannot have a nonconstant, meaningful sum, since the expression is formally $q-$periodic for any $q \in \mathbb{Q}$ and the sum should preserve this basic feature. Nonconstant functions with arbitrarily small periods are not Lebesgue measurable [47]. Since it is known that existence of nonmeasurable functions can proved only by using some form of the axiom of choice, no definable (such as "the sum of (1.46)") nonmeasurable function can be exhibited.

A good correspondence between functions and expansions is possible only by carefully restricting both. We will restrict the analysis to functions and expansions arising in differential or difference equations, and some few other concrete problems.

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Convergent series relate to their sums in a relations-preserving way. Can we associate to a divergent series a unique function by some generalized property-preserving summation process? The answer is no in general, as we have seen, and yes in many practical cases. Exploring this question will carry us through a number of interesting questions.

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In [33], Euler investigated the question of the possible sum of the formal series $s = 1 - 2 + 6 - 24 + 120 \cdots$, in fact extended to

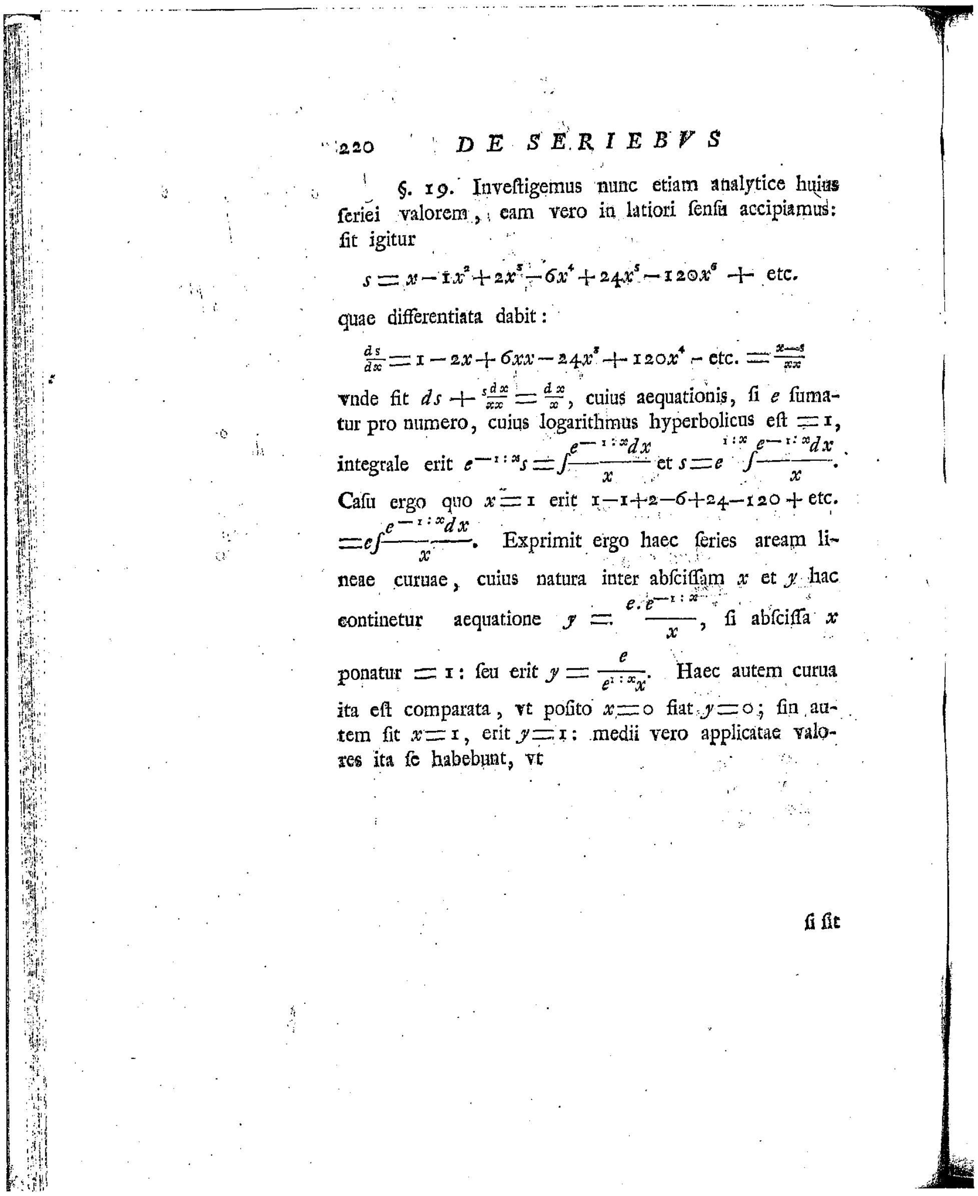
$$\tilde{f} := \sum_{k=0}^{\infty} k!(-z)^{k+1}, \quad z > 0 \tag{1.47}$$

In effect, Euler notes that $\tilde{f}$ satisfies the equation

$$z^2 y' + y = z \tag{1.48}$$

and thus $\tilde{f} = e^{1/z}\mathrm{Ei}(-1/z) + Ce^{1/z}$ (see Fig. 1.1), for some $C$, where $C$ must vanish since the series is formally small as $z \to 0^+$. Then, $\tilde{f} = e^{1/z}\mathrm{Ei}(-1/z)$, and in particular $s = e\mathrm{Ei}(-1)$. What does this argument show? At the very least, it proves that *if* there is a summation process capable of summing (1.47) to a function, in a way compatible with basic operations and properties, the function can only be $e^{1/z}\mathrm{Ei}(-1/z)$. In this sense, the sum is independent of the summation method.

Factorially divergent were already widely used at the turn of the 19th century for very precise astronomical calculations. As the variable, say $1/x$, becomes small, the first few terms of the expansion should provide a good approximation of the function. Taking for instance $x = 100$ and 5 terms in the



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§. 19. Inveſtigemus nunc etiam analytice huius ſeriei valorem, eam vero in latiori ſenſu accipiamus: ſit igitur

$$s = x - 1x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + \text{etc.}$$

quae differentiata dabit:

$$\tfrac{ds}{dx} = 1 - 2x + 6xx - 24x^3 + 120x^4 - \text{etc.} = \tfrac{x-s}{xx}$$

vnde fit $ds + \frac{s\,dx}{xx} = \frac{dx}{x}$, cuius aequationis, ſi $e$ ſumatur pro numero, cuius logarithmus hyperbolicus eſt $=1$, integrale erit $e^{-1:x}s = \int \frac{e^{-1:x}dx}{x}$ et $s = e^{1:x} \int \frac{e^{-1:x}dx}{x}$. Caſu ergo quo $x = 1$ erit $1 - 1 + 2 - 6 + 24 - 120 + \text{etc.}$ $= e \int \frac{e^{-1:x}dx}{x}$. Exprimit ergo haec ſeries aream lineae curuae, cuius natura inter abſciſſam $x$ et $y$ hac continetur aequatione $y = \frac{e . e^{-1:x}}{x}$, ſi abſciſſa $x$ ponatur $= 1$: ſeu erit $y = \frac{e}{e^{1:x}x}$. Haec autem curua ita eſt comparata, vt poſito $x = 0$ fiat $y = 0$; ſin autem ſit $x = 1$, erit $y = 1$: medii vero applicatae valores ita ſe habebunt, vt

ſi ſit

**FIGURE 1.1**: L. Euler, De seriebus divergentibus, *Novi Commentarii Academiae Scientiarum Petropolitanae (1754/55) 1760, p. 220*

asymptotic expansions (1.20) and (1.23) we get the value $2.715552711 \cdot 10^{41}$ for both $f_1(100) = 2.715552745 \ldots \cdot 10^{41}$ and $f_2(100) = f_1(100) - 3.05911 \cdots$. However, in using divergent series, there is a threshold in the accuracy of approximation, as it can be seen by comparing (1.20) and (1.23).

The two functions differ by a constant, which is exponentially smaller than each of them. The expected relative error cannot be better than exponentially small, at least for one of them. As we shall see, exponentially small relative errors (that is, absolute errors of order one) can be, for both of them, achieved by truncating the series at an optimal number of terms, *dependent on $x$* (*optimal truncation*), see Note 4.134 below. The absolute error in calculating $f_3(x) :=\mathrm{Ei}(x)$ by optimal truncations is even smaller, of order $x^{-1/2}$. Still, for fixed $x$, in such a calculation there is a built-in ultimate error, a final nonzero distance between the series and the function we want to calculate.

Cauchy [14] proved that optimal truncation in Stirling's formula gives errors of the order of magnitude of the least term, exponentially small relative to the function calculated. Stokes refined Cauchy summation to the least term, and discovered the "Stokes phenomenon": the behavior of a function described by a divergent series must change qualitatively as the direction in $\mathbb{C}$ varies, and furthermore, the change is first (barely) visible at what we now call Stokes lines.

But a general procedure of "true" summation was absent at the time. Abel, discoverer of a number of summation procedures of divergent series, labeled divergent series "an invention of the devil".

Later, the view of divergent series as somehow linked to specific functions and encoding their properties was abandoned (together with the concept of functions as specific rules). This view was replaced by the rigorous notion of an *asymptotic series*, associated instead to a vast family of functions via the rigorous Poincaré definition 1.10, which is precise and general, but specificity is lost even in simple cases.

## Some elements of Écalle's theory

In the 80's by Écalle discovered a vast class of functions, closed under usual operations (algebraic ones, differentiation, composition, integration and function inversion) whose properties are, at least in principle, easy to analyze: *the analyzable functions.* Analyzable functions are in a one-to-one isomorphic correspondence with generalized summable expansions, *transseries.*

What is the closure of simple functions under the operations listed? That is not easy to see if we attempt to construct the closure on the function side. Let's see what happens by repeated application of two operations, taking the reciprocal and integration.

$$1 \xrightarrow{\int} x \xrightarrow{\frac{1}{\cdot}} x^{-1} \xrightarrow{\int} \ln x$$

and $\ln x$ is not expressible in terms of powers, and so it has to be taken as a primitive object. Further,

$$\ln x \quad \xrightarrow{\frac{1}{\cdot}} \frac{1}{\ln x} \xrightarrow{\int} \int \frac{1}{\ln x} \tag{1.49}$$

and, within functions we would need to include the last integral as yet another primitive object, since the integral is nonelementary, and in particular it cannot be expressed as a finite combination of powers and logs. In this way, we generate an endless list of new objects.

**Transseries.** The way to obtain analyzable functions was in fact to first construct *transseries*, the closure of formal series under operations, which turns out to be a far more manageable task, and then find a general, well-behaved, summation procedure.

Transseries are surprisingly simple. They consists, roughly, in all formally asymptotic expansions in terms of powers, exponentials and logs, of ordinal length, with coefficients which have at most power-of-factorial growth. For instance, as $x \to \infty$, integrations by parts in (1.49), formally repeated infinitely many times, yields

$$\int \frac{1}{\ln x} \quad = \; x \sum_{k=1}^{\infty} \frac{k!}{(\ln x)^{k+1}}$$

(a divergent expansion). Other examples are:

$$e^{e^x + x^2} + e^{-x} \sum_{k=0}^{\infty} \frac{k!(\ln x)^k}{x^k} + e^{-x \ln x} \sum_{k=-1}^{\infty} \frac{k!^2 2^k}{x^{k/3}} \qquad x \to +\infty$$

$$\sum_{k=0}^{\infty} e^{-kx} \left( \sum_{j=0}^{\infty} \frac{c_{kl}}{x^k} \right)$$

Note how the terms are ordered decreasingly, with respect to $\gg$ (far greater than) from left to right. Transseries are constructed so that they are *finitely generated*, that is they are effectively (multi)series in a finite number of “bricks” (transmonomials), simpler combinations of exponentials powers and logs. The *generators* in the first and third transseries are $1/x$ and $e^{-x}$. Transseries contain, order by order, manifest asymptotic information.

Transseries, as constructed by Écalle, are the closure of series under a number of operations, including

(i) Algebraic operations: addition, multiplication and their inverses.

(ii) Differentiation and integration.

(iii) Composition and functional inversion.

However, operations (i), (ii) and (iii) are far from sufficient; for instance differential equations cannot be solved through (i)–(iii). Indeed, most ODEs *cannot* be solved by *quadratures*, *i.e.* by finite combinations of integrals of simple functions, but by limits of these operations. Limits though are not easily accommodated in the construction. Instead we can allow for

(iv) Solution of fixed point problems of *formally contractive mappings*, see §3.8.

Operation (iv) was introduced by abstracting from the way problems with a small parameter [4] are solved by successive approximations.

**Proposition.** Transseries are closed under (i)–(iv).

This will be shown in §4 and §4.9; it means many problems can be solved within transseries. It seems unlikely though that even with the addition (iv) do we obtain all that is needed to solve asymptotic problems; more needs to be understood.

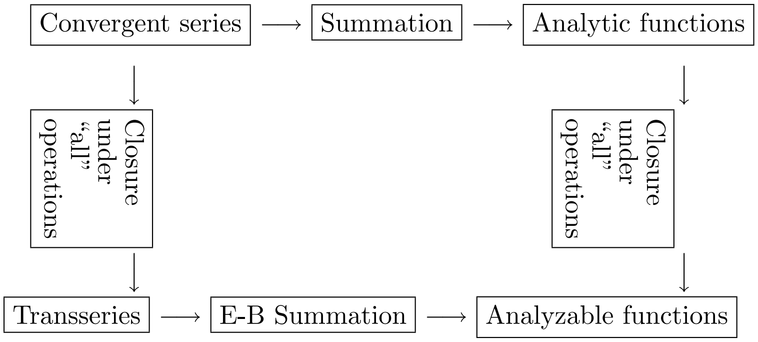
**Analyzable functions.** To establish a one-to-one isomorphic correspondence between a class of transseries and functions, Écalle also vastly generalized Borel summation.

Borel-Écalle-(BE) summation extends usual summation, it does not depend on how the transseries was obtained, and it is a perfect isomorphism between expansions and functions. The sum of an BE summable transseries is, by definition an analyzable function.

BE summable transseries are known to be closed under operations (i)–(iii) but not yet (iv). BE summability has been shown to apply generic systems of linear or nonlinear ODEs, PDEs (including the Schrödinger equation, Navier-Stokes) etc, Quantum Field Theory, KAM and so on. Some concrete theorems will be given later.

The representation by transseries is effective, the function associated to a transseries closely following the behavior expressed in the successive, ordered, terms of its transseries.

Determining the transseries of a function $f$ is the “analysis” of $f$, and transseriable functions are “analyzable ”, while the opposite process, reconstruction by BE summation of a function from its transseries is known as “synthesis”. We have the following diagram



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[4] The small parameter could be the independent variable itself.

This is the only known way to close functions under the listed operations.

### 1.1f Summation of a divergent series

If we write

$$n! = \int_0^\infty e^{-t} t^n dt$$

in (1.45), it becomes

$$\int_0^\infty \sum_{n=0}^\infty e^{-t} t^n x^{-n-1} dt = \int_0^\infty \frac{e^{-np}}{1+p} dp \tag{1.50}$$

provided we can interchange summation and integration, and sum the geometric series to $1/(1+p)$ for all values of $p$, not only for $|p| < 1$.

Upon closer examination, we see that another way to view the formal calculation leading to (1.50) is to say that we first performed a term-by-term inverse Laplace transform (cf. §2.2) of the series (the inverse Laplace transform of $n!x^{-n-1}$ being $p^n$), summed the $p$-series for small $p$ (to $(1+p)^{-1}$) *analytically continued* this sum on the whole of $\mathbb{R}^+$ and then took the Laplace transform of this result. Up to analytic continuations and ordinary convergent summations, what has been done in fact is the combination Laplace inverse-Laplace transform, which is the identity. In this sense, the emergent function should inherit the (many) formal properties that are preserved by analytic continuation and convergent summation. In particular, (1.50) is a solution of (1.43). The steps we have just described define Borel summation , which applies precisely when the above steps succeed.

# Chapter 2

# Review of some basic tools

## 2.1 The Phragmén-Lindelöf theorem

This result is very useful in obtaining information about the size of a function in a sector, when the only information available is on the edges. There are other formulations, specific other unbounded regions, such as strips and half-strips. We use the following setting.

**Theorem 2.1 (Phragmén-Lindelöf)** *Let* $U$ *be the open sector between two rays from the origin, forming an angle* $\pi/\beta$, $\beta > 1/2$. *Assume* $f$ *is analytic in* $U$, *and continuous on its closure, and for some* $C_1, C_2, M > 0$ *and* $\alpha \in (0, \beta)$ *it satisfies the estimates*

$$|f(z)| \le C_1 e^{C_2 |z|^{\alpha}}; \quad z \in U; \quad |f(z)| \le M; \quad z \in \partial U \tag{2.2}$$

*Then*

$$|f(z)| \le M; \quad z \in U \tag{2.3}$$

**PROOF** By a rotation we can make $U = \{z : 2|\arg z| < \pi/\beta\}$. Making a cut in the complement of $U$ we can define an analytic branch of the log in $U$ and, with it, an analytic branch of $z^{\beta}$. By taking $f_1(z) = f(z^{1/\beta})$, we can assume without loss of generality that $\beta = 1$ and $\alpha \in (0, 1)$ and then $U = \{z : |\arg z| < \pi/2\}$. Let $\alpha' \in (\alpha, 1)$ and consider the analytic function

$$e^{-C_2 z^{\alpha'}} f(z) \tag{2.4}$$

Since $|e^{-C_2 z^{\alpha'}}| < 1$ in $U$ (check) and $|e^{-C_2 z^{\alpha'} + C_2 z^{\alpha}}| \to 0$ as $|z| \to \infty$ on the half circle $|z| = R, \operatorname{Re} z \ge 0$ (check), the usual maximum modulus principle completes the proof. $\square$

**Exercise 2.5** *Assume* $f$ *is entire,* $|f(z)| \le C_1 e^{|az|}$ *in* $\mathbb{C}$ *and* $|f(z)| \le C e^{-|z|}$ *in a sector of opening more than* $\pi$. *Show that* $f$ *is identically zero. (A similar statement holds under much weaker assumptions, see Exercise 2.28.)*

## 2.2 Laplace and inverse Laplace transforms

Let $F \in L^1(\mathbb{R}^+)$ ($|F|$ is integrable on $[0,\infty)$). Then the Laplace transform

$$\mathcal{L}F := \int_0^\infty e^{-px}F(p)dp \tag{2.6}$$

is analytic in $\mathbb{H}$ and continuous in its closure, $\overline{\mathbb{H}}$. (Obviously, we could allow $Fe^{-|\alpha|p} \in L^1$ and then $\mathcal{L}F$ exists for $\mathrm{Re}\, x > |\alpha|$.)

**Proposition 2.7** *If $F \in L^1(\mathbb{R}^+)$ then $\mathcal{L}F$ is analytic $\mathbb{H}$ and continuous on the imaginary axis $\partial\mathbb{H}$, and $\mathcal{L}\{F\}(x) \to 0$ as $x \to \infty$ in $\mathbb{H}$.*

*Proof.* Continuity and analyticity are preserved by integration against a finite measure ($F(p)dp$). Equivalently, these properties follow by dominated convergence[1] =, as $\epsilon \to 0$, of $\int_0^\infty e^{-isp}(e^{-ip\epsilon}-1)F(p)dp$ and of $\int_0^\infty e^{-xp}(e^{-p\epsilon}-1)\epsilon^{-1}F(p)dp$ respectively, the last integral for $\mathrm{Re}\,(x) > 0$. The stated limit also follows easily from dominated convergence, if $|\arg(x) \pm \pi/2| > \delta$; the general case follows from the case $|\arg(x)| = \pi/2$ which is a consequence of the Riemann-Lebesgue lemma, see Proposition 3.55 below. □

**Remark 2.8** Extending $F$ on $\mathbb{R}^-$ by zero and using the continuity in $x$ proved in Proposition 2.7, we have $\mathcal{L}\{F\}(it) = \int_{-\infty}^\infty e^{-ipt}F(p)\mathrm{d}p = \hat{\mathcal{F}}F$. In this sense, the Laplace transform can be identified with the (analytic continuation of) the Fourier transform, restricted to functions vanishing on a half-line.

**First inversion formula.**

Let $\mathcal{H}$ denote the space of analytic functions in $\mathbb{H}$.

**Proposition 2.9** *(i) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{H}$ and $\|\mathcal{L}\{F\}\|_\infty \le \|F\|_1$.*
*(ii) $\mathcal{L} : L^1 \mapsto \mathcal{L}(L^1)$ is invertible, and the inverse is given by*

$$F(x) = \hat{\mathcal{F}}^{-1}\{\mathcal{L}\{F\}(it)\}(x) \tag{2.10}$$

*for ( $x \in \mathbb{R}^+$) where $\hat{\mathcal{F}}$ is the Fourier transform.*

**PROOF** Part (i) is immediate, since $|e^{-xp}| \le 1$. (ii) follows from Remark 2.8. ▯

**Lemma 2.11 (Uniqueness)** *Assume $F \in L^1(\mathbb{R}^+)$ and $\mathcal{L}F = 0$ for a set of $x$ with an accumulation point. Then $F = 0$ a.e.*

---

[1] See e.g. [47]. Essentially, if the functions $|f_n| \in L^1$ are bounded uniformly in $n$ by $g \in L^1$ and they converge pointwise (except possibly on a set of measure zero), then $\lim f_n \in L^1$ and $\lim \int f_n = \int \lim f_n$.

**PROOF** By analyticity, $\mathcal{L}F = 0$ in $\mathbb{H}$. The rest follows from Proposition 2.9. ∎

**Second inversion formula.**

Analytic functions in $\mathcal{H}$ with sufficient decay can be written as Laplace transforms.

**Proposition 2.12** *(i) Assume $f$ is analytic in an open sector $\mathbb{H}_\delta := \{x : |\arg(x)| < \pi/2 + \delta\}$, $\delta \geq 0$ and is continuous on $\partial\mathbb{H}_\delta$, and that for some $K > 0$ and any $x \in \overline{\mathbb{H}_\delta}$ we have*

$$|f(x)| \leq K(|x|^2 + 1)^{-1} \tag{2.13}$$

*Then $\mathcal{L}^{-1}f$ is well defined by*

$$F = \mathcal{L}^{-1}f = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}t\, e^{pt} f(t) \tag{2.14}$$

*and*

$$\int_0^\infty \mathrm{d}p\, e^{-px} F(p) = \mathcal{L}\mathcal{L}^{-1}f = f(x)$$

*In addition $\|\mathcal{L}^{-1}\{f\}\|_\infty \leq K/2$ and $\mathcal{L}^{-1}\{f\} \to 0$ as $p \to \infty$.*

*(ii) If $\delta > 0$ then $F = \mathcal{L}^{-1}f$ is analytic in the sector $S = \{p \neq 0 : |\arg(p)| < \delta\}$. In addition, $\sup_S |F| \leq K\pi$ and $F(p) \to 0$ as $p \to \infty$ in $S$.*

**PROOF**

(i) We have

$$\int_0^\infty \mathrm{d}p\, e^{-px} \int_{-\infty}^{\infty} i\mathrm{d}s\, e^{ips} f(is) = \int_{-\infty}^{\infty} i\mathrm{d}s\, f(is) \int_0^\infty \mathrm{d}p\, e^{-px} e^{ips} \tag{2.15}$$

$$= \int_{-i\infty}^{i\infty} f(z)(x-z)^{-1}\mathrm{d}z = 2\pi i f(x) \tag{2.16}$$

where we applied Fubini's theorem[2] and then pushed the contour of integration past $x$ to infinity. The norm is obtained by majorizing $|fe^{px}|$ by $K(|x^2| + 1)^{-1}$.

(ii) We have for any $\delta' < \delta$, by (2.13),

---

[2] This theorem addresses commutation of orders of integration, see [47]. Essentially, if $f \in L^1(A \times B)$, then $\int_{A\times B} f = \int_A \int_B f = \int_B \int_A f$.

$$\int_{-i\infty}^{i\infty} \mathrm{d}s\, e^{ps} f(s) = \left(\int_{-i\infty}^{0} + \int_{0}^{i\infty}\right) \mathrm{d}s\, e^{ps} f(s)$$
$$= \left(\int_{-i\infty e^{-i\delta'}}^{0} + \int_{0}^{i\infty e^{i\delta'}}\right) \mathrm{d}s\, e^{ps} f(s) \quad (2.17)$$

Given the exponential decay of the integrand, analyticity in (2.17) is clear.

For (ii) we note that (i) applies in $\bigcup\limits_{|\delta'|<\delta} e^{i\delta'}\mathbb{H}_0$. $\square$

Many cases can be reduced to (2.13) after transformations. For instance if $g = \sum_{j=1}^{N} a_j x^{-k_j} + f(x)$, with $k_j > 0$ and $f$ satisfies the other assumptions above, then $g$ is inverse Laplace transformable since $g - f$ is explicitly transformable.

□

**Proposition 2.18** *Let $F$ be analytic in the open sector $S_p = \{e^{i\phi}\mathbb{R}^+ : \phi \in (-\delta, \delta)\}$ and such that $|F(|x|e^{i\phi})| \le g(|x|) \in L^1[0,\infty)$. Then $f = \mathcal{L}F$ is analytic in the sector $S_x = \{x : |\arg(x)| < \pi/2 + \delta\}$ and $f(x) \to 0$ as $|x| \to \infty, \arg(x) = \theta \in (-\pi/2 - \delta, \pi/2 + \delta)$.*

**PROOF** Because of the analyticity of $F$ and the decay conditions for large $p$, the path of Laplace integration can be rotated by any angle $\phi \in (-\delta, \delta)$ without changing $(\mathcal{L}F)(x)$ (see also §4.4d ). This means Proposition 2.7 applies in $\cup_{|\phi|<\delta} e^{i\phi}\mathbb{H}$. □

**Note** that without further assumptions on $\mathcal{L}F$, $F$ is *not* necessarily analytic at $p = 0$.

### 2.2a Inverse Laplace space convolution

If $f$ and $g$ satisfy the assumptions of Proposition 2.12 then so does $fg$ and we have

$$\mathcal{L}^{-1} fg = (\mathcal{L}^{-1} f) * (\mathcal{L}^{-1} g) \quad (2.19)$$

where

$$(F * G)(p) := \int_0^p F(s)G(p-s)ds \quad (2.20)$$

This formula is easily checked by taking the Laplace transform of (2.20) and justifying the change of variables $p_1 = s, p_2 = p - s$.

Note that $\mathcal{L}(pF) = (\mathcal{L}F)'$.

We can draw interesting conclusions about $F$ from the rate of decay of $\mathcal{L}F$ alone.

**Proposition 2.21 (Lower bound on decay rates of Laplace transforms)** *Assume $F \in L^1(\mathbb{R}^+)$ and for some $\epsilon > 0$ we have*

$$\mathcal{L}F(x) = O(e^{-\epsilon x}) \quad \text{as} \quad x \to +\infty \tag{2.22}$$

*Then $F = 0$ on $[0, \epsilon]$.*

**PROOF** We write

$$\int_0^\infty e^{-px} F(p)dp = \int_0^\epsilon e^{-px} F(p)dp + \int_\epsilon^\infty e^{-px} F(p)dp \tag{2.23}$$

we note that

$$\Big| \int_\epsilon^\infty e^{-px} F(p)dp \Big| \le e^{-\epsilon x} \int_\epsilon^\infty |F(p)|dp \le e^{-p\epsilon} \|F\|_1 = O(e^{-\epsilon x}) \tag{2.24}$$

Therefore

$$g(x) = \int_0^\epsilon e^{-px} F(p)dp = O(e^{-\epsilon x}) \quad \text{as} \quad x \to +\infty \tag{2.25}$$

The function $g$ is entire (check). Let $h(x) = e^{\epsilon x} g(x)$. Then $h$ is entire and uniformly bounded for $x \in \mathbb{R}$ (since by assumption, for some $x_0$ and all $x > x_0$ we have $h \le C$ and by continuity $\max |h| < \infty$ on $[0, x_0]$). The function $h$ is bounded in $\mathbb{C}$ by $Ce^{2\epsilon|x|}$, for some $C > 0$, and it is manifestly bounded by $\|F\|_1$ for $x \in i\mathbb{R}$. By Phragmén-Lindelöf (first applied in the first quadrant and then in the fourth quadrant, with $\beta = 2, \alpha = 1$) $h$ is bounded in $\overline{\mathbb{H}}$. Now, for $x = -s < 0$ we have

$$e^{-s\epsilon} \int_0^\epsilon e^{sp} F(p)dp \le \int_0^\epsilon |F(p)| \le \|F\|_1 \tag{2.26}$$

Once more by Phragmén-Lindelöf (again applied twice) $h$, is bounded in the closed left half plane thus bounded in $\mathbb{C}$, and it is therefore a constant. But, by the Riemann-Lebesgue lemma, $h \to 0$ for $x = is$ when $s \to +\infty$. Thus $g = h \equiv 0$. Noting that $g = \mathcal{L}\chi_{[0,\epsilon]} F$ the result follows from (2.11). ◻

**Corollary 2.27** *Assume $F \in L^1$ and $\mathcal{L}F = O(e^{-AX})$ as $x \to +\infty$ for all $A > 0$. Then $F = 0$.*

**PROOF** This is straightforward. ◻

As we see, uniqueness of the Laplace transform can be reduced to estimates.

**Exercise 2.28 (*)** *Assume $f$ is analytic for $|z| > z_0$ in a sector $S$ of opening more than $\pi$ and that $|f(z)| \le Ce^{-|z|}$ in $S$, or that $f$ is bounded in $S$ and*

$|f(z)| \leq Ce^{-c\mathrm{Re}\,(z)}$ *if* $\mathrm{Re}\,(z) > z_0$ . *Show that* $f$ *is identically zero. Compare with Exercise 2.5.*

*(Hint: take a suitable inverse Laplace transform of* $f$ *and show that it is analytic in the unit disk and in some sector. Find a ray in the unit disk along which* $\mathcal{L}^{-1}f$ *vanishes.)*

See also Example 2 in §3.6.

# Chapter 3

# Classical asymptotics

## 3.1 Asymptotics of integrals: first results

**Example: Integration by parts and elementary truncation to the least term**. A solution of the differential equation

$$f' - 2xf + 1 = 0 \tag{3.1}$$

is related to the the complementary error function:

$$E(x) = e^{x^2} \int_x^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2} e^{x^2} \mathrm{erfc}(x) \tag{3.2}$$

Let us find the asymptotic behavior of $E(x)$ for $x \to +\infty$. One very simple technique is integration by parts, done in a way in which the integrated terms become successively smaller. A decomposition is sought such that in the identity $fdg = d(fg) - gdf$ we have $gdf \ll fdg$ in the region of interest. Note that there is no manifest perfect derivative in the integrand, but we can create a suitable one by writing $e^{-s^2} ds = -(2s)^{-1} d(e^{-s^2})$.

$$\begin{aligned} E(x) = \frac{1}{2x} - \frac{e^{x^2}}{2} \int_x^\infty \frac{e^{-s^2}}{s^2} ds = \frac{1}{2x} - \frac{1}{4x^3} + \frac{3e^{x^2}}{4} \int_x^\infty \frac{e^{-s^2}}{s^4} ds = ... \\ = \sum_{k=0}^{m-1} \frac{(-1)^k}{2\sqrt{\pi}} \frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} + \frac{(-1)^m e^{x^2} \Gamma(m+\frac{1}{2})}{\sqrt{\pi}} \int_x^\infty \frac{e^{-s^2}}{s^{2m}} ds \end{aligned} \tag{3.3}$$

On the other hand, we have, by L'Hospital

$$\left( \int_x^\infty \frac{e^{-s^2}}{s^{2m}} ds \right) \left( \frac{e^{-x^2}}{x^{2m+1}} \right)^{-1} \to \frac{1}{2} \text{ as } x \to \infty \tag{3.4}$$

and the last term in (3.3) is $O(x^{-2m-1})$. It is also clear that the remainder in (3.3) is alternating and thus

$$\sum_{k=0}^{m-1} \frac{(-1)^k}{2\sqrt{\pi}} \frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} \le E(x) \le \sum_{k=0}^{m} \frac{(-1)^k}{2\sqrt{\pi}} \frac{\Gamma(k+\frac{1}{2})}{x^{2k+1}} \tag{3.5}$$

if $m$ is even.

**Remark 3.6** *In Exercise 1.37, we conclude $F(z)$ has a Taylor series that at zero,*

$$\tilde{F}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2\sqrt{\pi}} \Gamma(k+\frac{1}{2}) z^{2m+1} \tag{3.7}$$

*and $F(z)$ is $C^\infty$ on $\mathbb{R}$ and analytic away from zero.*

**Exercise 3.8** Show that $z = 0$ is an isolated singularity of $F(z)$. Using Remark 1.31, show that $F$ is unbounded as 0 is approached along some directions in the complex plane.

**Exercise 3.9** Given $x$, find $m = m(x)$ so that the accuracy in approximating $E(x)$ by truncated series, see (3.5), is highest. Show that this $m$ is (approximately) the one that minimizes the $m$-th term of the series for the given $x$ ("least term"). For $x = 10$ the relative error in calculating $E(x)$ in this way is about $5.3 \cdot 10^{-42}\%$ (check).

**Notes.** (1) The series (3.7) is not related in any immediate way to the Laurent series of $F$ at 0. Laurent series converge. Think carefully about this distinction and why the positive index coefficients of the Laurent series and the coefficients of (3.7) do not coincide.

(2) The rate of convergence of the Laurent series of $F$ is slower as 0 is approached, quickly becoming numerically useless. By contrast, the precision gotten from (3.5) for $z = 1/x = 0.1$ is exceptionally good. However, of course the series used in (3.5) are divergent and cannot be used to calculate $F$ *exactly* $z \neq 0$, as explained in §1.1e .

### 3.1a Discussion: Laplace's method for solving linear ODEs with coefficients linear in $x$

Equations of the form

$$\sum_{k=0}^{n} (a_k x + b) y^{(k)} = 0 \tag{3.10}$$

can be solved through explicit integral representations of the form

$$\int_{\mathcal{C}} e^{-xp} F(p) dp \tag{3.11}$$

with $F$ expressible by quadratures and where $\mathcal{C}$ is a contour in $\mathbb{C}$, which has to be chosen subject to the following **conditions**:

- The integral (3.11) should be convergent, together with sufficiently many $x$-derivatives, and not identically zero.

- The function $e^{-xp}F(p)$ should vanish with sufficiently many derivatives at the endpoints, or more generally, the contributions from the endpoints when integrating by parts should cancel out.

Then it is clear that the equation satisfied by $F$ is first order linear homogeneous, and then it can be solved by quadratures. It is not very difficult to analyze this method in general, but this would be beyond the purpose of this course. We illustrate the method on Airy's equation

$$y'' = xy \tag{3.12}$$

Under the restrictions above we can check that $F$ satisfies the equation

$$p^2 F = F' \tag{3.13}$$

Then $F = \exp(p^3/3)$ and we get a solution in the form

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{-xp+p^3/3}dp \tag{3.14}$$

along some curve that crosse the real line. It is easy to check the restrictions for $x \in \mathbb{R}^+$, except for the fact that the integral is not identically zero. We can achieve this at the same time as finding the asymptotic behavior of the Ai function. Solutions of differential or difference equations can be represented in the form

$$F(x) = \int_a^b e^{xg(s)}f(s)ds \tag{3.15}$$

with simpler $g$ and $f$ in wider generality, as it will become clear in later chapters.

---

## 3.2 Laplace, stationary phase, saddle point methods and Watson's lemma

These deal with the behavior for large $x$ of integrals of the form (3.15). We distinguish three particular cases: (1) The case where all parameters are real (addressed by the so-called Laplace method); (2) The case where everything is real except for $x$ which is purely imaginary (stationary phase method) and (3) The case when $f$ and $g$ are analytic (steepest descent method–or saddle point method). In this latter case, the integral may also come as a contour integral along some path. In many cases, all three types of problems can be brought to a canonical form, to which Watson's lemma applies.

## 3.3 The Laplace method

Even when very little regularity can be assumed about the functions, we can still infer something about the large $x$ behavior of (3.15).

**Proposition 3.16** *If* $f(s) \in L^\infty([a,b])$ *then*

$$\lim_{x\to+\infty}\left(\int_a^b e^{xf(s)}ds\right)^{1/x} = e^{\|f\|_\infty}$$

**PROOF** This is simply the fact that $\|f\|_n \to \|f\|_\infty$ [47]. See also the note below. ∎

**Note**. The intuitive idea in estimating this kind of integrals is that if $x$ is large and $g$ has a unique absolute maximum, the absolute maximum in $s$ of $\phi(x;s) = \exp(xg(s))$ exceeds, for large $x$, by a large amount the value of $\phi$ at any point neighboring point. Then the contribution of the integral outside a tiny neighborhood of the maximum point is negligible.

In a neighborhood of the maximum point, both $f$ and $g$ are very well approximated by their local expansion. For example, assume the absolute maximum is at the left end, $x = 0$ and we have $f(0) \neq 0$ and $g'(0) = -\alpha < 0$. Then,

$$\int_0^a e^{xg(s)}f(s)ds \approx \int_0^a e^{xg(0)-\alpha xs}f(0)ds$$
$$\approx f(0)e^{xg(0)}\int_0^\infty e^{-\alpha xs}ds = f(0)e^{xg(0)}\frac{1}{\alpha x} \quad (3.17)$$

Watson's lemma, proved in the sequel, is perhaps the ideal way to make the previous argument rigorous, but for the moment we just convert the approximate reasoning into a proof following the same line of reasoning.

**Proposition 3.18** *(the case when $g$ is maximum at one endpoint). Assume $f$ is continuous on $[a,b]$, $f(a) \neq 0$, $g$ is in $C^1[a,b]$ and $g' < -\alpha < 0$ on $[a,b]$. Then*

$$J_x := \int_a^b f(s)e^{xg(s)}ds = \frac{f(a)e^{xg(a)}}{x|g'(a)|}(1+o(1)) \quad (x\to+\infty) \qquad (3.19)$$

**Note:** Since the derivative of $g$ enters in the final result, regularity is clearly needed.

**PROOF** Without loss of generality, we may assume $a = 0$, $b = 1$, $f(0) > 0$. Let $\epsilon$ be small enough and choose $\delta$ such that if $x < \delta$ we have $|f(x)-f(0)| < \epsilon$

and $|g'(x) - g'(0)| < \epsilon$. We write

$$\int_0^1 f(s)e^{xg(s)}ds = \int_0^\delta f(s)e^{xg(s)}ds + \int_\delta^1 f(s)e^{xg(s)}ds \tag{3.20}$$

the last integral in (3.20) is bounded by

$$\int_\delta^1 f(s)e^{xg(s)}ds \le \|f\|_\infty e^{xg(0)}e^{x(g(\delta)-g(0))} \tag{3.21}$$

For the middle integral in (3.20) we have

$$\begin{aligned}\int_0^\delta f(s)e^{xg(s)}ds &\le (f(0)+\epsilon)\int_0^\delta e^{x\left[g(0)+(g'(0)+\epsilon)s\right]}ds \\ &\le -\frac{e^{xg(0)}}{x}\frac{f(0)+\epsilon}{g'(0)+\epsilon}\left[1-e^{x\delta(g'(0)+\epsilon)}\right]\end{aligned} \tag{3.22}$$

Combining these estimates, as $x \to \infty$ we thus obtain

$$\limsup_{x\to\infty} xe^{-xg(0)}\int_0^1 f(s)e^{xg(s)}ds \le -\frac{f(0)+\epsilon}{g'(0)+\epsilon} \tag{3.23}$$

A lower bound is obtained in a similar way. Since $\epsilon$ is arbitrary, the result follows. ☐

When the maximum of $g$ is reached inside the interval of integration, sharp estimates require even more regularity.

**Proposition 3.24** *(Interior maximum) Assume $f \in C[-1,1]$, $g \in C^2[-1,1]$ has a unique absolute maximum (say at $x = 0$) and that $f(0) \neq 0$ (say $f(0) > 0$) and $g''(0) < 0$. Then*

$$\int_{-1}^1 f(s)e^{xg(s)}ds = \sqrt{\frac{2\pi}{x|g''(0)|}}f(0)e^{xg(0)}(1+o(1)) \quad (x \to +\infty) \tag{3.25}$$

**PROOF** The proof is similar to the previous one. Let $\epsilon$ be small enough and let $\delta$ be such that $|s| < \delta$ implies $|g''(s)-g''(0)| < \epsilon$ and also $|f(s)-f(0)| < \epsilon$. We write

$$\int_{-1}^1 e^{xg(s)}f(s)ds = \int_{-\delta}^\delta e^{xg(s)}f(s)ds + \int_{|s|\ge\delta} e^{xg(s)}f(s)ds \tag{3.26}$$

The last term will not contribute in the limit since by our assumptions for some $\alpha > 0$ and $|s| > \delta$ we have $g(s) - g(0) < -\alpha < 0$ and thus

$$e^{-xg(0)}\sqrt{x}\int_{|s|\ge\delta} e^{xg(s)}f(s)ds \le 2\sqrt{x}\|f\|_\infty e^{-x\alpha} \to 0 \text{ as } x \to \infty \tag{3.27}$$

On the other hand,

$$\int_{-\delta}^{\delta} e^{xg(s)} f(s)ds \le (f(0)+\epsilon) \int_{-\delta}^{\delta} e^{xg(0)+\frac{x}{2}(g''(0)+\epsilon)s^2} ds$$

$$\le (f(0)+\epsilon)e^{xg(0)} \int_{-\infty}^{\infty} e^{xg(0)+\frac{x}{2}(g''(0)+\epsilon)s^2} ds = \sqrt{\frac{2\pi}{|g''(0)-\epsilon|}}(f(0)+\epsilon)e^{xg(0)} \tag{3.28}$$

An inequality in the opposite direction follows in the same way, replacing $\le$ with $\ge$ and $\epsilon$ with $-\epsilon$ in the first line of (3.28), and then noting that

$$\frac{\int_{-a}^{a} e^{-xs^2} ds}{\int_{-\infty}^{\infty} e^{-xs^2} ds} \to 1 \quad \text{as } x \to \infty \tag{3.29}$$

as can be seen by changing variables to $u = sx^{-\frac{1}{2}}$. ☐

With appropriate decay conditions, the interval of integration does not have to be compact. For instance, let $J \subset \mathbb{R}$ be an interval (finite or not) and $[a,b] \subset J$.

**Proposition 3.30** *(Interior maximum, noncompact interval) Assume* $f \in C[a,b] \cap L^\infty(J)$, $g \in C^2[a,b]$ *has a unique absolute maximum at* $x = c$ *and that* $f(c) \ne 0$ *and* $g''(c) < 0$. *Assume further that* $g$ *is measurable in* $J$ *and* $g(c) - g(s) = \alpha + h(s)$ *where* $\alpha > 0$, $h(s) > 0$ *on* $J \setminus [a,b]$ *and* $e^{-h(s)} \in L^1(J)$. *Then,*

$$\int_A^B f(s)e^{xg(s)ds} = \sqrt{\frac{2\pi}{x|g''(c)|}} f(c)e^{xg(c)}(1+o(1)) \quad (x \to +\infty) \tag{3.31}$$

**PROOF** This case reduces to the compact interval case by noting that

$$\left| \sqrt{x} e^{-xg(c)} \int_{J\setminus[a,b]} e^{xg(s)} f(s)ds \right| \le \sqrt{x}\|f\|_\infty e^{-x\alpha} \int_J e^{-xh(s)} ds$$
$$\le Const.\sqrt{x}e^{-x\alpha} \to 0 \text{ as } x \to \infty \tag{3.32}$$

☐

*Example.* We see that the last proposition applies to the Gamma function by writing

$$n! = \int_0^\infty e^{-t}t^n dt = n^{n+1} \int_0^\infty e^{n(-s+\ln s)} ds \tag{3.33}$$

whence we get Stirling's formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1+o(1)); \quad n \to +\infty$$

## 3.4 Watson's lemma

In view of the wide applicability of BE summability as we shall see later, solutions to many problems admit representations as Laplace transforms

$$(\mathcal{L}F)(x) := \int_0^\infty e^{-xp} F(p)dp \tag{3.34}$$

For the error function note that

$$\int_x^\infty e^{-s^2} ds = x \int_1^\infty e^{-x^2u^2} du = \frac{x}{2} e^{-x^2} \int_0^\infty \frac{e^{-x^2 p}}{\sqrt{p+1}} dp$$

For the Gamma function, writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ in (3.33) we can make the substitution $t - \ln t = p$ in each integral and obtain (see §3.9c )

$$n! = n^{n+1} e^{-n} \int_0^\infty e^{-np} G(p)dp$$

Watson's lemma provides the asymptotic series at infinity of $(\mathcal{L}F)(x)$ in terms of the asymptotic series of $F(p)$ at zero.

**Lemma 3.35 (Watson's lemma)** *Let* $F \in L^1(\mathbb{R}^+)$ *and assume* $F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1+\beta_2-1}$ *as* $p \to 0^+$ *for some constants* $\beta_i$ *with* $\mathrm{Re}\,(\beta_i) > 0$, $i = 1, 2$. *Then, for* $a \le \infty$,

$$f(x) = \int_0^a e^{-xp} F(p)dp \sim \sum_{k=0}^\infty c_k \Gamma(k\beta_1 + \beta_2) x^{-k\beta_1 - \beta_2}$$

*along any ray* $\rho$ *in* $\mathbb{H}$.

**Remark 3.36** (i) Clearly, the asymptotic formula holds if $\int_0^\infty$ is replaced by $\int_0^a$, $a > 0$, since we can always extend $F$ and the integral by zero for $x > a$. (ii) The presence of $\Gamma(k\beta_1 + \beta_2)$ makes the $x$ series often divergent even when $F$ is analytic at zero. However, the asymptotic series of $f$ is still the term-by-term Laplace transform of the series of $F$ at zero, whether $a$ is finite or not or the series converges or not. This freedom in choosing $a$ shows that some information is lost.

**PROOF** Induction, using the conclusion of Lemma 3.37 below. □ ▯

**Lemma 3.37** *Let* $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\phi}$, $\rho > 0$, $\phi \in (-\pi/2, \pi/2)$ *and assume*

$$F(p) \sim p^\beta \quad \text{as } p \to 0^+$$

*with* $\mathrm{Re}\,(\beta) > -1$. *Then*

$$\int_0^\infty F(p)e^{-px}dp \sim \Gamma(\beta+1)x^{-\beta-1} \quad (\rho \to \infty)$$

**PROOF** If $U(p) = p^{-\beta}F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let $\chi_A$ be the characteristic function of the set $A$ and $\phi = \arg(x)$. We choose $C$ and $a$ positive so that $|F(p)| < C|p^\beta|$ on $[0,a]$. Since

$$\left|\int_a^\infty F(p)e^{-px}\mathrm{d}p\right| \le e^{-xa}\|F\|_1 \tag{3.38}$$

we have by dominated convergence, and after the change of variable $s = p/|x|$,

$$x^{\beta+1}\int_0^\infty F(p)e^{-px}\mathrm{d}p = e^{i\phi(\beta+1)}\int_0^\infty s^\beta U(s/|x|)\chi_{[0,a]}(s/|x|)e^{-se^{i\phi}}\mathrm{d}s$$
$$+ O(|x|^{\beta+1}e^{-xa}) \to \Gamma(\beta+1) \quad (|x| \to \infty) \tag{3.39}$$

$\square$

### 3.4a The Borel-Ritt lemma

Any asymptotic series at infinity is the asymptotic series in a half plane of some (vastly many in fact) entire functions. First a weaker result.

**Proposition 3.40** *Let* $\tilde{f}(z) = \sum_{k=0}^\infty a_k z^k$ *be a power series. There exists a function* $f$ *such that* $f(z) \sim \tilde{f}(z)$ *as* $z \to 0$.

**PROOF** The following elementary line of proof is reminiscent of optimal truncation of series. By Remark 1.31 we can assume, without loss of generality, that the series has zero radius of convergence. Let $z_0 > 0$ be small enough and for every $z$, $|z| < z_0$, define $N(z) = \max\{N : \forall\ n \le N,\ |a_n z^{n/2}| \le 2^{-n}$. We have $N(z) < \infty$, otherwise, by Abel's theorem, the series would have nonzero radius of convergence. Noting that for any $n$ we have $n \ln|z| \to -\infty$ as $|z| \to 0$ it follows that $N(z)$ is nonincreasing as $|z|$ decreases and that $N(z) \to \infty$ as $z \to 0$. Consider

$$f(z) = \sum_{j=0}^{N(z)} a_n z^n$$

Let $N$ be given and choose $z_N$; $|z_N| < 1$ such that $N(z_N) \ge N$. For $|z| < |z_N|$ we have $N(z) \ge N(z_N) \ge N$ and thus

$$\left|f(z) - \sum_{n=0}^N a_n z^n\right| = \left|\sum_{n=N+1}^{N(z)} a_n z^n\right| \le \sum_{j=N+1}^{N(z)} |z^{j/2}|2^{-j} \le |z|^{N/2+1/2}$$

Using Lemma 1.12, the proof follows. ☐

The function $f$ is certainly not unique. Given a power series there are many functions asymptotic to it. Indeed there are many functions asymptotic to the (identically) zero power series at zero, in any sectorial punctured neighborhood of zero in the complex plane, and even on the Riemann surface of the log on $\mathbb{C} \setminus \{0\}$, e.g. $e^{-x^{-1/n}}$ has this property in a sector of width $2n\pi$.

**Lemma 3.41 (Borel-Ritt)** *Given a formal power series* $\tilde{f} = \sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}}$ *there exists an entire function* $f(x)$*, of exponential order one (see proof below), which is asymptotic to* $\tilde{f}$ *in* $\mathbb{H}$*, i.e., if* $\phi \in (-\pi/2, \pi/2)$ *then*

$$f(x) \sim \tilde{f} \text{ as } x = \rho e^{i\phi}, \quad \rho \to +\infty$$

**PROOF** Let $\tilde{F} = \sum_{k=0}^{\infty} \frac{c_k}{(k-1)!} p^{k-1}$, let $F(p)$ be a function asymptotic to $\tilde{F}$ as in Proposition 3.40. Then clearly the function

$$f(x) = \int_0^1 e^{-xp} F(p) dp$$

is entire, bounded by $Const.e^{|x|}$ *i.e.* the exponential order is one, and, by Watson's lemma it has the desired properties.

☐

**Exercises.**

(1) How can this method be modified to give a function analytic in a sector of opening $2\pi n$ for an arbitrary fixed $n$ which is asymptotic to $\tilde{f}$?

(2) Assume $F$ is bounded on $[0, 1]$ and has an asymptotic expansion $F(t) \sim \sum_{k=0}^{\infty} c_k t^k$ as $t \to 0^+$. Let $f(x) = \int_0^1 e^{-xp} F(p) dp$ (a) Find necessary and sufficient conditions on $F$ such that $\tilde{f}$, the asymptotic power series of $f$ for large positive $x$, is a convergent series for $|x| > R > 0$. (b) Assume that $\tilde{f}$ *converges to* $f$. Show that $f$ is zero.

(c) Show that in case (a) if $F$ is analytic in a neighborhood of $[0, 1]$ then $f = \tilde{f} + e^{-x} \tilde{f}_1$ where $\tilde{f}_1$ is convergent for $|x| > R > 0$.

(3) The width of the sector in Proposition 3.41 cannot be extended to more than a half plane: Show that if $f$ is entire, of exponential order one, and bounded in a sector of opening exceeding $\pi$ then it is constant. (This follows immediately from the Phragmén-Lindelöf principle; an alternative proof can be derived from elementary properties of Fourier transforms and contour deformation.) The exponential order has to play a role in the proof: check that the function $\int_0^{\infty} e^{-px-p^2} dp$ is bounded for $\arg(x) \in (-\frac{3\pi}{4}, \frac{3\pi}{4})$. How wide can such a sector be made?

### 3.4b Laplace's method revisited

(i) **Absolute maximum at left endpoint with nonvanishing derivative**.

**Proposition 3.42** *Let $g$ be analytic (smooth) on $[a,b]$ where $g' < -\alpha < 0$. Then the problem of finding the large $x$ behavior of $F$ in (3.15) is analytically (respectively smoothly) conjugated to the canonical problem of the large $x$ behavior of*

$$\int_{g(a)}^{g(b)} e^{xs}H(s)ds = e^{xg(a)}\int_0^{g(a)-g(b)} e^{-xu}H(g(a)-u)du \tag{3.43}$$

*with $H(s) = f(\varphi(s))\varphi'(s)$.*

This just means that we can transform to (3.43) by analytic (smooth) changes of variable; in this case, the change is $g(s) = u$, $\varphi = g^{-1}$. The proof of smoothness is immediate, and we leave it to the reader. Note that we have not required that $f(0) \neq 0$ anymore. If $H$ is smooth and some derivative at zero is nonzero, Watson's lemma clearly provides the asymptotic expansion of the last integral in (3.43). The asymptotic series is dual, as in Lemma 3.35 to the series of $H$ at $g(a)$.

(ii) **Absolute maximum at an interior point with nonvanishing second derivative.**

**Proposition 3.44** *Let $g$ be analytic (smooth) on the interval $a \leq 0 \leq b$, $a < b$, where $g'' < -\alpha < 0$ and assume $g(0) = 0$. Then the problem of finding the large $x$ behavior of $F$ in (3.15) is analytically (respectively smoothly) conjugated to the canonical problem of the large $x$ behavior of*

$$\int_{-\sqrt{|g(a)|}}^{\sqrt{|g(b)|}} e^{-xu^2}H(u)du$$
$$= -\frac{1}{2}\int_0^{-|g(a)|} e^{-xv}H(-v^{\frac{1}{2}})v^{-\frac{1}{2}}dv + \frac{1}{2}\int_0^{|g(b)|} e^{-xv}H(v^{\frac{1}{2}})v^{-\frac{1}{2}}dv \tag{3.45}$$

*with $H(s) = f(\varphi(s))\varphi'(s)$, $\varphi^2(s) = -g(u)$; Watson's lemma applies to the last representation. If $g, f \in C^k$, then $\varphi \in C^{k-1}$ and $H \in C^{k-2}$.*

**PROOF** Note that near zero we have $g = -s^2h(s)$ where $h(0) = 1$. Thus $\sqrt{h}$ is well defined and analytic (smooth) near zero; we choose the usual branch and note that the implicit function theorem applies to the equation $s\sqrt{h}(s) = u$ throughout $[a,b]$. The rest is left to the reader. ☐

**Exercise 3.46** *Assume $H \in C^\infty$ and $a > 0$. Show that the asymptotic behavior of*

$$\int_{-a}^{a} e^{-xu^2}H(u)du \tag{3.47}$$

*is given by*

$$\sum_{l=0}^{\infty}\frac{1}{2l!}\int_{-\infty}^{\infty}H^{(2l)}(0)u^{2l}e^{-xu^2}du=\frac{1}{2}\sum_{l=0}^{\infty}\frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(l+1)}H^{(2l)}(0)x^{-\frac{1}{2}-l} \tag{3.48}$$

*(This is a formal series, not expected to converge, in general.) In other words, the classical asymptotic series is obtained by formal expansion of $H$ at the critical point $x=0$ and termwise integration, extending the limits of integration to infinity and odd terms do not contribute, by symmetry. The value of "a" does not enter the formula, so once more, information is lost.*

**Exercise 3.49** *Generalize (3.25) to the case when $g \in C^4[-1,1]$ and the first three derivatives vanish at the unique point of absolute maximum, $s=0$.*

**Exercise 3.50** ** Consider the problem (3.19) with $f$ and $g$ smooth and take $a=0$ for simplicity. Show that the asymptotic expansion of the integral equals the one obtained by the following formal procedure: we expand $f$ and $g$ in Taylor series at zero, replace $f$ in the integral by its Taylor series, keep $ng'(0)$ in the exponent, reexpand $e^{ng''(0)s^2/2!+\cdots}$ in series in $s$, and integrate the resulting series term by term. The contribution of a term $cs^m$ is $c(g'(0))^{-m-1}m!/x^{-m-1}$.*

**Exercise 3.51 (*)** *Consider now the inner maximum problem in the form (3.25), with $f$ and $g$ smooth at zero. Formulate and prove a procedure similar to the one in the previous problem. Odd terms give zero contribution. An even power $cs^{2m}$ gives rise to a contribution $c2^{m+1/2}\Gamma(m+1/2)(g''(0))^{-m-1/2}x^{-m-1/2}$.*

**Exercise 3.52 (*)** *Use Exercise (3.50) to show that the Taylor coefficients of the inverse function $\phi^{-1}$ can be obtained from the Taylor coefficients of $\phi$ in the following way. Assume $\phi'(0)=1$. We let $P_n(x)$, a polynomial in $x$, be the $n-th$ asymptotic coefficient of $e^{y\phi(x/y)}$ as $y\to\infty$. The desired coefficient is $\frac{1}{n!}\int_0^{\infty}e^{-x}P_{n+1}(x)dx$.*

**Remark 3.53** There is a relatively explicit function inversion formula, first found by Lagrange, and generalized in a number of ways. It is often called the Lagrange-Bürmann inversion formula [34]. It applies to analytic functions $f$ with nonvanishing derivative at the relevant point, and it now can be shown by elementary complex analysis means:

$$f^{-1}(z)=f^{-1}(z_0)+\sum_{n=1}^{\infty}\frac{d^{n-1}}{dw^{n-1}}\left(\frac{w-f^{-1}(z_0)}{f(w)-z_0}\right)^n\Big|_{w=f^{-1}(z_0)}\frac{(z-z_0)^n}{n!} \tag{3.54}$$

## 3.5 Oscillatory integrals and the stationary phase method

In this setting, an integral of a function against a rapidly oscillating exponential becomes small as the frequency of oscillation increases. Again we first look at the case where there is minimal regularity; the following is a version of the Riemann–Lebesgue lemma.

**Proposition 3.55** *Assume* $f \in L^1[0, 2\pi]$. *Then* $\int_0^{2\pi} e^{ixt} f(t)dt \to 0$ *as* $x \to \pm\infty$. *A similar statement holds in* $L^1(\mathbb{R})$.

It is enough to show the result on a set which is dense [1] in $L^1$. Since trigonometric polynomials are dense in the continuous functions on a compact set [2], say in $C[0, 2\pi]$ in the sup norm, and thus in $L^1[0, 2\pi]$, it suffices to look at trigonometric polynomials, thus (by linearity), at $e^{ikx}$ for fixed $k$; the latter integral can be expressed explicitly and gives

$$\int_0^{2\pi} e^{ixs} e^{iks} ds = O(x^{-1}) \text{ for large } x. \qquad \square$$

No rate of decay of the integral in the Proposition follows without further knowledge about the regularity of $f$. With some regularity we have the following characterization.

**Proposition 3.56** *For* $\eta \in (0, 1]$ *let the* $C^\eta[0, 1]$ *be the Hölder continuous functions of order* $\eta$ *on* $[0, 1]$, *i.e., the functions with the property that there is some* $C$ *such that for all* $x, x' \in [0, 1]$ *we have* $|f(x) - f(x')| \le C|x - x'|^\eta$.

*(i) We have*

$$f \in C^\eta[0, 1] \Rightarrow \left|\int_0^1 f(s)e^{ixs} ds\right| \le \frac{1}{2} C\pi^\eta x^{-\eta} + O(x^{-1}) \quad \text{as} \quad x \to \infty \tag{3.57}$$

*(ii) If* $f \in L^1(\mathbb{R})$ *and* $|x|^\eta f(x) \in L^1(\mathbb{R})$ *with* $\eta \in (0, 1]$, *then its Fourier transform* $\hat{f} = \int_{-\infty}^{\infty} f(s)e^{-ixs} ds$ *is in* $C^\eta(\mathbb{R})$.

*(iii) Let* $f \in L^1(\mathbb{R})$. *If* $x^n f \in L^1(\mathbb{R})$ *with* $n \in \mathbb{N}$ *then* $\hat{f}$ *is* $n - 1$ *times differentiable, with the* $n - 1$*th derivative Lipschitz continuous. If* $e^{|Ax|} f \in L^1(\mathbb{R})$ *then* $\hat{f}$ *extends analytically in a strip of width* $|A|$ *centered on* $\mathbb{R}$.

**PROOF** (i) We have as $x \to \infty$ ($\lfloor \cdot \rfloor$ denotes the integer part)

---

[1] A set of functions $f_n$ which, collectively, are arbitrarily close to any function in $L^1$. Using such a set we can write

$$\int_0^{2\pi} e^{ixt} f(t)dt = \int_0^{2\pi} e^{ixt}(f(t) - f_n(t))dt + \int_0^{2\pi} e^{ixt} f_n(t)dt$$

and the last two integrals can be made arbitrarily small.

[2] One can associate the density of trigonometric polynomials with approximation of functions by Fourier series.

$$\left|\int_0^1 f(s)e^{ixs}ds\right| =$$
$$\left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds+\int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right|+O(x^{-1})$$
$$=\left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}}\big(f(s)-f(s+\pi/x)\big)e^{ixs}ds\right|+O(x^{-1})$$
$$\leq \sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor} C\left(\frac{\pi}{x}\right)^{\eta}\frac{\pi}{x}\leq \frac{1}{2}C\pi^{\eta}x^{-\eta}+O(x^{-1}) \quad (3.58)$$

(ii) We see that

$$\left|\frac{\hat{f}(s)-\hat{f}(s')}{(s-s')^{\eta}}\right| = \left|\int_{-\infty}^{\infty}\frac{e^{ixs}-e^{ixs'}}{x^{\eta}(s-s')^{\eta}}x^{\eta}f(x)dx\right| \leq \int_{-\infty}^{\infty}\left|\frac{e^{ixs}-e^{ixs'}}{(xs-xs')^{\eta}}\right|\left|x^{\eta}f(x)\right|dx \quad (3.59)$$

is bounded. Indeed, by elementary geometry we see that for $|\phi_1-\phi_2|<\pi$ we have

$$|\exp(i\phi_1)-\exp(i\phi_2)|\leq|\phi_1-\phi_2|\leq|\phi_1-\phi_2|^{\eta} \quad (3.60)$$

while for $|\phi_1-\phi_2|\geq\pi$ we see that

$$|\exp(i\phi_1)-\exp(i\phi_2)|\leq 2\leq 2|\phi_1-\phi_2|^{\eta}$$

(iii) Follows in the same way as (ii), using dominated convergence. ☐

**Exercise 3.61** *Complete the details of this proof. Show that for any* $\eta\in(0,1]$ *and all* $\phi_{1,2}\in\mathbb{R}$ *we have* $|\exp(i\phi_1)-\exp(i\phi_2)|\leq\sqrt{2}|\phi_1-\phi_2|^{\eta}$.

**Note.** In Laplace type integrals Watson's lemma implies that it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral whereas in Fourier-like integrals, the considerably weaker decay (3.57) is optimal as seen in the exercise below.

**Exercise 3.62 (*)** *(a)* Consider the function $f$ given by the *lacunary trigonometric series* $f(z)=\sum_{k=2^n,n\in\mathbb{N}}k^{-\eta}e^{ikz}$, $\eta\in(0,1)$. Show that $f\in C^{\eta}[0,2\pi]$. One way is to write $\phi_{1,2}$ as $a_{1,2}2^{-p}$, use the first inequality in (3.60) to estimate the terms in $f(\phi_1)-f(\phi_2)$ with $n<p$ and the simple bound $2/k^{\eta}$ for $n\geq p$. Then it is seen that $\int_0^{2\pi}e^{-iks}f(s)ds=2\pi k^{-\eta}$ and the decay of the Fourier transform is exactly given by (3.57).

(b) Use Proposition 3.56 and the result in Exercise 3.62 to show that the function $f(t)=\sum_{k=2^n,n\in\mathbb{N}}k^{-\eta}t^k$, analytic in the open unit disk, has no analytic continuation across the unit circle, that is, the unit circle is a *barrier of singularities for* $f$.

**Note 3.63** *Dense non-differentiability is the only way one can get poor decay, see also Exercise 3.71.*

**Notes.** In part (i), compactness of the interval is crucial. In fact, the Fourier transform of an $L^2(\mathbb{R})$ entire function may not necessarily decrease pointwise. Indeed, the function $\hat{f}(x) = 1$ on the interval $[n, n+e^{-n^2}]$ for $n \in \mathbb{N}$ and zero otherwise is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and further has the property that $e^{|Ax|}\hat{f} \in L^1(\mathbb{R})$ for any $A \in \mathbb{R}$, and thus $\mathcal{F}^{-1}\hat{f}$ is entire. Thus $\hat{f}$ is the Fourier transform of an entire function, it equals $\mathcal{F}^{-1}\hat{f}$ a.e., and nevertheless it does not decay pointwise as $x \to \infty$. Evidently the issue here is poor behavior of $f$ at infinity, otherwise integration by parts would show decay.

**Proposition 3.64** *Assume* $f \in C^n[a,b]$. *Then we have*

$$\int_a^b e^{ixt} f(t)dt = e^{ixa}\sum_{k=1}^{n} c_k x^{-k} + e^{ixb}\sum_{k=1}^{n} d_k x^{-k} + o(x^{-n})$$

$$= e^{ixt}\left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n}\right)\Bigg|_a^b + o(x^{-n}) \quad (3.65)$$

**PROOF** This follows by integration by parts and the Riemann-Lebesgue lemma since

$$\int_a^b e^{ixt} f(t)dt = e^{ixt}\left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1}\frac{f^{(n-1)}(t)}{(ix)^n}\right)\Bigg|_a^b + \frac{(-1)^n}{(ix)^n}\int_a^b f^{(n)}(t)e^{ixt}dt \quad (3.66)$$

□

**Corollary 3.67** *(1) Assume* $f \in C^\infty[0, 2\pi]$ *is periodic with period* $2\pi$. *Then* $\int_0^{2\pi} f(t)e^{int} = o(n^{-m})$ *for any* $m > 0$ *as* $n \to +\infty, n \in \mathbb{Z}$.

*(2) Assume* $f \in C_0^\infty[a,b]$, *a smooth function which vanishes with all derivatives at the endpoints; then* $\hat{f}(x) = \int_a^b f(t)e^{ixt} = o(x^{-m})$ *for any* $m > 0$ *as* $x \to +\infty$.

**Exercise 3.68** Show that if $f$ is analytic in a neighborhood of $[a,b]$ but not entire, then both series in (3.65) have zero radius of convergence.

**Exercise 3.69** In Corollary 3.67 (2) show that $\limsup_{x\to\infty} e^{\epsilon|x|}|\hat{f}(x)| = \infty$ for any $\epsilon > 0$ unless $f = 0$.

**Exercise 3.70** For smooth $f$, the interior of the interval does not contribute because of cancellations: rework the argument in the proof of Proposition 3.56 under smoothness assumptions. If we write $f(s+\pi/x) = f(s) + f'(s)(\pi/x) + \frac{1}{2}f''(c)(\pi/x)^2$ cancellation is manifest.

**Exercise 3.71** Show that if $f$ is piecewise differentiable and the derivative is in $L^1$, then the Fourier transform is $O(x^{-1})$.

### 3.5 .1 Oscillatory integrals with monotonic phase

**Proposition 3.72** *Let the real valued functions $f \in C^m[a,b]$ and $g \in C^{m+1}[a,b]$ and assume $g' \neq 0$ on $[a,b]$. Then*

$$\int_a^b f(t)e^{ixg(t)}dt = e^{ixg(a)}\sum_{k=1}^{m} c_k x^{-k} + e^{ixg(b)}\sum_{k=1}^{m} d_k x^{-k} + o(x^{-m}) \qquad (3.73)$$

*as $x \to \pm\infty$, where the coefficients $c_k$ and $d_k$ can be computed by Taylor expanding $f$ and $g$ at the endpoints of the interval of integration.*

This essentially follows from Proposition 3.42, since the problem is amenable by smooth transformations to the setting of Proposition 3.64. Carry out the details.

## 3.5a Stationary phase method

In general, the asymptotic behavior of oscillatory integrals of the form (3.73) comes from:

- endpoints;
- stationary points;
- singularities of $f$ or $g$.

We consider now the case when $g(s)$ has a stationary point inside the interval $[a,b]$. Then, the main contribution to the integral on the lhs of (3.73) comes from a neighborhood of the stationary point of $g$ since around that point the oscillations that make the integral small are less rapid.

We have the following result:

**Proposition 3.74** *Assume $f, g$ are real valued $C^\infty[a,b]$ functions and that $g'(c) = 0$ $g''(x) \neq 0$ on $[a,b]$. Then for any $m \in \mathbb{N}$ we have*

$$\int_a^b f(s)e^{ixg(s)}ds = e^{ixg(c)}\sum_{k=1}^{2m} c_k x^{-k/2}$$
$$+ e^{ixg(a)}\sum_{k=1}^{m} d_k x^{-k} + e^{ixg(b)}\sum_{k=1}^{m} e_k x^{-k} + o(x^{-m}) \quad (3.75)$$

*for large $x$, where the coefficients of the expansion can be calculated by Taylor expansion around $a, b$ and $c$ of the integrand as follows from the proof. In particular, we have*

$$c_1 = \sqrt{\frac{2\pi i}{g''(c)}}f(c)$$

**PROOF** Again, by smooth changes of variables, the problem is amenable to the problem of the behavior of

$$J = \int_{-a}^{a} H(u)e^{ixu^2}du \quad (3.76)$$

which is given, as we will see in a moment, by

$$J \sim \sum_{k\ge 0}\left(e^{-ixa^2}\int_{-a}^{i\infty e^{i\pi/4}} \frac{H^{(k)}(-a)}{k!}(u+a)^k e^{ixu^2}du\right.$$
$$\left.-e^{ixa^2}\int_{a}^{i\infty e^{i\pi/4}} \frac{H^{(k)}(a)}{k!}(u-a)^k e^{ixu^2}du + \int_{-\infty e^{-i\pi/4}}^{i\infty e^{i\pi/4}} \frac{H^{(2k)}(0)}{2k!}u^{2k}e^{ixu^2}du\right) \quad (3.77)$$

in the sense that $J$ minus a finite number of terms of the series is small on the scale of the last term kept.

For a conveniently small $\epsilon$ we break the integral and are left with estimating the three integrals

$$J_1 = \int_{-a}^{-\epsilon} H(u)e^{ixu^2}du; \quad J_3 = \int_{\epsilon}^{a} H(u)e^{ixu^2}du; \quad J_2 = \int_{-\epsilon}^{\epsilon} H(u)e^{ixu^2}du$$

By smooth changes of variables, $J_1$ turns into

$$\int_{\epsilon^2}^{a^2} H_1(v)e^{ixv}dv \quad (3.78)$$

where $H, H_1$ are smooth. Proposition 3.64 applies to the integral (3.78); $J_3$ is treated similarly. For the second integral we write

$$J_2 - \sum_{l=0}^{m} \frac{H^{(k)}(0)}{k!} \int_{-\epsilon}^{\epsilon} u^k e^{ixu^2} du$$

$$= \int_{-\epsilon}^{\epsilon} u^{m+1} e^{ixu^2} F(u) du = \int_0^{\epsilon^2} v^{\frac{m-1}{2}} F_1(v) e^{ixv} du \quad (3.79)$$

where $F_1$ is smooth. We can integrate by parts $m/2$ times in the last integral. Thus, combining the results from the two cases, we see that $J$ has an asymptotic series in powers of $x^{-1/2}$. Since there exists an asymptotic series, we know it is unique. Then, the series of $J$ cannot of course depend on an arbitrarily chosen parameter $\epsilon$. Thus, we do not need to keep any endpoint terms at $\pm\epsilon$: they cancel out. □

**Note** It is easy to see that in the settings of Watson's lemma and of Propositions 3.64, 3.72 and 3.74 the asymptotic expansions are differentiable, in the sense that the integral transforms are differentiable and their derivative is asymptotic to the formal derivative of the associated expansion.

### 3.5b Analytic integrands

In this case, contour deformation is used to transform oscillatory exponentials into decaying ones. A classical result in this direction is the following.

**Proposition 3.80 (Fourier coefficients of analytic functions)** *Assume $f$ is periodic of period $2\pi$, analytic in the strip $\{z : |\mathrm{Im}\,(z)| < R\}$ and continuous in its closure. Then the Fourier coefficients $c_n = (2\pi)^{-1} \int_0^{2\pi} e^{int} f(t) dt$ are $o(e^{-|n|R})$ for large $|n|$. Conversely, if $c_n = o(e^{-|n|R})$, then $f$ is analytic in the given strip.*

**PROOF** We take $n > 0$, the opposite case being very similar. By analyticity we have

$$\int_0^{2\pi} e^{int} f(t) dt = \int_0^{iR} e^{int} f(t) dt + \int_{iR}^{iR+2\pi} e^{int} f(t) dt - \int_{2\pi}^{2\pi+iR} e^{int} f(t) dt$$

The first and last integrals on the rhs cancel by periodicity while the middle one equals

$$e^{-nR} \int_0^{2\pi} e^{ins} f(s+iR) ds = o(e^{-nR}) \quad \text{as} \quad n \to \infty$$

The converse is straightforward. □

### 3.5c Examples

*Example 1.* Consider the problem of finding the asymptotic behavior of the integral

$$I(n) = \int_{-\pi}^{\pi} \frac{e^{-int}}{2-e^{it}} dt := \int_{-\pi}^{\pi} F(t)dt$$

as $n \to \infty$. We see by Corollary 3.67 that $J = o(x^{-m})$ for any $m \in \mathbb{N}$. Proposition 3.80 tells us more, namely that the integral is exponentially small. But both methods only give us *upper bounds* for the decay, and no precise estimates.

In this simple example however we could simply expand convergently the integrand and use dominated convergence:

$$\int_{-\pi}^{\pi} \frac{e^{-int}}{2-e^{it}} = \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} 2^{-k-1} e^{-it(n-k)} = \sum_{k=0}^{\infty} \int_{-\pi}^{\pi} 2^{-k-1} e^{-it(n-k)} = 2^{-n}\pi$$

In case $n < 0$ we get $I(n) = 0$. If we have $x \notin \mathbb{N}$ instead of $n$ we could try same, but in this case we end up with

$$i(e^{-2\pi ix} - 1) \sum_{k=0}^{\infty} \frac{(-2)^{-k-1}}{x-k}$$

which needs further work to extract an asymptotic behavior in $x$.

We can alternatively apply a more general method to estimate the integral, using deformation of contour. The point is to try to express $J$ in terms of integrals along paths of constant phase of $e^{-int}$. Then Watson's lemma would be applicable. Note that $F$ is analytic in $\mathbb{C} \setminus \{-i \ln 2 + 2k\pi\}_{k\in\mathbb{Z}}$ and meromorphic in $\mathbb{C}$. Furthermore, as $N \to \infty$ we have $F(t - iN) \to 0$ exponentially fast. This allows us to push the contour of integration down, in the following way. We have

$$\oint_C F(t)dt = 2\pi i \operatorname{Res}\Big(F(t); t = -i \ln 2\Big) = -\pi 2^{-x}$$

where the contour $C$ of integration is an anticlockwise rectangle with vertices $-\pi, \pi, -iN + \pi, -iN - \pi$ with $N > \ln 2$. As $N \to \infty$ the integral over the segment from $-iN + \pi$ to $-iN - \pi$ goes to zero exponentially fast, and we find that

$$\int_{-\pi}^{\pi} F(t)dt = \int_{-\pi}^{-\pi-i\infty} F(t)dt - \int_{\pi}^{\pi-i\infty} F(t)dt + \pi 2^{-x}$$

$$I(x) = -i(e^{ix\pi} - e^{-ix\pi}) \int_0^{\infty} \frac{e^{-xs}}{2+e^s} ds + \pi 2^{-x} = 2 \sin \pi x \int_0^{\infty} \frac{e^{-xs}}{2+e^s} ds + \pi 2^{-x}$$

Watson's lemma now applies and we have

$$\int_0^\infty \frac{e^{-xs}}{2+e^s}ds \sim \frac{1}{3x}-\frac{1}{9x^2}-\frac{1}{27x^3}+\frac{1}{27x^4}+\frac{5}{81x^5}-\frac{7}{243x^6}+...$$

and thus

$$I(x) \sim 2\sin \pi x\left(\frac{1}{3x}-\frac{1}{9x^2}-\frac{1}{27x^3}+\frac{1}{27x^4}+\frac{5}{81x^5}-\frac{7}{243x^6}+...\right) \tag{3.81}$$

*whenever the prefactor in front of the series is not too small.* More generally, the difference between $I(x)$ and the $m$-th truncate of the expansion is $o(x^{-m})$. Or, the function on the left hand side can be decomposed in two functions using Euler's formula, each of which has a nonvanishing asymptotic expansion. This is the way to interpret similar asymptotic expansions, which often occur in the theory of special functions, when the expansions involve trigonometric functions. But none of these characterizations tells us what happens when the prefactor *is* small. Does the function vanish when $\sin \pi x = 0$? Not for $x > 0$. Another reason to be careful with relations of the type (3.81).

**Exercise 3.82** *Make use of the results in this section to find the behavior as* $y \to +\infty$ *of*

$$\sum_{k=0}^{\infty} \frac{a^k}{y+k}; \quad (|a|<1)$$

### 3.5c .1 Note on exponentially small terms

In our case we have more information: if we add the term $\pi 2^{-x}$ to the expansion and write

$$I(x) \sim 2\sin \pi x\left(\frac{1}{3x}-\frac{1}{9x^2}-\frac{1}{27x^3}+\frac{1}{27x^4}+\frac{5}{81x^5}-\frac{7}{243x^6}+...\right)+\pi 2^{-x} \tag{3.83}$$

then the expansion is valid when $x \to +\infty$ along the positive integers, a rather trivial case since only $\pi 2^{-x}$ survives. But we have trouble interpreting the expansion (3.83) when $x$ is not an integer! The expression (3.83) is not of the form (1.7) nor can we extend the definition to allow for $\pi 2^{-x}$ since $2^{-x}$ is asymptotically smaller than any term of the series, and no number of limits as in Definition 1.10 would reveal it. We cannot subtract the *whole* series preceding the exponential from $I(x)$ to see "what is left", since the series has zero radius of convergence. (The $k$-th coefficient is, by Watson's lemma, $k!$ times the corresponding Maclaurin coefficient of the function $(2+e^s)^{-1}$ and this function is not entire.)

We may nevertheless have the feeling that (3.83) is correct "somehow". Indeed it is, in the sense that (3.83) is the complete transseries of $J$, as it will become clear after we study more carefully BE summability.

*

## 3.6 Steepest descent method

Consider the problem of finding the large $x$ behavior of an integral of the form

$$\int_C f(s)e^{xg(s)}ds \tag{3.84}$$

where $g$ is analytic and $f$ is meromorphic (more general singularities can be allowed) in a domain in the complex plane containing the contour $C$ and $x$ is a large parameter.

As in the Example 1 on p. 50, the key idea is to use deformation of contour to bring the integral to one which is suitable to the application of the Laplace method. We can assume without loss of generality that $x$ is real and positive.

(A) Let $g = u + iv$ and let us first look at the simple case where $C'$ is a curve such that $v = K$ is constant along it. Then

$$\int_{C'} f(s)e^{xg(s)}ds = e^{xiK}\int_{C'} f(s)e^{xu(s)}ds = e^{xiK}\int_0^1 f(\gamma(t))e^{xu(\gamma(t))}\gamma'(t)dt$$

is in a form suitable for Laplace's method.

The method of steepest descent consists in using the meromorphicity of $f$, analyticity of $g$ to deform the contour of integration such that modulo residues, the original integral can be written as a sum of integrals of the type $C'$ mentioned. The name steepest descent comes from the following remark. The lines of $v$ =constant are perpendicular to the direction of $\nabla v$. As a consequence of the Cauchy-Riemann equations we have $\nabla u \cdot \nabla v = 0$ and thus the lines $v$ =constant are lines of steepest variation of $u$ therefore of $|e^{xg(s)}|$. On the other hand, the best way to control the integral is to go along the descent direction. The direction of steepest descent of $u$ is parallel to $-\nabla u$. Thus the steepest descent lines are the integral curves of the ODE system

$$\dot{x} = -u_x(x, y); \quad \dot{y} = -u_y(x, y) \tag{3.85}$$

We first look at some examples, and then discuss the method in more generality.

*Example 1.* The Bessel function $J_0(\xi)$ can be written as $\frac{1}{\pi}\mathrm{Re}\, I$, where

$$I = \int_{-\pi/2}^{\pi/2} e^{i\xi\cos t}dt \tag{3.86}$$

Suppose we would like to find the behavior of $J_0(\xi)$ as $\xi \to +\infty$. It is convenient to find the steepest descent lines by plotting the phase portrait of the flow (3.85), which in our case is

$$\dot{x} = -\cos x \sinh y; \quad \dot{y} = -\sin x \cosh y \tag{3.87}$$

and which is easy to analyze by standard ODE means. Consequently, we write

$$I=\int_{-\pi/2}^{-\pi/2+i\infty}e^{i\xi\cos t}dt+\int_{\gamma}e^{i\xi\cos t}dt+\int_{\pi/2}^{\pi/2-i\infty}e^{i\xi\cos t}dt \tag{3.88}$$

as shown in Fig 3.1.

All the curves involved in this decomposition of $I$ are lines of constant imaginary part of the exponent, and the ordinary Laplace method can be applied to find their asymptotic behavior for $\xi\to+\infty$ (note also that the integral along the curve $\gamma$, called Sommerfeld contour, is the only one contributing to $J_0$, the other two being purely imaginary, as it can be checked by making the changes of variable $t=-\pi/2\pm is$). Then, the main contribution to the integral comes from the point along $\gamma$ where the real part of the exponent is maximum, that is $z=0$. We then expand $\cos t=1-t^2/2+t^4/4!+\cdots$ keep the first two terms in the exponent and expand the rest out:

$$\int_{\gamma}e^{i\xi\cos t}dt\sim e^{ix}\int_{\gamma}e^{-i\xi t^2/2}(1+i\xi t^4/4!+\cdots)dt$$
$$\sim\int_{\infty e^{3i\pi/4}}^{\infty e^{-i\pi/4}}e^{-i\xi t^2/2}(1+i\xi t^4/4!+\cdots)dt \tag{3.89}$$

and integrate term by term. Justifying this rigorously would amount to redoing parts of the proofs of theorems we have already dealt with. Whenever possible, Watson's lemma is a shortcut, often providing more information as well. We will use it for (3.86) in Example 4.

*

*Example 2.* We know by Watson's lemma that for a function $F$ which has a nontrivial power series at zero, $\mathcal{L}F=\int_0^\infty e^{-xp}F(p)dp$ decreases algebraically as $x\to\infty$. We also know by Proposition 2.21 that regardless of $F\not\equiv 0\in L^1$, $\mathcal{L}F$ cannot decrease superexponentially. What happens if $F$ has a rapid oscillation near zero? Consider for $x\to+\infty$ the integral

$$I:=\int_0^\infty e^{-xp}\cos(1/p)dp \tag{3.90}$$

It is convenient to write

$$I=\mathrm{Re}\int_0^\infty e^{-xp}e^{-i/p}dp=\mathrm{Re}\,I_1 \tag{3.91}$$

To bring this problem to the steepest descent setting, we make the substitution $p=t/\sqrt{x}$. Then $I_1$ becomes

$$I_1=x^{-1/2}\int_0^\infty e^{-\sqrt{x}(t+i/t)}e^{-i/p}dp \tag{3.92}$$

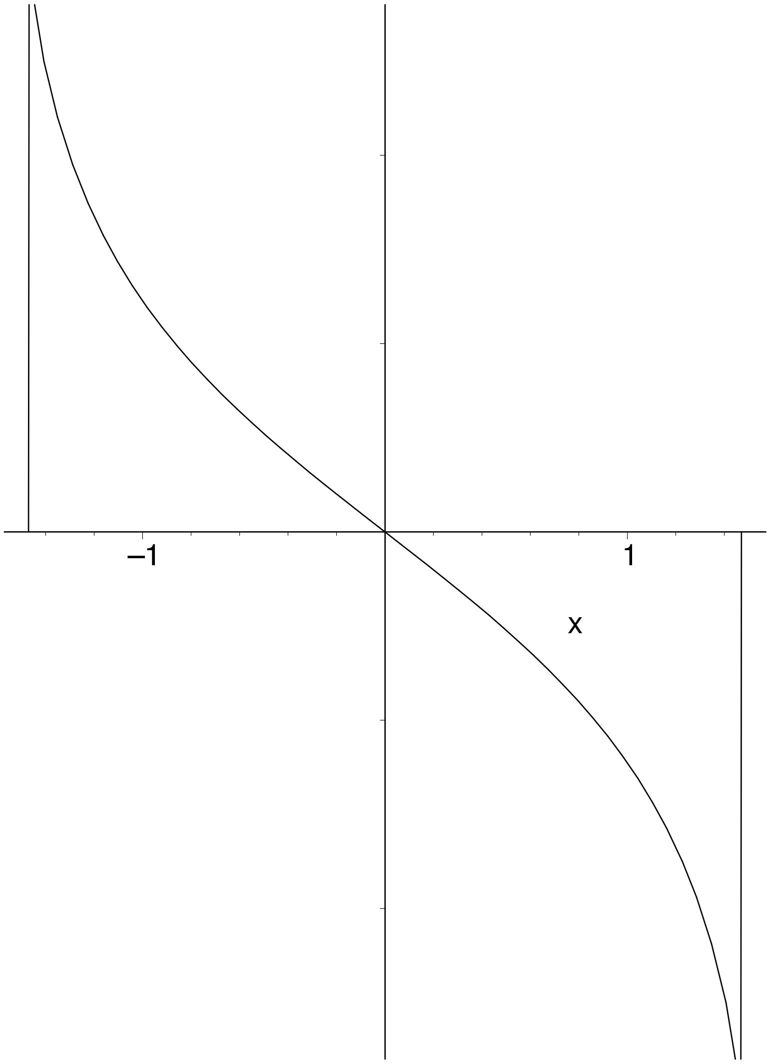

**FIGURE 3.1**: Relevant contours for $J_0$

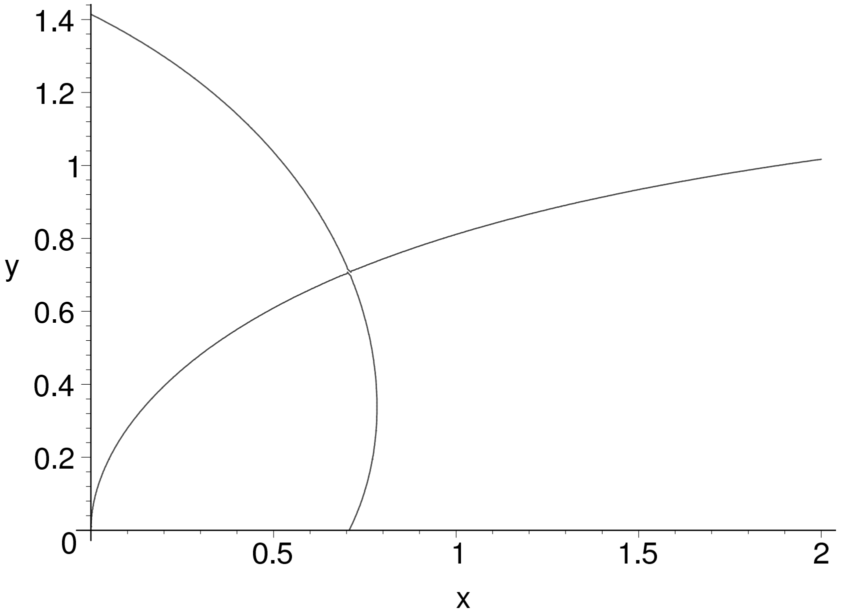

**FIGURE 3.2**: Constant phase lines for $t + i/t$ passing through the saddle point $t = \sqrt{i}$.

The constant imaginary part lines of interest now are those of the function $t + i/t$. This function has saddle points at $(t + i/t)' = 0$ i.e. $t = \pm\sqrt{i}$. We see that $t = \sqrt{i} = t_0$ is a maximum point for $-\mathrm{Re}\, g := -\mathrm{Re}\,(t + i/t)$ and the main contribution to the integral is from this point. We have, near $t = t_0$ $g = g(t_0) + \frac{1}{2}g''(t_0)(t - t_0)^2 + \cdots$ and thus

$$I_1 \sim x^{-1/2} e^{-\sqrt{2}(1+i)\sqrt{x}} \int_{-\infty}^{\infty} \exp\left[\left(-\frac{1}{2} + \frac{i}{2}\right)\sqrt{2x}(t - t_0)^2\right] \tag{3.93}$$

and the behavior of the integral is, roughly, $e^{-\sqrt{x}}$, decaying faster than powers of $x$ but slower than exponentially. The calculation can be justified mimicking the reasoning in Proposition 3.24. But this integral too can be brought to a form suitable for Watson's lemma.

**Exercise 3.94** *Finish the calculations in this example.*

## *3.6a Further discussion of steepest descent lines

Assume for simplicity that $g$ is nonconstant entire and $f$ is meromorphic. We can let the points on the curve $C = (x_0(\tau), y_0(\tau));\ \tau \in [0, 1]$ evolve with

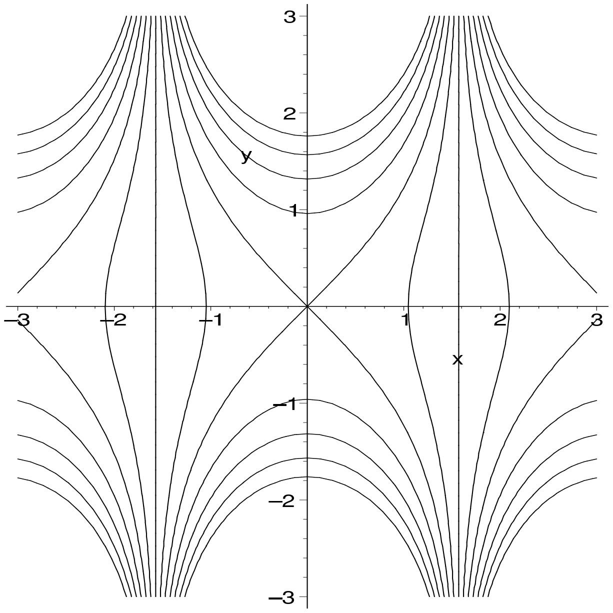

**FIGURE 3.3**: Steepest descent lines for $\mathrm{Re}\,[i\cos(x+iy)]$

(3.85) keeping the endpoints fixed. More precisely, at time $t$ consider the curve $t \mapsto C(t) = C_1 \cup C_2 \cup C_3$ where $C_1 = (x(s,x_0(0)), y(s,y_0(0));\ s \in [0,t)$, $C_2 = x(t,x_0(\tau)), y(t,y_0(\tau)),\ \tau \in (0,1))$ and $C_3 = (x(s,x_0(1)), y(s,y_0(1));\ s \in [0,t)$. Clearly, if no poles of $f$ are crossed,

$$\int_C f(s)e^{xg(s)}ds = \int_{C(t)} f(s)e^{xg(s)}ds \tag{3.95}$$

We can see that $z(t,x_0(\tau)) = (x(t,x_0(\tau)), y(t,x_0(\tau)))$ has a limit as $t \to \infty$ on the Riemann sphere, since $u$ is strictly decreasing along the flow:

$$\frac{d}{dt}u\Big(x(t),y(t)\Big) = -u_x^2 - u_y^2 \tag{3.96}$$

There can be no closed curves along which $v = K$ =const. or otherwise we would have $v \equiv K$ since $v$ is harmonic. Thus steepest descent lines extend to infinity. They may pass through *saddle points* of $u$ (and $g$: $\nabla u = 0 \Rightarrow g' = 0$) where their direction can change non-smoothly. These are equilibrium points of the flow (3.85).

Define $\mathcal{S}$ as the smallest forward invariant set with respect to the evolution (3.85) which contains $(x_0(0), y_0(0))$, all the limits in $\mathbb{C}$ of $z(t,x_0(\tau))$ and the descent lines originating at these points. The set $\mathcal{S}$ is a union of steepest descent curves of $u$, $\mathcal{S} = \cup_{j=1}^{n} C_j$ and, if $s_j$ are poles of $f$ crossed by the curve $C(t)$ we have, under suitable convergence assumptions[3],

$$\int_C f(s)e^{xg(s)}ds = \sum_{j=1}^{n'\leq n} \int_{C_n} f(s)e^{xg(s)}ds + 2\pi i \sum_j \mathrm{Res}(f(s)e^{xg(s)})_{s=s_j} \quad (3.97)$$

and the situation described in (A) above has been achieved.

One can allow for branch points of $f$, each of which adds a contributions of the form

$$\int_C \delta f(s)e^{xg(s)}ds$$

where $C$ is a cut starting at the branch point of $f$, along a line of steepest descent of $g$, and $\delta f(s)$ is the jump across the cut of $f$.

### 3.6b Reduction to Watson's lemma

It is often more convenient to proceed as follows.

We may assume we are dealing with a simple smooth curve. We assume $g' \neq 0$ at the endpoints (the case of vanishing derivative is illustrated shortly on an example). Then, possibly after an appropriate small deformation of $C$ we have $g' \neq 0$ along the path of integration $C$ and $g$ is invertible in a small enough neighborhood $\mathcal{D}$ of $C$. We make the change of variable $g(s) = -\tau$ and note that the image of $C$ is smooth and has at most finitely many self-intersections. We can break this curve into piecewise smooth, simple curves. If the pieces are small enough, they are homotopic (see footnote on p. 165) to straight lines; we get

$$\sum_{n=1}^{N} \int_{c_n}^{c_{n+1}} f\left(s(\tau)\right) e^{-x\tau} \frac{ds}{d\tau} d\tau \quad (3.98)$$

We calculate each integral in the sum separately. Without loss of generality we take $n = 1$, $c_1 = 0$ and $c_2 = i$:

$$I_1 = \int_0^i f(s(\tau))e^{-x\tau}s'(\tau)d\tau \quad (3.99)$$

---

[3] Convergence assumptions are required, as can be seen by applying the described procedure to very simple integral

$$\int_0^i e^{xe^{-z}}dz$$

.

The lines of steepest descent for $I_1$ are horizontal, towards $+\infty$. Assuming suitable analyticity and growth conditions and letting $H(\tau) = f(s(\tau))s'(\tau)$ we get that $I_1$ equals

$$\begin{aligned} I_1 = \int_0^{\infty} e^{-x\tau} H(\tau)d\tau - \int_i^{i+\infty} H(\tau)e^{-x\tau}d\tau \\ - 2\pi i \sum_j \operatorname{Res}\left(H(\tau)e^{-x\tau}\right)_{s=s_j} + \sum_j \int_{d_j}^{d_j+\infty} \delta H(\tau)e^{-x\tau}d\tau \end{aligned} \tag{3.100}$$

where the residues come from poles of $H$ in the strip $S = \{x + iy : x > 0, y \in [0, 1]\}$, while $d_j$ are branch points of $H$ in $S$, assumed integrable, and $\delta H$ denotes the jump of $H$ across the branch cut. If more convenient, one can alternatively subdivide $C$ such that $g'$ is nonzero on the (open) subintervals.
*Example 4.* In the integral (3.86) we have, using the substitution $\cos t = i\tau$,

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} e^{i\xi\cos t}dt = 2\int_0^{\pi/2} e^{i\xi\cos t}dt = -2i\int_{-i}^{0} \frac{e^{-\xi\tau}}{\sqrt{1+\tau^2}}d\tau = 2i\int_0^{\infty} \frac{e^{-\xi\tau}}{\sqrt{1+\tau^2}}d\tau \\ - 2i\int_{-i}^{-i+\infty} \frac{e^{-\xi\tau}}{\sqrt{1+\tau^2}}d\tau = 2i\int_0^{\infty} \frac{e^{-\xi\tau}}{\sqrt{1+\tau^2}}d\tau - 2ie^{i\xi}\int_0^{\infty} \frac{e^{-\xi s}}{\sqrt{-2is+s^2}}ds \end{aligned} \tag{3.101}$$

to which Watson's lemma applies.
**Exercise.** Find the asymptotic behavior for large $x$ of

$$\int_{-1}^{1} \frac{e^{ixs}}{s^2+1}ds$$

*

The integral in (3.14) can be brought to Watson's lemma setting by simple changes of variables. First we put $p = q\sqrt{x}$ and get

$$\operatorname{Ai}(x) = \frac{1}{2\pi i}x^{1/2}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{-x^{3/2}(q-q^3/3)}dq \tag{3.102}$$

We see that $(q - q^3/3)' = 1 - q^2 = 0$ iff $q = \pm 1$. We now choose the contour of integration to pass through $q = 1$. It is natural to substitute $q = 1 + z$ and then the integral becomes

$$\operatorname{Ai}(x) = \frac{e^{-\frac{2}{3}x^{3/2}}x^{1/2}}{2\pi i}\Big[\int_{\infty e^{-\pi i/3}}^{0} e^{-x^{3/2}(z^2+z^3/3)}dz + \int_0^{\infty e^{\pi i/3}} e^{-x^{3/2}(z^2+z^3/3)}dz\Big] \tag{3.103}$$

Along each path, the equation $z^2+z^3/3 = s$ has a unique well defined solution $z_{1,2}(s)$ where we choose $\arg(z_1) \to \pi/2$, as $s \to 0^+$. As $z_1 \to \infty e^{\pi i/3}$ we have

$s \to \infty$ tangent to $\mathbb{R}^+$. We can homotopically deform the contour and write

$$\mathrm{Ai}(x) = \frac{e^{-2/3x^{3/2}}x^{1/2}}{2\pi i}\Big[\int_0^\infty e^{-sx^{3/2}}\frac{dz_1}{ds}ds - \int_0^\infty e^{-sx^{3/2}}\frac{dz_2}{ds}ds\Big] \tag{3.104}$$

where the analysis proceeds as in the Gamma function case, inverting $z^2+z^3/2$ near zero and calculating the expansion to any number of orders.

**Exercise 3.105 (*)** *Complete the details of the analysis and show that*

$$Ai(x) = \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}}(1+o(1)) \quad (x \to +\infty) \tag{3.106}$$

*§3.9b .*

Again, once we know that an asymptotic series exists, and it is differentiable (by Watson's lemma), to obtain the first few terms of the asymptotic series it is easier to deal directly with the differential equation, see also [6], pp. 101. We can proceed as follows. The expansion is not a power series, but its logarithmic derivative is. We then substitute $y(x) = e^{w(x)}$ in the equation (a simple instance of the WKB method, discussed later), we get $(w')^2 + w'' = x$, and for a power series we expect $w'' \ll (w')^2$ (check that this would be true if $w$ is a differentiable asymptotic power series; see also p. 146 and 140), and set the iteration scheme

$$(w')_{n+1} = -\sqrt{x - (w'_{n+1})'}$$

Then,

$$w' = -\sqrt{x} - \frac{1}{4x} + \frac{5}{32}x^{-5/2} - \frac{15}{64}x^{-4} + \frac{1105}{2048}x^{-11/2} - \cdots$$

It follows that

$$y \sim Const.e^{-\frac{2}{3}x^{3/2}}\left(1 - \frac{5}{48}x^{-3/2} + \frac{385}{4608}x^{-3} - \frac{85085}{663552}x^{-9/2} + \cdots\right)$$

and the constant is obtained by comparing to (3.106).

**The Bessel equation** is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{3.107}$$

For $\nu = 0$

$$xy'' + y' + xy = 0 \tag{3.108}$$

to which Laplace's method applies. We get

$$(p^2Y)' - pY + Y' = 0 \ \Rightarrow Y = C(p^2+1)^{-1/2} \tag{3.109}$$

We get solutions by taking contours from $+\infty$, around a singularity and back to infinity in

$$\int_{\mathcal{C}} \frac{e^{-xp}}{\sqrt{p^2+1}} dp \tag{3.110}$$

or around both branch points.

**Exercise 3.111 (*)** *Find the relations between these integrals (we know that there are exactly two linearly independent solutions to (3.108)).*

To find the asymptotic behavior of an integral starting at $\infty + i - i\epsilon$, going around $x = i$ and then to $\infty + i + i\epsilon$, we note that this integral equals

$$2\int_i^{\infty+i} \frac{e^{-xp}}{\sqrt{p^2+1}} dp = 2e^{-ix}\int_0^{\infty} \frac{e^{-xs}}{\sqrt{s^2+2is}} ds$$
$$\sim e^{-ix}\sqrt{\pi}\left[\frac{1-i}{\sqrt{x}} + \frac{1}{8}\frac{1+i}{x^{3/2}} - \frac{9}{128}\frac{1-i}{x^{5/2}} + \cdots\right] \tag{3.112}$$

by Watson's lemma.

**Exercise 3.113 (*)** *Using the binomial formula, find the general term in the expansion (3.112).*

## 3.7 Application: Asymptotics of Taylor coefficients of analytic functions

There is dual relation between the behavior of the Taylor coefficients of an analytic function and the structure of its singularities in the complex plane. There exist very general results, with mild assumptions on the coefficients, and these are known as Tauberian/Abelian theorems [53]

We will study a few examples in which detailed information about the singularities is known, and then complete asymptotics of the coefficients can be found.

**Proposition 3.114** Assume $f$ is analytic in the open disk of radius $R + \epsilon$ with $N$ cuts at $z_n = Re^{i\phi_n}$ towards infinity, and in a neighborhood of $z_n$ $f$ has a convergent Puiseux series ("convergent" can be replaced with "asymptotic", see Note 3.116 below) [4]

$$f(z) = (z - z_n)^{\beta_1^{[n]}} A_1^{[n]}(z) + ... + (z - z_n)^{\beta_m^{[n]}} A_m^{[n]}(z) + A_{m+1}^{[n]}(z)$$

---

[4] A convergent series in terms of integer or noninteger powers of the $z - z_n$¿

where $A_1^{[n]}, ..., A_{m+1}^{[n]}$ are analytic in a neighborhood of $z = z_n$ (and we can assume $\beta_i^{[n]} \notin \mathbb{N} \cup \{0\}$). With $c_k = f^{(k)}(0)/k!$, we have

$$c_k \sim R^{-k} \sum_{l=1}^{N} e^{-ik\phi_l} \left( k^{-\beta_1^{[l]}-1} \sum_{j=0}^{\infty} \frac{c_{j;1}^{[l]}}{k^j} + ... + k^{-\beta_m^{[l]}-1} \sum_{j=0}^{\infty} \frac{c_{j;m}^{[l]}}{k^j} \right) \tag{3.115}$$

where the coefficients $c_{j;m}^{[n]}$ can be calculated from the Taylor coefficients of the functions $A_1^{[n]}, ..., A_m^{[n]}$, and conversely, this asymptotic expansion determines the functions $A_1^{[n]}, ..., A_m^{[n]}$.

**Note 3.116** We can relax the condition of convergence of the Puiseux series, replacing it with a condition of asymptoticity, where the $A_j^{[k]}$ become integer power series, with a slight modification of the proof: an arbitrary but finite numbers of terms of the asymptotic series are subtracted out and the contribution of the remainder is estimated straightforwardly.

**PROOF** We have

$$c_k = \frac{1}{2\pi i} \oint \frac{f(s)}{s^{k+1}} ds$$

where the contour is a small circle around the origin. This contour can be deformed, by assumption, to the dotted contour in the figure. The integral around the circle of radius $R + \epsilon$ can be estimated by

$$\frac{1}{2\pi} \left| \oint_{C_{R+\epsilon}} \frac{f(s)}{s^{k+1}} ds \right| \le \|f\|_\infty (R+\epsilon)^{-k-1} = O((R+\epsilon)^{-k-1})$$

and does not participate in the series (3.115), since it is smaller than $R^{-k}$ times any power of $k$, as $k \to \infty$. Now the contribution from each singularity is of the form

$$\frac{1}{2\pi} \int_{B_l} \frac{f(s)}{s^{k+1}} ds$$

where $B_l$ is an open dotted box around the branch cut at $Re^{i\phi_l}$ as in the figure, so it is enough to determine the contribution of one of them, say $z_1$. By the substitution $f_1(z) = f(Re^{i\phi_1} z)$, we reduce ourselves to the case $R = 1$, $\phi = 0$. We omit for simplicity the superscript "${}^{[1]}$".

The integral along $B_1$ is a sum of integrals of the form

$$\frac{1}{2\pi i} \int_C (s-1)^\beta A(s) s^{-k-1} ds \tag{3.117}$$

We can restrict ourselves to the case when $\beta$ is not an integer, the other case being calculable by residues.

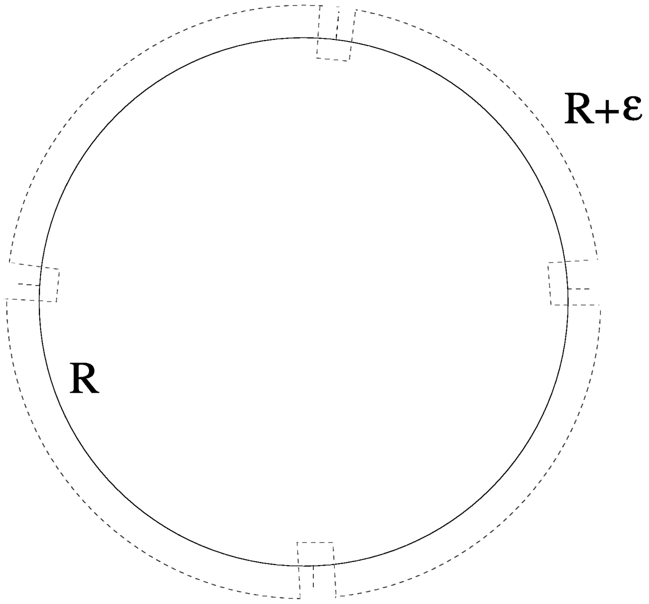

**FIGURE 3.4**: Deformation of the Cauchy contour.

Assume first that (i) $\mathrm{Re}\,(\beta) > -1$. We then have

$$\frac{1}{2\pi i}\int_C (s-1)^\beta A(s)s^{-k-1}ds = -e^{\pi i\beta}\frac{\sin(\pi\beta)}{\pi}\int_1^{1+\epsilon}(s-1)^\beta A(s)s^{-k-1}ds \tag{3.118}$$

with the branch choice $\ln(s-1) > 0$ for $s \in (1,\infty)$. It is convenient to change variables to $s = e^u$. The rhs of (3.118) becomes

$$-e^{\pi i\beta}\frac{\sin(\pi\beta)}{\pi}\int_0^{\ln(1+\epsilon)} u^\beta\left(\frac{e^u-1}{u}\right)^\beta A(e^u)e^{-ku}du \tag{3.119}$$

where $A(e^u)$ and $[u^{-1}(e^u-1)]^\beta$ are analytic at $u=0$, the assumptions of Watson's lemma are satisfied and we thus have

$$\int_C (s-1)^\beta A(s)s^{-k-1}ds \sim k^{-\beta-1}\sum_{j=0}^\infty \frac{d_j}{k^j} \tag{3.120}$$

where the $d_j$ can be calculated straightforwardly from the Taylor coefficients of $A[u^{-1}(e^u-1)]^\beta$. The proof when $\mathrm{Re}\,\beta \le -1$ is by induction. Assume that (3.120) holds for all for $\mathrm{Re}\,(\beta) > -m$ with $1 \le m_0 \le m$. One integration by parts gives

$$\int_C (s-1)^{\beta} A(s) s^{-k-1} ds = \frac{(s-1)^{\beta+1}}{\beta+1} A(s) s^{-k-1}|_C$$
$$- \frac{1}{\beta+1} \int_C (s-1)^{\beta+1} [A(s) s^{-k-1}]' ds = O((R+\epsilon)^{-k-1})$$
$$+ k \frac{1+1/k}{\beta+1} \int_C (s-1)^{\beta+1} A(s) s^{-k-2} ds - \frac{1}{\beta+1} \int_C (s-1)^{\beta+1} A'(s) s^{-k-1} ds \tag{3.121}$$

By assumption, (3.120) applies with $\beta+1 \leftrightarrow \beta$ to both integrals in the last sum and the proof is easily completed. ☐

**Exercise 3.122 (*)** *Carry out the details of the argument sketched in Note 3.116.*

### 3.7 .1 Finding the location of singularities from the leading asymptotic expansion of Taylor coefficients

**Exercise 3.123** Assume that the Maclaurin coefficients $c_n$ of $f$ at zero have the asymptotic behavior $c_n = an^{-2} + O(n^{-3})$. It is clear that $f$ is singular on the unit circle. Show that one singularity is necessarily placed at $z = 1$. Hint: Consider the function $g(z) = f' - a\sum_{n=1}^{\infty} n^{-1}z^{n-1}$. Show that $g$ is bounded at $z = 1$ while $\sum_{n=1}^{\infty} n^{-1}z^{n-1}$ is not.

---

## 3.8 Banach spaces and the contractive mapping principle

In rigorously proving asymptotic results about *solutions* of various problems, where the solution is not given in closed form, the contractive mapping principle is a handy tool. Once an asymptotic expansion solution has been found, if we use a truncated expansion as a quasi-solution, the remainder should be small. As a result, the complete problem becomes one to which the truncation is an exact solution modulo small errors (usually involving the unknown function). Therefore, most often, asymptoticity can be shown rigorously by rewriting this latter equation as a fixed point problem of an operator which is the identity plus a correction of tiny norm. Some general guidelines on how to construct this operator are discussed in §3.8b . It is desirable to go through the rigorous proof, whenever possible – this should be straightforward when the asymptotic solution has been correctly found–, one reason being that this quickly signals errors such as omitting important terms, or exiting the region of asymptoticity. We discuss, for completeness, a few basic features of

Banach spaces. There is a vast literature on the subject; see e.g. [43]. Familiar examples of Banach spaces are the $n$-dimensional euclidian vector spaces $\mathbb{R}^n$. A norm exists in a Banach space, which has the essential properties of a length: scaling, positivity except for the zero vector which has length zero and the triangle inequality (the sum of the lengths of the sides of a triangle is no less than the length of the third one). Once we have a norm, we can define limits, by reducing the notion to that in $\mathbb{R}$: $x_n \to x$ iff $\|x - x_n\| \to 0$. A normed vector space $\mathcal{B}$ is a Banach space if it is complete, that is every sequence with the property $\|x_n - x_m\| \to 0$ uniformly in $n, m$ (a Cauchy sequence) has a limit in $\mathcal{B}$. Note that $\mathbb{R}^n$ can be thought of as the space of functions defined on the set of integers $\{1, 2, ..., n\}$. If the domain contains infinitely many points, then the Banach space is usually infinite-dimensional. An example is $L^\infty[0,1]$, the space of bounded functions on $[0,1]$ with the norm $\|f\| = \sup_{[0,1]} |f|$. A function $L$ between two Banach spaces which is linear, $L(x+y) = Lx + Ly$, is bounded (-or continuous) if $\|L\| := \sup_{\|x\|=1} \|Lx\| < \infty$. Assume $\mathcal{B}$ is a Banach space and that $S$ is a closed subset of $\mathcal{B}$. In the *induced topology* (i.e, in the same norm), $S$ is a complete normed space. Assume $\mathcal{M} : S \mapsto \mathcal{B}$ is a (linear or nonlinear) operator with the property that for any $x, y \in S$ we have

$$\|\mathcal{M}(y) - \mathcal{M}(x)\| \le \lambda \|y - x\| \tag{3.124}$$

with $\lambda < 1$. Such operators are called **contractive**. Note that if $\mathcal{M}$ is linear, this just means that the norm of $\mathcal{M}$ is less than one.

**Theorem 3.125** *Assume $\mathcal{M} : S \mapsto S$, where $S$ is a closed subset of $\mathcal{B}$ is a contractive mapping. Then the equation*

$$x = \mathcal{M}(x) \tag{3.126}$$

*has a unique solution in $S$.*

**PROOF** Consider the sequence $\{x_j\}_j \in \mathbb{N}$ defined recursively by

$$\begin{aligned} x_0 &= x_0 \in S \\ x_1 &= \mathcal{M}(x_0) \\ &\cdots \\ x_{j+1} &= \mathcal{M}(x_j) \\ &\cdots \end{aligned} \tag{3.127}$$

We see that

$$\|x_{j+2} - x_{j+1}\| = \|\mathcal{M}(x_{j+1}) - \mathcal{M}(x_j)\| \le \lambda \|x_{j+1} - x_j\| \le \cdots \le \lambda^j \|x_1 - x_0\| \tag{3.128}$$

Thus,

$$\|x_{j+p+2} - x_{j+2}\| \le \left(\lambda^{j+p} + \cdots \lambda^j\right) \|x_1 - x_0\| \le \frac{\lambda^j}{1-\lambda} \|x_1 - x_0\| \tag{3.129}$$

and $x_j$ is a Cauchy sequence, and it thus converges, say to $x$. Since by (3.124) $\mathcal{M}$ is continuous, passing the equation for $x_{j+1}$ in (3.127) to the limit $j \to \infty$ we get

$$x = \mathcal{M}(x) \tag{3.130}$$

that is existence of a solution of (3.126). For uniqueness, note that if $x$ and $x'$ are two solutions of (3.126), by subtracting their equations we get

$$\|x - x'\| = \|\mathcal{M}(x) - \mathcal{M}(x')\| \le \lambda \|x - x'\| \tag{3.131}$$

implying $\|x - x'\| = 0$, since $\lambda < 1$. $\Box$

**Note 3.132** *Note that contractivity and therefore existence of a solution of a fixed point problem depends on the norm. An adapted norm needs to be chosen for this approach to give results.*

**Exercise 3.133** *Show that if* $L$ *is a linear operator from the Banach space* $\mathcal{B}$ *into itself and* $\|L\| < 1$ *then* $I - L$ *is invertible, that is* $x - Lx = y$ *has always a unique solution* $x \in \mathcal{B}$*. "Conversely", assuming that* $I - L$ *is not invertible, then in whatever norm* $\|\cdot\|_*$ *we choose to make the same* $\mathcal{B}$ *a Banach space, we must have* $\|L\|_* \ge 1$ *(why?).*

## 3.8a Fixed points and vector valued analytic functions

A theory of analytic functions from a Banach space to itself can be constructed by almost exactly following the usual construction of analytic functions. For the construction to work, we need the usual vector space operations and a topology in which they are continuous. In a Banach algebra setting [5] – multiplication is continuous, in the sense that $\|f \cdot g\| \le \|f\|\|g\|$. We can define a derivative in the usual way, by writing $F(f+\epsilon g) = F(f) + \epsilon L_f g + o(\epsilon)$, $\epsilon \in \mathbb{C}$ small, where $L_f =: \partial_f F$ is a linear operator, define an integral in the usual way, as a limit of a sum, or using appropriately generalized measure theory. Cauchy's formula is valid for complex-differentiable (analytic) functions. A detailed presentation is found in [27] and [36], but the basic facts are simple enough for the reader to redo the necessary proofs. An immediate recasting of the contractive mapping principle is that

**Remark 3.134** *In the context of Theorem 3.125 we have, equivalently: If* $\mathcal{N} : S^2 \mapsto S$ *is analytic in* $f$ *and* $\|\partial_f \mathcal{N}\| < \lambda < 1$ *for* $f, g$ *in* $S$*, then the equation* $f = \mathcal{N}(f, g)$*, where* $\|\partial_f \mathcal{N}\| < \lambda < 1$ *in* $S$ *has a unique fixed point in* $S$*.*

---

[5] A Banach algebra is a Banach space of functions endowed with multiplication which is distributive, associative and continuous in the Banach norm.

Indeed, if $\|h\| = \delta$ we have $h = \delta h_1$ with $\|h_1\| = 1$

$$\|\mathcal{N}(g, f + h) - \mathcal{N}(g, f)\| \le \int_0^\delta \left\| \frac{\partial \mathcal{N}}{\partial f}\Big(g, f + t h_1\Big) \right\| dt \le \lambda\delta$$

The implicit function theorem could be restated abstractly in a similar setting.

### 3.8b Choice of the contractive map

An equation can be rewritten in a number of equivalent ways. In solving an asymptotic problem, as a general guidelines we mention:

- The final operator $\mathcal{N}$ should not contain derivatives (of highest order), small differences, or other operations poorly behaved with respect to asymptotics, and it should only depend on the sought-for solution in a formally negligible way. The latter condition should be, in a first stage, checked for consistency: the discarded terms, calculated using the first order approximation, should indeed turn out to be small.
- To obtain an equation where the discarded part is manifestly small it often helps to write the sought-for solution as the sum of the first few terms of the approximation, plus an exact remainder, say $\delta$. The equation for $\delta$ is usually more contractive. It also becomes, up to small corrections, linear.
- The norms should reflect as well as possible the expected growth/decay tendency of the solution itself and the spaces are spaces where this solution lives.
- All freedom in the solution has been accounted for, that is, the final equation is expected to have a unique solution.

**Note 3.135** The contractive mapping and implicit function results above are trivially equivalent, and the difficulty in proving an asymptotic result virtually never lies here, but in finding the contractive reformulation, and the adequate spaces and norms.

---

## 3.9 Examples

### 3.9a Linear differential equations in Banach spaces

Consider the equation

$$Y'(t) = L(t)Y; \quad Y(0) = Y_0 \tag{3.136}$$

in a Banach space $X$, where $L(t) : X \to X$ is linear, norm continuous in $t$ and uniformly bounded,

$$\max_{t\in[0,\infty)} \|L(t)\| < L \tag{3.137}$$

Then the problem (3.136) has a global solution on $[0,\infty)$ and $\|Y\|(t) \le C_\epsilon e^{(L+\epsilon)t}$.

**PROOF** By comparison with the case when $X = \mathbb{R}$, the natural growth is indeed $Ce^{Lt}$, so we rewrite (3.136) as an integral equation, in a space where the norm reflects this possible growth. Consider the space of continuous functions $Y : [0,\infty) \mapsto \mathcal{B}$ in the norm

$$\|Y\|_{\infty,L} = \sup_{t\in[0,\infty)} e^{-Lt/\lambda}\|Y(t)\| \tag{3.138}$$

with $\lambda < 1$ and the auxiliary equation

$$Y(t) = Y_0 + \int_0^t L(s)Y(s)ds \tag{3.139}$$

which is well defined on $\mathcal{B}$ and is contractive there since

$$e^{-Lt/\lambda}\left|\int_0^t L(s)Y(s)ds\right| \le Le^{-Lt/\lambda}\int_0^t e^{Ls/\lambda}\|Y\|_{\infty,L}ds = \lambda\|Y\|_{\infty,L} \tag{3.140}$$

□

### 3.9b A Puiseux series for the asymptotics of the Gamma function

We choose a simple example which can be dealt with in a good number of other ways, yet containing some features of more complicated singular problems. Suppose we need to find the solutions of the equation $x - \ln x = t$ for $t$ (and $x$) close to 1. The implicit function theorem does not apply to $F(x,t) = x - \ln x - t$ at $(1,1)$. We then attempt to find a simpler equation that approximates well the given one in the singular regime, that is we look for *asymptotic simplification*, and then we try to present the full problem as a perturbation of the approximate one. We write $x = 1+z, t = 1+s$, expand the left side in series for small $z$, and retain only the first nonzero term. The result is $z^2/2 \approx s$. There are two solutions, thus effectively two different problems when $s$ is small. Keeping all terms, we treat the cubic and higher powers of $z$ as corrections. We look at one choice of sign, the other one being very similar, and write

$$z = \left(2s + \frac{2z^3}{3} - \frac{z^4}{2} + \frac{2z^5}{5} + \cdots\right)^{1/2} = (2s + \epsilon(z))^{1/2} \tag{3.141}$$

where $\epsilon(z)$ is expected to be small. We then have

$$z = \left(2s + O(z^3)\right)^{1/2} = \Big(2s + O(s^{3/2})\Big)^{1/2} \tag{3.142}$$

hence

$$z = \left(2s + \Big[(2s)^{1/2} + O(s^{3/2})\Big]^3 /3\right)^{1/2} = \left(2s + \frac{4\sqrt{2}}{3}s^{3/2} + O(s^2)\right)^{1/2} \tag{3.143}$$

and further,

$$z = \left(2s + \frac{4\sqrt{2}}{3}s^{3/2} + \frac{2s^2}{3} + O(s^{5/2})\right)^{1/2} = \sqrt{2s} + \frac{2s}{3} + \frac{\sqrt{2}}{18}s^{3/2} - \frac{2s^2}{135} + O(s^{5/2}) \tag{3.144}$$

etc., where in fact the emerging series converges, as shown in the Exercise 3.146 below. Here $z$ should be close to $\sqrt{2s}$; we set $s = w^2/2$ and $z = wZ$ and get

$$Z = \left(1 + \frac{2}{3}wZ^3 - \frac{1}{2}w^2Z^4 + \frac{2}{5}w^3Z^5 + \cdots\right)^{1/2} \tag{3.145}$$

**Exercise 3.146** *Show that if $\epsilon$ is small enough, then (3.145) is contractive in the sup norm in a ball of radius $\epsilon$ centered at 1 in the space of functions $Z$ analytic in $w$ for $|w| \le \epsilon$. Show thus that $z$ is analytic in $\sqrt{s}$ for small $s$.*

Once the behavior of the solutions has been clarified, we may sometimes gain in simplicity, or more global information, by returning to the implicit function theorem, but properly applied. Which one is better depends on the problem and on taste. The contraction mapping principle is often more natural, especially when the topology, suggested by the problem itself, is not one of the common ones. We take $t = \tau^2/2$ and write $z^2/2 + (z - \ln(1+z) - z^2/2) =: z^2/2(1 + z\phi(z)) = \tau^2/2$ and (differentiating $z\phi$, reintegrating and changing variables) we get

$$z\sqrt{1 - z\phi(z)} = \pm\tau; \quad \phi(z) = \int_0^1 \frac{\sigma^2 d\sigma}{1 + z\sigma} \tag{3.147}$$

with the usual choice of branch for the square root. *The implicit function theorem clearly applies, at $(0,0)$, to the functions $F(z,w) := z\sqrt{1 - z\phi(z)}$ $\pm w$.* The first few terms of the series are easily found from the fixed point equation by repeated iteration, as in §3.9g ,

$$z = \frac{1}{\sqrt{2}}\tau + \frac{1}{12}\tau^2 - \frac{\sqrt{2}}{72}\tau^3 + \frac{13}{4320}\tau^4 + \cdots \tag{3.148}$$

### 3.9c The Gamma function

We start from the representation

$$n! = \int_0^\infty t^n e^{-t} dt = n^{n+1} \int_0^\infty e^{-n(s-\ln s)} ds \tag{3.149}$$

We can now use the results in §3.9b and Watson's lemma to find the behavior of $n!$. With $s = 1 + z, z - \ln(1 + z) = u^2$, $dz = F(u)du$ we have

$$F(u) = \sqrt{2} + \frac{4}{3}u + \frac{\sqrt{2}}{6}u^2 - \frac{8}{135}\,u^3 + \frac{\sqrt{2}}{216}\,u^4 + \frac{8}{2835}\,u^5 - \cdots \tag{3.150}$$

**Exercise 3.151 (*)** *Note the pattern of signs:* $++--\cdots$. *Show that this pattern continues indefinitely.*

We have, using Exercise 3.46,

$$\int_0^\infty e^{-n(s-\ln s)} ds \sim n^n e^{-n} \sqrt{2} \int_{-\infty}^\infty \left(1 + \frac{u^2}{6} + \cdots\right) e^{-nu^2} du \tag{3.152}$$

or

$$n! \sim \sqrt{2\pi n} n^n e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + ...\right) \tag{3.153}$$

### 3.9d Linear meromorphic differential equations. Regular and irregular singularities

A linear meromorphic $m$-th order differential equation has the canonical form

$$y^{(m)} + B_{m-1}(x)y^{(m-1)} + ... + B_0(x)y = B(x) \tag{3.154}$$

where the coefficients $B_j(x)$ are meromorphic near $x_0$. We note first that any equation of the form (3.154) can be brought to a homogeneous meromorphic of order $n = m + 1$

$$y^{(n)} + C_{n-1}(x)y^{(n-1)} + ... + C_0(x)y = 0 \tag{3.155}$$

by applying $B(x)\frac{d}{dx}\frac{1}{B(x)}$ to (3.154). We want to look at the possible singularities of the solutions $y(x)$ of this equation. Note first that by the general theory of linear differential equations (or by a simple fixed point argument) if all coefficients are analytic at a point $x_0$ then the general solution is also analytic. Such a point is called regular point. Solutions of linear ODEs can only be singular because of singularities of the *equation*.

The main distinction is made with respect to the type of local solutions, whether they can be expressed as convergent asymptotic series (regular singularity) or not (irregular one).

**Theorem 3.156 (Frobenius)** *If near the point $x = x_0$ the coefficients $C_{n-j}$, $j = 1, ..., n$ can be written as $(x-x_0)^{-j}A_{n-j}(x)$ where $A_{n-j}$ are analytic, then there is a fundamental system of solutions in the form of convergent Frobenius series:*

$$y_m(x) = (x - x_0)^{r_m} \sum_{j=0}^{N_m} (\ln(x - x_0))^j B_{j;m}(x) \tag{3.157}$$

*where $B_{j;m}$ are analytic in an open disk centered at $x_0$ with radius equal to the distance from $x_0$ to the first singularity of $A_j$. The powers $r_m$ are solutions of the* indicial *equation*
$r(r-1)\cdots(r-n+1) + A_{n-1}(x_0)r(r-1)\cdots(r-n+2) + \ldots + A_0(x_0) = 0$
*Furthermore, logs appear only in the resonant case, when two (or more) characteristic roots $r_m$ differ by an integer.*

A straightforward way to prove the theorem is by induction on $n$. We can take $x_0 = 0$. Let $r_M$ be one of the indicial equation solutions. A transformation of the type $y = x^{r_M} f$ reduces the equation (3.155) to an equation of the same type, but where one characteristic root is zero. One can show that there is an analytic solution $f_0$ of this equation by inserting a power series, identifying the coefficients and estimating the growth of the coefficients. The substitution $f = f_0 \int g(s)ds$ gives an equation for $g$ which is of the same type as (3.155) but of order $n - 1$. This completes the induction step. For $n = 1$, the result is trivial.

We will not go into the details of the general case but instead we illustrate the approach on the simple equation

$$x(x-1)y'' + y = 0 \tag{3.158}$$

around $x = 0$. The indicial equation is $r(r-1) = 0$ (a resonant case). Substituting $y_0 = \sum_{k=0}^{\infty} c_k x^k$ in the equation and identifying the powers of $x$ yields the recurrence

$$c_{k+1} = \frac{k^2 - k + 1}{k(k+1)} c_k \tag{3.159}$$

with $c_0 = 0$ and $c_1$ arbitrary. By linearity we may take $c_1 = 1$ and by induction we see that $0 < c_k < 1$. Thus the power series has radius of convergence at least 1. The radius of convergence is in fact exactly one as it can be seen applying the ratio test and using (3.159); the series converges exactly up to the nearest singularity of (3.158).

**Exercise 3.160** *What is the asymptotic behavior of $c_k$ as $k \to \infty$?*

We let $y_0 = y_0 \int g(s)ds$ and get for $g$ the equation

$$g' + 2\frac{y_0'}{y_0} g = 0 \tag{3.161}$$

and, by the previous discussion, $2y_0'/y_0 = 2/x + A(x)$ with $A(x)$ is analytic. The point $x = 0$ is a regular singular point of (3.161) and in fact we can check that $g(x) = C_1 x^{-2} B(x)$ with $C_1$ an arbitrary constant and $B(x)$ analytic at $x = 0$. Thus $\int g(s)ds = C_1(a/x + b\ln(x) + A_1(x)) + C_2$ where $A_1(x)$ is analytic at $x = 0$. Undoing the substitutions we see that we have a fundamental set of solutions in the form $\{y_0(x), B_1(x) + B_2(x)\ln x\}$ where $B_1$ and $B_2$ are analytic.

A converse of this theorem also holds, namely

**Theorem 3.162 (Fuchs)** *If a meromorphic linear differential equation has, at $x = x_0$, a fundamental system of solutions in the form (3.157), then $x_0$ is a regular singular point of the equation.*

For *irregular* singularities, at least one formal solution contains divergent power series and/or exponentially small (large) terms. The way divergent power series are generated by the higher order of the poles is illustrated below.

*Example.* Consider the equation

$$y' + x^{-2}y = 1 \tag{3.163}$$

which has an irregular singularity at $x = 0$ since the order of the pole in $C_0 = x^{-2}$ exceeds the order of the equation. Substituting $y = \sum_{k=0}^{\infty} c_k x^k$ we get $c_0 = c_1 = 0$, $c_2 = 1$ and in general the recurrence

$$c_{k+1} = -kc_k$$

whence $c_k = (-1)^k (k-1)!$ and the series has zero radius of convergence. (It is useful to compare this recurrence with the one obtained if $x^{-2}$ is replaced by $x^{-1}$ or by 1.) The associated homogeneous equation $y' + x^{-2}y = 0$ has the general solution $y = Ce^{1/x}$ with an exponential singularity at $x = 0$.

## 3.9e Spontaneous singularities: the Painlevé's equation $\mathrm{P_I}$

In nonlinear differential equations, the solutions may be singular at points $x$ where the equation is regular. For example, the equation

$$y' = y^2 + 1$$

has a one parameter family of solutions $y(x) = \tan(x + C)$; each solution has infinitely many poles. Since the location of these poles depends on $C$, thus on the solution itself, these singularities are called *movable* or *spontaneous*.

Let us analyze local singularities of the Painlevé equation $\mathrm{P_I}$,

$$y'' = y^2 + x \tag{3.164}$$

We look at the local behavior of a solution that blows up, and will find solutions that are meromorphic but not analytic. In a neighborhood of a point where $y$ is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)x^{p-2} = A^2p^2$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(x-x_0)^{-2}$ : $y = 6(x-x_0)^{-2} + c_{-1}(x-x_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -x_0/10, c_3 = -1/6$ and $c_4$ is undetermined, thus free. Choosing a $c_4$, all others are uniquely determined. To show that there indeed is a convergent such power series solution, we follow the remarks in §3.8b . Substituting $y(x) = 6(x-x_0)^{-2} + \delta(x)$ where for consistency we should have $\delta(x) = o((x-x_0)^{-2})$ and taking $x = x_0 + z$ we get the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{3.165}$$

Note now that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (3.165) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximately by a linear equation. It is then natural to separate out the large terms from the small terms and write a fixed point equation for the solution based on this separation. We write (3.165) in the form

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \tag{3.166}$$

and integrate as if the right side was known. This leads to an equivalent integral equation. Since all unknown terms on the right side are chosen to be *relatively smaller*, by construction this integral equation is expected to be contractive. The indicial equation for the Euler equation corresponding to the left side of (3.166) is $r^2 - r - 12 = 0$ with solutions $4, -3$. By the method of variation of parameters we thus get

$$\begin{aligned} \delta = \frac{D}{z^3} - \frac{1}{10}x_0z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_0^z s^{-3}\delta^2(s)ds \\ = -\frac{1}{10}x_0z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \end{aligned} \tag{3.167}$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$; $C$ is arbitrary. To find $\delta$ formally, we would simply iterate (3.167) in the following way: We take $r := \delta^2 = 0$ first and obtain $\delta_0 = -\frac{1}{10}x_0z^2 - \frac{1}{6}z^3 + Cz^4$. Then we take $r = \delta_0^2$ and compute $\delta_1$ from (3.167) and so on. This yields:

$$\delta = -\frac{1}{10}x_0z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + \ldots \tag{3.168}$$

This series is actually convergent. To see that, we scale out the leading power of $z$ in $\delta$, $z^2$ and write $\delta = z^2 u$. The equation for $u$ is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z su^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \quad (3.169)$$

It is straightforward to check that, given $C_1$ large enough (compared to $x_0/10$ etc.) there is an $\epsilon$ such that this is a contractive equation for $u$ in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \epsilon$. We conclude that $\delta$ is analytic and that $y$ is meromorphic near $x = x_0$.

**Note**. The Painlevé property discussed in §3.9f , requires that $y$ is globally meromorphic, and we did *not* prove this. That indeed $y$ is globally meromorphic is in fact true, but the proof is delicate (see e.g. [1]). Generic equations fail even the local Painlevé property. For instance, for the simpler, autonomous, equation

$$f'' + f' + f^2 = 0 \quad (3.170)$$

the same analysis yields a local behavior starting with a double pole, $f \sim -6z^{-2}$. Taking $f = -6z^{-2} + \delta(z)$ with $\delta = o(z^{-2})$ again leads to a nearly linear equation for $\delta$ which can be solved by convergent iteration, using arguments similar to the ones above. The iteration is (for some $a \neq 0$)

$$\delta = \frac{6}{5z} + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_a^z s^{-3}\delta^2(s)ds \quad (3.171)$$

but now the leading behavior of $\delta$ is larger, $\frac{6}{5z}$. Iterating in the same way as before, we see that this will eventually produce logs in the expansion for $\delta$ (it first appears in the second integral, thus in the form $z^4 \ln z$). We get

$$\delta = \frac{6}{5z} + \frac{1}{50} + \frac{z}{250} + \frac{7z^2}{5000} + \frac{79}{75000}z^3 - \frac{117}{2187500}z^4\ln(z) + Cz^4 + \ldots \quad (3.172)$$

where later terms will contain higher and higher powers of $\ln(z)$. This is effectively a series in powers of $z$ and $\ln z$ a simple example of a transseries, which is convergent as can be straightforwardly shown using the contractive mapping method, as above.

**Note 3.173** *Eq. (3.170) does not have the Painlevé property, see § 3.9f below. This log term shows that infinitely many solutions can be obtained just by analytic continuation around one point, and suggests the equation is not integrable.*

## *3.9f Discussion: the Painlevé property.

Painlevé studied the problem of finding differential equations, now called equations with the Painlevé property , whose only *movable* singularities are poles[6]. There are no restriction on the behavior at singular points of *the equation.* The solutions of such an equation have a common Riemann surface simple enough we can hope to understand globally. The motivation of this study apparently goes back to Fuchs, who had the intuition that through such equations one should be able to construct interesting, well-behaved new functions.

We note that the Painlevé property guarantees some form of integrability of the equation, in the following sense. If we denote by $Y(x; x_0; C_1, C_2)$ the solution of the differential equation $y'' = F(x, y, y')$ with initial conditions $y(x_0) = C_1, y'(x_0) = C_2$ we see that $y(x_1) = Y(x_1; x; y(x), y'(x))$ is formally constant along trajectories and so is $y'(x_1) = Y'(x_1; x; y(x), y'(x))$. This gives thus two constants of motion in $\mathbb{C}$ provided the solution $Y$ is well defined almost everywhere in $\mathbb{C}$, i.e., if $Y$ is meromorphic.

On the contrary, "randomly occurring" movable branch-points make the inversion process explained above ill defined.

This does not of course entail that there is no constant of motion. However, the presence of spontaneous branch-points does have the potential to prevent the existence of well-behaved constants of motions for the following reason. Suppose $y_0$ satisfies a meromorphic (second order, for concreteness) ODE and $K(x; y, y')$ is a constant of motion. If $x_0$ is a branch point for $y_0$, then $y_0$ can be continued past $x_0$ by avoiding the singular point, or by going around $x_0$ any number of times before moving away. This leads to different branches $(y_0)_n$ of $y_0$, all of them, by simple analytic continuation arguments, solutions of the same ODE. By the definition of $K(x; y, y')$ however, we should have $K(x; (y_0)_n, (y_0)'_n) = K(x; y_0, y'_0)$ for all $n$, so $K$ assumes the same value on this infinite set of solutions. We can proceed in the same way around other branch points $x_1, x_2, ...$ possibly returning to $x_0$ from time to time. Generically, we expect to generate a family of $(y_0)_{n_1,...n_j}$ which is dense in the phase space. This is an expectation, to be proven in specific cases. To see whether an equation falls in this generic class M. Kruskal introduced a test of nonintegrability, the *poly-Painlevé test* which measures indeed whether branching is "dense". Properly interpreted and justified the Painlevé property measures whether an equation is integrable or not.

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[6] There is no complete agreement on what the Painlevé property should require and Painlevé himself apparently oscillated among various interpretations; certainly movable branch points are not allowed, but often the property is understood to mean that all solutions are single-valued on a common Riemann surface.

### 3.9g Irregular singularity of a nonlinear differential equation

As another example, consider the equation

$$y' + y = x^{-1} + y^3 + xy^5 \tag{3.174}$$

with the restriction $y \to 0$ as $x \to +\infty$. Exact solutions exist for special classes of equations, and (3.174) does not (at least not manifestly) belong to any of them. However, formal asymptotic series solutions, as $x \to \infty$, are usually easy to find. If $y$ is small and power-like, then $y', y^3 \ll y$ and a first approximation is $y_1 \approx 1/x$. Then $y_2 \approx 1/x + y_1^3 + xy_1^5 - y_1'$. A few iterations quickly yield (see Appendix 3.9i )

$$y(x) = x^{-1} + x^{-2} + 3\,x^{-3} + 13\,x^{-4} + 69\,x^{-5} + 428\,x^{-6} + O\left(x^{-7}\right) \tag{3.175}$$

To find a contractive mapping reformulation, we have to find what can be dropped in a first approximation. Though the derivative is formally small, as we discussed in §3.8b , it cannot be discarded when a rigorous proof is sought. Since $f$ and $1/x$ are both formally larger than $f'$ they cannot by discarded either. Thus the approximate equation can only be

$$y' + y = x^{-1} + E(f) \tag{3.176}$$

where the "error term" $E$ is just $f^3 + xf^5$. An equivalent integral equation is obtained by solving (3.176) as though $E$ was known,

$$y = g_0 + \mathcal{N}(y)$$
$$g_0(x) = y(x_0)e^{-(x-x_0)} + e^{-x}\int_{x_0}^{x} \frac{e^s}{s}ds;\ \ \mathcal{N}(y) = e^{-x}\int_{x_0}^{x} e^s\left[y^3(s) + sy^5(s)\right]ds \tag{3.177}$$

say with $x, x_0 \in \mathbb{R}^+$ (a sector in $\mathbb{C}$ can be easily accommodated). Now, the expected behavior of $y$ is, from (3.175) $x^{-1}(1 + o(1))$. We take the norm $\|y\| = \sup_{x\ge x_0} |xy(x)|$ and $S$ the ball $\{y : (x_0, \infty) : \|y\| < a\}$ where $a > 1$ (we have to allow it to be slightly bigger than 1, by (3.175)).

To evaluate the norms of the operators involved in (3.177) we need the following relatively straightforward result.

**Lemma 3.178** *For $x > x_0 > m$ we have*

$$e^{-s}\int_{x_0}^{x} e^s s^{-m} ds \le |1 - m/x_0|^{-1}x^{-m}$$

**PROOF** In a sense, the proof is by integration by parts: for $x > x_0 > m$ we have

$$e^x x^{-m} \le |1 - m/x_0|^{-1}(e^x x^{-m})'$$

and the result follows by integration.

**Exercise 3.179** (i) Show that, if $a > 1$ and if $x_0$ is sufficiently large, then $\mathcal{N}$ is well defined on $S$ and contractive there. Thus (3.177) has a unique fixed point in $S$. How small can you make $x_0$?

(ii) A slight variation of this argument can be used to prove the validity of the expansion (3.175). If we write $y = y_N + \delta(x)$ where $y_N$ is the sum of the first $N$ terms of the formal power series of $y$, then, by construction, $y_N$ will satisfy the equation up to errors of order $x^{-N-1}$. Write an integral equation for $\delta$ and show that $\delta$ is indeed $O(x^{-N-1})$. See also §3.9h below.

□

## 3.9h Proving the asymptotic behavior of solutions of nonlinear ODEs: an example

Consider the differential equation

$$y' - y = x^{-2} - y^3 \tag{3.180}$$

for large $x$. If $f$ behaves like a power series in inverse powers of $x$ then $y'$ and $y^3$ are small, and we can proceed as in §3.9g to get, formally,

$$y(x) \sim -x^{-2} + 2\,x^{-3} - 6\,x^{-4} + 24\,x^{-5} - 119\,x^{-6} + 708\,x^{-7} - 4926\,x^{-8} + \cdots \tag{3.181}$$

How do we prove this rigorously? One way is to truncate the series in (3.181) to $n$ terms, say the truncate is $y_n$, and look for solutions of (3.180) in the form $y(x) = y_n(x) + \delta(x)$. For $\delta(x)$ we write a contractive equation in a space of functions with norm $\sup_{x>x_0} |x^{n+1}\delta(x)|$.

**Exercise 3.182** *Carry out the construction above and show that there is a solution with an asymptotic power series starting as in (3.181).*

Alternatively, we can write an integral equation just for $y$, as in §3.9g and show that it is contractive in a space of functions with norm $\sup_{x>x_0} |x^2 y(x)|$. Then, knowing that it is a contraction, we can iterate the operator a given number of times, with controlled errors. First,

$$\begin{aligned} e^x \int_x^\infty e^{-s} s^{-2} ds = e^x \frac{1}{x} \int_1^\infty e^{-xs} s^{-2} ds = \frac{1}{x} \int_0^\infty e^{-xs} (1+s)^{-2} ds \\ \sim \frac{1}{x^2} - \frac{2}{x^3} + \frac{6}{x^4} - \frac{24}{x^5} + \cdots \end{aligned} \tag{3.183}$$

Then,

$$y(x) = e^x \int_\infty^x e^{-s} s^{-2} ds - e^x \int_\infty^x e^{-s} y(s)^3 ds \tag{3.184}$$

together with contractivity in the chosen norm implies

$$y(x) = e^x \int_\infty^x e^{-s}s^{-2}ds + e^x \int_\infty^x e^{-s}O(s^{-6})ds$$
$$= \frac{1}{x^2} - \frac{2}{x^3} + \frac{6}{x^4} - \frac{24}{x^5} + O(x^{-6}) \quad (3.185)$$

We can use (3.185) and (3.183) in (3.184) to obtain the asymptotic expansion of $y$ to $O(x^{-10})$, and by induction, to all orders.

**Exercise 3.186** *Based on (3.185) and (3.183) show that* $y$ *has an asymptotic power series in* $\mathbb{H}$*. In particular, the asymptotic series is differentiable (why?).*

To find the power series of $y$, we can also note that the asymptotic series must be a formal power series solution of (3.180) (why?). Say we want five terms of the expansion. Then we insert $y = a_2x^{-2} + a_3x^{-3} + a_4x^{-4} + a_5x^{-5} + a_6x^{-6}$ in (3.180) and solve for the coefficients. We get

$$\frac{1+a_2}{x^2} + \frac{2a_2+a_3}{x^3} + \frac{3a_3+a_4}{x^4} + \frac{4a_4+a_5}{x^5} + \frac{a_6+5a_5+a_2^3}{x^6} = 0 \quad (3.187)$$

and it follows immediately that

$$a_2 = -1, a_3 = 2, a_4 = -6, a_5 = 24, a_6 = -119 \quad (3.188)$$

Note that the signs alternate! This is true to all orders and it follows from Watson's lemma, after BE summation.

### 3.9i Appendix: some computer algebra calculations

Figure 3.5 shows a way to solve $y' + y = x^{-1} + y^3 + xy^5$ by asymptotic power series, using Maple11."%" is Maple shortcut for "the previous expression". The input line following Eq. (4) is copied and pasted without change. In practice one would instead return to the line after Eq. (4) and re-run it as many times as needed. Of course, a do loop can be easily programmed, but there is no point in that unless a very high number of terms is needed.

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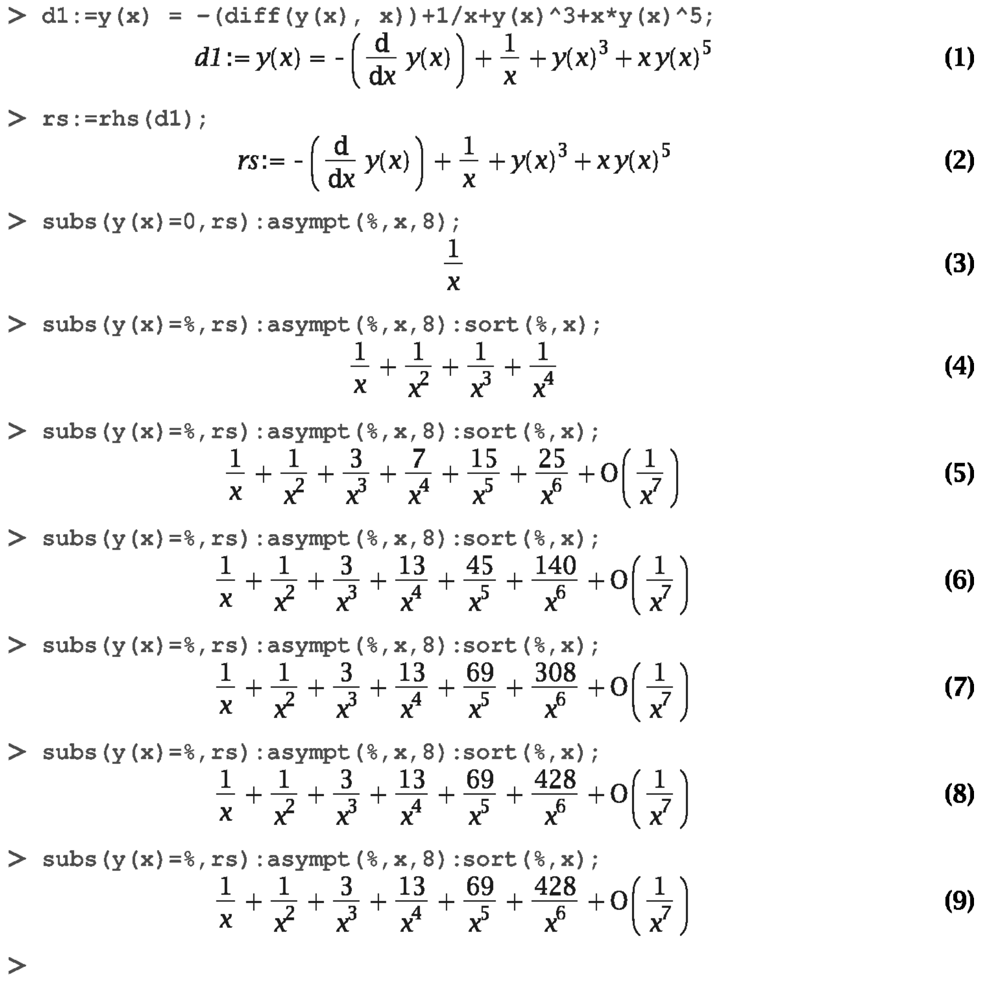
## 3.10 Singular perturbations

### 3.10a Introduction to the WKB method

In problems depending analytically on a small parameter, internal or external, the dependence of the solution on this parameter may be analytic (*regular perturbation*) or not (*irregular perturbation*). Ordinary differential



```
> d1:=y(x) = -(diff(y(x), x))+1/x+y(x)^3+x*y(x)^5;
```

$$d1 := y(x) = -\left(\frac{d}{dx}\,y(x)\right) + \frac{1}{x} + y(x)^3 + x\,y(x)^5 \tag{1}$$

```
> rs:=rhs(d1);
```

$$rs := -\left(\frac{d}{dx}\,y(x)\right) + \frac{1}{x} + y(x)^3 + x\,y(x)^5 \tag{2}$$

```
> subs(y(x)=0,rs):asympt(%,x,8);
```

$$\frac{1}{x} \tag{3}$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} \tag{4}$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{7}{x^4} + \frac{15}{x^5} + \frac{25}{x^6} + O\left(\frac{1}{x^7}\right) \tag{5}$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{13}{x^4} + \frac{45}{x^5} + \frac{140}{x^6} + O\left(\frac{1}{x^7}\right) \tag{6}$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{13}{x^4} + \frac{69}{x^5} + \frac{308}{x^6} + O\left(\frac{1}{x^7}\right) \tag{7}$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{13}{x^4} + \frac{69}{x^5} + \frac{428}{x^6} + O\left(\frac{1}{x^7}\right) \tag{8}$$

```
> subs(y(x)=%,rs):asympt(%,x,8):sort(%,x);
```

$$\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{13}{x^4} + \frac{69}{x^5} + \frac{428}{x^6} + O\left(\frac{1}{x^7}\right) \tag{9}$$

```
>
```

**FIGURE 3.5**: Maple11 output.

equations depending on a small parameter are singularly perturbed when the perturbation is such that, in a formal series solution, the highest derivative is formally small. In this case, in a formal successive approximation scheme the highest derivative is first discarded, then it appears in the corrections and it it thereby iterated upon. This, as we have seen in many examples, leads to divergent expansions. Furthermore, there should exist formal solutions other than power series, since the procedure above obviously yields a space of solutions of dimensionality strictly smaller than the degree of the equation.

An example is the Schrödinger equation

$$-\epsilon^2\psi'' + V(x)\psi - E\psi = 0 \tag{3.189}$$

for small $\epsilon$, which will be studied in more detail later. In an $\epsilon$-power series, $\psi''$ is subdominant [7]. The leading approximation would be $(V(x) - E)\psi = 0$ or $\psi = 0$ which is not an admissible solution.

Similarly, in

$$z^2 f' + f = z^2 \quad (z \text{ near zero}) \tag{3.190}$$

the presence of $z^2$ in front of $f'$ makes $f'$ subdominant if $f \sim z^p$ for some $p$. In this sense the Airy equation (3.209) below, is also singularly perturbed, at $x = \infty$. It turns out that in many of these problems the behavior of solutions is exponential in the parameter, generically yielding level one transseries, studied in the sequel, of the form $Qe^P$ where $P$ and $Q$ have algebraic behavior in the parameter. An exponential substitution of the form $f = e^w$ should then make the leading behavior algebraic.

## 3.10b Singularly perturbed Schrödinger equation. Setting and heuristics

We look at (3.189) under the assumption that $V \in C^\infty(\mathbb{R})$ and would like to understand the behavior of solutions for small $\epsilon$.

### 3.10b .1 Heuristics

Assume $V \in C^\infty$ and that the equation $V(x_0) = E$ has finitely many solutions.

Applying the WKB transformation $\psi = e^w$ (for further discussions on the reason this works, and for generalizations see §4.9, 41, pp. 140, 41 , and pp. 146) we get

$$-\epsilon^2 {w'}^2 - \epsilon^2 w'' + V(x) - E = -\epsilon^2 {w'}^2 - \epsilon^2 w'' + U(x) = 0 \tag{3.191}$$

---

[7] Meaning that it is asymptotically much less than other terms in the equation.

where, near an $x_0$ where

$$U(x_0) \neq 0 \tag{3.192}$$

the only consistent balance[8] is between $-\epsilon^2 w'^2$ and $V(x) - E$ with $\epsilon^2 w''$ much smaller than both. For that to happen we need

$$\epsilon^2 U^{-1} h' \ll 1 \quad \text{where} \quad h = w' \tag{3.193}$$

We place the term $\epsilon^2 h'$ on the right side of the equation and set up the iteration scheme

$$h_n^2 = \epsilon^{-2} U - h'_{n-1}; \quad h_{-1} = 0 \tag{3.194}$$

or

$$h_n = \pm \frac{\sqrt{U}}{\epsilon} \sqrt{1 - \frac{\epsilon^2 h'_{n-1}}{U}}; \quad h_{-1} = 0 \tag{3.195}$$

Under the condition (3.193) the square root can be Taylor expanded around 1,

$$h_n = \pm \frac{\sqrt{U}}{\epsilon} \left( 1 - \frac{1}{2} \epsilon^2 \frac{h'_{n-1}}{U} - \frac{1}{8} \epsilon^4 \left( \frac{h'_{n-1}}{U} \right)^2 + \cdots \right) \tag{3.196}$$

We thus have

$$h_0 = \pm \epsilon^{-1} U^{1/2} \tag{3.197}$$

$$h_1 = \pm \epsilon^{-1} U^{1/2} \left( 1 \pm \epsilon^2 \frac{h'_0}{U} \right) = \pm \epsilon^{-1} U^{1/2} - \frac{1}{4} \frac{U'}{U} \tag{3.198}$$

$$h_2 = \pm \epsilon^{-1} U^{1/2} - \frac{1}{4} \frac{U'}{U} + \epsilon \left( - \frac{5}{32} \frac{(U')^2}{U^{5/2}} + \frac{1}{8} \frac{U''}{U^{3/2}} \right) \tag{3.199}$$

and so on. We can check that the procedure is formally sound if $\epsilon^2 U^{-1} h'_0 \ll 1$ or

$$\epsilon U' U^{-3/2} \ll 1 \tag{3.200}$$

Formally we have

$$w = \pm \epsilon^{-1} \int U^{1/2}(s) ds - \frac{1}{4} \ln U + \cdots \tag{3.201}$$

and thus

$$\psi \sim U^{-1/4} e^{\pm \epsilon^{-1} \int U^{1/2}(s) ds} \tag{3.202}$$

---

[8] As the parameter, $\epsilon$ in our case, gets small, various terms in the equation contribute unevenly. Some become relatively large (the dominant ones) and some are small (the subdominant ones). If no better approach is presented, one tries all possible combinations, and rules out those which lead to conclusions inconsistent with the size assumptions made. The approach roughly described here is known as the method of dominant balance [6]. It is efficient but heuristic and has to be supplemented by rigorous proofs at a later stage of the analysis.

If we include the complete series in powers of $\epsilon$ in (3.202) we get

$$\psi \sim \exp\left(\pm\epsilon^{-1}\int U^{1/2}(s)ds\right)U^{-1/4}\left(1+\epsilon F_1(x)+\epsilon^2 F_2(x)+...\right) \qquad (3.203)$$

There are two possibilities compatible with our assumption about $x_0$, namely $V(x_0) > E$ and $V(x_0) < E$. In the first case there is (formally) an exponentially small solution and an exponentially large one, in the latter two rapidly oscillating ones.

The points where (3.200) fails are called *turning points.* Certainly if $|U(x_1)| > \delta$, then (3.200) holds near $x_1$, for $\epsilon$ small enough (depending on $\delta$). In the opposite direction, assume $U'U^{-3/2} = \phi$ is bounded; integrating from $x_0+\epsilon$ to $x$ we get $-2(U(x)^{-1/2}+U(x_1)^{-1/2}) = \int\phi(s)ds$, and thus $U(x_0+\epsilon)^{-1/2}$ is uniformly bounded near $x_0$. For instance if $U$ has a simple root at $x=0$, the only one that we will consider here (but multiple roots are not substantially more difficult) then condition (3.200) reads

$$x \gg \epsilon^{2/3} \qquad (3.204)$$

The region where this condition holds is called *outer* region.

### 3.10c Formal reexpansion and matching

Often, on the edge of validity, the growth structure of the terms of the series, (or, more generally, transseries), suggests the type of expansion it should be matched to in an adjacent region; thereby this suggests what approximation should be used in the equation as well, in a new regime. We can observe this interesting phenomenon of matching by reshuffling, on the formal solutions of (3.189) close to a turning point. We assumed that $U \in C^\infty$ and $U$ has finitely many zeros. Suppose $U(0) = 0$ and $U'(0) = a > 0$. If we look at the expansion (3.199) in a neighborhood of $x=0$ and approximate $U$ by its Taylor series at zero $U(x) = ax + bx^2 + ...$

$$h_2 = \frac{\sqrt{ax}}{\epsilon}\left(1+\frac{bx}{2a}\right) + -\frac{1}{4x}\left(1+\frac{bx}{a}+\cdots\right) - \frac{5z}{32\sqrt{ax^5}}\left(1-\frac{bx}{10a}+\cdots\right) \qquad (3.205)$$

and in general we would get

$$h_n = \frac{\sqrt{x}}{\epsilon}\left(y_0+\xi y_1+\xi^2 y_2+\cdots\right); \quad \xi = \frac{\epsilon}{x^{3/2}} \qquad (3.206)$$

where

$$y_j = a_{j0} + a_{j1}x + a_{j2}x^2 + \cdots = a_{j0} + a_{j1}\frac{\epsilon^{2/3}}{\xi^{2/3}} + a_{j2}\frac{\epsilon^{4/3}}{\xi^{4/3}} + \cdots \qquad (3.207)$$

We note that now the expansion has *two* small parameters, $\epsilon$ and $x$; these cannot be chosen small *independently*: the condition if $\xi \ll 1$ has to be satisfied to make asymptotic sense of (3.206). This would carry us down to values of $x$ such that, say, $\xi \ll 1/\ln|\epsilon|$. §6 is devoted to the study of matching, at the level of transseries.

### 3.10d The equation in the inner region; matching subregions

In a small region where (3.200) fails, called *inner* region, a different approximation will be sought. We see that $V(x)-E = V'(0)x+x^2h(x) =: \alpha x+x^2h(x)$ where $h(x) \in C^\infty(\mathbb{R})$. We then write

$$-\epsilon^2\psi'' + \alpha x = -x^2h(x)\psi \tag{3.208}$$

and treat the rhs of (3.208) as a small perturbation. The substitution $x = \epsilon^{2/3}t$ makes the leading equation an Airy equation:

$$-\psi'' + \alpha t\psi = -\epsilon^{2/3}t^2h(\epsilon^{2/3}t)\psi \tag{3.209}$$

which is a regularly perturbed equation! For a perturbation method to apply, we merely need that $x^2h(x)\psi$ in (3.208) is much smaller than the lhs, roughly requiring $x \ll 1$. This shows that the inner and outer regions overlap, there is a subregion *–the matching region–* where both expansions apply, and where, by equating them, the free constants in each of them can be linked. In the matching region, *maximal* balance occurs, in that a larger number of terms participate in the dominant balance. Indeed, if we examine (3.191) near $x = 0$, we see that $w'^2 \gg w''$ if $\epsilon^{-2}x \gg \epsilon^{-1}x^{-1/2}$, where we used (3.197). In the transition region, all terms in the middle expression in (3.191) participate equally.

### 3.10e Outer region. Rigorous analysis

We first look at a region where $U(x)$ is bounded away from zero. We will write $U = F^2$.

**Proposition 3.210** *Let $F \in C^\infty(\mathbb{R})$, $F^2 \in \mathbb{R}$, and assume $F(x) \neq 0$ in $[a,b]$. Then for small enough $\epsilon$ there exists a fundamental set of solutions of (3.189) in the form*

$$\psi_\pm = \Phi_\pm(x;\epsilon)\exp\left[\pm\epsilon^{-1}\int F(s)ds\right] \tag{3.211}$$

*where $\Phi_\pm(x;\epsilon)$ are $C^\infty$ in $\epsilon > 0$.*

**PROOF** We show that there exists a fundamental set of solutions in the form

$$\psi_\pm = \exp\left[\pm\epsilon^{-1}R_\pm(x;\epsilon)\right] \tag{3.212}$$

where $R_{\pm}(x;\epsilon)$ are $C^{\infty}$ in $\epsilon$. The proof is by rigorous WKB.

Note first that linear independence is immediate, since for small enough $\epsilon$ the ratio of the two solutions cannot be a constant, given their $\epsilon$ behavior.

We take $\psi = e^{w/\epsilon}$ and get, as before, to leading order $w' = \pm F$. We look at the plus sign case, the other case being similar. It is then natural to substitute $w' = F + \delta$; we get

$$\delta' + 2\epsilon^{-1}F\delta = -F' - \epsilon^{-1}\delta^2 \tag{3.213}$$

which we transform into an integral equation by treating the rhs as if it was known and integrating the resulting linear inhomogeneous differential equation. Setting $H = \int F$ the result is

$$\delta = -e^{-\frac{2H}{\epsilon}}\int_a^x F'(s)e^{\frac{2H(s)}{\epsilon}}ds - \frac{1}{\epsilon}e^{-\frac{2H}{\epsilon}}\int_a^x \delta^2(s)e^{\frac{2H(s)}{\epsilon}}ds =: J(\delta) =: \delta_0 + N(\delta) \tag{3.214}$$

We assume that $F > 0$ on $(a,b)$, the case $F < 0$ being very similar. The case $F \in i\mathbb{R}$ is not too different either, as we will explain at the end.

Let now $\|F'\|_\infty = A$ in $(a,b)$ and assume also that $\min_{s\in(a,b)}|U(s)| = B^2 > 0$.

**Lemma 3.215** *For small $\epsilon$, the operator $J$ is contractive in a ball $\mathcal{B} := \{\delta : \|\delta\|_\infty \le 2AB^{-1}\epsilon\}$*

**PROOF** i) Preservation of $\mathcal{B}$. We have

$$|\delta_0(x)| \le Ae^{-\frac{2}{\epsilon}H(x)}\int_a^x e^{\frac{2}{\epsilon}H(s)}ds$$

By assumption, $H$ is increasing on $(a,b)$ and $H' \ne 0$ and thus, by the Laplace method, cf. Proposition 3.18, for small $\epsilon$ we have (since $H' = \sqrt{U}$),

$$|\delta_0(x)| \le 2Ae^{-\frac{2}{\epsilon}H(x)}\frac{e^{\frac{2}{\epsilon}H(x)}}{\frac{2}{\epsilon}H'(x)} \le \epsilon AB^{-1}$$

**Note** We need this type of estimates to be uniform in $x \in [a,b]$ as $\epsilon \to 0$. To see that this is the case, we write

$$\int_a^x e^{\frac{2}{\epsilon}H(s)}ds = \int_a^x e^{\frac{2}{\epsilon}H(s)}\frac{2F(s)}{\epsilon}\frac{\epsilon}{2F(s)}ds$$
$$\le \frac{\epsilon}{2B}e^{\frac{2}{\epsilon}H(s)}\Big|_a^x \le \frac{\epsilon}{2B}e^{\frac{2}{\epsilon}H(x)} \tag{3.216}$$

Similarly,

$$\left|\frac{1}{\epsilon}e^{-\frac{2H}{\epsilon}}\int_a^x \delta^2(s)e^{\frac{2H(s)}{\epsilon}}ds\right| \le 2\epsilon^2A^2B^{-3}$$

and thus, for small $\epsilon$ and $\delta \in \mathcal{B}$ we have

$$J(\delta) \le \epsilon^{-1}AB^{-1} + 2\epsilon^2A^2B^{-3} \le 2\epsilon AB^{-1}$$

ii) *Contractivity.* We have, with $\delta_1, \delta_2 \in \mathcal{B}$, using similarly Laplace's method,

$$|J(\delta_2) - J(\delta_1)| \le \frac{1}{\epsilon}e^{-\frac{2H}{\epsilon}}\int_a^x |\delta_2(s) - \delta_1(s)||\delta_2(s) + \delta_1(s)|e^{\frac{2H(s)}{\epsilon}}ds$$
$$\le \frac{2\epsilon A}{B^2}\|\delta_2 - \delta_1\| \quad (3.217)$$

and thus the map is contractive for small enough $\epsilon$.

□

**Note**. We see that the conditions of preservation of $\mathcal{B}$ and contractivity allow for a dependence of $(a, b)$ on $\epsilon$. Assume for instance that $a, b > 0$ and $V(x) = E$ has no root in $[a, b + \gamma)$ with $\gamma > 0$, and that $a$ is small. Assume further that $V(0) = E$ is a simple root, $V'(0) = \alpha \neq 0$. Then for some $C > 0$ we have $B \ge Cm^2a^2$ and the condition of contractivity reads

$$\frac{\epsilon^2|\alpha|}{|\alpha|^3} < 1$$

i.e. $a > (\epsilon/|\alpha|)^{2/3}$ and for small enough $\epsilon$ this is also enough to ensure preservation of $\mathcal{B}$. We thus find that the equation $\delta = J(\delta)$ has a unique solution and that, furthermore, $\|\delta\| \le const.\epsilon$. Using this information and (3.217) which implies

$$\|J(\delta)\| \le \frac{\epsilon A}{B^2}2AB^{-1}\epsilon$$

we easily get that, for some constants $C_i > 0$ independent on $\epsilon$,

$$|\delta - \delta_0| \le C_1\epsilon|\delta| \le C_1\epsilon|\delta_0| + C_1\epsilon|\delta - \delta_0|$$

and thus

$$|\delta - \delta_0| \le C_2\epsilon|\delta_0|$$

and thus, applying again Laplace's method we get

$$\delta \sim \frac{-\epsilon F'}{2F} \quad (3.218)$$

which gives

$$\psi \sim \exp\left(\pm\epsilon^{-1}\int U^{1/2}(s)ds\right)U^{-1/4}$$

The proof of the $C^\infty$ dependence on $\epsilon$ can be done by induction, using (3.218) to estimate $\delta^2$ in the fixed point equation, to get an improved estimate on $\delta$, etc.

In the case $F \in i\mathbb{R}$, the proof is the same, by using the stationary phase method instead of the Laplace method.

$\Box$

### 3.10f Inner region. Rigorous analysis

By rescaling the independent variable we may assume without loss of generality that $\alpha = 1$ in (3.209) which we rewrite as

$$-\psi'' + t\psi = -\epsilon^{2/3}t^2h_1(\epsilon^{2/3}t)\psi := f(t) \tag{3.219}$$

which can be transformed into an integral equation in the usual way,

$$\psi(t) = -\mathrm{Ai}(t)\int^t f(s)\mathrm{Bi}(s)ds + \mathrm{Bi}(t)\int^t f(s)\mathrm{Ai}(s)ds + C_1\mathrm{Ai}(t) + C_2\mathrm{Bi}(t) \tag{3.220}$$

where Ai, Bi are the Airy functions, with the asymptotic behavior

$$\mathrm{Ai}(t) \sim \frac{1}{\sqrt{\pi}}t^{-1/4}e^{-\frac{2}{3}t^{\frac{3}{2}}};\ \ \mathrm{Bi}(t) \sim \frac{1}{\sqrt{\pi}}t^{-1/4}e^{\frac{2}{3}t^{\frac{3}{2}}} \tag{3.221}$$

and

$$|t^{-1/4}\mathrm{Ai}(t)| < const.,\ \ |t^{-1/4}\mathrm{Bi}(t)| < const. \tag{3.222}$$

as $t \to -\infty$. In view of (3.221) we must be careful in choosing the limits of integration in (3.220). It is important to ensure that the second term does not have a fast growth as $t \to \infty$, and for this purpose we need to integrate from $t$ toward infinity in the associated integral. For that, we ensure that the maximum of the integrand is achieved *at or near the variable endpoint of integration.* Then Laplace's method shows that the leading contribution to the integral comes from the variable endpoint of integration as well, which allows for the opposite exponentials to cancel out. We choose to look at an interval in the original variable $x \in I_M = [-M, M]$ where we shall allow for $\epsilon$-dependence of $M$. We then write the integral equation with concrete limits in the form below, which we analyze in $I_M$.

$$\psi(t) = -\mathrm{Ai}(t)\int_0^t f(s)\mathrm{Bi}(s)ds+$$
$$\mathrm{Bi}(t)\int_M^t f(s)\mathrm{Ai}(s)ds + C_1\mathrm{Ai}(t) + C_2\mathrm{Bi}(t) = J\psi + \psi_0 \quad (3.223)$$

**Proposition 3.224** *For some positive const., if $\epsilon$ is small enough (3.223) is contractive in the sup norm if $M \le const.\epsilon^{2/5}$.*

**PROOF** Using the Laplace method we see that for $t > 0$ we have

$$t^{-1/4}e^{-\frac{2}{3}t^{\frac{3}{2}}}\int_0^t s^{-1/4}e^{\frac{2}{3}s^{\frac{3}{2}}}ds \le const.(|t|+1)^{-1}$$

and also

$$t^{-1/4}e^{\frac{2}{3}t^{\frac{3}{2}}}\int_t^M s^{-1/4}e^{-\frac{2}{3}s^{\frac{3}{2}}}ds \le t^{-1/4}e^{\frac{2}{3}t^{\frac{3}{2}}}\int_t^\infty s^{-1/4}e^{-\frac{2}{3}s^{\frac{3}{2}}}ds$$
$$\le const.(|t|+1)^{-1} \quad (3.225)$$

and thus for a constant independent of $\epsilon$, using (3.221) we get

$$|J\psi(t)| \le const.\epsilon^{2/3}(|t|+1)^{-1}\sup_{s\in[0,t]}|\psi(s)|$$

for $t > 0$. For $t < 0$ we use (3.222) and obtain

$$\left|\mathrm{Ai}(t)\int_M^t f(s)\mathrm{Bi}(s)ds\right| \le (1+|t|)^{-1/4}\sup_{s\in[-t,0]}|f(s)|(const. + \int_t^0 s^{-1/4}ds$$

and get for a constant independent of $\epsilon$

$$|J\psi(t)| \le const.\epsilon^{2/3}(1+|t|)^{5/2} \le const.\epsilon^{2/3}(\epsilon^{-2/3}M)^{5/2} < 1$$

We see that for small enough $\epsilon$, the regions where the outer and inner equations are contractive overlap. This allows for performing asymptotic matching in order to relate these two solutions. For instance, from the contractivity argument it follows that

$$\psi = (1-J)^{-1}\psi_0 = \sum_{k=0}^{\infty} J^k\psi_0$$

giving a power series asymptotics in powers of $\epsilon^{2/3}$ for $\psi$.

$\square$

### 3.10g Matching

We may choose for instance $x = const.\epsilon^{1/2}$ for which the inner expansion (in powers of $\epsilon^{2/3}$) and the outer expansion (in powers of $\epsilon$) are valid at the same time. We assume that $x$ lies in the oscillatory region for the Airy functions (the other case is slightly more complicated).

We note that in this region of $x$ the coefficient of $\epsilon^k$ of the outer expansion will be large, of order $(U'U^{-3/2})^k \sim \epsilon^{-3k/4}$. A similar estimate holds for the terms of the inner expansion. Both expansions will thus effectively be expansions in $\epsilon^{-1/4}$. Since they represent the same solution, they must agree and thus the coefficients of the two expansions are linked. This determines the constants $C_1$ and $C_2$ once the outer solution is prescribed.

## 3.11 WKB on a PDE

Consider now a a parabolic PDE, say the heat equation.

$$\psi_t = \psi_{xx} \tag{3.226}$$

The fact that the principal symbol is degenerate (there are fewer $t$ than $x$ derivatives) has an effect similar to that of a singular perturbation. If we attempt to solve the PDE by a power series

$$\psi = \sum_{k=0}^{\infty} t^k F_k(x) \tag{3.227}$$

this series will generically have zero radius of convergence. Indeed, the recurrence relation for the coefficients is $F_k = F''_{k-1}/k$ whose solution, $F_k = F_0^{(2k)}/k!$ behaves like $F_k \sim k!$ for large $k$, if $F$ is analytic but not entire.

Generally, exponential solutions are expected too.[9] If we take $\psi = e^w$ in (3.226) we get

$$w_t = w_x^2 + w_{xx} \tag{3.228}$$

where the assumption of algebraic behavior of $w$ is expected to ensure $w_x^2 \gg w_{xx}$ and so the leading equation is approximately

$$w_t = w_x^2 \tag{3.229}$$

---

[9] The reason will be better understood after Borel summation methods have been studied. Divergence means that the Borel transform of the formal solution is nontrivial: it has singularities. Upon Laplace transforming it, paths of integration on different sides of the singularities give different results, and the differences are exponentially small.

which can be solved by characteristics. We take $w_x = u$ and get for $u$ the quasilinear equation

$$u_t = 2uu_x \tag{3.230}$$

with a particular solution $u = -x/(2t)$, giving $w = -x^2/(4t)$. We thus take $w = -x^2/(4t) + \delta$ and get for $\delta$ the equation

$$\delta_t + \frac{x}{t}\delta_x + \frac{1}{2t} = \delta_x^2 + \delta_{xx} \tag{3.231}$$

where we have separated the relatively small terms to the rhs. We would normally solve the leading equation (the lhs of (3.231)) and continue the process, but for this equation we note that $\delta = -\frac{1}{2}\ln t$ solves not only the leading equation, but the full equation (3.231). Thus

$$w = -\frac{x^2}{4t} - \frac{1}{2}\ln t \tag{3.232}$$

which gives the classical heat kernel

$$\psi = \frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}} \tag{3.233}$$

This exact solvability is of course rather accidental, but a perturbation approach formally works in a more PDE general context.

# Chapter 4

# Analyzable functions and transseries

As we have seen, there is an important distinction between asymptotic expansions and asymptotic series. The operator $f \mapsto \mathcal{A}_p(f)$ which associates to $f$ its asymptotic power series is linear as seen in §1.1c . But it has a nontrivial kernel ($\mathcal{A}_p(f) = 0$ for many nonzero functions), and thus the description through asymptotic power series is fundamentally *incomplete*. There is no unambiguous way to determine a function from its classical asymptotic series alone. On the other hand, the operator $f \mapsto \mathcal{A}(f)$ which associates to $f$ its asymptotic *expansion* has zero kernel, but it is still false that $\mathcal{A}(f) = \mathcal{A}(g)$ implies $f = g$ ($\mathcal{A}$ is *not* linear, see Remark 1.24). The description of a function through its asymptotic *expansion* is also incomplete.

---

## 4.1 Analytic function theory as a toy model of the theory of analyzable functions

Let $A$ denote the set of germs of analytic functions at $z = 0$, let $\mathbb{C}[[z]]$ be the space of formal series in $z$ with complex coefficients, of the form $\sum_{k=0}^{\infty} c_k z^k$, and define $\mathbb{C}_c[[z]]$ as the subspace of series with nonzero radius of convergence. The Taylor series at zero of a function in $A$ is also its asymptotic series at zero. Moreover, the map $\mathcal{T} : A \mapsto \mathbb{C}_c[[z]]$, the Taylor expansion operator, is an isomorphism and its inverse $\mathcal{T}^{-1} = \mathcal{S}$ is simply the operator of summation of series in $\mathbb{C}_c[[z]]$. $\mathcal{T}$ and $\mathcal{S}$ commute with most function operations defined on $\mathcal{A}$. For instance we have, with $\tilde{f}, \tilde{f}_1$ and $\tilde{f}_2$ in $\mathbb{C}_c[[z]]$

$$\mathbf{1}.\quad \mathcal{S}\{\alpha\tilde{f}_1 + \beta\tilde{f}_2\} = \alpha\mathcal{S}\tilde{f}_1 + \beta\mathcal{S}\tilde{f}_2; \qquad \mathbf{2}.\quad \mathcal{S}\{\tilde{f}_1\tilde{f}_2\} = \mathcal{S}\tilde{f}_1\mathcal{S}\tilde{f}_2;$$
$$\mathbf{3}.\quad \mathcal{S}\{\tilde{f}^\star\} = \left\{\mathcal{S}\tilde{f}\right\}^\star; \qquad \mathbf{4}.\quad \mathcal{S}\{\tilde{f}\}' = \left\{\mathcal{S}\tilde{f}\right\}';$$
$$\mathbf{5}.\ \mathcal{S}\left\{\int_0^x \tilde{f}\right\} = \int_0^x \mathcal{S}\tilde{f}; \qquad \mathbf{6}.\quad \mathcal{S}\{\tilde{f}_1 \circ \tilde{f}_2\} = \mathcal{S}\tilde{f}_1 \circ \mathcal{S}\tilde{f}_2; \qquad \mathbf{7}.\quad \mathcal{S}1 = 1 \tag{4.1}$$

where $\tilde{f}^\star(z) = \overline{\tilde{f}}(\overline{z})$. All this is standard analytic function theory.

Convergent summation, $\mathcal{S}$, is such a good isomorphism between $A$ and $\mathbb{C}_c[[z]]$, that usually no distinction is made between formal (albeit convergent) expansions and their sums which are actual functions. There does not even exist a notational distinction between a convergent series, as a series, and its sum as a number. Yet we can see there is a distinction, at least until we have proven convergence.

**Consequences of the isomorphism to solving problems.** As a result of the isomorphism, whenever a problem can be solved in $\mathbb{C}_c[[z]]$, $\mathcal{S}$ provides an actual solution of the same problem. For example, if $\tilde{y}$ is a formal solution of the equation

$$\tilde{y}' = \tilde{y}^2 + z \tag{4.2}$$

as a series in powers of $z$, with nonzero radius of convergence, and we let $y = \mathcal{S}\tilde{y}$ we may write, using (4.1),

$$\left(\tilde{y}' = \tilde{y}^2 + z\right) \Leftrightarrow \left(\mathcal{S}\left\{\tilde{y}'\right\} = \mathcal{S}\left\{\ \tilde{y}^2\right\} + z\right) \Leftrightarrow \left(y' = y^2 + z\right)$$

i.e. $\tilde{y}$ is a formal solution of (4.2) iff $y$ is an actual solution. The same reasoning would work in many problems with analytic coefficients for which solutions $\tilde{y} \in C_C[[z]]$ can be found.

On the other hand, if we return to the example in Remark 1.24, $f_1$ and $f_2$ differ by a constant $C$, coming from the lower limit of integration, and this $C$ is lost in the process of calculating the asymptotic expansion. To have a complete description, clearly we must account for $C$. It is then natural to try to write instead

$$f_{1,2} \ \text{"} \sim \text{"}\ e^x \tilde{f} + C_{1,2} \tag{4.3}$$

However, Note 1.22 shows $C_{1,2}$ cannot be defined through (1.11); $C_{1,2}$ cannot be calculated as $f_{1,2} - e^x\tilde{f}$ since $\tilde{f}$ does not converge. The right side of (4.3) becomes for now a purely formal object, in the sense that it does not connect to an actual function in any obvious way; (4.3) is perhaps the simplest nontrivial instance of a transseries.

It is the task of the theory of analyzable functions to interpret in a natural and rigorous way expansions such as (4.3), so that expansions and functions are into a one-to-one correspondence. An isomorphism like (4.1) holds in much wider generality.

Some ideas of the theory of analyzable functions can be traced back to Euler as seen in §1.1e , Cauchy, Borel who found the first powerful technique to deal with divergent expansions, and by Dingle and Berry who substantially extended optimal truncation methods.

In the early 80's exponential asymptotics became a field of its own, with the a number of major discoveries of Écalle, the theory of transseries and analyzable functions, and a very comprehensive generalization of Borel summation.

**Setting of the problem**. One operation is clearly missing from both $A$ and $\mathbb{C}_c[[z]]$ namely division, and this severely limits the range of problems that can be solved in either $A$ or $\mathbb{C}_c[[z]]$. The question is then, which spaces $A_1 \supset A$ and $S_1 \supset \mathbb{C}_c[[z]]$ are closed under all function operations, including division, and are such that an extension of $\mathcal{T}$ is an isomorphism between them? (See also §1.1e ). Because of the existence of an isomorphism between $A_1$ and the formal expansions $S_1$ the functions in $A_1$ were called *formalizable*). Exploring the limits of formalizability is at the core of the modern theory of analyzable functions. See also § 1.1e .

In addition to the obvious theoretical interest, there are many important practical applications. One of them, for some generic classes of differential systems where it has been worked out, is the possibility of solving problems starting from formal expansions, which are easy to obtain (usually algorithmically), and from which the isomorphism produces, constructively, actual solutions.

We start by looking at expansions as formal algebraic objects, to understand their structure and operations with them.

### 4.1a Formal asymptotic power series

**Definition 4.4** *For $x \to \infty$, an asymptotic power series (APS) is a formal structure of the type*

$$\sum_{i\in\mathbb{N}} \frac{c_i}{x^{k_i}} \tag{4.5}$$

*We assume that $k_i > k_j$ if $i > j$ and that there is no accumulation point of the $k_i$* [1]

In particular, there is a *smallest* power $k_j \in \mathbb{Z}$, possibly negative. We usually arrange that $c_1 \neq 0$, and then $j = 1$.

**Examples**. (1) Integer power series, i.e. series of the form

$$\sum_{k=M}^{\infty} \frac{c_k}{x^k} \tag{4.6}$$

(2) Multiseries, *finitely generated* power series, of the form

$$\sum_{k_i\ge M} \frac{c_{k_1,k_2,\dots,k_n}}{x^{\alpha_1 k_1+\dots+\alpha_n k_n}} \tag{4.7}$$

for some $M \in \mathbb{Z}$ and $n \in \mathbb{N}$, where $\alpha_1 > 0, \dots, \alpha_n > 0$. Its generators are the *monomials* $x^{-\alpha_1}, \dots, x^{-\alpha_n}$.

**Proposition 4.8** *A series of the form (4.7) can be rearranged as an APS.*

---

[1] $\mathbb{N}$ could be replaced by an ordinal. However, for power series, under the nonaccumulation point assumption, there would be no added generality.

**PROOF** For the proof we note that for any $m \in \mathbb{N}$, the set

$$\{(k_1, k_2, ..., k_n) \in \mathbb{Z}^n : k_i \geq M \text{ for } 1 \leq i \leq n \text{ and } \sum_{i=1}^{n} \alpha_i k_i \leq m\}$$

is finite. Indeed, $k_i$ are bounded below, $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i k_i \to \infty$ if at least one of the sequences $\{k_{i_j}\}$ is unbounded. Thus, there are finitely many distinct powers $x^{-p}$, $p$ between $m$ and $m+1$.

**Exercise 4.9** *As a consequence show that:*
*(1)*

$$\inf\{\nu : \nu = \alpha_1 k_1 + \cdots + \alpha_n k_n \textit{ for some } k_1, ..., k_n \geq M\} < \infty$$

*(2) The set*

$$J := \{\nu : \nu = \alpha_1 k_1 + \cdots + \alpha_n k_n \textit{ for some } k_1, ..., k_n \geq M\}$$

*is countable with no accumulation point. Furthermore $J$ can be linearly ordered*

$$\nu_1 < \nu_2 < \cdots < \nu_k < \cdots$$

*and all the sets*

$$J_i := \{k_1, ..., k_j \geq M : \nu_i = \alpha_1 k_1 + \cdots + \alpha_n k_n\}$$

*are finite.*

*Complete the proof of the proposition.*

$\square$

Thus (4.7) can be written in the form (4.5). The largest term in a series $S$ is *the dominance of* $S$:

**Definition 4.10 (of** *Dom***)** *If $S$ is a nonzero APS of the type (4.5) we define $Dom(S)$ to be $c_{i_1} x^{-k_{i_1}}$ where $i_1$ is the first $i$ in (4.5) for which $c_i \neq 0$ (as noted above, we usually arrange $c_1 \neq 0$). We write $Dom(S) = 0$ iff $S = 0$.*

### 4.1a .1 Operations with APS

**Note 4.11** *The following operations are defined in a natural way and have the usual properties: $+, -, \times, /$ differentiation and composition $S_1 \circ S_2$ where $S_2$ is a series such that $k_1 < 0$. For composition and division, see note after Proposition 4.18. For instance,*

$$\sum_{k=0}^{\infty} \frac{c_j}{x^{\nu_j}} \sum_{l=0}^{\infty} \frac{d_l}{x^{\eta_l}} = \sum_{k,l=0}^{\infty} \frac{c_j d_l}{x^{\nu_j + \eta_l}} \tag{4.12}$$

***Exercise 4.13 (*)*** *Show that the last series in (4.12) can be written in the form (4.5).*

**Exercise 4.14 (*)** *Show that finitely generated power series are closed under the operations mentioned above.*

### 4.1a .2 Asymptotic order relation

If $C_1, C_2 \neq 0$, we naturally write (remember that $x \to +\infty$ and the definition of $\ll$ in (1.8) and (1.9))

$$C_1 x^p \ll C_2 x^q \quad \text{iff} \quad p < q$$

**Definition 4.15** *For two nonzero APSs $S_1$, $S_2$ we write $S_1 \gg S_2$ iff Dom$(S_1) \gg$ Dom$(S_2)$.*

**Proposition 4.16** *Dom$(S_1 S_2) =$ Dom$(S_1)$Dom$(S_2)$, and if Dom$(S) \neq$ const then Dom$(S') =$ Dom$(S)'$.*

**PROOF** Exercise. ∎

Thus we have

**Proposition 4.17** *(See note(4.11)).*
*(i) $S_1 \ll T$ and $S_2 \ll T$ imply $S_1 + S_2 \ll T$ and for any nonzero $S_3$ we have $S_1 S_3 \ll S_2 S_3$.*
*(ii) $S_1 \gg T_1$ and $S_2 \gg T_2$ imply $S_1 S_2 \gg T_1 T_2$.*
*(iii) $S \ll T$ implies $\frac{1}{S} \gg \frac{1}{T}$.*
*(iv) $S \ll T \ll 1$ implies $S' \ll T' \ll 1$ and $1 \ll S \ll T$ implies $S' \ll T'$ (prime denotes differentiation). Also, $s \ll 1 \Rightarrow s' \ll s$ and $L \gg 1 \Rightarrow L \gg L'$. $S' \gg T'$ and $T \gg 1$ implies $S \gg T$. Also $1 \gg S' \gg T'$ implies $S \gg T$.*
*(v) There is the following trichotomy for two nonzero APSs : $S \ll T$ or $S \gg T$ or else $\frac{S}{T} - C \ll 1$ for some constant $C$.*

**PROOF** Exercise. ∎

**Proposition 4.18** *Any nonzero APS $S$ can be uniquely decomposed in the following way*

$$S = L + C + s$$

*where $C$ is a constant and $L$ and $s$ are APS, with the property that $L$ has nonzero coefficients only for* positive *powers of $x$ ($L$ is purely large) and $s$ has nonzero coefficients only for* negative *powers of $x$ ($s$ is purely small; this is the same as, simply, small).*

**PROOF** Exercise. ∎

**Exercise 4.19 (*)** *Show that any nonzero series can be written in then form $S = D(1 + s)$ where $D =$Dom$(S)$ and $s$ is a small series.*

**Exercise 4.20** *Show that the large part of a series has only finitely many terms.*

**Exercise 4.21 (*)** *Show that for any coefficients* $a_1, ..., a_m, ...$ *and* small *series* $s$ *the formal expression*

$$1 + a_1 s + a_2 s^2 + \cdots \tag{4.22}$$

*defines a formal power series. (A proof in a more general setting is given in §4.9.)*

**Note 4.23** *Let* $S$ *be a nonzero series and* $D = C_1 x^{-\nu_1} = Dom(S)$. *We define* $1/D = (1/C_1)x^{\nu_1}$ *and*

$$\frac{1}{S} = \frac{1}{D}(1 - s + s^2 - s^3 \cdots) \tag{4.24}$$

*and more generally*

$$S^\beta := C_1^\beta x^{-\nu_1 \beta}\left(1 + \beta s + \frac{1}{2}\beta(\beta-1)s^2 + \cdots\right) \tag{4.25}$$

*The composition of two series* $S = \sum_{k=0}^{\infty} s_k x^{-\nu_k}$ *and* $L$ *where* $L$ *is large is defined as*

$$S \circ L := \sum_{k=0}^{\infty} s_k L^{-\nu_k} \tag{4.26}$$

**Exercise 4.27 (*)** *Show that (4.26) defines a formal power series which can be written in the form (4.5).*

### Examples

**Proposition 4.28** *The differential equation*

$$y' + y = \frac{1}{x} + y^3 \tag{4.29}$$

*has a unique solution as an APS which is purely small.*

**PROOF** For the existence part, note that direct substitution of a formal integer power series $y_0 = \sum_{k=1}^{\infty} c_k x^{-k}$ leads to the recurrence relation $c_1 = 1$ and for $k \geq 2$,

$$c_k = (k-1)c_{k-1} + \sum_{k_1+k_2+k_3=k; k_i \geq 1} c_{k_1} c_{k_2} c_{k_3}$$

for which direct induction shows the existence of a solution, and we have

$$y_0 = \frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{12}{x^4} + \frac{60}{x^5} + \cdots$$

For uniqueness assume $y_0$ and $y_1$ are APS solutions and let $\delta = y_1 - y_0$. Then $\delta$ satisfies

$$\delta' + \delta = 3y_0^2\delta + 3y_0\delta^2 + \delta^3 \tag{4.30}$$

Since by assumption $\delta \ll 1$ we have $\mathrm{Dom}(\delta') \ll \mathrm{Dom}(\delta)$ and similarly $\mathrm{Dom}(3y_0^2\delta + 3y_0\delta^2 + \delta^3) \ll \mathrm{Dom}(\delta)$. But this implies $\mathrm{Dom}(\delta) = 0$ and thus $\delta = 0$. There are further formal solutions, not within APS but as more general *transseries* containing exponentially small terms.

□

### 4.1a .3 The exponential

**Proposition 4.31** *If $s$ is a purely small series then the equation $y' = s'y$ (corresponding intuitively to $y = e^s$) has APS solutions of the form $C + s_1$ where $s_1$ is small. If we choose $C = 1$ then $s_1 = s_{1;1}$ is uniquely defined.*

We define, according to the previous proposition, $e^s = 1 + s_{1;1}$; $1 + s_{1,;1}$ is simply the familiar Maclaurin series of the exponential.

But $e^x$ is not definable in terms of APS.

**Proposition 4.32** *The differential equations $f' = \pm x$ $(*)$ have no nonzero APS solution.*

**PROOF** By Proposition 4.17, if $f \neq 0$ is an APS, then $f' \ll f$, so $f' = \pm f$ is not possible. □

Thus we adjoin a solution of (*) as a new *formal monomial* $e^x$ [2], determined by the equation only up to a multiplicative constant, derive its properties and check for consistency of the extension (meaning in this case that the new element is compatible with the structure it was adjoined to). Monomials are by definition positive, so we postulate $e^x > 0$.[3] Then, from (*) we see that $e^x > \mathit{const}$ and inductively, $e^x > \mathit{const}\,x^n$ for any $n$. Thus $e^x \gg x^n$ for all $n$. Consistency of the definition is a consequence of the existence of transseries, constructed in detail in §4.9 (with a sketch in §4.2b ).

More generally, if $L$ is a large series, say purely large, then we have to *define* the composition $\exp(L(x))$ as a new symbol (it cannot be constructed based on APS and $e^x$). To preserve the properties of differentiation we should have $(e^L)' = L'e^L$. Then $e^L$ is a solution of the equation $f' = L'f$; since $(e^{L_1}e^{L_2})$ is a solution of $f' = (L_1' + L_2')f$, for consistency, we should define $e^{L_1}e^{L_2} = e^{L_1+L_2}$. If $L_1 = -L_2$ then $e^{L_1}e^{L_2} = \mathit{const}$ which, by symmetry, should be one.

**Remark 4.33** The general solution of $y' + y = 0$ is $Ce^{-x}$. Indeed, we may multiply by $e^x$ and get $(ye^x)' = 0$, i.e. $ye^x = C$ or $y = Ce^{-x}$.

---

[2] The existence of a function solution to (*) is not relevant here, since APSs are not functions.

[3] We know that this cannot be inconsistent with the equation and order relation, since it is *true* for the actual exponential.

**Definition 4.34** *In general if $S = L+C+s$ we write $e^S = C(1+s_{1;1})e^L$ where $e^L$ is to be thought of as a primary symbol, subject to the further definitions $e^{L_1+L_2} = e^{L_1}e^{L_2}$ and $(e^L)' = L'e^L$.*

After we have adjoined these new elements, the general solution of

$$f' + f = x^{-1} \tag{4.35}$$

is

$$\tilde{y}_0 + Ce^{-x} := \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} + Ce^{-x} \tag{4.36}$$

Indeed, if $\tilde{y}$ is any solution of (4.35) then $\tilde{f} = \tilde{y} - \tilde{y}_0$ satisfies the homogeneous equation $f' + f = 0$. The rest follows from Remark 4.33.

To formally solve (4.35) within power series, only algebraic operations and differentiation are needed. However, within the differential field [4] , *generated* by $1/x$, [5] (4.36) has no nonzero solution, as see in the exercise below.

**Exercise 4.37 (*)** Find a differential field, containing $1/x$,in which (4.35) has no solution: Note that the space of functions which are meromorphic at infinity form a differential field. Show that this is the case with convergent power series of the form $\sum_{k\ge k_0} c_k x^{-k}$ with $k_0 \in \mathbb{Z}$ (possibly negative). Complete the proof.

This shows again that the space of transseries has to have enough many "innate" objects, or else simple equations cannot be solved. If there are too many though, association with true functions is endangered. It is a delicate balance.

### 4.1a .4 Exponential power series (EPS)

A simple example of EPS is a formal expression of the type

$$\sum_{i,j=1}^{\infty} \frac{c_{ij}}{e^{\lambda_i x} x^{k_j}} \tag{4.38}$$

where $\lambda_i$ are increasing in $i$ and $k_j$ are increasing in $j$. Again the usual operations are well defined on EPS (except for composition, to be defined later, together with transseries).

The order relation, compatible with the discussion in § 4.1a .3, is defined by $e^{\lambda_1 x}x^{k_2} \gg e^{\lambda_3 x}x^{k_4}$ iff $\lambda_1 > \lambda_3$ or if $\lambda_1 = \lambda_3$ and $k_2 > k_4$. Consistent with

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[4] That is, roughly, a differential algebra with a consistent division. A (commutative) differential algebra is a structure endowed with the algebraic operations $+, -\times$, multiplication by constants and differentiation, and with respect to these operations behave as expected, see e.g. [39].

[5] There is a minimal differential field containing $1/x$, by definition the one *generated by* $1/x$.

this order relation it is then natural to reorder the expansion(4.38) as follows

$$\sum_{i=1}^{\infty} e^{-\lambda_i x} \sum_{j=1}^{\infty} \frac{c_{ij}}{x^{k_j}} \tag{4.39}$$

Then we can still define the dominance of a structure of the form (4.38).

As another simple example of an EPS, let us find the formal antiderivative of $e^{x^2}$. We write $y' = e^{x^2}$ and, for instance by WKB we see that $y$ is of order $e^{x^2}$. We write $y = ge^{x^2}$ and get

$$g' + 2xg = 1 \tag{4.40}$$

where, a power series solution can be found by noting that, within APSs $g'/x \ll g$, and we write

$$g = \frac{1}{2x} - \frac{1}{2x} g' \tag{4.41}$$

and by formal iteration we get

$$\tilde{g}_0 = \frac{1}{2x} + \frac{1}{4x^3} + \frac{1 \cdot 3}{8x^5} + \frac{1 \cdot 3 \cdot 5}{16x^5} \cdots = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k x^{2k+1}} \quad ((-1)!! = 1) \tag{4.42}$$

(compare with §3.1). The general solution is $\tilde{g}_0$ plus the general solution of the associated homogeneous equation $g' + 2xg = 0$, $Ce^{-x^2}$. Thus

$$EPS\left(\int e^{x^2}\right) = e^{x^2} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k x^{2k+1}} + C, \quad x \to \infty$$

### 4.1a .5 Exponential power series solutions for (4.29)

To show how transseries arise naturally as solutions of ODEs we continue the formal analysis of (4.29).

To simplify notation, we drop the tildes from formal asymptotic expansions. We have obtained, in Proposition 4.17 a formal series solution (4.29), $y_0$. We look for possible further solutions. We take $y = y_0 + \delta$. The equation for $\delta$ is (4.30) where we search for solutions $\delta \ll 1$, in which assumption the terms on the right side of the equation are subdominant (see footnote 8 on Page 80). We have $\delta' + \delta(1 + o(1)) = 0$ thus $\delta = Ce^{-x+o(x)}$and this suggests the substitution $\delta = e^w$. We get

$$w' + 1 = 3y_0^2 + 3y_0 e^w + e^{2w}$$

and since $e^w = \delta \ll 1$ the dominant balance (footnote 8, Page 80) is between the terms on the left side, thus $w = -x + C + w_1$ and we get

$$w_1' = 3y_0^2 + 3y_0 e^{-x} e^{w_1} + e^{-2x+2w_1}$$

We have $y_0 e^{-x} e^{w_1} = y_0 \delta = y_0 e^{-x+o(x)}$. Since $y_0 e^{-x} e^{w_1} \ll x^{-n}$ for any $n$ and thus $w_1' = O(x^{-2})$ then $w_1 = O(x^{-1})$. Thus, $e^{w_1} = 1 + w_1 + w_1^2/2 + ...$ and consequently $3y_0 e^{-x} e^{w_1} + e^{-2x+2w_1}$ is negligible with respect to $y_0^2$. Again by dominant balance, to leading order, $w_1' = 3y_0^2$ and thus $w_1 = \int 3y_0^2 + w_2 := \phi_1 + w_2$ ($\phi_1$ is a formal power series). It follows that, to leading order, we have

$$w_2' = 3y_0 e^{-x}$$

and thus $w_2 = \phi_2 e^{-x}$ where $\phi_2$ is a power series. Continuing this process of iteration, we can see inductively that $w$ must be of the form

$$w = -x + \sum_{k=0}^{\infty} \phi_k e^{-kx}$$

where $\phi_k$ are formal power series, which means

$$y = \sum_{k=0}^{\infty} e^{-kx} y_k \tag{4.43}$$

where $y_k$ are also formal power series. Having obtained this information, it is more convenient to plug in(4.43) directly in the equation and solve for the unknown series $y_k$. We get the system

$$\begin{gathered} y_0' + y_0 = x^{-1} + y_0^3 \\ y_1' = 3y_0^2 y_1 \\ \cdots \\ y_k' - ky_k - 3y_0^2 y_k = 3y_0 \sum_{k_1+k_2=k; k_i \ge 1} y_{k_1} y_{k_2} + \sum_{k_1+k_2+k_3=k; k_i \ge 1} y_{k_1} y_{k_2} y_{k_3} \\ \cdots \end{gathered} \tag{4.44}$$

(Check that for a given $k$, the sums contain finitely many terms.) We can easily see by induction that this system of equations does admit a solution where $y_k$ are integer power series. Furthermore, $y_1$ is defined up to an arbitrary multiplicative constant, and there is no further freedom in $y_k$, whose equation can be solved by our usual iteration procedure, after placing the subdominant term $y_k'$ on the rhs. We note that all equations for $k \ge 1$ are *linear inhomogeneous.* The fact that high-order equations are linear is a general feature in perturbation theory.

Choosing then $y_0$ in such a way that $y_1^{[1]} = 1 + ax^{-1} + ...$ we have $y_1 = Cy_1^{[1]}$. By the special structure of the rhs of the general equation in (4.44) we see that if $y_k^{[1]}$ is the solution with the choice $y_1 = y_1^{[1]}$ we see, by induction, that the solution when $y_1 = Cy_1^{[1]}$ is $C^k y_k^{[1]}$. Thus the general formal solution of (4.29) in our setting should be

$$\sum_{k=0}^{\infty} C^k y_k^{[1]} e^{-kx}$$

where $y_0^{[1]} = y_0$.

**Exercise 4.45 (*)** Complete the details in the previous analysis: show that the equation for $y_1$ in (4.44) has a one parameter family of solutions of the form $y_1 = c(1+s_1)$ where $s_1$ is a small series, and that this series is unique. Show that for $k > 1$, given $y_0, ..., y_{k-1}$, the equation for $y_k$ in (4.44) has a unique small series solution. Show that there exists exactly a one parameter family of solutions general formal exponential-power series solution of the form (4.43) of (4.29).

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## 4.2 Transseries

### 4.2a Remarks about the form of asymptotic expansions

The asymptotic expansions seen in the previous examples have the common feature that they are written in terms of powers of the variable, exponentials and logs, e.g.

$$\int_x^{\infty} e^{-s^2} ds \sim e^{-x^2}\left(\frac{1}{2x} - \frac{1}{4x^2} + \frac{5}{8x^3} - ...\right) \tag{4.46}$$

$$n! \sim \sqrt{2\pi} e^{n\ln n - n + \frac{1}{2}\ln n}\left(1 + \frac{1}{12n} + ...\right) \tag{4.47}$$

$$\int_1^x \frac{e^t}{t} dt \sim e^x \left(\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + ...\right) \tag{4.48}$$

### 4.2b Construction of transseries: a first sketch

Transseries are studied carefully in §4.9. They are finitely generated asymptotic combinations of powers, exponentials and logs and are defined inductively. In the case of a power series, finite generation means that the series is an integer multiseries in $y_1, ..., y_n$ where $y_j = x^{-\beta_j}, \mathrm{Re}\,(\beta_j) > 0$. Examples are (4.38), (3.81) and (1.25); a more involved one would be

$$\ln\ln x + \sum_{k=0}^{\infty} e^{-k\exp(\sum_{k=0}^{\infty} k! x^{-k})}$$

A single term in a transseries is a transmonomial.

1. A term of the form $m = x^{-\alpha_1 k_1 - \ldots - \alpha_n k_n}$ with $\alpha_i > 0$ is a level zero **(trans)monomial**.

2. Real transseries of level zero are simply finitely generated *asymptotic* power series. That is, given $\alpha_1, ..., \alpha_n$ with $\alpha_i > 0$ a level zero transseries is a sum of the form

$$S = \sum_{k_i \ge M_i} c_{k_1,\ldots,k_n} x^{-\alpha_1 k_1 - \ldots - \alpha_n k_n} \tag{4.49}$$

   with $c_{M_1,\ldots M_n} \neq 0$ where $M_1, ..., M_n$ are *integers*, positive or negative; the terms of $S$ are therefore nonincreasing in $k_i$ and bounded *above* by $O(x^{-\alpha_1 M_1 - \ldots - \alpha_n M_n})$.

3. $x^{-\alpha_1 M_1 - \ldots - \alpha_n M_n}$ is the leading order, $c_{M_1,\ldots M_n}$ is the leading constant and $c_{M_1,\ldots M_n} x^{-\alpha_1 M_1 - \ldots - \alpha_n M_n}$ is the dominance of (4.49), $\mathrm{Dom}(S)$.

   **Note.** When we will construct transseries more carefully, we will denote $\boldsymbol{\mu}_{\mathbf{k}} =: \mu_1^{k_1} \cdots {\mu_n}^{k_n}$ the monomial $x^{-k_1\alpha_1 - \cdots - k_n\alpha_n}$. We note that $\mathbf{k} \mapsto \boldsymbol{\mu}_{\mathbf{k}}$ defines a morphism between $\mathbb{Z}^n$ and the Abelian multiplicative group generated by $\mu_1, ..., \mu_n$.

4. The lower bound for $k_i$ easily implies that there are only finitely many terms with the same monomial. Indeed, the equation $\alpha_1 k_1 + ... + \alpha_n k_n = p$ does not have solutions if $\mathrm{Re}\,(\alpha_i) k_i > |p| + \sum_{j \neq i} |\alpha_j||M_j|$.

5. A level zero transseries can be decomposed as $L + \mathit{const} + s$ where $L$, which could be zero, is the purely large part in the sense that it contains only large monomials, and $s$ is small.

   If $S \neq 0$ we can write uniquely

$$S = \mathit{const}\, x^{-\alpha_1 M_1 - \ldots - \alpha_n M_n}(1 + s)$$

   where $s$ is small.

6. Operations are defined on level zero transseries in a natural way. The product of level zero transseries is a level zero transseries where as in (4) above the lower bound for $k_i$ entails that there are only finitely many terms with the same monomial in the product.

7. It is easy to see that the expression $(1 - s)^{-1} := 1 - s + s^2 - ...$ is well defined and this allows definition of division via

$$1/S = \mathit{const}^{-1} x^{\alpha_1 M_1 + \ldots + \alpha_n M_n}(1 - s)^{-1}$$

8. A transmonomial is small if $m = o(1)$ and large if $1/m$ is small. $m$ is neither large nor small iff $m = 1$ which happens iff $-\alpha_1 k_1 - ... - \alpha_n k_n = 0$; this is a degenerate case.

9. It can be checked that level zero transseries form a differential field. Composition $S_1(S_2)$ is also well defined whenever $S_2$ is a *large* transseries.

In a more abstract language that we will use later, for a given set of monomials $\mu_1, ..., \mu_n$ and the multiplicative group $\mathcal{G}$ generated by them, a transseries of level zero is a function defined on $\mathbb{Z}^n$ with values in $\mathbb{C}$, with the property that for some $\mathbf{k}_0$ we have $F(\mathbf{k}) = \mathbf{0}$ if $\mathbf{k} < \mathbf{k}_0$.

More general transseries are defined inductively; in a first step exponentials of purely large level zero series are level one transseries.

It is convenient to first construct transseries without logs and then define the general ones by composition to the right with an iterated log.

10. **Level one**. The exponential $e^x$ has no asymptotic *power* series at infinity (Proposition 4.32) and $e^x$ is taken to be its own expansion. It is a new element. More generally, $e^L$ with $L$ purely large (positive or negative) is a new element.

11. A level one transmonomial is of the form $\mu = me^L$ where $m$ is a level zero transmonomial and $L$ is a purely large level zero transseries. $\mu$ is *large* if the leading constant of $L$ is positive and small otherwise. If $L$ is large and positive then $e^L$ is, by definition, much larger than any monomial of level zero. We define naturally $e^{L_1}e^{L_2} = e^{L_1+L_2}$. Note that in our convention both $x$ and $-x$ are *large* transseries.

12. A level one transseries is of the form

$$S = \sum_{k_i \geq M_i} c_{k_1,...,k_n} \mu_1^{-k_1} \cdots \mu_n^{-k_n} := \sum_{\mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \tag{4.50}$$

   where $\mu_i$ are *large* level one transmonomials.

   With the operations defined naturally as above, level one transseries form a differential field.

13. We define, for a *small* transseries, $e^s = \sum_{k=0}^{\infty} s^k/k!$. If $s$ is of level zero, then $e^s$ is of level zero too. Instead, we cannot expand $e^L$, where $L$ is purely large.

14. Differentiation is defined inductively: with $(x^a)' = ax^{a-1}$, and the steps to be carried by induction are $(fg)' = f'g + fg'$, $\left(\sum_{\mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}}\right)' = \sum c_{\mathbf{k}} \left(\boldsymbol{\mu}^{\mathbf{k}}\right)'$ and $(e^L)' = L'e^L$.

15. The construction proceeds similarly, by induction and a general exponential-free transseries is one obtained at *some level* of the induction. They form a differential field.

16. In general, transseries have an exponential level (height) which is the highest order of composition of the exponential (and similarly a logarithmic depth) $\exp(\exp(x^2)) + \ln x$ has height 2 and depth 1. Height and depth are required to be finite. That is, for instance, the expression

    $$f = e^{-x} + e^{-e^{x}} + e^{-e^{e^{x}}} + \cdots \tag{4.51}$$

    (as a *function series* (4.51) converges uniformly on $\mathbb{R}$, to a $C^\infty$ function) [6] is not a valid transseries.

17. It can be shown, by induction, that $S' = 0$ *iff* $S = const.$

18. *Dominance*: If $S \neq 0$ then there is a largest transmonomial $\mu_1^{-k_1} \cdots \mu_n^{-k_n}$ in $S$, with nonzero coefficient, $C$. Then $\mathrm{Dom}(S) = C\mu_1^{-k_1} \cdots \mu_n^{-k_n}$. If $S$ is a nonzero transseries, then $S = \mathrm{Dom}(S)(1 + s)$ where $s$ is small, i.e., all the transmonomials in $s$ are small. A base of monomials can then be chosen such that all $M_i$ in $s$ are positive; this is shown in §4.9.

19. Topology.

    (a) If $\tilde{S}$ is the space of transseries generated by the monomials $\mu_1, ..., \mu_n$ then, by definition, the sequence $S^{[j]}$ converges to $S$ given in (4.50) if for any $\mathbf{k}$ there is a $j_0 = j_0(\mathbf{k})$ such that $c_{\mathbf{k}}^{[j]} = c_{\mathbf{k}}$ for all $j \geq j_0$.

    (b) In this topology, addition and multiplication are continuous, but multiplication by scalars is not.

    (c) It is easy to check that any Cauchy sequence is convergent and transseries form a complete linear topological space.

    (d) Contractive mappings: A function (operator) $\mathcal{A} : \tilde{S} \to \tilde{S}$ is contractive if for some $\alpha < 1$ and any $S_1, S_2 \in \tilde{S}$ we have $\mathrm{Dom}(\mathcal{A}(S_1) - \mathcal{A}(S_2)) \leq \alpha\, \mathrm{Dom}(S_1 - S_2)$.

    (e) *Fixed point theorem.* It can be proved in the usual way that if $\mathcal{A}$ is contractive, then the equation $S = S_0 + \mathcal{A}(S)$ has a unique fixed point.

    *Examples* –This is a convenient way to show the existence of multiplicative inverses. It is enough to invert $1 + s$ with $s$ small. We choose a basis such that all $M_i$ in $s$ are positive. Then the equation $y = 1 - sy$ is contractive.

---

[6] It turns out that the Taylor series of $f$ has zero radius of convergence everywhere, with $|f^{(m)}|$ exceeding $e^{m \ln m \ln \ln m}$.

–Differentiation is contractive on level zero transseries (multiseries). This is intuitively obvious, since every power in a series decreases by one.

–The equation $y = 1/x - y'$ is contractive within level zero transseries; It has a unique solution.

–The inhomogeneous Airy equation

$$y = \frac{1}{x} + \frac{1}{x}y''$$

also has a unique solution, namely

$$y = \left[\sum_{k=0}^{\infty}\left(\frac{1}{x}\frac{d^2}{dx^2}\right)^k\right]\frac{1}{x} = \frac{1}{x} + \frac{2}{x^4} + \frac{40}{x^7} + \cdots \tag{4.52}$$

20. If $L_n = \log(\log(\ldots\log(x)))$ $n$ times, and $T$ is an exponential-free transseries then $T(L_n)$ is a general transseries. Transseries form a differential field, furthermore closed under integration, composition to the right with large transseries, and many other operations; this closure is proved as part of the general induction.

21. The theory of differential equations in transseries has many similarities with the usual theory. For instance it is easy to show, using an integrating factor and 17 above that the equation $y' = y$ has the general solution $Ce^x$ and that the Airy equation $y'' = xy$ that we looked at already, has at most two linearly independent solutions. We will find two such solutions in the examples below.

**Note 4.53** *Differentiation is* **not contractive** *on the space of power series* **at zero**, *or at any point* $z_0 \in \mathbb{C}$, *but* **only** *on asymptotic series at infinity. Note that* $d/dz = d/d(1/x) = -x^2 d/dx$.

### 4.2c Examples of transseries solution: a nonlinear ODE

To find a formal power series solution of

$$y' + y = x^{-2} + y^3 \tag{4.54}$$

we proceed as usual, separating out the dominant terms, in this case $y$ and $x^{-2}$. We get the iterations scheme, contractive on level zero transseries,

$$y_{[n]}(x) - x^{-2} = y^3_{[n-1]} - y'_{[n-1]} \tag{4.55}$$

with $y_{[0]} = 0$. After a few iterations we get

$$\tilde{y}(x) = x^{-2} + 2\,x^{-3} + 6\,x^{-4} + 24\,x^{-5} + 121\,x^{-6} + 732\,x^{-7} + 5154\,x^{-8} + \cdots \tag{4.56}$$

To find further solutions, since contractivity shows there are no further level zero transseries, we look for higher order corrections. We write $y = \tilde{y} + \delta =: y_0 + \delta$ and obtain (4.30). Since $y_0$ and $\delta$ are small, to leading order the equation is $\delta' + \delta = 0$. Thus $\delta = Ce^{-x}$. Including the power correction $3y_0^2\delta$ we get $\delta = Cy_1e^{-x}$ where $y_1$ is a power series. Clearly, to next order of approximation, we need to take into account $3y_0\delta^2$ which is roughly $3C^2y_0y_1^2e^{-2x}$. This introduces a correction of type $C^2y_2e^{-2x}$ to $\delta$, with $y_2$ a power series, and continuing, we get, through the nonlinearity $\delta = \sum_{k=1}^{\infty} C^k e^{-kx} y_k$, a level one transseries. To show uniqueness we can write the equation for $\delta$ in a contractive way, which is better done within the rigorous theory of transseries (cf. Exercise 4.232). For further analysis of this transseries see §5.3b .

*Example 2.* To find a formal solution for the Gamma function recurrence $a_{n+1} = na_n$, we look directly for transseries of level at least one, $a_n = e^{f_n}$ (since it is clear that no power series would satisfy the recurrence). Thus $f_{n+1} = \ln n + f_n$. It is clear that $f_{n+1} - f_n \ll f_n$; this suggests writing $f_{n+1} = f_n + f_n' + \frac{1}{2}f_n'' + ...$ and, taking $f' = h$ we get the equation

$$h_n = \ln n - \frac{1}{2}h_n' - \frac{1}{6}h_n'' - ... \tag{4.57}$$

(which is contractive in the space of transseries of zero level, see also Note4.53). We get

$$h = \ln n - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} ...$$

and thus

$$f_n = n\ln n - n - \frac{1}{2}\ln n + \frac{1}{12n} - \frac{1}{360n^3} ... + C$$

## 4.3 Solving equations in terms of Laplace transforms

Let us now consider again the Airy equation

$$y'' = xy \tag{4.58}$$

We divide by $\exp(\frac{2}{3}x^{3/2})$ and change variable $\frac{2}{3}x^{3/2} = s$ to ensure that the transformed function has an asymptotic series with power-one of the factorial divergence. The need for that will be clarified later, see §4.7.

Taking then $y(x) = e^{\frac{2}{3}x^{3/2}}h(\frac{2}{3}x^{3/2})$ we get

$$h'' + \Big(2 + \frac{1}{3s}\Big)h' + \frac{1}{3s}h = 0 \tag{4.59}$$

and with $H = \mathcal{L}^{-1}(h)$ we get

$$p(p-2)H' = \frac{5}{3}(1-p)H$$

The solution is

$$H = Cp^{-5/6}(2-p)^{-5/6}$$

and it can be easily checked that any integral of the form

$$h = \int_0^{\infty e^{i\phi}} e^{-ps} H(p) dp$$

for $\phi \neq 0$ is a solution of (4.59) yielding the expression

$$f = e^{\frac{2}{3}x^{3/2}} \int_0^{\infty e^{i\phi}} e^{-\frac{2}{3}x^{3/2}p} p^{-5/6}(2-p)^{-5/6} dp \tag{4.60}$$

for a solution of the Airy equation. A second solution can be obtained in a similar way, replacing $e^{\frac{2}{3}x^{3/2}}$ by $e^{-\frac{2}{3}x^{3/2}}$, or by taking the difference between two integrals of the form (4.60). Note what we did here is *not* Laplace's method of solving linear ODEs. Examine the differences.

For Example 2 above, factorial divergence suggests taking inverse Laplace transform of $g_n = f_n - (n \ln n - n - \frac{1}{2}\ln n)$.
The recurrence satisfied by $g$ is

$$g_{n+1} - g_n = q_n = 1 - \left(\frac{1}{2} + n\right)\ln\left(1 + \frac{1}{n}\right) = -\frac{1}{12n^2} + \frac{1}{12n^3} + ...$$

First note that $\mathcal{L}^{-1}q = p^{-2}\mathcal{L}^{-1}q''$ which can be easily evaluated by residues since

$$q'' = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}\Big(\frac{1}{(n+1)^2} + \frac{1}{n^2}\Big)$$

Thus, with $\mathcal{L}^{-1}g_n := G$ we get

$$(e^{-p} - 1)G(p) = \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2}$$

$$g_n = \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2} + 1\right)e^{-p}}{p^2(e^{-p} - 1)} e^{-np} dp$$

(It is easy to check that the integrand is analytic at zero; its Taylor series is $\frac{1}{12} - \frac{1}{720}p^2 + O(p^3)$.)

The integral is well defined, and it easily follows that

$$f_n = C + n(\ln n - 1) - \frac{1}{2}\ln n + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)} e^{-np}dp$$

solves our recurrence. The constant $C = \frac{1}{2}\ln(2\pi)$ is most easily obtained by comparing with Stirling's series (3.153) and we thus get the identity

$$\ln\Gamma(n+1) = n(\ln n - 1) - \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi) + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)} e^{-np}dp \tag{4.61}$$

which holds with $n$ replaced by $z \in \mathbb{C}$ as well.

This represents, as it will be clear from the definitions, the Borel summed version of Stirling's formula.

**Exercise 4.62 (*)** *Prove formula (1.2); find a bound for* $C$ *when* $|z| < 1/2$.

Other recurrences can be dealt with in the same way. One can calculate $\sum_{j=1}^{n} j^{-1}$ as a solution of the recurrence

$$s_{n+1} - s_n = \frac{1}{n}$$

Proceeding as in the Gamma function example, we have $f' - \frac{1}{n} = O(n^{-2})$ and the substitution $s_n = \ln n + g_n$ yields

$$g_{n+1} - g_n = \frac{1}{n} + \ln\left(\frac{n}{n+1}\right)$$

and in the same way we get

$$f_n = C + \ln n + \int_0^\infty e^{-np}\left(\frac{1}{p} - \frac{1}{1-e^{-p}}\right)dp$$

where the constant can be obtained from the initial condition, $f_1 = 0$,

$$C = -\int_0^\infty e^{-p}\left(\frac{1}{p} - \frac{1}{1-e^{-p}}\right)dp$$

which, by comparison with the usual asymptotic expansion of the harmonic sum also gives an integral representation for the Euler constant,

$$\gamma = \int_0^\infty e^{-p}\left(\frac{1}{1-e^{-p}} - \frac{1}{p}\right)dp$$

Comparison with (4.61) gives

$$\sum_{j=1}^{n-1}\frac{1}{j} - \gamma = \ln n + \int_0^\infty e^{-np}\left(\frac{1}{p} - \frac{1}{1-e^{-p}}\right)dp = \frac{\Gamma'(n)}{\Gamma(n)} \tag{4.63}$$

*Exercise: The Zeta function.* Use the same strategy to show that

$$(n-1)!\zeta(n) = \int_0^\infty p^{n-1}\frac{e^{-p}}{1-e^{-p}}dp = \int_0^1 \frac{\ln^{n-1} s}{1-s}ds \tag{4.64}$$

### 4.3 .1 The Euler-Maclaurin summation formula

Assume $f(n)$ does not increase too rapidly with $n$ and we want to find the asymptotic behavior of

$$S(n+1) = \sum_{k=k_0}^{n} f(k) \tag{4.65}$$

for large $n$. We see that $S(k)$ is the solution of the difference equation

$$S(k+1) - S(k) = f(k) \tag{4.66}$$

To be more precise, assume $f$ has a level zero transseries as $n \to \infty$. Then we write $\tilde{S}$ for the transseries of $S$ which we seek at level zero. Then $\tilde{S}(k+1) - \tilde{S}(k) = \tilde{S}'(k) + \tilde{S}''(k)/2 + ... + \tilde{S}^{(n)}(k)/k! + ... = \tilde{S}' + L\tilde{S}$ where

$$L = \sum_{k=2}^{\infty} \frac{1}{j!}\frac{d^j}{dk^j} \tag{4.67}$$

is contractive on $\mathcal{T}_0$ (check) and thus

$$\tilde{S}'(k) = f(k) - L\tilde{S} \tag{4.68}$$

has a unique solution,

$$\tilde{S}' = \sum_{j=0}^{\infty}(-1)^j L^j f =: \frac{1}{1-L} f \tag{4.69}$$

(check that there are no transseries solutions of higher level). From the first few terms, or using successive approximations, we get

$$\tilde{S}'(k) = f(k) - \frac{1}{2}f'(k) + \frac{1}{12}f''(k) - \frac{1}{720}f^{(4)}(k) + \cdots = \sum_{j=0}^{\infty} C_j f^{(j)}(k) \tag{4.70}$$

Examining the way the $C_j's$ are obtained, it is clear that they do not depend on $f$. Then it suffices to look at some particular $f$ for which the sum can be calculated explicitly. If $n > 0$ we have

$$\sum_{k=0}^{\infty} e^{-k/n} = \frac{1}{1-e^{-1/n}} \tag{4.71}$$

while, by one of the definitions of the Bernoulli numbers we have

$$\frac{z}{1-e^{-z}} = \sum_{k=0}^{\infty}(-1)^j \frac{B_j}{j!} z^j \tag{4.72}$$

**Exercise 4.73** *Using these identities, determine the coefficients $C_j$ in (4.70).*

By integration we get

$$S(k) \sim \int_{k_0}^{k} f(s)ds + C + \sum_{j=0}^{\infty} \frac{B_{j+1}}{(j+1)!} f^{(j)}(k) \tag{4.74}$$

Rel. (4.74) is called the Euler-Maclaurin sum formula.

**Exercise 4.75 (*)** *Complete the details of the calculation involving the identification of coefficients in the Euler-Maclaurin sum formula.*

**Exercise 4.76** *Find for which values of $a > 0$ the series*

$$\sum_{k=1}^{\infty} \frac{e^{i\sqrt{k}}}{k^a}$$

*is convergent.*

**Exercise 4.77 (*)** *Prove the Euler-Maclaurin sum formula in the case $f$ is $C^\infty$ by first looking at the integral $\int_n^{n+1} f(s)ds$ and expanding $f$ in Taylor by $s = n$. Then correct $f$ to get a better approximation etc.*

That (4.74) gives the correct asymptotic behavior in fairly wide generality is proved, for example, in [19].

We will prove here, under stronger assumptions, a stronger result which implies (4.74). The conditions are often met in applications, after changes of variables, as our examples showed.

**Lemma 4.78** *Assume $f$ has a Borel summable expansion at $0^+$ (in applications $f$ is often analytic at $0$) and $f(z) = O(z^2)$. Then $f(1/n) = \int_0^\infty F(p)e^{-np}dp$, $F(p) = O(p)$ for small $p$ and*

$$\sum_{k=n_0}^{n-1} f(1/n) = \int_0^\infty e^{-np} \frac{F(p)}{e^{-p}-1} dp - \int_0^\infty e^{-n_0 p} \frac{F(p)}{e^{-p}-1} dp \tag{4.79}$$

**PROOF** We seek a solution of (4.66) in the form $S = C + \int_0^\infty H(p)e^{-kp}dp$, or, in other words we inverse Laplace transform the equation (4.66). We get

$$(e^{-p} - 1)H = F \Rightarrow H(p) = \frac{F(p)}{e^{-p}-1} \tag{4.80}$$

and the conclusion follows by taking the Laplace transform which is well defined since $F(p) = O(p)$, and imposing the initial condition $S(k_0) = 0$. □

### 4.3a A second order ODE: the Painlevé equation $\mathrm{P_I}$

$$\frac{d^2y}{dx^2} = 6y^2 + x \tag{4.81}$$

We first look for formal solutions. As a transseries of level zero it is easy to see that the only possible balance is $6y^2 + x = 0$ giving

$$y \sim \pm \frac{i}{\sqrt{6}}\sqrt{x}$$

We choose one of the signs, say + and write

$$\tilde{y}_0 = \frac{i}{\sqrt{6}}\sqrt{x - \tilde{y}_0''} = \frac{i}{\sqrt{6}}\left(\sqrt{x} - \frac{\tilde{y}_0''}{2\sqrt{x}} - \frac{1}{8x^{3/2}}(\tilde{y}_0'')^2 \cdots\right) \tag{4.82}$$

By iteration we get

$$\tilde{y}_0 = \frac{i}{\sqrt{6}}\left(\sqrt{x} + \frac{i\sqrt{6}}{48x^2} + \frac{49i}{768x^{9/2}} \cdots\right) \tag{4.83}$$

To find the solution as a Laplace transform of a function with a convergent series at the origin, we need to ensure that the formal series is Gevrey one, see §4.5 and §4.7. The growth of the coefficients of the $x$ series can be estimated from their recurrence, but there are better ways to proceed, for instance using the duality with the type of possible small exponential corrections. The reason behind this duality will be explained in §4.7.

**Exercise 4.84** *Let* $\tilde{y} = \tilde{y}_0 + \delta$ *be a transseries solution to (4.82). Show (for instance by WKB) that* $\ln\delta = \frac{4}{5}\sqrt{2i}6^{1/4}x^{5/4}(1 + o(1)$

Equivalently, still heuristically for the moment, we note that the series is obtained, by and large, by repeated iteration of $d^2/(\sqrt{x}dx^2)$. This applied to power series, and insofar as the ensuing divergence of coefficients is concerned, is equivalent to repeated iteration of $d/(x^{1/4}dx) \sim d/dx^{5/4}$. Iteration of $d/dt$ on analytic nonentire functions produces Gevrey one series (§4.5), and thus the natural variable is $t = x^{5/4}$. This variable appears, as mentioned before, in the exponential corrections, see Exercise 4.84. We let

$$t = \frac{(-24x)^{5/4}}{30};\ y(x) = \sqrt{\frac{-x}{6}}\left(1 - \frac{4}{25t^2} + h(t)\right)$$

$\mathrm{P_I}$ becomes

$$h'' + \frac{1}{t}h' - h - \frac{1}{2}h^2 - \frac{392}{625t^4} = 0 \tag{4.85}$$

If we write $h(t) = \int_0^\infty H(p)e^{-tp}dp$, then the equation for $H$ is

$$(p^2 - 1)H(p) = \frac{196}{1875}p^3 + \int_0^p sH(s)ds + \frac{1}{2}H * H \tag{4.86}$$

where convolution is defined by (2.20). We will study convolution equations of the form (4.86) on an example in § 5.3a and in general in §5.

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## 4.4 Borel transform, Borel summation

The formal Laplace transform, still denoted $\mathcal{L} : \mathbb{C}[[p]] \mapsto \mathbb{C}[[x^{-1}]]$ is defined by

$$\mathcal{L}\{s\} = \mathcal{L}\left\{\sum_{k=0}^{\infty} c_k p^k\right\} = \sum_{k=0}^{\infty} c_k \mathcal{L}\{p^k\} = \sum_{k=0}^{\infty} c_k k! x^{-k-1} \tag{4.87}$$

(with $\mathcal{L}\{p^{\alpha-1}\} = \Gamma(\alpha)x^{-\alpha}$ the definition extends straightforwardly to noninteger power series).

### 4.4a The Borel transform $\mathcal{B}$

The **Borel transform**, $\mathcal{B} : \mathbb{C}[[x^{-1}]] \mapsto \mathbb{C}[[p]]$ is the (formal) inverse of the operator $\mathcal{L}$ in (4.87). This is a transform on the space of formal series. By definition, for a monomial we have

$$\mathcal{B}\frac{\Gamma(s+1)}{x^{s+1}} = p^s \tag{4.88}$$

in $\mathbb{C}$ (more precisely, on the universal covering of $\mathbb{C} \setminus \{0\}$, see footnote on p. 226) to be compared with the inverse Laplace transform,

$$\mathcal{L}^{-1}\frac{\Gamma(s+1)}{x^{s+1}} = \begin{cases} p^s & \text{for } \mathrm{Re}\, p > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.89}$$

(for $\mathrm{Re}\,(p) \le 0$ the contour in (2.14) can be pushed to $+\infty$).

Because the $k-th$ coefficient of $\mathcal{B}\{\tilde{f}\}$ is smaller by a factor $k!$ than the corresponding coefficient of $\tilde{f}$, $\mathcal{B}\{\tilde{f}\}$ may converge even if $\tilde{f}$ does not. Note that $\mathcal{LB}$ is the identity operator, on series. If $\mathcal{B}\{\tilde{f}\}$ is *convergent* and $\mathcal{L}$ is the actual Laplace transforms, we effectively get an identity-like operator from Gevrey one series to functions.

These two facts account for the central role played by $\mathcal{LB}$, the operator of Borel summation in the theory of analyzable functions . See also the diagram on p. 26.

### 4.4b Definition of Borel summation and basic properties

Series of the form $\tilde{f} = \sum_{k=0}^{\infty} c_{\mathbf{k}} x^{-\beta_1 k_1 - \ldots - \beta_m k_m - r}$ with $\mathrm{Re}\,(\beta_j) > 0$ frequently arise as formal solutions of differential systems. We will first analyze the case $m = 1, r = 1, \beta = 1$ but the theory extends without difficulty to more general series.

Borel summation is relative to a direction, see Definition 4.111. The same formal series $\tilde{f}$ may yield different functions by Borel summation in different directions.

Borel summation along $\mathbb{R}^+$ consists in three operations, assuming (2) and (3) are possible:

1. Borel transform, $\tilde{f} \mapsto \mathcal{B}\{\tilde{f}\}$.

2. Convergent summation of the series $\mathcal{B}\{\tilde{f}\}$ and analytic continuation along $\mathbb{R}^+$ (denote the continuation by $F$ and by $\mathcal{D}$ an open set in $\mathbb{C}$ containing $\mathbb{R}^+ \cup \{0\}$ where $F$ is analytic).

3. Laplace transform, $F \mapsto \int_0^{\infty} F(p) e^{-px} \mathrm{d}p =: \mathcal{L}\mathcal{B}\{\tilde{f}\}$, which requires exponential bounds on $F$, defined in some half plane $\mathrm{Re}\,(x) > x_0$.

**Note 4.90** Slightly more generally, the formal inverse Laplace transform (Borel transform, $\mathcal{B}$) of a small zero level transseries, that is of a small multiseries, is defined, roughly, as the *formal multiseries* obtained by term-by-term inverse Laplace transform,

$$\mathcal{B} \sum_{\mathbf{k}>0} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\mathbf{a}} = \sum_{\mathbf{k}>0} c_{\mathbf{k}} p^{\mathbf{k}\cdot\mathbf{a}-1} / \Gamma(\mathbf{k}\cdot\mathbf{a}) \tag{4.91}$$

The definition of Borel summation for multiseries as in (4.91) is the same, replacing analyticity at zero with ramified analyticity.

*

The *domain* of Borel summation is the subspace $S_{\mathcal{B}}$ of series for which the conditions for the steps above are met. For 3 we can require that for some constants $C_F$, $\nu_F$ we have $|F(p)| \le C_F e^{\nu_F p}$. Or we can require that $\|F\|_\nu < \infty$ where, for $\nu > 0$ we define

$$\|F\|_\nu := \int_0^{\infty} e^{-\nu p} |F(p)| dp \tag{4.92}$$

**Remark 4.93** *The results above can be rephrased for more general series of the form* $\sum_{k=0}^{\infty} c_k x^{-k-r}$ *by noting that for* $\mathrm{Re}\,(\rho) > -1$ *we have*

$$\mathcal{L} p^{\rho} = x^{-\rho-1} \Gamma(\rho+1)$$

*and thus*

$$\mathcal{B}\left(\sum_{k=0}^{\infty} c_k x^{-k-r}\right) = c_0 \frac{p^{r-1}}{\Gamma(r)} + \frac{p^{r-1}}{\Gamma(r)} * \mathcal{B}\left(\sum_{k=1}^{\infty} c_k x^{-k}\right)$$

*Furthermore, Borel summation naturally extends to series of the form*

$$\sum_{k=-M}^{\infty} c_k x^{-k-r}$$

*where* $M \in \mathbb{N}$ *by defining*

$$\mathcal{LB}\left(\sum_{k=-M}^{\infty} c_k x^{-k-r}\right) = \sum_{k=-M}^{0} c_k x^{-k-r} + \mathcal{LB}\left(\sum_{k=0}^{\infty} c_k x^{-k-r}\right)$$

*More general powers can be allowed, replacing analyticity in* $p$ *with analyticity in* $p^{\beta_1}, ..., p^{\beta_m}$.

Simple examples of Borel summed series are series that indeed come from the Laplace transform of analytic functions, as in (4.61), (4.63), (4.113) and (4.128).

We note that $L^1_\nu := \{f : \|f\|_\nu < \infty\}$ forms a Banach space, and it is easy to check that

$$L^1_\nu \subset L^1_{\nu'} \text{ if } \nu' > \nu \tag{4.94}$$

and that

$$\|F\|_\nu \to 0 \text{ as } \nu \to \infty \tag{4.95}$$

the latter statement following from dominated convergence.

**Note 4.96** A *function* $f$ is sometimes called Borel summable (by slight abuse of language), if it analytic and suitably decaying in a half plane (say $\mathbb{H}$), and its inverse Laplace transform $F$ is analytic in a neighborhood of $\mathbb{R}^+ \cup \{0\}$. Such functions are clearly into a one-to-one correspondence with their asymptotic series. Indeed, if the asymptotic series coincide, then their Borel transforms –convergent–coincide, and their analytic continuation is the same in a neighborhood of $\mathbb{R}^+ \cup \{0\}$. The two functions are equal.

### 4.4b .1 Note on exponentially small corrections

Note 4.96 shows that we can define corrections to divergent expansions, within the realm of Borel summable series. For instance we can represent $f$ by $\tilde{f}$ (a power series) plus $Ce^{-x}$, iff $f - Ce^{-x}$ is Borel summable.

## 4.4c Further properties of Borel summation

**Proposition 4.97** *(i)* $S_{\mathcal{B}}$ *is a differential field,*[7] *and* $\mathcal{LB} : S_{\mathcal{B}} \mapsto \mathcal{LB}S_{\mathcal{B}}$ *commutes with all these operations, that is, it is a differential algebra isomorphism.*

---

[7] with respect to formal addition, multiplication, and differentiation of power series.

*(ii) If $S_c \subset S_{\mathcal{B}}$ denotes the differential algebra of convergent power series, and we identify a convergent power series with its sum, then $\mathcal{LB}$ is the identity on $S_c$.*

*(iii) In addition, for $\tilde{f} \in S_B$, $\mathcal{LB}\{\tilde{f}\} \sim \tilde{f}$ as $|x| \to \infty$, $\mathrm{Re}\,(x) > 0$.*

For the proof, we need to look more closely at convolutions.

**Definition 4.98 (Inverse Laplace space convolution)** *If $F, G \in L^1_{loc}$ then*

$$(F * G)(p) := \int_0^p F(s)G(p-s)ds \tag{4.99}$$

Assuming exponential bounds at infinity we have (cf (2.19))

$$\mathcal{L}(F * G) = \mathcal{L}F\,\mathcal{L}G \tag{4.100}$$

**Lemma 4.101** *The space of functions which are in $L^1[0,\epsilon)$ for some $\epsilon > 0$ and real-analytic on $(0,\infty)$ is closed under convolution. If $F$ and $G$ are exponentially bounded then so is $F * G$. If $F, G \in L^1_\nu$ then $F * G \in L^1_\nu$.*

**PROOF** The statement about $L^1$ follows easily from Fubini's theorem. Writing

$$\int_0^p f_1(s)f_2(p-s)ds = p\int_0^1 f_1(pt)f_2(p(1-t))dt \tag{4.102}$$

analyticity is manifest. Clearly, if $|F_1| \le C_1 e^{\nu_1 p}$ and $|F_2| \le C_2 e^{\nu_2 p}$, then

$$|F_1 * F_2| \le C_1 C_2 p\, e^{(\nu_1+\nu_2)p} \le C_1 C_2\, e^{(\nu_1+\nu_2+1)p}$$

Finally, we note that

$$\int_0^\infty e^{-\nu p}\left|\int_0^p F(s)G(p-s)ds\right|dp \le \int_0^\infty e^{-\nu s}e^{-\nu(p-s)}\int_0^p |F(s)||G(p-s)|dsdp$$
$$= \int_0^\infty\int_0^\infty e^{-\nu s}|F(s)|e^{-\nu\tau}|G(\tau)|d\tau = \|F\|_\nu\|G\|_\nu \tag{4.103}$$

by Fubini. ◻

To show multiplicativity, we use §4.7b . Analyticity and exponential bounds of $|F * G|$ follow from Lemma 4.101. Consequently, $F * G$ is Laplace transformable, and the result follows from (4.100).

**PROOF of Proposition 4.97** We have to show that if $\tilde{f}$ is a Borel summable series, then so is $1/\tilde{f}$. We have $f = Cx^m(1+s)$ for some $m$ where $s$ is a small series.

We want to show that

$$1 - s + s^2 - s^3 + \cdots \tag{4.104}$$

is Borel summable, or that

$$-s + s^2 - s^3 + \cdots \tag{4.105}$$

is Borel summable. Let $\mathcal{B}s = H$. We examine the function series

$$S = -H + H * H - H^{*3} + \cdots \tag{4.106}$$

where $H^{*n}$ is the self convolution of $H$ $n$ times. Each term of the series is analytic, by Lemma 4.101. Let $K$ be an arbitrary compact subset of $\mathcal{D}$. If $\max_{p\in K} |H(p)| = m$, then it is easy to see that

$$|H^{*n}| \le m^n 1^{*n} = m^n \frac{p^{n-1}}{(n-1)!} \tag{4.107}$$

Thus the function series in (4.106) is absolutely and uniformly convergent in $K$ and the limit is analytic. Let now $\nu$ be large enough so that $\|H\|_\nu < 1$ (see (4.95)). Then the series in (4.106) is norm convergent, thus an element of $L^1_\nu$.

**Exercise 4.108** *Check that* $(1 + \mathcal{L}H)(1 + \mathcal{L}S) = 1$.

It remains to show that the asymptotic expansion of $\mathcal{L}(F * G)$ is indeed the product of the asymptotic series of $\mathcal{L}F$ and $\mathcal{L}G$. This is, up to a change of variable, a consequence of Lemma 1.32.

(ii) Since $\tilde{f}_1 = \tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k-1}$ is convergent, then $|c_k| \le CR^k$ for some $C, R$ and $F(p) = \sum_{k=0}^{\infty} c_k p^k/k!$ is entire, $|F(p)| \le \sum_{k=0}^{\infty} CR^k p^k/k! = Ce^{Rp}$ and thus $F$ is Laplace transformable for $|x| > R$. By dominated convergence we have for $|x| > R$,

$$\mathcal{L}\{\sum_{k=0}^{\infty} c_k p^k/k!\} = \lim_{N\to\infty} \mathcal{L}\{\sum_{k=0}^{N} c_k p^k/k!\} = \sum_{k=0}^{\infty} c_k x^{-k-1} = f(x)$$

(iii) This part follows simply from Watson's lemma, cf. § 3.4. ☐

### 4.4c .1 Convergent series composed with Borel summable series

**Proposition 4.109** *Assume $A$ is an analytic function in the disk of radius $\rho$ centered at the origin, $a_k = A^{(k)}(0)/k!$, and $\tilde{s} = \sum s_k x^{-k}$ is a small series which is Borel summable along $\mathbb{R}^+$. Then the formal power series obtained by reexpanding*

$$\sum a_k s^k$$

*in powers of $x$ is Borel summable along $\mathbb{R}^+$.*

**PROOF** Let $S = \mathcal{B}s$ and choose $\nu$ be large enough so that $\|S\|_\nu < \rho^{-1}$ in $L^1_\nu$. Then

$$\|F\|_\nu := \|A(*S)\|_\nu := \|\sum_{k=0}^{\infty} a_k S^{*k}\|_\nu \leq \sum_{k=0}^{\infty} a_k \|S\|_\nu^k \leq \sum_{k=0}^{\infty} a_k \rho^k < \infty \qquad (4.110)$$

thus $A(*S) \in L^1_\nu$. Similarly, $A(*S)$ is in $L^1_\nu([0,a))$ and in $\mathcal{A}_{K,\nu}\nu([0,a))$ (see (4) on p. 154) for any $a$. $\square$

### 4.4c .2 Directionality of Borel sums

In general, a Laplace transform depends on the direction of the ray.

**Definition 4.111** *The Borel sum of a series in the direction* $\phi$ *(*$\arg x = \phi$*),* $(\mathcal{LB})_\phi \tilde{f}$ *is by convention, the Laplace transform of* $\mathcal{B}\tilde{f}$ *along the ray* $xp \in \mathbb{R}^+$*, that is* $\arg(p) = -\phi$*:*

$$\left(\mathcal{LB}\right)_\phi \tilde{f} = \int_0^{\infty e^{-i\phi}} e^{-px} F(p)dp = \mathcal{L}_{-\phi}F = \mathcal{L}F(\cdot e^{-i\phi}) \qquad (4.112)$$

We can also say that Borel summation of $\tilde{f}$ along the ray $\arg(x) = \phi$ is defined as the ($\mathbb{R}^+$) Borel summation of $\tilde{f}(xe^{i\phi})$.

For example, we have

$$\mathcal{LB}_\phi \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} = \mathcal{L}_{-\phi}\{(1-p)^{-1}\} = \begin{cases} \mathrm{e}^{-x}(\mathrm{Ei}(x) - \pi i) & \text{for } \phi \in (-\pi, 0) \\ \mathrm{e}^{-x}\mathrm{Ei}(x) & \text{for } \phi = 0 \\ \mathrm{e}^{-x}(\mathrm{Ei}(x) + \pi i) & \text{for } \phi \in (0, \pi) \end{cases} \qquad (4.113)$$

The middle formula does not follow by Borel summation, but rather by BE summation and uses an elementary instance of medianization. See also the discussion in §4.4f . Medianization will be considered in higher generality in §5, and it reduces in this simple case to taking the Cauchy principal part of the integral along $\mathbb{R}^+$.

### 4.4d Stokes phenomena and Laplace transforms: an example

The change in behavior of a Borel summed series as the direction in $\mathbb{C}$ changes is conveniently determined by suitably changing the contour of integration of the Laplace transform. We illustrate this on a simple case:

$$f(x) := \int_0^{\infty} \frac{e^{-px}}{1+p} dp \qquad (4.114)$$

We seek to find the asymptotic behavior of the analytic continuation of $f$ for large $x$, along different directions in $\mathbb{C}$. A simple estimate along a large arc of circle shows that, for $x \in \mathbb{R}^+$ we also have

$$f(x) = \int_0^{\infty e^{-i\pi/4}} \frac{e^{-px}}{1+p} dp \tag{4.115}$$

Then the functions given in (4.114) and (4.115) agree in $\mathbb{R}^+$ thus they agree everywhere they are analytic. Furthermore, the expression (4.115) is analytic for $\arg x \in (-\pi/4, 3\pi/4)$ and by the very definition of analytic continuation $f$ admits analytic continuation in a sector $\arg(x) \in (-\pi/2, 3\pi/4)$. Now we take $x$ with $\arg x = \pi/4$ and note that along this ray, by the same argument as before, the integral equals

$$f(x) = \int_0^{\infty e^{-\pi i/2}} \frac{e^{-px}}{1+p} dp \tag{4.116}$$

We can continue this rotation process until $\arg(x) = \pi - \epsilon$ where we have

$$f(x) = \int_0^{\infty e^{-\pi i + i\epsilon}} \frac{e^{-px}}{1+p} dp \tag{4.117}$$

which is now manifestly analytic for $\arg(x) \in (\pi/2 - \epsilon, 3\pi/2 - \epsilon)$. To proceed further, we collect the residue at the pole:

$$\int_0^{\infty e^{-\pi i - i\epsilon}} \frac{e^{-px}}{1+p} dp - \int_0^{\infty e^{-\pi i + i\epsilon}} \frac{e^{-px}}{1+p} dp = 2\pi i e^x \tag{4.118}$$

and thus

$$f(x) = \int_0^{\infty e^{-\pi i - i\epsilon}} \frac{e^{-px}}{1+p} dp - 2\pi i e^x \tag{4.119}$$

which is manifestly analytic for $\arg(x) \in (\pi/2+\epsilon, 3\pi/2+\epsilon)$. We can now freely proceed with the analytic continuation in similar steps until $\arg(x) = 2\pi$ and get

$$f(xe^{2\pi i}) = f(x) - 2\pi i e^x \tag{4.120}$$

The function has *nontrivial monodromy*[8] *at infinity.*

We also note that by Watson's lemma, as long as $f$ is simply equal to a Laplace transform, $f$ has an asymptotic series in a half-plane. Relation (4.119) shows that this ceases to be the case when $\arg(x) = \pi$. This line is called a **Stokes line**. The exponential, "born" there is smaller than the terms of the series until $\arg(x) = 3\pi/2$.

In solutions to nonlinear problems, most often infinitely many exponentials are born on Stokes lines.

---

[8] Change in behavior along a closed curve containing a singularity.

At $\arg(x) = 3\pi/2$ the exponential becomes the dominant term of the expansion. This latter direction is called an **antistokes line**. In solutions to nonlinear problems, all exponentials collected at a Stokes line become large at the same time, and this is usually a source of singularities of the represented function, see §4.4e . Sometimes, the Stokes phenomenon designates, more generally the change in the asymptotic expansion of a function when the direction towards the singular point is changed, especially if it manifests itself by the appearance of oscillatory exponentials at antistokes lines.

In the example above, it so happens that the function itself is not single-valued. Taking first take first $x \in \mathbb{R}^+$, we write

$$\begin{aligned} f(x) = e^{-x}\int_1^\infty \frac{e^{-xt}}{t}dt = e^{-x}\int_x^\infty \frac{e^{-s}}{s}ds = e^{-x}\left(\int_x^1 \frac{e^{-s}}{s}ds + \int_1^\infty \frac{e^{-s}}{s}ds\right) \\ = e^{-x}\left(C_1 + \int_x^1 \frac{e^{-s}}{s}ds\right) = e^{-x}\left(C_1 + \int_x^1 \frac{e^{-s}-1}{s}ds - \ln x\right) \\ = e^{-x}\left(\text{entire} - \ln x\right) \quad (4.121) \end{aligned}$$

However, the Stokes phenomenon is *not* due to the multivaluedness of the function but rather to the divergence of the asymptotic series, as seen from the following remark.

**Remark 4.122** *Assume $f$ is analytic outside a compact set and is asymptotic to $\tilde{f}$ as $|x| \to \infty$ (in any direction). Then $\tilde{f}$ is convergent.*

**PROOF** By the change of variable $x = 1/z$ we move the analysis at zero. The existence of an asymptotic series as $z \to 0$ implies in particular that $f$ is bounded at zero. Since $f$ is analytic in $\mathbb{C} \setminus \{0\}$ then zero is a removable singularity of $f$, and thus the asymptotic series, which as we know is unique, must coincide with the Taylor series of $f$ at zero, a convergent series. □

The exercise below also shows that the Stokes phenomenon is not due to multivaluedness.

**Exercise 4.123 (*)** *(1) Show that the function $f(x) = \int_x^\infty e^{-s^2}ds$ is entire.*
*(2) Note that*

$$\int_x^\infty e^{-s^2}ds = \frac{1}{2}\int_{x^2}^\infty \frac{e^{-t}}{\sqrt{t}}dt = \frac{1}{2x}\int_1^\infty \frac{e^{-x^2u}}{\sqrt{u}}du = \frac{e^{-x^2}}{2x}\int_0^\infty \frac{e^{-x^2p}}{\sqrt{1+p}}dp \quad (4.124)$$

*Do a similar analysis to the one in the text and identify the Stokes and antistokes lines for $f$. Note that the "natural variable" now is $x^2$.*

See (6.1) for the form of the Stokes phenomenon for generic linear ODEs.

### 4.4e Nonlinear Stokes phenomena and formation of singularities

Let us now look at (4.61). If we take $n$ to be a complex variable, then the Stokes lines are those for which after deformation of contour of the integral

$$\int_0^\infty \frac{1-\frac{p}{2}-\left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)}e^{-np}dp \tag{4.125}$$

in (4.61), which is manifestly a Borel sum of a series, will run into singularities of the denominator. This happens when $n$ is purely imaginary. Take first $n$ on the ray $\arg n = -\pi/2+\epsilon$. We let

$$F(p)=\frac{1-\frac{p}{2}-\left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)}$$

We rotate the contour of integration to $\arg p = \pi/2-\epsilon$, compare with the integral to the left of the imaginary line and get the representation

$$\begin{aligned}\int_0^{\infty e^{i\pi/2-i\epsilon}} F(p)e^{-np}dp &= \int_0^{\infty e^{i\pi/2+i\epsilon}} F(p)e^{-np}dp \\ +2\pi i\sum_{j\in\mathbb{N}}\mathrm{Res}F(p)e^{-np}|_{p=2j\pi i} &+ \int_0^{\infty e^{i\pi/2+i\epsilon}} F(p)e^{-np}dp + \sum_{j\in\mathbb{N}}\frac{1}{je^{2nj\pi i}} \\ &= \int_0^{\infty e^{i\pi/2+i\epsilon}} F(p)e^{-np}dp - \ln(1-\exp(-2n\pi i))\end{aligned} \tag{4.126}$$

where the sum is convergent when $\arg n = -\pi/2+\epsilon$, and thus, when $\arg n = -\pi/2+\epsilon$ we can also write

$$\Gamma(n+1)=\frac{1}{1-\exp(-2n\pi i)}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\exp\left(\int_0^{\infty e^{i\pi/2+i\epsilon}} F(p)e^{-np}dp\right) \tag{4.127}$$

from which it is manifest that for $\arg x \neq \pi$ $\Gamma(x+1)$ is analytic and Stirling's formula holds for large $|x|$, while along $\mathbb{R}^-$ it is meromorphic, with simple poles at all negative integers.

We see that for $\arg(n)\in(-\pi,-\pi/2)$ the exponential, present in the Borel summed formula, is *beyond all orders*, and would not participate in any classical asymptotic expansion of $\Gamma(n+1)$.

We also see that the poles occur on the antistokes line of $\Gamma$, and essentially as soon as exponential corrections are *classically* visible, they produce singularities. This is typical indeed when there are infinitely many singularities in Borel space, generically the case for nonlinear ODEs that we will study in §6. This is also the case in difference equations. We also note that, after reexpansion of the log, the middle expression in (4.126) is a Borel summed series plus a *transseries* in $n$ (although now we allow $n$ to be complex).

Conversely then, $\tilde{f}$ Borel sums to

$$\mathcal{LB}\tilde{f} = \begin{cases} \ln\Gamma(n+1) - \ln\left[\sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n\right] & \arg(n \in (-\pi/2, \pi/2) \\ \ln\Gamma(n+1) - \ln\left[\sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n\right] - \ln(1-e^{-2n\pi i}) & \arg n \in \left(-\pi, -\dfrac{\pi}{2}\right) \\ \ln\Gamma(n+1) - \ln\left[\sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n\right] - \ln(1-e^{2n\pi i}) & \arg n \in \left(\dfrac{\pi}{2}, \pi\right) \end{cases} \tag{4.128}$$

and the lines $\arg n = \pm\pi/2$ are Stokes lines. We note that nothing special happens to the function on these lines, while the asymptotic series has the same shape. The discontinuity lies in the link between the series and its BE sum.

## 4.4f Limitations of classical Borel summation

The need of extending Borel summation to BE summation arises because the domain of definition of Borel summation is not wide enough; series occurring in even simple equations are not always Borel summable. A formal solution of $f' + f = 1/x$ is $\tilde{f} = \sum_{k=0}^{\infty} k! x^{-k-1}$. Then, since $\sum \mathcal{B}\tilde{f} = (1-p)^{-1}$ is *not* Laplace transformable, because of the nonintegrable singularity at $p = 1$, $\tilde{f}$ is not Borel summable.

While in a particular context one can avoid the singularity by slightly rotating the contour of $\mathcal{L}$ in the complex plane, there is clearly no *one* ray of integration that would allow for arbitrary location of the singularities of general formal solutions of say differential equations.

We cannot impose restrictions on the *location* of singularities in Borel plane without giving up trivial changes of variable such as $x' = ax$.

If the ray of integration has to depend on $\tilde{f}$, then linearity of the summation operator becomes a serious problem (and so is commutation with complex conjugation).

Écalle has found general averages of analytic continuations in Borel plane, which do not depend on the origin of the formal series, such that, replacing the Laplace transform along a *singular rays* with averages of Laplace transforms of these continuations, the properties of Borel summation are preserved, and its domain is vastly widened. The fact that such averages exist is nontrivial, though many averages are quite simple and explicit.

Multiplicativity of the summation operator is the main difficulty that is overcome by these special averages. Perhaps surprisingly, convolution (the image of multiplication through $\mathcal{L}^{-1}$), *does not* commute in general with analytic continuation along curves passing between singularities! (see §5.12a ).

A simplified form of medianization, the balanced average, which works for generic ODEs (but not in the generality of Écalle's averages) is discussed in

§5.10.

Mixtures of different factorial rates of divergence in the same series when present, usually preclude classical Borel summation as well. Acceleration and multisummation (the latter considered independently, from a cohomological point of view by Ramis, see; see also §8.2), universal processes too, were introduced by Écalle to deal with this problem in many contexts. Essentially BE summation is Borel summation, supplemented by averaging and acceleration when needed.

(Also, the domain of definition of classical Borel summation does not, of course, include transseries, but this is not a serious problem since the definition $\mathcal{LB}\exp(ax) = \exp(ax)$ solves it.)

---

## 4.5 Gevrey classes, least term truncation, and Borel summation

Let $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k}$ be a formal power series, with power-one of factorial divergence, and let $f$ be a function asymptotic to it. The definition (1.11) provides estimates of the value of $f(x)$ for large $x$, within $o(x^{-N})$, $N \in \mathbb{N}$, which are, as we have seen, insufficient to determine a unique $f$ associated to $\tilde{f}$. Simply widening the sector in which (1.11) is required cannot change this situation since, for instance, $\exp(-x^{1/m})$ is beyond all orders of $\tilde{f}$ in a sector of angle almost $m\pi$.

*If*, however, by truncating the power series at some suitable $N(x)$ instead of a fixed $N$, we can achieve exponentially good approximations in a sector of width more than $\pi$, then uniqueness is ensured, by Exercise 2.28.

This leads us to the notion of Gevrey asymptotics.

**Gevrey asymptotics.**

$$\tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k}, \quad x \to \infty$$

is by definition Gevrey of order $1/m$, or Gevrey-$(1/m)$ if

$$|c_k| \le C_1 C_2^k (k!)^m$$

for some $C_1, C_2$ [5]. There is an immediate generalization to noninteger power series. Taking $x = y^m$ and $\tilde{g}(y) = \tilde{f}(x)$, then $\tilde{g}$ is Gevrey-1 and we will focus on this case. Also, the corresponding classification for series in $z$, $z \to 0$ is obtained by taking $z = 1/x$.

**Remark 4.129** (a) The Gevrey order of the series $\sum_k (k!)^r x^{-k}$, where $r > 0$, is the same as that of $\sum_k (rk)! x^{-k}$. Indeed, if $\epsilon > 0$ we have, by Stirling's formula,

$$Const\ (1+\epsilon)^{-k} \le (rk)!/(k!)^r \sim Const\ k^{\frac{1}{2}-r} \le Const\ (1+\epsilon)^k$$

*

**Definition 4.130** *Let $\tilde{f}$ be Gevrey-1 . A function $f$ is* Gevrey-1 asymptotic *to $\tilde{f}$ as $x \to \infty$ in a sector $S$ if for some $C_3, C_4, C_5$, and all $x \in S$ with $|x| > C_5$ and all $N$ we have*

$$|f(x) - \tilde{f}^{[N]}| \le C_1 C_2^{N+1} |x|^{-N-1} (N+1)! \tag{4.131}$$

*i.e.* if the error $f - \tilde{f}^{[N]}$ is of the same form as the first omitted term in $\tilde{f}$.

Note the *uniformity requirement* in $N$ and $x$; this plays a crucial role.

**Remark 4.132 (Exponential accuracy)** *If $\tilde{f}$ is Gevrey-1 and $f$ is Gevrey-1 asymptotic to $\tilde{f}$ then $f$ can be approximated by $\tilde{f}$ with exponential precision in the following way. Let $N = \lfloor |x/C_2| \rfloor$ ($\lfloor \cdot \rfloor$ is the integer part); then for any $C > C_2$ we have*

$$f(x) - \tilde{f}^{[N]}(x) = O(|x|^{-1/2} e^{-|x|/C}), \qquad |x| \text{ large}) \tag{4.133}$$

Indeed, letting $|x| = NC_2 + \epsilon$ with $\epsilon \in [0,1)$ and applying Stirling's formula we have

$$N!(N+1)C_2^N |NC_2 + \epsilon|^{-N-1} = O(|x|^{1/2} e^{-|x|/C_2})$$

□

**Note 4.134** *Optimal truncation*, see e.g. [18], is in a sense a refined version of Gevrey asymptotics. It requires *optimal constants* in addition to an improved form of Rel. (4.131). In this way the imprecision of approximation of $f$ by $\tilde{f}$ turns out to be smaller than the largest of the exponentially small corrections allowed by the problem where the series originated. Thus the cases in which uniqueness is ensured are more numerous. Often, optimal truncation means stopping near the least term of the series, and this is why this procedure is also known as *summation to the least term.*

### 4.5a Connection between Gevrey asymptotics and Borel summation

The following theorem goes back to Watson [35].

**Theorem 4.135** *Let $\tilde{f} = \sum_{k=2}^{\infty} c_k x^{-k}$ be a Gevrey-1 series and assume the function $f$ is analytic for large $x$ in $S_{\pi+} = \{x : |\arg(x)| < \pi/2 + \delta\}$ for some $\delta > 0$ and Gevrey-1 asymptotic to $\tilde{f}$ in $S_{\pi+}$. Then*

*(i) $f$ is unique.*

*(ii)* $\tilde{f}$ *is Borel summable in any direction* $e^{i\theta}\mathbb{R}^+$ *with* $|\theta| < \delta$ *and* $f = \mathcal{L}\mathcal{B}_\theta \tilde{f}$.

*(iii)* $\mathcal{B}(\tilde{f})$ *is analytic (at* $p = 0$ *and) in the sector* $S_\delta = \{p : \arg(p) \in (-\delta, \delta)\}$, *and Laplace transformable in any closed subsector.*

*(iv) Conversely, if* $\tilde{f}$ *is Borel summable along any ray in the sector* $S_\delta$ *given by* $|\arg(x)| < \delta$, *and if* $\mathcal{B}\tilde{f}$ *is uniformly bounded by* $e^{\nu|p|}$ *in any closed subsector of* $S_\delta$, *then* $f$ *is Gevrey-1 with respect to its asymptotic series* $\tilde{f}$ *in the sector* $|\arg(x)| \le \pi/2 + \delta$.

**Note.**In particular, when the assumptions of the theorem are met, Borel summability follows using only *asymptotic estimates*.

The Nevanlinna-Sokal theorem [50] weakens the conditions sufficient for Borel summability, requiring essentially estimates in a half plane only. It was originally formulated for expansions at zero, essentially as follows:

**Theorem 4.136 (Nevanlinna-Sokal)** Let $f$ be analytic in $C_R = \{z : \mathrm{Re}\,(1/z) > R^{-1}\}$ and satisfy the estimates

$$f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z) \tag{4.137}$$

with

$$|R_N(z)| \le A\sigma^N N!|z|^N \tag{4.138}$$

uniformly in $N$ and in $z \in C_R$. Then $B(t) = \sum_{n=0}^{\infty} a_n t^n/n!$ converges for $|t| < 1/\sigma$ and has analytic continuation to the strip-like region $S_\sigma = \{t : \mathrm{dist}(t, \mathbb{R}^+) < 1/\sigma\}$, satisfying the bound

$$|B(t)| \le K \exp(|t|/R) \tag{4.139}$$

uniformly in every $S_{\sigma'}$ with $\sigma' > \sigma$. Furthermore, $f$ can be represented by the absolutely convergent integral

$$f(z) = z^{-1} \int_0^\infty e^{-t/z} B(t) dt \tag{4.140}$$

for any $z \in C_R$. Conversely, if $B(t)$ is a function analytic in $S_{\sigma''}$ ($\sigma'' < \sigma$) and there satisfying (4.139), then the function $f$ defined by (4.140) is analytic in $C_R$, and satisfies (4.137) and (4.138) [with $a_n = B^{(n)}(t)|_{t=0}$] uniformly in every $C_{R'}$ with $R' < R$.

**Note 4.141** Let us point out first a possible pitfall in proving Theorem 4.135. Inverse Laplace transformability of $f$ and analyticity away from zero in some sector follow immediately from the assumptions. What does not follow immediately is analyticity of $\mathcal{L}^{-1}f$ at zero. On the other hand, $\mathcal{B}\tilde{f}$ clearly converges to an analytic function near $p = 0$. But there is no guarantee that $\mathcal{B}\tilde{f}$ has anything to do with $\mathcal{L}^{-1}f$! This is where Gevrey estimates enter.

**PROOF of Theorem 4.135**

(i) Uniqueness clearly follows once we prove (ii).

(ii) and (iii) By a simple change of variables we arrange $C_1 = C_2 = 1$. The series $\tilde{F}_1 = \mathcal{B}\tilde{f}$ is convergent for $|p| < 1$ and defines an analytic function, $F_1$. By Proposition 2.12, the function $F = \mathcal{L}^{-1}f$ is analytic for $|p| > 0, |\arg(p)| < \delta$, and $F(p)$ is analytic and uniformly bounded if $|\arg(p)| < \delta_1 < \delta$. We now show that $F$ is analytic for $|p| < 1$. (A proof different from the one below is seen in §4.5a .1.) Taking $p$ real, $p \in [0, 1)$ we obtain in view of (4.131) that

$$\begin{aligned}|F(p) - \tilde{F}^{[N-1]}(p)| &\le \int_{-i\infty+N}^{i\infty+N} d|s| \left|f(s) - \tilde{f}^{[N-1]}(s)\right| e^{\mathrm{Re}\,(ps)} \\ &\le N!e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{|x+iN|^N} = N!e^{pN}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+N^2)^{N/2}} \\ &\le \frac{N!e^{pN}}{N^{N-1}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{(\xi^2+1)^{N/2}} \le CN^{3/2}e^{(p-1)N} \to 0 \text{ as } N\to\infty \end{aligned} \tag{4.142}$$

for $0 \le p < 1$. Thus $\tilde{F}^{[N-1]}(p)$ converges. Furthermore, the limit, which by definition is $F_1$, is seen in (4.142) to equal $F$, the inverse Laplace transform of $f$ on $[0, 1)$. Since $F$ and $F_1$ are analytic in a neighborhood of $(0, 1)$, $F = F_1$ wherever *either* of them is analytic [9]. The domain of analyticity of $F$ is thus, by (ii), $\{p : |p| < 1\} \cup \{p : |p| > 0, |\arg(p)| < \delta\}$.

(iv) Let $|\phi| < \delta$. We have, by integration by parts,

$$f(x) - \tilde{f}^{[N-1]}(x) = x^{-N}\mathcal{L}\frac{d^N}{dp^N}F \tag{4.143}$$

On the other hand, $F$ is analytic in $S_a$, some $a = a(\phi)-$neighborhood of the sector $\{p : |\arg(p)| < |\phi|\}$. Estimating Cauchy's formula on an $a-$circle around the point $p$ with $|\arg(p)| < |\phi|$ we get

$$|F^{(n)}(p)| \le N!a(\phi)^{-N}\|F(p)\|_{\infty;S_a}$$

Thus, by (4.143), with $|\theta| \le |\phi|$ chosen so that $\gamma = \cos(\theta - \arg(x))$ is maximal we have

$$\begin{aligned}\left|f(x) - \tilde{f}^{[N]}\right| &= \left|x^{-N}\int_0^{\infty\exp(-i\theta)} F^{(N)}(p)e^{-px}dp\right| \\ &\le N!a^{-N}|x|^{-N}\|Fe^{\nu|p|}\|_{\infty;S_a}\int_0^{\infty} e^{-px+\nu|p|\gamma}dp \\ &= \mathit{const.}N!a^{-N}\gamma^{-1}|x|^{-N-1}\|F\|_{\infty;S_a}\end{aligned} \tag{4.144}$$

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[9] Here and elsewhere we identify a function with its analytic continuation.

for large enough $x$.

$\Box$

### 4.5a .1 Sketch of the proof of Theorem 4.136

We can assume that $f(0) = f'(0) = 0$ since subtracting out a finite number of terms of the asymptotic expansion does not change the problem. Then, we take to $x = 1/z$ (essentially, to bring the problem to our standard setting). Let

$$F = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(1/x)e^{px}dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(x)e^{px}dx$$

We now want to show analyticity in $S_\sigma$ of $F$. That, combined with the proof of Theorem 4.135 completes the argument.

We have

$$f(x) = \sum_{j=2}^{N-1}\frac{a_j}{x^j} + R_N(x)$$

and thus,

$$F(p) = \sum_{j=2}^{N-1}\frac{a_j p^{j-1}}{(j-1)!} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R_N(1/x)e^{px}dx$$

and thus

$$|F^{(N-2)}(p)| = \left|\frac{a_{N-1}}{N-1} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{N-2}R_N(1/x)e^{px}dx\right| \le A_2\sigma^N N!;\ \ p\in\mathbb{R}^+$$

and thus $|F^{(n)}(p)/n!| \le A_3 n^2\sigma^n$, and the Taylor series of $F$ at any point $p_0 \in \mathbb{R}^+$ converges, by Taylor's theorem, to $F$, and the radius of convergence is $\sigma$. The bounds at infinity follow in the usual way: let $c = R^{-1}$. Since $f$ is analytic for $\operatorname{Re} x > c$ and is uniformly bounded for $\operatorname{Re} x \ge c$, we have

$$\left|\int_{c-i\infty}^{c+i\infty} f(1/x)e^{px}dx\right| \le K_1 e^{cp}\int_{-\infty}^{\infty}\frac{dx}{x^2+1} \le K_2 e^{cp} \tag{4.145}$$

for $p \in \mathbb{R}^+$. In the strip, the estimate follows by combining (4.145) with the local Taylor formula.

**Note 4.146** As we see, *control over the analytic properties of $\mathcal{B}\tilde{f}$ near $p = 0$ is essential to Borel summability and, it turns out,* BE *summability.* Mere inverse Laplace transformability of a function with a given asymptotic series, in however large a sector, does not ensure Borel summability of the series. We know already that for any power series, for instance one that is not Gevrey of finite order, we can find a function $f$ analytic and asymptotic to it in more than a half plane (in fact, many functions). Then $(\mathcal{L}^{-1}f)(p)$ exists, and is analytic in an open sector in $p$, origin not necessarily included. Since the series is not Gevrey of finite order, it can't be Borel summable. What goes wrong is the behavior of $\mathcal{L}^{-1}f$ at zero.

## 4.6 Borel summation as analytic continuation

There is another interpretation showing that Borel summation should commute with all operations. Returning to the example $\sum k!(-x)^{-k-1}$, we can consider the more general sum

$$\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta k + \beta)}{x^{k+1}} \tag{4.147}$$

which for $\beta = 1$ agrees with (1.45). For $\beta = i$, (4.147) *converges* if $|x| > 1$, and the sum is, using the integral representation for the Gamma function and dominated convergence,

$$\int_0^{\infty} \frac{e^{-px}}{1 + p^{\beta} x^{\beta - 1}} dp \tag{4.148}$$

Analytic continuation of (4.148) back to $\beta = i$ becomes precisely (1.50).

**Exercise 4.149** *Complete the details in the calculations above. Show that continuation to $i$ and to $-i$ give the same result (1.50).*

Thus Borel summation should commute with all operations with which analytic continuation does. This latter commutation is very general, and comes under the umbrella of the vaguely stated "principle of permanence of relations" which can hardly be formulated rigorously without giving up some legitimate "relations".

**Exercise 4.150 (*)** *Complete the proof of Theorem 4.136.*

## 4.7 Notes on Borel summation

BE works very well in problems such as (singular) differential, partial differential, and difference equations. We have seen that (1) due to the closure of transseries under operations, often formal transseries solutions exist; (2) they usually diverge due to the presence of repeated differentiation in the process of generating them. Some of the reasons why Borel summation is natural in these classes of problems are sketched in §4.7b .

But Borel summation has been successfully applied in problems in which divergence is generated in other ways. There is likely quite a bit more to understand and room for developing the theory.

**Note 4.151** Summation naturally extends to transseries. In practice however, rarely does one need to Borel sum several levels of a transseries[10]: once the lowest level has been summed, usually the remaining sums are convergent; we will then not address multilevel summation in this book.

## 4.7a Choice of critical time

1. To ensure Borel summability, we change the independent variable so that, with respect to the new variable $t$ the coefficient $c_\alpha$ of $t^{-\alpha}$ is $\Gamma(\alpha+1)$, up to geometrically bounded prefactors. This is the order of growth implied by Watson's lemma. The order of growth is intimately linked to the form of free small exponential corrections, as explained in §4.7a .1 below. If these are of the form $e^{-x^q}$ then divergence is usually like $(n!)^{1/q}$. The variable should then be chosen to be $t=x^q$: this $t$ is the *critical time.*

2. The choice of "$t$" and preparation of the equation are essential however for the process to succeed, see 2 in § 4.7a .1. Since $(d/dt)^{2n}=(d^2/dt^2)^n$ iterating the $n-th$ order derivative on a power series produces power one of the factorial divergence, regardless of $n$. For instance, iterating the operator in (4.68) produces the same divergence as iterating simply $D$. This partly explains the similarity in summability properties between differential and difference equations. As another example, in (4.52), the divergence is the same as that produced by iteration of $d/(x^{1/2}dx)\sim d/d(x^{3/2})$ so $t=x^{3/2}$. See also §4.3a and 1 in § 4.7a .1 and §4.7b .

3. Often series fail to be Borel summable because of singularities in the direction of the Laplace transforms, or different types of factorial divergence being mixed together. $\mathsf{BE}$ summation extends Borel summation and addresses these difficulties. See §4.4f .

### 4.7a .1 Critical time, rate of divergence, exponentially small corrections.

1. (a) If we expect the solutions of an equation to be summable with respect to a power of the variable, then the possible freedom in choosing the Laplace contour in the complex domain should be compatible with the type of freedom in the solutions.

   In returning to the original space, the Laplace transform $f(x;\phi)=\int_0^{\infty e^{i\phi}}F(p)e^{-xp}dp$ can be taken along any nonsingular direction if exponential bounds exist at infinity. For instance we can take the Laplace transform of $(1-p)^{-1}$ along any ray other than $\mathbb{R}^+$, and obtain a solution of $f'+f=1/x$. An upper half plane transform differs from a lower

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[10] That is, terms appearing in higher iterates of the exponential.

half plane transform precisely by $2\pi i e^{-x}$. Now, if $F$ is analytic in a neighborhood of $\mathbb{R}^+$ but is not entire, then, by Watson's lemma $f(x;\phi)$ has an asymptotic series in $x^{-1}$ of Gevrey order exactly one, difference between two Laplace transforms on two sides of a singularity $p_0$ is, to leading order of the form $e^{-p_0 x}$.

Thus, if the Gevrey order of a formal solution is $k$, we need to take $x^{1/k}$ as a variable, otherwise the discussion above shows that proper Borel summation cannot succeed.

(b) Conversely then, if the difference between two solutions is of the form $e^{-x^q}$ then divergence of the formal series is expected to be of factorial type $(n!)^{1/q}$.

2. It is crucial to perform Borel summation in the adequate variable. If the divergence is not fully compensated (undersummation), then obviously we are still left with a divergent series. "Oversummation", the result of overcompensating divergence usually leads to superexponential growth of the transformed function. The presence of singularities in Borel plane is in fact a good sign. For equation (4.40), the divergence is like $\sqrt{n!}$. The equation is oversummed if we inverse Laplace transform it in $x$; what we get is

$$2H' - pH = 0; \quad H(0) = 1/2 \tag{4.152}$$

   and thus $H = \frac{1}{2}e^{p^2/4}$. There are no singularities anymore but we have superexponential growth; this combination is a sign of oversummation. Superexponential growth is in certain ways worse than the presence of singularities. Close to the real line, there is no obvious way of taking the Laplace transform.

3. In some cases, a simple change of variable in the $x$-plane, as we have seen, can solve the problem. Here, the freedom (difference between solutions) in (4.40) $Ce^{-x^2}$. The critical time is $t = x^2$ . We then take $g = h(x^2)$. The equation for $h$ is

$$h' + h = \frac{1}{2\sqrt{t}} \tag{4.153}$$

4. If the solutions of an equation are summable, then it is expected that the transformed equation should be more regular. In this sense, Borel summation is a regularizing transformation; see also § 7.2 where this feature is very useful.

   In the case of (4.153) it becomes

$$-pH + H = p^{-1/2}\pi^{-1/2} \tag{4.154}$$

   an algebraic equation, with algebraic singularities. The transformed function is more regular.

### 4.7b Borel summation and differential and difference systems

We recall that, in differential systems, a problem is singularly perturbed if the highest derivative does not participate in the dominant balance; the highest derivative is then discarded to leading order. It reappears eventually, in higher order corrections. Then higher and higher corrections contain derivatives of increasing order. For analytic nonentire functions, by Cauchy's formula, $f^{(n)}$ grows roughly like $const^n n!$.

It is then natural to diagonalize the main part of the operator producing divergence. For instance, in (4.40) it is $d/(2xdx) = d/dx^2 := d/dt$: then, by definition, in transformed space, $d/dt$ becomes multiplication by the dual variable $p$. Repeated differentiation corresponds to repeated multiplication by $p$. The latter operation produces geometric growth/decay and thus nonzero radia of convergence of the expansion.

The operator $d/dt$ is diagonalized by the Fourier transform. In an asymptotic problem, say for a large variable $t$, we need to keep $t$ large in the transform process. The Fourier transform on a vertical contour in the complex domain is in fact an inverse Laplace transform, cf. also Remark 2.8.

In this sense, the pair $(\mathcal{L},\mathcal{L}^{-1})$, in appropriate coordinates, is canonically associated to summation of formal solutions of singularly perturbed equations.

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## 4.8 Borel transform of the solutions of an example ODE, (4.55)

For differential equations there exist general results on the Borel summability of formal transseries solutions, see §5. The purpose now is to illustrate a strategy of proof that is convenient and which applies to a reasonably large class of settings.

It would be technically awkward to prove based on the formal series alone that its Borel transform extends analytically along the real line and that is has the required exponential bounds towards infinity.

A better approach is to control the Borel transform of $\tilde{y}$ via the equation it satisfies. This equation is the formal inverse Laplace transform of (4.55), namely, setting $Y = \mathcal{B}\tilde{y}$

$$-pY + Y = p + Y * Y * Y := p + Y^{*3} \tag{4.155}$$

We then show that the equation (4.155) has a (unique) solution which is analytic in a neighborhood of the origin together with a sector centered on $\mathbb{R}^+$ in which this solution has exponential bounds. Thus $Y$ is Laplace transformable, and immediate verification shows that $y = \mathcal{L}Y$ satisfies (4.54). Furthermore, since the Maclaurin series $S(Y)$ formally satisfies (4.155) then the formal Laplace (inverse Borel) transform $\mathcal{B}^{-1}SY$ is a *formal* solution of (4.54), and thus equals $\tilde{y}$ since this solution, as we proved in many similar settings is unique. But since $SY = \mathcal{B}\tilde{y}$ it follows that $\tilde{y}$ is Borel summable, and the Borel sum solves (4.54).

The transformed equations are expected to have analytic solutions, therefore to be more regular than the original ones.

Further analysis of the convolution equations reveals the detailed analytic structure of $\mathcal{B}\tilde{y}$, including the position and type of singularities, needed in understanding the Stokes phenomena in the actual solutions.

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## *4.9 Appendix : Rigorous construction of transseries

This section can be largely omitted at a first reading except when rigor, further information, or precise definitions and statements are needed.

Écalle's original construction is summarized in [31]. Alternative constructions are given in [3], [37]. An interesting recent extension is [32].

This section is based on [21] and it provides the proofs needed to back up the statements in §4.2a . Following the steps involved in the rigorous construction of transseries is needed in order to fully understand their structure, the reason why so many problems have transseries solutions, as well as the various limitations of transseries.

### 4.9a Abstracting from §4.2b

1. Let $(\mathcal{G}, \cdot, \ll)$ be a finitely generated, totally ordered (any two elements are comparable) Abelian group, with generators $\mu_1, \mu_2, ...\mu_n$, such that $\ll$ is compatible with the group operations, that is, $g_1 \ll g_2$ and $g_3 \ll g_4$ implies $g_1 g_3 \ll g_2 g_4$, and so that $1 \gg \mu_1 \gg \cdots \gg \mu_n$. This is the case when $\mu_i$ are transmonomials of level zero.
2. We write $\mu_{\mathbf{k}} = \mu^{\mathbf{k}} := \mu_1^{k_1} \cdots \mu_n^{k_n}$.

   **Lemma 4.156** *Consider the partial order relation on* $\mathbb{Z}^n$, $\mathbf{k} > \mathbf{m}$ *iff* $k_i \geq m_i$ *for all* $i = 1, 2, ..., n$ *and at least for some* $j$ *we have* $k_j > m_j$. *If* $B \subset A = \{\mathbf{k} \in \mathbb{Z}^n : \mathbf{k} \geq \mathbf{m}\}$, *then there is no infinite nonascending chain in* $B$. *That, is there is no infinite sequence in* $B, b_n \neq b_m$ *for* $n \neq m$, *and* $b_{n+1} \not> b_n$ *for all* $n$.

   **PROOF** Assume there is an infinite nonascending sequence, $\{\mathbf{k}(m)\}_{m\in\mathbb{N}}$. Then at least for some $i \in \{1, 2, ..., n\}$ the sequence $\{k_i(m)\}_{m\in\mathbb{N}}$ must have infinitely many distinct elements. Since the $k_i(m)$ are bounded below, then the set $\{k_i(m)\}_{m\in\mathbb{N}}$ is unbounded above, and we can extract a strictly increasing subsequence $\{k_i(m_l)\}_{l\in\mathbb{N}}$. We now take the sequence $\{\mathbf{k}(m_l)\}_{l\in\mathbb{N}}$. At least for some $j \neq i$ the set $k_j(m_l)$ needs to have infinitely many elements too. Indeed if the sets $\{k_j(m_l); j \neq i\}$ are finite, we can split $\{\mathbf{k}(m_l)\}_{l\in\mathbb{N}}$ into a finite set of subsequences, in each of which all $k_j(m_l)$, $j \neq i$, are constant while $k_i$ is strictly increasing. But every such subsequence would be strictly decreasing, which is impossible. By finite induction we can extract a subsequence $\{\mathbf{k}(m_t)\}_{t\in\mathbb{N}}$ of $\mathbf{k}(m)\}_{m\in\mathbb{N}}$ in which all $k_l(m_t)$ are increasing, a contradiction. $\square$

**Remark.** This is a particular, much easier result of J. Kruskal's tree theorem which we briefly mention here. A relation is well-founded if and only if there is no countable infinite descending sequence $\{x_j\}_{\in\mathbb{N}}$ of elements of $X$ such that $x_{n+1}Rx_n$ for every $n \in \mathbb{N}$. The relation $R$ is a quasiorder if it is reflexive and transitive. Well-quasi-ordering is a well-founded quasi-ordering such that that there is no sequence $\{x_j\}_{\in\mathbb{N}}$ with $x_i \not\leq x_j \forall i < j$. A tree is a collection of vertices in which any two vertices are connected by exactly one line. *J. Kruskal's tree theorem states that the set of finite trees over a well-quasi-ordered set is well-quasi-ordered.*

3. *Exercises. (1) Show that the equation* $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{l}$ *has only finitely many solutions in the set* $\{\mathbf{k} : \mathbf{k} \geq \mathbf{m}\}$.

   *(2) Show that for any* $\mathbf{l} \in \mathbb{R}^n$ *there can only be finitely many* $p \in \mathbb{N}$ *and* $\mathbf{k}_j \in \mathbb{R}^n, j = 1, ..., p$ *such that* $\mathbf{k}_1 + \mathbf{k}_2 + \cdots \mathbf{k}_p = \mathbf{l}$.

**Corollary 4.157** *For any set* $B \subset A = \{\mathbf{k} \in \mathbb{Z}^n : \mathbf{k} \geq \mathbf{m}\}$ *there is a set* $B_1 = \mathrm{mag}(B)$ *with* **finitely many elements**, *such that* $\forall b \in B \setminus B_1$ *there exists* $b_1 \in B_1$ *such that* $b_1 < b$.

Consider the set of all elements which are not greater than other elements of $B$, $B_1 = \{b_1 \in B | b \neq b_1 \Rightarrow b \not> b_1\}$. In particular, no two elements of $B_1$ can be compared with each-other. But then, by Lemma 4.156 this set cannot be infinite since it would contain an infinite nonascending chain.

Now, if $b \in B \setminus B_1$, then by definition there is a $b' > b$ in $B$. If $b' \in B_1$ there is nothing to prove. Otherwise there is a $b'' > b'$ in $B$. Eventually some $b^{(k)}$ must belong to $B_1$, finishing the proof, otherwise $b < b' < ...$ would form an infinite nonascending chain.

4. For any $\mathbf{m} \in \mathbb{Z}^n$ and any set $B \subset \{\mathbf{k} | \mathbf{k} \geq \mathbf{m}\}$, the set $A = \{\mu_{\mathbf{k}} | \mathbf{k} \in B\}$ has a largest element with respect to $>$. Indeed, if such was not the case, then we would be able to construct an infinitely ascending sequence.

**Lemma 4.158** *No set of elements of* $\mu_{\mathbf{k}} \in \mathcal{G}$ *such that* $\mathbf{k} \geq \mathbf{m}$ *can contain an infinitely* ascending *chain, that is a sequence of the form*

$$g_1 \ll g_2 \ll \cdots$$

**PROOF** For such a sequence, the corresponding $\mathbf{k}$ would be strictly nonascending, in contradiction with Lemma 4.156. $\square$

5. It follows that for any $\mathbf{m}$ every $B \subset A_{\mathbf{m}} = \{g \in \mathcal{G} | g = \mu_{\mathbf{k}}; \mathbf{k} \geq \mathbf{m}\}$ is well ordered (every subset has a largest element) and thus $B$ can be indexed by ordinals. By this we mean that there exists a set of ordinals $\Omega$ (or, which is the same, an ordinal) which is in one-to-one correspondence with $B$ and $g_\beta \ll g_{\beta'}$ if $\beta > \beta'$.

6. If $A$ is as in 4, and *if* $g \in \mathcal{G}$ has a *successor* in $A$, that is, there is a $\tilde{g} \in A$, $g \gg \tilde{g}$ then it has an *immediate successor*, the largest element in the subset of $A$ consisting of all elements less than $g$. There may not be an immediate *predecessor* though, as is the case of $e^{-x}$ in $A_1 = \{x^{-n}, n \in \mathbb{N}\} \cup \{e^{-x}\}$. Note also that, although $e^{-x}$ has infinitely many predecessors, there is no infinite ascending chain in $A_1$.

**Lemma 4.159** *For any* $g \in \mathcal{G}$, *and* $\mathbf{m} \in \mathbb{Z}^n$, *there exist finitely many (distinct)* $\mathbf{k} \geq \mathbf{m}$ *such that* $\mu_{\mathbf{k}} = g$.

**PROOF** Assume the contrary. Then for at least one $i$, say $i = 1$ there are infinitely many $k_i$ in the set of $(\mathbf{k})_i$ such that $\mu_{\mathbf{k}} = g$. As in Lemma 4.166, we can extract a strictly increasing subsequence. But then, along it, $\mu_1^{k_2} \cdots \mu_n^{k_n}$ would form an infinite strictly ascending sequence, a contradiction. $\square$

7. For any coefficients $c_{\mathbf{k}} \in \mathbb{R}$, consider the formal multiseries, which we shall call *transseries* over $\mathcal{G}$,

$$T = \sum_{\mathbf{k} \in \mathbb{Z}^n; \mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \mu_{\mathbf{k}} \tag{4.160}$$

Transseries actually needed in analysis are constructed in the sequel, with a particular inductive definition of generators $\mu_k$.

8. *More generally a* **transseries over** $G$ *is a sum which can be written in the form (4.160) for some (fixed)* $n \in \mathbb{N}$ *and for some* **some choice of generators** $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^n$.

9. The fact that a transseries $s$ is small does *not* mean that the corresponding $\mu_{\mathbf{k}}$ have positive $\mathbf{k}$; $s$ could contain terms such as $xe^{-x}$ of $x^{\sqrt{2}}x^{-2}$ etc. But positiveness can be *arranged* by a suitable choice of generators as follows Lemma 4.166.

10. **Note** It is important that a transseries is defined over a set of the form $A_{\mathbf{m}}$. For instance, in the group $\mathcal{G}$ with two generators $x^{-1}$ and $x^{-\sqrt{2}}$ an expression of the form

$$\sum_{\{(m,n)\in\mathbb{Z}^2 \mid m\sqrt{2}+n>0\}} c_{mn}x^{-m\sqrt{2}-n} \tag{4.161}$$

is not acceptable (the possible powers in (4.161 is dense) and the behavior of a function whose "asymptotic expansion" is (4.161) is not at all clear.)

**Exercise 4.162** *Consider the numbers the form* $m\sqrt{2}+n$*, where* $m, n \in \mathbb{Z}$*. It can be shown, for instance using continued fractions, that one can choose a subsequence from this set such that* $s_n \uparrow 1$*. Show that* $\sum_n x^{-s_n}$ *is not a transseries over any group of monomials of order zero.*

Expressions similar to the one in the exercise do appear in some problems in discrete dynamics. The very fact that transseries are closed under many operations, including solutions of ODEs, shows that such functions are "highly transcendental".

11. Given $\mathbf{m} \in \mathbb{Z}^n$ and $g \in \mathcal{G}$, the set $S_g = \{\mathbf{k}|\mu_{\mathbf{k}} = g\}$ contains, by Lemma 4.159 finitely many elements (possibly none). Thus the constant $d(g) = \sum_{\mathbf{k}\in S_g} c_{\mathbf{k}}$ is well defined. By 4 there is a largest $g = g_1$ in the set $\{\mu_{\mathbf{k}}|d(g) \neq 0\}$, unless all coefficients are zero. We call this $g_1$ the magnitude of $T$, $g_1 = \text{mag}(T)$, and we write $\text{dom}(T) = d(g_1)g_1 = d_1g_1$.

12. By 5, the set $\{g = \mu_{\mathbf{k}}|\mathbf{k} \geq \mathbf{m}\}$ can be indexed by ordinals, and we write

$$T = \sum_{\beta\in\Omega} d_\beta g_\beta \tag{4.163}$$

where $g_\beta \ll g_{\beta'}$ if $\beta > \beta'$. By convention, the first element in (4.163), $d_1g_1$ is nonzero.

**Convention.** To simplify the notation and terminology, we will say, with some abuse of language, that a group element $g_\beta$ appearing in (4.163) belongs to $T$.

Whenever convenient, we can also select the elements of $d_\beta g_\beta$ in $T$ with nonzero coefficients. As a subset of a well ordered set, it is well ordered too, by a set of ordinals $\tilde{\Omega} \subset \Omega$ and we can write

$$T = \sum_{\beta \in \tilde{\Omega}} d_\beta g_\beta \tag{4.164}$$

where all $d_\beta$ are nonzero.

13. **Notation** To simplify the exposition we will denote by $A_{\mathbf{m}}$ the set $\{\mu_{\mathbf{k}} | \mathbf{k} \geq \mathbf{m}\}$, $\mathbf{K}_{\mathbf{m}} = \{\mathbf{k} | \mathbf{k} \geq \mathbf{m}\}$ and $\mathcal{T}_{A_{\mathbf{m}}}$ the set of transseries over $A_{\mathbf{m}}$.

14. Any transseries can be written in the form

$$T = L + c + s = \sum_{\beta \in \Omega; g_\beta \gg 1} d_\beta g_\beta + c + \sum_{\beta \in \Omega; g_\beta \ll 1} d_\beta g_\beta \tag{4.165}$$

where $L$ is called a purely large transseries, $c$ is a constant and $s$ is called a small transseries.

Note that $L, c$ and $s$ are transseries since, for instance, the set $\{\beta \in \Omega; g_\beta \ll 1\}$ is a subset of ordinals, thus an ordinal itself.

**Lemma 4.166** *If $\mathcal{G}$ is finitely generated, if $A_{\mathbf{m}} \subset \mathcal{G}$ and $s$ is a small transseries over $A_{\mathbf{m}}$ we can always assume, for an $n \geq n'$ that the generators $\nu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^{n'}$ are such that for all $\nu_{\mathbf{k}'} \in s$ we have $\mathbf{k}' > 0$.*

$$s = \sum_{\mathbf{k} \geq \mathbf{m}} \mu_{\mathbf{k}} c_{\mathbf{k}} = \sum_{\beta \in \tilde{\Omega}} d_\beta g_\beta = \sum_{\mathbf{k}' > 0} \nu_{\mathbf{k}'} c'_{\mathbf{k}'} \tag{4.167}$$

**PROOF** In the first sum on the left side we can retain only the set of indices $I$ such that $\mathbf{k} \in I \Rightarrow \mu_{\mathbf{k}} = g_\beta$ has nonzero coefficient $d_\beta$. In particular, since all $g_\beta \ll 1$, we have $\mu_{\mathbf{k}} \ll 1$ $\forall \mathbf{k} \in I$. Let $I_1 = \mathrm{Mag}(I)$. We adjoin to the generators of $\mathcal{G}$ all the $\nu_{\mathbf{k}'} = \mu_{\mathbf{k}}$ with $\mathbf{k}' \in I_1$. The new set of generators is still finite and for all $\mathbf{k} \in I$ there is a $\mathbf{k}' \in \mathrm{Mag}(I)$ such that $\mathbf{k} \geq \mathbf{k}'$ and $\mu_{\mathbf{k}}$ can be written in the form $\nu^1_{\mathbf{k}'} \mu_{\mathbf{l}}$ where all $\mathbf{l} \geq 0$. ☐

**Remark**. After the construction, generally, there will be nontrivial relations between the generators. But nowhere do we assume that generators are relation-free, so this creates no difficulty.

15. An algebra over $\mathcal{G}$ can be defined as follows. Let $A$ and $\tilde{A}$ be well ordered sets in $\Omega$. The set of pairs $(\beta, \tilde{\beta}) \in A \times \tilde{A}$ is well ordered (check!). For every $g$, the equation $g_\beta \cdot g_{\tilde{\beta}} = g$ has finitely many solutions. Indeed, otherwise there would be an infinite sequence of $g_\beta$ which cannot be ascending, thus there is a subsequence of them which is strictly descending. But then, along that sequence, $g_{\tilde{\beta}}$ would be strictly ascending;

then the set of corresponding ordinals $\tilde{\beta}$ would form an infinite strictly descending chain, which is impossible. Thus, in

$$T \cdot \tilde{T} := \sum_{\gamma \in A \times \tilde{A}} g_\gamma \sum_{g_\beta \cdot g_{\tilde{\beta}} = g_\gamma} d_\beta d_{\tilde{\beta}} \tag{4.168}$$

the inner sum contains finitely many terms.

16. We denote by $\mathcal{T}_{\mathcal{G}}$ the algebra of transseries over $\mathcal{G}$. $\mathcal{T}_{\mathcal{G}}$ is a commutative algebra with respect to $(+, \cdot)$. We will see in the sequel that $\mathcal{T}_{\mathcal{G}}$ is in fact a field. We make it an ordered algebra by writing

$$T_1 \ll T_2 \Leftrightarrow \operatorname{mag}(T_1) \ll \operatorname{mag}(T_2) \tag{4.169}$$

and writing

$$T > 0 \Leftrightarrow \operatorname{dom}(T) > 0 \tag{4.170}$$

17. **Product form**. With the convention $\operatorname{dom}(0) = 0$, any transseries can be written in the form

$$T = \operatorname{dom}(T)(1 + s) \tag{4.171}$$

where $s$ is small (check).

18. Embeddings (cf. footnote on p. 156). If $\mathcal{G}_1 \subset \mathcal{G}$, we write that $\mathcal{T}_{\mathcal{G}_1} \subset \mathcal{T}_{\mathcal{G}}$ in the natural way.

19. **Topology** on $\mathcal{T}_{\mathcal{G}}$. We consider a sequence of transseries over a *common* set $A_{\mathbf{m}}$ of elements of $\mathcal{G}$, indexed by the ordinal $\Omega$.

$$\{T^{[j]}\}_{j \in \mathbb{N}};\ T^{[j]} = \sum_{\beta \in \beta} d_\beta^{[j]} g_\beta^{[j]}$$

**Definition.** We say that $T^{[j]} \to 0$ as $j \to \infty$ if for any $\beta \in \Omega$ there is a $j(\beta)$ such that the coefficient $d_\beta^{[j]} = 0$ for all if $j > j(\beta)$.

Thus the transseries $T^{[j]}$ must be *eventually depleted of all coefficients.* This aspect is very important. The mere fact that $\operatorname{dom}(S) \to 0$ does not suffice. Indeed the sequence $\sum_{k>j} x^{-k} + je^{-x}$, though "rapidly decreasing" is not convergent according to the definition, and probably should not converge in any reasonable topology.

20. Equivalently, the sequence $T^{[j]} \to 0$ is convergent if there is a representation such that

$$T^{[j]} = \sum_{\mathbf{k} \geq \mathbf{m}} c_{\mathbf{k}}^{[j]} \mu_{\mathbf{k}} \tag{4.172}$$

and in the sum $\mu_{\mathbf{k}} = g$ has only one solution (we know that such a choice is possible), and $\min\{|k_1| + \cdots + |k_n| : c_{\mathbf{k}}^{[j]} \neq 0\} \to 0$ as $j \to \infty$.

21. Let $\mu_1, ..., \mu_n$ be any generators for $\mathcal{G}$, $\mathbf{m} \in \mathbb{Z}^d$, as in 5 and $T_j \in \mathcal{T}_{A_{\mathbf{m}}}$ a sequence of transseries. Let $N_j := \min\{k_1 + ... + k_n | \mu_1^{p_1} \cdots \mu_n^{p_n} \in T_j\}$. Note that we can write min since, by Lemma 4.156, the minimum value is attained (check this!). If $N_j \to \infty$ then $T_j \to 0$. Indeed, if this was not the case, then there would exist a $g_\beta$ such that $g_\beta \in T_j$ with $d_\beta \neq 0$ for infinitely many $j$. Since $N_j \to \infty$ there is a sequence $\mu_{\mathbf{k}} \in A_{\mathbf{m}}$ such that $k_1 + ... + k_n \to \infty$ and $\mu_{\mathbf{k}} = g_\beta$. This would yield an infinite set of solutions of $\mu_{\mathbf{k}} = g_\beta$ in $A_{\mathbf{m}}$, which is not possible. The function $\max\{e^{-|k_1|+\cdots+|k_n|} : \sum_{\mu_k=g} c_{\mathbf{k}} \neq 0\}$ is a semimetric (it satisfies all properties of the metric except the triangle inequality) which induces the same topology.

    More generally, transseries are a subset of functions $f$ defined on $\mathcal{G}$ with real values and for which there exists a $\mathbf{k}_0(f) = \mathbf{k}_0$ such that $f(g_{\mathbf{k}}) = 0$ for all $\mathbf{k} < \mathbf{k}_0$. On these functions we can define a topology by writing $f^{[j]} \to 0$ if $\mathbf{k}_0(f^{[j]})$ does not depend on $j$ and for any $g_\beta$ there is an $N$ we have $f^{[n]}(g_\beta) = 0$ for all $n > N$. The first restriction is imposed to disallow, say, the convergence of $x^n$ to zero, which would not be compatible with a good structure of transseries.

22. This topology is metrizable. For example we can proceed as follows. Let $A_{\mathbf{m}}$ be the common set over which the transseries are defined. The elements of $\mathcal{G}$ are countable. We choose any counting on $A_{\mathbf{m}}$. We then identify transseries over $A_{\mathbf{m}}$ with the space $\mathcal{F}$ of real-valued functions defined on the natural numbers. We define $d(f,g) = 1/n$ where $n$ is the least integer such that $f(n) \neq g(n)$ and $d(f,f) = 0$. The only property that needs to be checked is the triangle inequality. Let $h \in \mathcal{F}$. If $d(g,h) \geq 1/n$, then clearly $d(f,g) \leq d(f,h) + d(h,g)$. If $d(g,h) < 1/n$ then $d(f,h) = 1/n$ and the inequality holds too.

23. The topology cannot come from a norm, since in general $a_n\mu \not\to 0$ as $a_n \to 0$.

24. We also note that the topology is *not* compatible with the order relation. For example $s_n = x^{-n} + e^{-x} \to e^{-x}$ as $n \to \infty$, $s_n \gg e^{-\sqrt{x}}$ for all $n$ while $e^{-x} \not\gg e^{-\sqrt{x}}$. The same argument shows that there is no distance compatible with the order relation.

25. In some sense, there is no "good" topology compatible with the order relation $\ll$. Indeed, if there was one, then the sequences $s_n = x^{-n}$ and $t_n = x^{-n} + e^{-x}$ which are interlaced in the order relation should have the same limit, but then addition would be discontinuous[11].

26. Giving up compatibility with asymptotic order allows us to ensure continuity of most operations of interest.

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[11] This example was pointed out by G. Edgar.

*Exercise. Show that a Cauchy sequence in $\mathcal{T}_{A_{\mathbf{m}}}$, is convergent, and $\mathcal{T}_{A_{\mathbf{m}}}$ is a topological algebra.*

27. If $\mathcal{G}$ is finitely generated, then for any small transseries

$$s = \sum_{\beta\in\Omega:g_\beta\ll 1} d_\beta g_\beta \tag{4.173}$$

we have $s^j \to 0$ as $j \to \infty$.

**PROOF** Indeed, by Lemma 4.166 we may assume that the generators of $\mathcal{G}$, $\mu_1, ..., \mu_n$, are chosen such that all $\mathbf{k} > 0$ in $s$. Let $g \in \mathcal{G}$. The terms occurring in the formal sum of $s^j$ are of the form $const.\mu_1^{l_1^1+...+l_1^j} \cdots \mu_n^{l_n^1+\cdots l_n^j}$ where $l_m^s \geq 0$ and at least one $l_j^s > 0$. Therefore $l_1^1 + ... + l_1^j \to \infty$ and $\sum_{l=1..M} s^l \to 0$ by 21 for any $j, M \to \infty$. $\Box$

As a side remark, finite generation is not needed at this point. More generally, let $A \subset \mathcal{G}$ be well ordered. It follows from J. Kruskal's theorem that the set $\tilde{A} \supset A$ of all products of elements of $A$ is also well quasi-ordered.

**Note 4.174** The sum $\sum_{k=0}^{\infty} c_k s^k$ might belong to a space of transseries defined over a larger, but still finite, number of generators. For instance, if

$$\frac{1}{xe^x+1} = \frac{1}{xe^x(1+xe^{-x})} = \frac{e^{-x}}{x}\sum_{j=0}^{\infty}(-1)^j x^j e^{-jx} \tag{4.175}$$

then the generators of (4.175) can be taken to be $x^{-1}, e^{-x}, xe^{-x}$ but certainly cannot stay $e^{-x}, x^{-1}$ since then the powers of $x^{-1}$ would be unbounded below.

28. In particular if $f(\mu) := \sum_{k=0}^{\infty} c_k \mu^k$ is a formal series and $s$ is a small transseries, then

$$f(s) := \sum_{k=0}^{\infty} c_k s^k \tag{4.176}$$

is well defined.

**Exercise 4.177** *Show that $f$ is continuous, in the sense that if $s = s^{[n]} \to 0$ as $n \to \infty$, then $f(s^{[n]}) \to c_0$.*

29. If $T_1 \gg T_2$, $T_3 \ll T_1$ and $T_4 \ll T_2$ then $T_1 + T_3 \gg T_2 + T_4$. Indeed, $\text{mag}(T_1 + T_3)$=$\text{mag}(T_1)$ and $\text{mag}(T_2 + T_4)$=$\text{mag}(T_2)$.

30. It is easily checked that $(1+s)\cdot 1/(1+s)=1$, where

$$\frac{1}{1+s} := \sum_{j\ge 0}(-1)^j s^j \tag{4.178}$$

More generally we define

$$(1+s)^a = 1 + a\,s + \frac{a(a-1)}{2}s^2 + \cdots$$

31. Writing $S = \mathrm{dom}(S)(1+s)$ we define $S^{-1} = \mathrm{dom}(S)^{-1}(1+s)^{-1}$.

32. If $\mu^r$ is defined for a real $r$, we then adjoin $\mu^r$ to $\mathcal{G}$ and define

$$T^r := d_1^r g_1^r (1+s)^r$$

33. If $\mu_j \mapsto \mu_j'$ is a differentiation, defined from the generators $\mu_j$ into $\mathcal{T}_{\mathcal{G}}$, where we assume that differentiation is compatible with the relations between the generators, we can extend it by $(g_1 g_2)' = g_1' g_2 + g_1 g_2'$, $1' = 0$ to the whole of $\mathcal{G}$ and by linearity to $\mathcal{T}_{\mathcal{G}}$,

$$\left(\sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}\mu_{\mathbf{k}}\right)' = \sum_{j=1}^n \mu_j' \sum_{\mathbf{k}\in\mathbb{Z}^n} k_j c_{\mathbf{k}} \mu_1^{k_1}\cdots\mu_j^{k_j-1} \tag{4.179}$$

and the latter sum is a well defined finite sum of transseries.

*Exercise. Show that with these operations, $\mathcal{T}_{\mathcal{G}}$ is a differential field.*

34. If $s$ is a small series, we define

$$e^s = \sum_{k\ge 0}\frac{s^k}{k!} \tag{4.180}$$

*Exercise. Show that $e^s$ has the usual properties with respect to multiplication and differentiation.*

35. **Transseries are limits of finite sums**. We let $\mathbf{m}\in\mathbb{Z}^n$ and $\mathbf{M}_p = (p,p,...,p)\in\mathbb{N}^n$. Note that

$$T_p := \sum_{g_\beta=\mu_{\mathbf{k}};\mathbf{m}\le\mathbf{k}\le\mathbf{M}_p;\beta\in\Omega} d_\beta g_\beta \xrightarrow[p\to\infty]{} \sum_{\beta\in\Omega} d_\beta g_\beta$$

Indeed, it can be checked that $d(T_p,T)\to 0$ as $p\to\infty$.

36. More generally, let $\mathcal{G}$ be finitely generated and $\mathbf{k}_0\in\mathbb{Z}$. Assume $s_{\mathbf{k}}\to 0$ as $\mathbf{k}\to\infty$. Then, for any sequence of real numbers $c_{\mathbf{k}}$, the sequence

$$\sum_{\mathbf{k}_0\le\mathbf{k}\le\mathbf{M}_p} c_{\mathbf{k}} s_{\mathbf{k}} \tag{4.181}$$

where $\mathbf{M}_p = (p, ..., p), p \in \mathbb{N}$ is Cauchy and the limit

$$\lim_{p\to\infty} \sum_{\mathbf{k}_0 \le \mathbf{k} \le \mathbf{M}_p} c_{\mathbf{k}} s_{\mathbf{k}} \tag{4.182}$$

is well defined. In particular, for a given transseries

$$T \triangleleft \mathbf{s} = \sum d_{\mathbf{k}} s_{\mathbf{k}} \tag{4.183}$$

we define the **transcomposition**

$$T \triangleleft \mathbf{s} = \sum_{\mathbf{k} \ge \mathbf{k}_0} d_{\mathbf{k}} s_{\mathbf{k}} \tag{4.184}$$

37. As an example of transcomposition, we see that transseries are closed under right pseudo-composition with *large* (not necessarily purely large) transseries $\mathbf{T} = T_i; i = 1, 2, ..., n$ by

$$T_1(1/\mathbf{T}) = \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}} \mathbf{T}^{-\mathbf{k}} \tag{4.185}$$

if

$$T_1 = \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}} \mu^{\mathbf{k}}$$

(cf. 27) We should mention that at this level of abstractness pseudo-composition may not behave as a composition, for instance it may not be compatible with chain rule in differentiation.

38. **Asymptotically contractive operators.** Contractivity is usually defined in relation to a metric, but given a topology, contractivity depends on the metric while convergence does not. There is apparently no natural metric on transseries.

**Definition 4.186** *Let first $J$ be a linear operator from $\mathcal{T}_{A_{\mathbf{m}}}$ or from one of its subspaces, to $A_{\mathbf{k}}$,*

$$JT = J \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}} \mu_{\mathbf{k}} = \sum_{\mathbf{k} \ge \mathbf{m}} c_{\mathbf{k}} J \mu_{\mathbf{k}} \tag{4.187}$$

*Then $J$ is called asymptotically contractive on $\tilde{\mathcal{A}}_{\mathbf{m}}$ if*

$$J\mu_{\mathbf{j}} = \sum_{\mathbf{p} > 0} c_{\mathbf{p}} \mu_{\mathbf{j}+\mathbf{p}} \tag{4.188}$$

**Remark 4.189** Contractivity depends on the set of generators. We can, more generally say that an operator is contractive is there is an extension of the set of generators such that Condition (4.188) holds.

**Remark 4.190** *It can be checked that contractivity holds if*

$$J\mu_{\mathbf{j}} = \sum_{\mathbf{p}>0} c_{\mathbf{p}}\mu_{\mathbf{j}+\mathbf{p}}(1+s_{\mathbf{j}}) \tag{4.191}$$

*where* $s_{\mathbf{j}}$ *are small transseries.*

**Exercise 4.192** *Check that for any* $\mu_{\mathbf{j}}$ *we have*

$$\sup_{p>0} \sum_{k=n}^{n+p} J^k \mu_{\mathbf{j}} \to 0$$

*as* $n \to \infty$*.*

We then have

$$JT = \sum_{\mathbf{k}\geq\mathbf{m}} J\mu_{\mathbf{k}} \tag{4.193}$$

**Definition 4.194** *The linear or nonlinear operator* $J$ *is (asymptotically) contractive on the set* $A \subset A_{\mathbf{m}}$ *if* $J : A \mapsto A$ *and the following condition holds. Let* $T_1$ *and* $T_2$ *in* $A$ *be arbitrary and let*

$$T_1 - T_2 = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}}\mu_{\mathbf{k}} \tag{4.195}$$

*Then*

$$J(T_1) - J(T_2) = \sum_{\mathbf{k}\geq\mathbf{m}} c'_{\mathbf{k}}\mu_{\mathbf{k}+\mathbf{p}_{\mathbf{k}}}(1+s_{\mathbf{k}}) \tag{4.196}$$

*where* $\mathbf{p}_{\mathbf{k}} > 0$ *and* $s_{\mathbf{k}}$ *are small.*

**Remark 4.197** The sum of asymptotically contractive operators is contractive; the composition of contractive operators, whenever defined, is contractive.

**Theorem 4.198** *(i) If* $J$ *is linear and contractive on* $\mathcal{T}_{A_{\mathbf{m}}}$ *then for any* $T_0 \in \mathcal{T}_{A_{\mathbf{m}}}$ *the fixed point equation* $T = JT + T_0$ *has a unique solution* $T \in \mathcal{T}_{A_{\mathbf{m}}}$*.*

*(ii) In general, if* $A \subset A_{\mathbf{m}}$ *is closed and* $J : A \mapsto A$ *is a (linear or nonlinear) contractive operator on* $A$*, then* $T = J(T)$ *has a unique solution is* $A$*.*

**PROOF** (ii): the sequence $T_{n+1} = J(T_n)$ is convergent since for some coefficients $c_{j,\mathbf{k}}$ we have

$$J^q(T) - J(T) = \sum_{\mathbf{k} \geq m} c_{j,\mathbf{k}} \mu_{\mathbf{k}+q\mathbf{p_k}} \to 0$$

as $q \to \infty$. Uniqueness is immediate. $\square$

39. When working with transseries we often encounter this fixed point problem in the form $X = Y + \mathcal{N}(X)$, where $Y$ is given, $X$ is the unknown $Y$ is given, and $\mathcal{N}$ is "small".

    *Exercise. Show the existence of a unique inverse of* $(1+s)$ *where* $s$ *is a small transseries, by showing that the equation* $T = 1 - sT$ *is contractive.*

40. For example $\partial$ is contractive on transseries of level zero (see also Note 4.53). This is clear since in every monomial the power of $x$ decreases by one. But note that $\partial$ is not contractive anymore if we add "terms beyond all orders", e.g., $(e^{-x^2})' = -2xe^{-x^2} \gg e^{-x^2}$.

    We cannot expect any contractivity of $\partial$ in general, since if $T_1$ is the level zero solution of $T = 1/x - T'$ then $T_1 + Ce^{-x}$ is a solution for any $C$ so uniqueness fails.

41. One reason the WKB method works near irregular singularities, where exponential behavior is likely, is that is reduces the level of the transseries to zero, where $\partial$ is contractive. Iterated exponentials almost never occur in differential/difference equations, and then the substitution $y = e^w$ achieves the level reduction.

42. We take the union

$$\mathcal{T} = \bigcup_{\mathcal{G}} \mathcal{T}_{\mathcal{G}}$$

    with the natural embeddings. It can be easily checked that $\mathcal{T}$ is a differential field too. The topology is that of inductive limit, namely a sequence of transseries converges if they all belong to some $\mathcal{T}_{\mathcal{G}}$ and they converge there.

43. One can check that algebraic operations, exponentiation, composition with functions for which composition is defined, are continuous wherever the functions are "$C^\infty$".

**Exercise 4.199** *Let* $T \in A_{\mathbf{m}}$. *Show that the set* $\{T_1 \in A_{\mathbf{m}} | T_1 \ll T\}$ *is closed.*

## 4.9b General logarithmic-free transseries

## 4.9c Inductive assumptions

1. We have constructed level zero transseries. Transseries of any level are constructed inductively, level by level.

   Since we have already studied the properties of abstract multiseries, the construction is relatively simple, all we have to do is essentially watch for consistency of the definitions at each level.

2. Assume finitely generated transseries of level at most $n$ have already been constructed. We assume a number of properties, and then build level $n+1$ transseries and show that these properties are conserved.

   (a) Transmonomials $\mu_j$ of order at most $N$ are totally ordered, with respect to two order relations, $\ll$ and $<$. Multiplication is defined on the transmonomials, it is commutative and compatible with the order relations.

   (b) For a set of $n$ small transmonomials, a transseries of level at most $N$ is defined as expression of the form (4.160).

   It follows that the set $\{g = \mu_{\mathbf{k}} | \mathbf{k} \geq \mathbf{m}\}$ can be indexed by ordinals, and we can write the transseries in the form (4.163). The decomposition (4.165) then applies.

   It also follows that two transseries are equal iff their corresponding $d_\beta$ coincide.

   The order relations on transseries of level $N$ are defined as before, $T \gg 1$ if, by definition $g_1 \gg 1$, and $T > 0$ iff $d_1 > 0$.

   Transseries of level at most $N$ are defined as the union of all $\mathcal{T}_{\mathcal{A}_{\mathbf{m}}}$ where $A_{\mathbf{m}}$ is as before.

   (c) A transmonomial or order at most $N$ is of the form $x^a e^L$ where $L$ is a purely large or null transseries of level $N-1$, and $e^L$ is defined recursively. There are no transseries of level $-1$, so for $N = 1$ we take $L = 0$.

   *Exercise. Show that any transmonomial is of the form* $x^a e^{L_1} e^{L_2} \cdots e^{L_j}$ *where* $L_j$ *are of order exactly* $j$ *meaning that they are of order* $j$ *but not of lower order.*

   (d) For any transmonomial, $(x^a e^L)^r$ is defined as $x^{ar} e^{rL}$ where the ingredients have already been defined. It may be a adjoined to the generators of $\mathcal{G}$ and then, as in the previous section, $T^r$ is well defined.

   (e) By definition, $x^a e^L = e^L x^a$ and $x^{a_1} e^{L_1} x^{a_2} e^{L_2} = x^{a_1+a_2} e^{L_1+L_2}$. Furthermore $e^{L_1} \gg x^a e^{L_2}$ for any $a$ if $L_1 > 0$ is a purely large transseries of level strictly higher than the level of $L_2$.

(f) There is a differentiation with the usual properties on the generators, compatible with the group structure and equivalences. We have $(x^a e^L)' = ax^{a-1}x^L + x^a L' e^L$ where $L'$ is a (finitely generated) transseries of level at most $N-1$.
We define

$$T' = \sum_{\mathbf{k}\in\mathbb{Z}^n;\mathbf{k}\geq\mathbf{M}} c_{\mathbf{k}} \left[(x^{-\mathbf{k}\cdot\boldsymbol{\alpha}})' e^{-\mathbf{L}\cdot\boldsymbol{\beta}} + x^{-\mathbf{k}\cdot\boldsymbol{\alpha}}(e^{-\mathbf{L}\cdot\boldsymbol{\beta}})'\right] \tag{4.200}$$

where, according to the definition of differentiation, (4.200) is a finite sum of products of transseries of level at most $N$.
We have $T' = 0$ iff $T = const$. If $\operatorname{dom}(T_{1,2}) \neq const.$, *then* $T_1 \ll T_2$ implies $T_1' \ll T_2'$.

3. It can be checked by induction that $T > 0, T \gg 1$ implies $T' > 0$. In this sense, differentiation is compatible with the order relations.

4. It can then be checked that differentiation has the usual properties.

5. if $c$ is a constant, then $e^c$ is a constant, the usual exponential of $c$, and if $L + c + s$ is the decomposition of a transseries of level $N-1$ we write $e^{L+c+s} = e^L e^c e^s$ where $e^s$ is *reexpanded* according to formula (4.180) and the result is a *transseries* of level $N$.

   We convene to write $e^T$, for any $T$ transseries of level at most $N$ *only* in this reexpanded form.

   Then it is always the case that $e^T = T_1 e^{L_1}$ where $T_1$ and $L_1$ are transseries of level $N-1$ and $L_2$ is purely large or zero. The transseries $e^T$ is finitely generated, with generators $e^{-L_1}$ , if $L_1 > 0$ or $e^{L_1}$ otherwise, together with all the generators of $L_1$.

   Sometimes it is convenient to adjoin to the generators of $T$ all the generators in the exponents of the transmonomials in $T$, and then the generators in exponents in the exponents of the transmonomials in $T$ etc. Of course, this process is finite, and we end up with a finite number of generators, which we will call *the complete set of generators* of $T$.

6. This **defines** the exponential of any transseries of level at most $N-1$ if $L \neq 0$ and the exponential of any transseries of level at most $N$ if $L = 0$. We can check that $e^{T_1} = e^{T_2}$ iff $T_1 = T_2$.

7. If all transseries of level $N$ are written in the canonical form (4.163) then $T_1 = T_2$ iff all $g_\beta$ at all levels have exactly the same coefficients. Transseries, in this way, have a unique representation in a strong sense.

8. The space of transseries of level $N$, $\mathcal{T}^{[N]}$, is defined as the union of all spaces of transseries over finitely generated groups of transmonomials of level $N$.

$$\mathcal{T}^{[N]} = \bigcup_{\mathcal{G}_N} \mathcal{T}_{\mathcal{G}_N}$$

with the inductive limit topology.

9. The abstract theory of transseries we have developed in the previous section applies. In particular we have by definition $1/(1-s) = \sum_j s^j$ and $1/T = 1/\operatorname{dom}(T)(1+s)^{-1}$ and transseries of level $N$ form a differential field closed under the contractive mappings.

10. Note that transseries of order $N$ are closed under the contractive mapping principle.

### 4.9d Passing from step $N$ to step $N+1$

1. We now proceed in defining transseries of level at most $N+1$. We have to check that the construction preserves the properties in §4.9c .

2. For any purely large transseries of level $N$ we define $x^a e^L$ to equal the already defined transmonomial of order $N$. If $L$ is a (finitely generated) purely large transseries of level exactly $N$ we define a new objects, $x^a e^L$, a transmonomial of order $N+1$, having the properties

   (a) $e^0 = 1$.

   (b) $x^a e^L = e^L x^a$.

   (c) $x^{a_1} e^{L_1} x^{a_2} e^{L_2} = x^{a_1+a_2} e^{L_1+L_2}$.

   (d) If $L > 0$ is a purely large transseries of level exactly $N$ then $e^L \gg x^a$ for any $a$.

   *Exercise. Show that if $L_1$ and $L_1$ are purely large transseries and the level of $L_1$ strictly exceeds the level of $L_2$, then $e^{L_1} \gg x^a e^{L_2}$ for any $a$.*

   Note that $L_1 \pm L_2$ may be of lower level but it is either purely large or else zero; $L_1 L_2$ is purely large.

   **Note 4.201** At this stage, no meaning is given to $e^L$, or even to $e^x$; they are treated as primitives. There are possibly many models of this construction. By BE summation, a subclass of transseries is isomorphically associated to set of functions. The symbol $e^x$ corresponds to the usual exponential, convergent multiseries will correspond to their sums etc.

3. If $\alpha > 0$ and $L$ is a *positive* transseries of level $N$ we define a generator of order $N$ to be $\mu = x^{-\alpha} e^{-L}$. We choose a number of generators $\mu_1, ..., \mu_n$, and define the Abelian multiplicative group generated by them, with the multiplication rule just defined. We can check that $\mathcal{G}$ is a totally ordered, of course finitely generated, Abelian group, and that the order relation is compatible with the group structure.

4. We can now define transseries over $\mathcal{G} = \mathcal{G}^{[N+1]}$ as in §4.9.

5. We define transseries of order $N+1$ to be the union over all $\mathcal{T}_{\mathcal{G}^{[N+1]}}$, with the natural embeddings. We denote these transseries by $\mathcal{T}^{[N+1]}$.

6. *Compatibility of differentiation with the order relation.* We have already assumed that this is the case for transseries of level at most $N$. (i) We first show that it holds for transmonomials of level $N+1$. If $L_1 - L_2$ is a positive transseries, then $(x^a e^{L_1})' \gg (x^b e^{L_2})'$ follows directly from the formula of differentiation, the fact that $e^{L_1 - L_2}$ is large and the induction hypothesis. If $L_1 = L_2$ then $a > b$ and the property follows from the fact that $L_1$ is either zero, or else $L \gg x^\beta$ for some $\beta > 0$ for some positive $\beta$ (check!).

   (ii) For the general case we note that

   $$\left(\sum_\beta d_\beta \mu_\beta\right)' = \sum_\beta d_\beta \mu_\beta'$$

   and $\mu_{\beta_1}' \ll \mu_{\beta_2}'$ if $\beta_1 > \beta_2$. Then $\operatorname{dom}(T)' = (\operatorname{dom}(T))'$ and the property follows.

7. Differentiation is continuous. Indeed, if $T^{[m]} \to 0$,

   $$T^{[m]} = \sum_{\mathbf{k} \geq \mathbf{m}} c_{\mathbf{k}}^{[m]} x^{\mathbf{k}\cdot\mathbf{a}} e^{-\mathbf{k}\cdot\mathbf{L}} \to 0 \quad \text{as} \quad m \to \infty$$

   where the transseries $L_1, ..., L_n$ are purely large, then

   $$(T^{[m]})' = \frac{1}{x} \sum_{\mathbf{k} \geq \mathbf{m}} (\mathbf{k} \cdot \mathbf{a}\, c_{\mathbf{k}}^{[m]}) x^{\mathbf{k}\cdot\mathbf{a}} e^{-\mathbf{k}\cdot\mathbf{L}} - \mathbf{L}' \cdot \sum_{\mathbf{k} \geq \mathbf{m}} (\mathbf{k} c_{\mathbf{k}}^{[m]}) x^{\mathbf{k}\cdot\mathbf{a}} e^{-\mathbf{k}\cdot\mathbf{L}}$$

   and the rest follows from continuity of multiplication and the definition of convergence.

8. Therefore, if a property of differentiation holds for finite sums of transmonomials, then it holds for transseries.

9. By direct calculation, if $\mu_1, \mu_2$ are transmonomials of order $N+1$ then $(\mu_1\mu_2)' = \mu_1'\mu_2 + \mu_1\mu_2'$. Then, one can check by usual induction, the product rule holds for finite sums of transmonomials. Using 8 the product rule follows for general transseries.

### 4.9d .1 Composition

10. Composition *to the right* with a *large* (not necessarily purely large) transseries $T$ of level $m$ is defined as follows.

The power of a transseries $T = x^a e^L(1+s)$ is defined by $T^p = x^{ap}e^{pL}(1+s)^p$, where the last expression is well defined and $(T^p)' = pT'T^{p-1}$ (check).

The exponential of a transseries is defined, inductively, in the following way.

$$T = L + c + s \Rightarrow e^T = e^L e^c e^s = Se^L e^c \tag{4.202}$$

where $S$ is given in (4.180).

A general exponential-free transseries of level zero has the form

$$T_0 = \sum_{\mathbf{k}\ge\mathbf{m}} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\boldsymbol{\alpha}} \tag{4.203}$$

where $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^{+n}$ for some $n$.

Then we take $\mathbf{T} = (T^{\alpha_1}, ..., T^{\alpha_n})$ and define $T_0(1/T)$ by (4.185); $T_0(1/T)$ has level $m$. If the sum (4.203) contains finitely many terms, it is clear that $[T_0(1/T)]' = T_0'(1/T)T'$. By continuity, this is true for a general $T_0$ of level zero.

11. Assume that composition with $T$ has been defined for all transseries of level $N$. It is assumed that this composition is a transseries of level $N + m$. Then $L(T) = L_1 + c_1 + s_1$ (it is easily seen that $L(T)$ is not necessarily purely large). Then

$$(x^a e^L) \circ (T) := T^a e^{L(T)} = x^b(1 + s_1(T))e^{L_1(T)} \tag{4.204}$$

where $L_1(T)$ is purely large. Since $L_1$ has level $N+m$, then $(x^a e^L)\circ(T)$ has level $N+m+1$. We have $(e^{L_1})' = L_1' e^{L_1}$ and the chain rule follows by induction and from the sum and product rules.

**Exercise 4.205** *If $T^{[n]}$ is a sequence of transseries, then $e^{T^{[n]}}$ is a not necessarily a valid sequence of transseries. But if it is, then there is an $L_0$ such that $L^{[n]} = L_0$ for all large $n$. If $e^{T^{[n]}}$ is a sequence of transseries and $T^{[n]} \to 0$, then $e^{T^{[n]}} \to 1$.*

12. The exponential is continuous. This follows from the Exercise 4.205 and Exercise 4.177.

13. Take now a general *large* transseries of level $N + 1$ and write $T = x^a e^L(1 + s)$; let

$$t = \sum_{\mathbf{k}\ge\mathbf{m}} x^{-\mathbf{k}\cdot\boldsymbol{\alpha}} e^{-\mathbf{k}\cdot\mathbf{l}} \tag{4.206}$$

Then $t(T)$ is well defined as the limit of the following finite sum with generators $x^{-|a\alpha_j|}, x^{-\alpha_j}e^{-l_j(T)}, e^{-l_j(T)}$; $, j = 1, ..., n$:

$$t(T) = \sum_{\mathbf{M}_p \geq \mathbf{k} \geq \mathbf{m}} x^{-a(\mathbf{k}\cdot\boldsymbol{\alpha})} e^{-\mathbf{k}\cdot\mathbf{l_1}(T)}(1 + \mathbf{s}(T)) \tag{4.207}$$

14. The chain rule holds by continuity.
15. The general theory we developed in §4.9 applies and guarantees that the properties listed in §4.9c hold (check!).

### 4.9d .2 Small transseries as infinitesimals; expansions beyond all orders

16. Let $T$ be a transseries of level $N$ over $\mathcal{G}$ and $dx$ a small transseries with dominance $e^{-L}$ where $L$ is a positive large transseries of level $N+p$, $p > 0$. Then $(T(x+dx) - T(x))/dx = T'(x) + s(T)$ where $s(T)$ is a small transseries of level $N+p$.

    The proof is by induction on the level. By linearity and continuity it is enough to prove the statement for transmonomials. We have

    $$(x+dx)^a e^{-L_1(x+dx)} = x^a(1+dx/x)^a e^{L_1(x)+L_1'(x)dx+s(L)}$$

    where $L_1'dx$ is a small transseries (since $L_1 e^{-L}$ is small) and $s(L_1)$ is of level $N+p$. The claim follows after reexpansion of the two terms in the product. Note that $dx$ must be far less than all terms in $T$; $dx \ll 1$ is not enough.

**Exercise 4.208** *Show that, under the same assumptions that*

$$T(x+dx) = \sum_{j=0}^{\infty} T^{(n)}(x)\frac{dx^n}{n!} \tag{4.209}$$

In this sense, transseries behave like analytic functions.

### 4.9d .3 An inequality helpful in WKB analysis.

**Proposition 4.210** *If $L \gg 1$ then $L'' \ll (L')^2$ (or, which is the same, $L' \ll L^2$).*

**PROOF** If $L = x^a e^{L_1}$ where $L_1 \neq 0$ then $L_1$ is purely large, then the dominance of $L'$ is of the form $x^b e^{L_1}$, whereas the dominance of $L$ is of the form $x^a e^{2L_1}$ and the property is obvious. If $L_1 = 0$ the property is obvious as well. ☐

In WKB analysis this result is mostly used in the form (4.212 below.

**Exercise 4.211** *Show that if* $T \gg 1$, $T$ *positive or negative, we have*

$$\mathrm{dom}[(e^T)^{(n)}] = \mathrm{dom}[(T')^n e^T] \tag{4.212}$$

## 4.9e General logarithmic-free transseries

These are simply defined as

$$\mathcal{T}_e = \bigcup_{N \in \mathbb{N}} \mathcal{T}^{[N]} \tag{4.213}$$

with the natural embeddings.

The general theory we developed in §4.9 applies to $\mathcal{T}_e$ as well. Since any transseries belongs to some level, any finite number of them share some level. There are no operations defined which involve infinitely many levels, because they would involve infinitely many generators. Then, the properties listed in §4.9c hold in $\mathcal{T}_e$ (check!).

## 4.9f Écalle's notation

- $\sqcup$ —small transmonomial.
- $\sqcap$ —large transmonomial.
- $\square$ —any transmonomial, large or small.
- $\sqcup\!\sqcup$ —small transseries.
- $\sqcap\!\sqcap$ —large transseries.
- $\square\!\square$ —any transseries, small or large.

### 4.9f .1 Further properties of transseries

*Definition.* The level $l(T)$ of $T$ is $n$ if $T \in \mathcal{T}^{[n]}$ and $T \notin \mathcal{T}^{[n-1]}$.

### 4.9f .2 Further properties of differentiation

We denote $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}x}$

**Corollary 4.214** *We have* $\mathcal{D}T = 0 \iff T = \mathrm{Const}.$

**PROOF** We have to show that if $T = L + s \neq 0$ then $T' \neq 0$. If $L \neq 0$ then for some $\beta > 0$ we have $L + s \gg x^\beta + s$ and then $L' + s' \gg x^{\beta-1} \neq 0$. If instead $L = 0$ then $(1/T) = L_1 + s_1 + c$ and we see that $(L_1 + s_1)' = 0$ which, by the above, implies $L_1 = 0$ which gives $1/s = s_1$, a contradiction. $\square$

**Proposition 4.215** *Assume* $T = L$ *or* $T = s$. *Then:*
*(i) If* $l(\operatorname{mag}(T)) \geq 1$ *then* $l(\operatorname{mag}(T^{-1}T')) < l(\operatorname{mag}(T))$.
*(ii)* $\operatorname{dom}(T') = \operatorname{dom}(T)'(1+s)$.

**PROOF** Straightforward induction. ☐

### 4.9f .3 Transseries with complex coefficients

Complex transseries $\mathcal{T}_{\mathbb{C}}$ are constructed in a similar way as real transseries, replacing everywhere $L_1 > L_2$ by $\operatorname{Re} L_1 > \operatorname{Re} L_2$. Thus there is only one order relation in $\mathcal{T}_{\mathbb{C}}$, $\gg$. Difficulties arise when exponentiating transseries whose dominant term is imaginary. Operations with complex transseries are then limited. We will only use complex transseries in contexts that will prevent these difficulties.

### 4.9f .4 Differential systems in $\mathcal{T}_e$

The theory of differential equations in $\mathcal{T}_e$ is similar in many ways to the corresponding theory for functions.

*Example.* The general solution of the differential equation

$$f' + f = 1/x \tag{4.216}$$

in $\mathcal{T}_e$ (for $x \to +\infty$) is $T(x;C) = \sum_{k=0}^{\infty} k! x^{-k} + Ce^{-x} = T(x;0) + Ce^{-x}$.

The particular solution $T(x;0)$ is the unique solution of the equation $f = 1/x - \mathcal{D}f$ which is manifestly contractive in the space of level zero transseries.

Indeed, the fact that $T(x;C)$ is a solution follows immediately from the definition of the operations in $\mathcal{T}_e$ and the fact that $e^{-x}$ is a solution of the homogeneous equation.

To show uniqueness, assume $T_1$ satisfies (4.216). Then $T_2 = T_1 - T(x;0)$ is a solution of $\mathcal{D}T + T = 0$. Then $T_2 = e^x T$ satisfies $\mathcal{D}T_2 = 0$ i.e., $T_2 = \text{Const}$.

## 4.9g The space $\mathcal{T}$ of general transseries

We define

$$\log_n(x) = \underbrace{\log\log \dots \log(x)}_{n \text{ times}} \tag{4.217}$$

$$\exp_n(x) = \underbrace{\exp\exp \dots \exp(x)}_{n \text{ times}} \tag{4.218}$$

$$\tag{4.219}$$

with the convention $\exp_0(x) = \log_0(x) = x$.

We write $\exp(\log x) = x$ and then any log-free transseries can be written as $T(x) = T \circ \exp_n(\log_n(x))$. This defines right composition with $\log_n$ in this trivial case, as $T_1 \circ \log_n(x)) = (T \circ \exp_n) \circ \log_n(x) := T(x)$.

More generally, we define $\mathcal{T}$, the space of general transseries , as a set of formal compositions

$$\mathcal{T} = \{T \circ \log_n : T \in \mathcal{T}_e\}$$

with the algebraic operations and inequalities (symbolized below by $\odot$) inherited from $\widetilde{\widetilde{\mathcal{T}}}$ by

$$(T_1 \circ \log_n) \odot (T_2 \circ \log_{n+k}) = [(T_1 \circ \exp_k) \odot T_2] \circ \log_{n+k} \tag{4.220}$$

and using (4.220), differentiation is defined by

$$\mathcal{D}(T \circ \log_n) = x^{-1} \left[ (\prod_{k=1}^{n-1} \log_k)^{-1} \right] (\mathcal{D}T) \circ \log_n$$

**Proposition 4.221** *$\mathcal{T}$ is an ordered differential field, closed under restricted composition.*

**PROOF** Exercise. ◻

The logarithm of a transseries. This is defined by first considering the case when $T \in \mathcal{T}_e$ and then taking right composition with iterated logs.

If $T = c\,\mathrm{mag}(T)(1+s) = cx^a e^L(1+s)$ then we define

$$\log(T) = \log(\,\mathrm{mag}(T)) + \log c + \log(1+s) = a \log x + L + \log c + \log(1+s) \tag{4.222}$$

where $\log c$ is the usual log, while $\log(1+s)$ is defined by expansion which we know is well defined on small transseries.

1. If $L \gg 1$ is large, then $\log L \gg 1$ and if $s \ll 1$, then $\log s \gg 1$.

### 4.9g .1 Integration

**Proposition 4.223** *$\mathcal{T}$ is closed under integration.*

**PROOF** The idea behind the construction of $\mathcal{D}^{-1}$ is the following: we first find an invertible operator $J$ which is *to leading order* $\mathcal{D}^{-1}$; then the equation for the correction will be contractive. Let $T = \sum_{\mathbf{k} \ge \mathbf{k}_0} \mu^{\mathbf{k}} \circ \log_n$. To unify the treatment, it is convenient to use the identity

$$\int_x T(s)ds = \int_{\log_{n+2}(x)} \left(T \circ \exp_{n+2}\right)(t) \prod_{j \le n+1} \exp_j(t) \mathrm{d}t \; = \int_{\log_{n+2}(x)} T_1(t) \mathrm{d}t$$

where the last integrand, $T_1(t)$ is a log-free transseries and moreover

$$T_1(t) = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}}\mu_1^{k_1}\cdots\mu_M^{k_M} = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}}e^{-k_1L_1-\ldots-k_ML_M}$$

The case $\mathbf{k}=0$ is trivial and it thus suffices to find $\partial^{-1}e^{\pm L}$, where $n = l(L) \geq 1$ where $L > 0$. We analyze the case $\partial^{-1}e^{-L}$, the other one being similar. Then $L \gg x^m$ for any $m$ and thus also $\partial L \gg x^m$ for all $m$. Therefore, since $\partial e^{-L} = -(\partial L)e^{-L}$ we expect that $\operatorname{dom}(\partial^{-1}e^{-L}) = -(\partial L)^{-1}e^{-L}$ and we look for a $\Delta$ so that

$$\partial^{-1}e^{-L} = -\frac{e^{-L}}{\partial L}(1+\Delta) \tag{4.224}$$

Then $\Delta$ should satisfy the equation

$$\Delta = -\frac{\partial^2 L}{(\partial L)^2} - \frac{\partial^2 L}{(\partial L)^2}\Delta + (\partial L)^{-1}\partial\Delta \tag{4.225}$$

Since $s_1 = 1/L'$ and $s_2 = L''/(L')^2$ are small, by Lemma 4.166, there is a set of generators in which all the magnitudes of $s_{1,2}$ are of the form $\boldsymbol{\mu}^{\mathbf{k}}$ with $\mathbf{k} > 0$. By Proposition 4.210 and Exercise 4.199, (4.225) is contractive and has a unique solution in the space of transseries with the complete set of generators of $L$ and $x^{-1}$ and $\Delta \ll L$ and the generators constructed above. For the last term, note that if $\Delta = \sum c_\omega e^{-L_\omega}$ and $L = e^{L_1}$ , then $\Delta'/L' = \sum c_\omega L'_\omega e^{-L_\omega}e^{-L_1}$ and $L'_\omega e^{-L} = \mu_\omega \ll 1$.

$\Box$

1. Since the equation is contractive, it follows that $\operatorname{mag}(\Delta) = \operatorname{mag}(L''/L'^2)$.

In the following we also use the notation $\partial T = T'$ and we write $\mathcal{P}$ for the antiderivative $\partial^{-1}$ constructed above.

**Proposition 4.226** *$\mathcal{P}$ is an antiderivative without constant terms, i.e,*

$$\mathcal{P}T = L + s$$

**PROOF** This follows from the fact that $\mathcal{P}e^{-L} \ll 1$ while $P(e^L)$ is purely large, since all small terms are of lower level. Check! $\Box$

**Proposition 4.227** *We have*

$$\begin{aligned}
\mathcal{P}(T_1+T_2) &= \mathcal{P}T_1 + \mathcal{P}T_2\\
(\mathcal{P}T)' = T;\ \mathcal{P}T' &= T_{\overline{0}}\\
\mathcal{P}(T_1T_2') &= (T_1T_2)_{\overline{0}} - \mathcal{P}(T_1'T_2)\\
T_1 \gg T_2 &\implies \mathcal{P}T_1 \gg \mathcal{P}T_2\\
T > 0 \textit{ and } T \gg 1 &\implies \mathcal{P}T > 0
\end{aligned} \tag{4.228}$$

*where*

$$T=\sum_{\mathbf{k}\geq\mathbf{k}_0}c_{\mathbf{k}}\mu^{\mathbf{k}}\implies T_{\bar{0}}=\sum_{\mathbf{k}\geq\mathbf{k}_0;\mathbf{k}\neq 0}c_{\mathbf{k}}\mu^{\mathbf{k}}$$

**PROOF** Exercise. ◻

There exists only one $\mathcal{P}$ with the properties (4.228), for any two would differ by a constant.

**Remark 4.229** *Let $s_0\in\mathcal{T}$. The operators defined by*

$$J_1(T)=\mathcal{P}(e^{-x}(\text{Const.}+s_0)T(x)) \tag{4.230}$$

$$J_2(T)=e^{\pm x}x^{\sigma}\mathcal{P}(x^{-2}x^{-\sigma}e^{\mp x}(\text{Const.}+s_0)T(x)) \tag{4.231}$$

*are contractive on $\mathcal{T}$.*

**PROOF** For (4.230) it is enough to show contractivity of $\mathcal{P}(e^{-x}\cdot)$. If we assume the contrary, that $T'\not\ll Te^{-x}$ it follows that $\log T\not\gg 1$. We know that if $\log T$ is small then $\operatorname{mag}(T)=c$, $c$ constant. But if $\operatorname{mag}(T)=c$ then the property is immediate. The proof of (4.230) is very similar.

◻

**Exercise 4.232** In Eq. (4.54) let $\tilde{y}_0$ be a level zero transseries solution, and let $y=y_0+\delta$ be the general transseries solution. If $\delta'/\delta=L+c+s$, show that $L=0$ and $c=-1$. Then $\delta=Ce^{-x}(1+s_1)$. Show that the equation for $s_1$ is contractive in the space of small transseries of any fixed level.

# Chapter 5

## Borel summability in differential equations

In §5.3 we look at an ODE example which illustrates some key ingredients needed in a more general analysis. §5.9 contains a number of results which hold for generic systems of differential equations; the proofs, outlined in §5.10 and given in detail (with some simplifying assumptions) in §5.11 rely on methods that can be adapted to other problems.

Using the singularity structure in Borel plane, one can generically reconstruct a system of ODEs from just one, possibly formal, solution, see Remark 5.73.

### 5.1 Convolutions revisited

We have

$$\mathcal{L}\left[f * g\right] = (\mathcal{L}f)(\mathcal{L}g) \tag{5.1}$$

Furthermore,

$$\mathcal{L}\left[f * (g * h)\right] = \mathcal{L}[f]\mathcal{L}\left[g * h\right] = \mathcal{L}[f]\mathcal{L}[g]\mathcal{L}[h] = \mathcal{L}\left[(f * g) * h\right] \tag{5.2}$$

and since the Laplace transform is injective, we get

$$f * (g * h) = (f * g) * h \tag{5.3}$$

and convolution is associative. Similarly, it is easy to see that

$$f * g = g * f, \qquad f * (g + h) = f * g + f * h \tag{5.4}$$

Some spaces arise naturally and are well suited for the study of convolution equations.

(1) Let $\nu \in \mathbb{R}^+$ and define $L^1_\nu := \{f : \mathbb{R}^+ : f(p)e^{-\nu p} \in L^1(\mathbb{R}^+)\}$; then the norm $\|f\|_\nu$ is defined as $\|f(p)e^{-\nu p}\|_1$ where $\|\cdot\|_1$ denotes the $L^1$ norm.

We recall that $L^1_\nu$ is a Banach algebra with respect to convolution, see Proposition 4.101.

We see that the norm is the Laplace transform of $|f|$ evaluated at large argument $\nu$, and it is, in this sense, a Borel dual of the sup norm in the original space.

(2) We say that $f$ is in $L^1_\nu(\mathbb{R}^+e^{i\phi})$ if $f_\phi := f(te^{i\phi}) \in L^1_\nu$. Convolution along $\mathbb{R}^+e^{i\phi}$ reads

$$(f*g)(p) = \int_0^p f(s)g(p-s)ds =$$
$$e^{i\phi}\int_0^{|p|e^{i\phi}} f(te^{i\phi})g(e^{i\phi}(|p|-t)dt = e^{i\phi}(f_\phi * g_\phi)(|p|e^{i\phi}) \quad (5.5)$$

It is clear that $L^1_\nu(\mathbb{R}^+e^{i\phi})$ is also a Banach algebra with respect to convolution.

(3) Similarly, we say that $f \in L^1_\nu(S)$ where $S = \{te^{i\phi} : t \in \mathbb{R}^+, \phi \in (a,b)\}$ if $f \in L^1_\nu(\mathbb{R}^+e^{i\phi})$ for all $\phi \in (a,b)$. We define $\|f\|_{\nu,S} = \sup_{\phi\in(a,b)} \|f\|_{L^1_\nu(\mathbb{R}^+e^{i\phi})}$. $L^1_\nu(S)$ is also a Banach algebra.

(4) The $L^1_\nu$ spaces can be restricted to an initial interval along a ray, or a compact subset of $S$, restricting the norm to an appropriate set. For instance,

$$L^1_\nu([0,1]) = \left\{ f : \int_0^1 e^{-\nu s}|f(s)|ds < \infty \right\} \quad (5.6)$$

These spaces are Banach algebras as well. Obviously, if $A \subset B$, $L^1_\nu(B)$ is naturally embedded (cf. footnote on pp. 156) in $L^1_\nu(A)$.

(5) Another important space is $\mathcal{A}_{K;\nu}(\mathcal{E})$, the space of analytic functions analytic in a star-shaped neighborhood $\mathcal{E} \in \mathbb{C}$ in the ball $\{p : |p| \le K$ of the interval $[0, K']$ in the norm $(\nu \in \mathbb{R}^+)$

$$\|f\| = K \sup_{p\in\mathcal{E}} \left| e^{-\nu|p|} f(p) \right|$$

**Note.** This norm is topologically equivalent with the sup norm (convergent sequences are the same), but better suited for controlling exponential growth.

**Proposition 5.7** *The space $\mathcal{A}_{K;\nu}$ is a Banach algebra with respect to convolution.*

**PROOF** Analyticity of convolution is proved in the same way as Lemma 4.101. For continuity we let $|p| = P$, $p = Pe^{i\phi}$ and note that

$$\left|Ke^{-\nu P}\int_0^p f(s)g(p-s)ds\right| = \left|Ke^{-\nu P}\int_0^P f(te^{i\phi})g((P-t)e^{i\phi})dt\right|$$
$$= \left|K^{-1}\int_0^P Kf(te^{i\phi})e^{-\nu t}Kg((P-t)e^{i\phi})e^{-\nu(P-t)}dt\right|$$
$$\leq K^{-1}\|f\|\|g\|\int_0^K d|t| = \|f\|\|g\| \quad (5.8)$$

$\square$

Note that $\mathcal{A}_{K;\nu} \subset L^1_\nu(\mathcal{E})$.
(6) Finally, we note that the space $\mathcal{A}_{K,\nu;0}(\mathcal{E}) = \{f \in \mathcal{A}_{K,\nu}(\mathcal{E}) : f(0) = 0\}$ is a closed subalgebra of $\mathcal{A}_{K,\nu}$.

**Remark 5.9** *In the spaces* $L^1_\nu$, $\mathcal{A}_{K;\nu}$, $\mathcal{A}_{K,\nu;0}$ *etc. we have, for a bounded function* $f$,

$$\|fg\| \leq \|g\| \max |f|$$

### 5.1a Spaces of sequences of functions

In Borel summing not simply series but transseries it is convenient to look at sequences of vector-valued functions belonging to one ore more of the spaces introduced before. We let

$$\mathbf{y} = \{\mathbf{y_k}\}_{\mathbf{k}\succ 0}\,; \quad \mathbf{k} \in \mathbb{Z}^m,\ \mathbf{y_k} \in \mathbb{C}^n \quad (5.10)$$

and

$$\mathbf{Y} = \{\mathbf{Y_k}\}_{\mathbf{k}\succ\mathbf{0}} \quad (5.11)$$

For instance if $m = 1$ we define

$$L^1_{\nu,\mu} = \{\mathbf{Y} \in (L^1_\nu)^{\mathbb{N}} : \sum_{k=1}^{\infty} \mu^{-k}\|\mathbf{Y}_k\|_\nu < \infty\} \quad (5.12)$$

We introduce the following convolution on $L^1_{\nu,\mu}$

$$(\mathbf{F}\ast\mathbf{G})_k = \sum_{j=1}^{n-1} \mathbf{F}_j * \mathbf{G}_{k-j} \quad (5.13)$$

**Exercise 5.14** *Show that*

$$\|\mathbf{F}\ast\mathbf{G}\|_{\nu,\mu} \leq \|\mathbf{F}\|_{\nu,\mu}\|\mathbf{G}\|_{\nu,\mu} \quad (5.15)$$

*and* $(L^1_{\nu,\mu}, +, \ast, \|\ \|_{\nu,\mu})$ *is a Banach algebra.*

## 5.2 Focusing spaces and algebras

An important property of the norms (1)–(4) and (6) in §5.1 is that for any $f$ we have $\|f\| \to 0$ as $\nu \to \infty$. For $L^1_\nu$ for instance this is an immediate consequence of dominated convergence.

A family of norms $\|\|_\nu$ depending on a parameter $\nu \in \mathbb{R}^+$ is **focusing** if for any $f$ with $\|f\|_{\nu_0} < \infty$ we have

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{5.16}$$

(We note that norms can only be focusing in Banach algebras *without* identity, since $\|I\| \le \|I\|\|I\|$ implies $\|I\| \ge 1$.) This feature, when present, is important since it allows us to choose $\nu$ such that nonlinear terms in an equation are arbitrarily small.

Let $\mathcal{V}$ be a linear space and $\{\|\|_\nu\}$ a family of norms satisfying (5.16). For each $\nu$ we define a Banach space $\mathcal{B}_\nu$ as the completion of $\{f \in \mathcal{V} : \|f\|_\nu < \infty\}$. Enlarging $\mathcal{V}$ if needed, we may assume that $\mathcal{B}_\nu \subset \mathcal{V}$. For $\alpha < \beta$, (5.16) shows $\mathcal{B}_\alpha$ is naturally embedded in $\mathcal{B}_\beta$[1] . Let $\mathcal{F} \subset \mathcal{V}$ be the projective limit of the $\mathcal{B}_\nu$. That is to say

$$\mathcal{F} := \bigcup_{\nu>0} \mathcal{B}_\nu \tag{5.17}$$

where a sequence is convergent if it converges in *some* $\mathcal{B}_\nu$. We call $\mathcal{F}$ a **focusing space**.

Consider now the case when $(\mathcal{B}_\nu, +, *, \|\|_\nu)$ are commutative Banach algebras. Then $\mathcal{F}$ inherits a structure of a commutative algebra, in which $*$ is continuous. We say that $(\mathcal{F}, *, \|\|_\nu)$ is a **focusing algebra**.

**Examples.** The spaces $\bigcup_{\nu>0} L^1_\nu$ and $\bigcup_{\nu>0} \mathcal{A}_{K;\nu;0}$ and $L^1_{\nu,\mu}$ are focusing algebras. The last space is focusing as $\nu \to \infty$ and/or $\mu \to \infty$.

An extension to distributions, very useful in studying singular convolution equations, is the space of staircase distributions $\mathcal{D}'_{m,\nu}$, see §5.13.

**Remark 5.18** The following result is immediate. Let $A, B$ be any sets and assume that the equation $f(x) = 0$ is well defined and has a unique solution $x_1$ in $A$, a unique solution $x_2$ in $B$ and a unique solution $x_3$ in $A \cap B$. Then $x_1 = x_2 = x_3 = x$. In particular, if $A \subset B$ then $x \in A \cap B$. This is useful when we want to show that one solution has a number of different properties: analyticity, boundedness, etc. and we do not want to complicate the norms. See *e.g.* Proposition 5.20 below.

---

[1] That is, we can naturally identify $\mathcal{B}_\alpha$ with a subset of $\mathcal{B}_\beta$ which is isomorphic to it.

## 5.3 Borel summation of the formal solutions to (4.54).

### 5.3a Borel summability of the asymptotic series solution

Since we have a Banach algebra structure in Borel plane, differential equations become effectively algebraic equations (with multiplication represented by *), much easier to deal with.

The analysis of (4.54) captures some important part of what is needed in a general setting. Its formal inverse Laplace is

$$-pY + Y = p + Y^{*3}; \ \Leftrightarrow Y = \frac{p}{1-p} + \frac{1}{1-p}Y^{*3} := \mathcal{N}(Y) \tag{5.19}$$

where $\mathcal{L}^{-1}y = Y$ and $Y^{*3} = Y * Y * Y$.

Let $[a,b] \in (0, 2\pi)$, and $S = \{p : \arg(p) \in (a,b)$, $S_K = \{p \in S : |p| < K\}$, $B = \{p : |p| < a < 1\}$.

**Proposition 5.20** *(i) For large enough $\nu$, Eq. (5.19) has a unique solution in the following spaces: $L^1_\nu(S), L^1_\nu(S_K), \mathcal{A}_{\nu,0}(S_K \cup B)$. (ii) There is a solution $Y$ which is analytic in $S \cup B$ and it is Laplace transformable along any direction in $S$. The Laplace transform is a solution of (4.54).*

**PROOF** The proof is the same for all these spaces, since they generate focusing algebras. Choose $\epsilon$ small enough. Then for large enough $\nu$ we have

$$\left\| \frac{p}{1-p} \right\|_\nu < \epsilon/2 \tag{5.21}$$

Let $\mathfrak{B}$ be the ball of radius $\epsilon$ in the norm $\nu$. Then if $F \in \mathfrak{B}$ we have

$$\|\mathcal{N}(F)\|_\nu \le \left\| \frac{p}{1-p} \right\|_\nu + \max \left| \frac{1}{p-1} \right| \|Y\|^3_\nu = \epsilon/2 + c\epsilon^3 \le \epsilon \tag{5.22}$$

if $\epsilon$ is small enough (that is, if $\nu$ is large). Furthermore, for large $\nu$, $\mathcal{N}$ is contractive in $\mathfrak{B}$ for we have, for small $\epsilon$,

$$\|\mathcal{N}(F_1) - \mathcal{N}(F_2)\|_\nu \le c\|F_1^{*3} - F_2^{*3}\|_\nu = c\|(F_1 - F_2) * (F_1^{*2} + F_1 * F_2 + F_2^{*2})\|_\nu$$
$$\le c\|(F_1 - F_2)\|_\nu (3\epsilon^2) < \epsilon \tag{5.23}$$

(ii) We have the following embeddings: $L^1_\nu(S) \subset L^1_\nu(S_K), \mathcal{A}_{\nu,0}(S_K \cup B) \subset L^1_\nu(S_K)$. Thus, by Remark 5.18, there exists a unique solution $Y$ of (5.19) which belongs to all these spaces.

Thus $Y$ is analytic in $S$ and in $L^1_\nu(S)$, in particular it is Laplace transformable. The Laplace transform is a solution of (4.54) as it is easy to check.

It also follows that the formal power series solution $\tilde{y}$ of (4.54) is Borel summable in any sector not containing $\mathbb{R}^+$, which is a Stokes line. We have, indeed, $\mathcal{B}\tilde{y} = Y$ (check!). ☐

## 5.3b Borel summation of the transseries solution

If we substitute

$$\tilde{y} = \tilde{y}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{y}_k \tag{5.24}$$

in (4.54) and equate the coefficients of $e^{-kx}$ we get the system of equations

$$\tilde{y}_k' + (1 - k - 3\tilde{y}_0^2)\tilde{y}_k = 3\tilde{y}_0 \sum_{j=1}^{k-1} \tilde{y}_j \tilde{y}_{k-j} + \sum_{j_1+j_2+j_3=k; j_i \geq 1} \tilde{y}_{j_1} \tilde{y}_{j_2} \tilde{y}_{j_3} \tag{5.25}$$

The equation for $y_1$ is linear and homogeneous:

$$\tilde{y}_1' = 3\tilde{y}_0^2 \tilde{y}_1 \tag{5.26}$$

Thus

$$\tilde{y}_1 = Ce^{\tilde{s}}; \quad \tilde{s} := \int_{\infty}^{x} 3\tilde{y}_0{}^2(t)dt \tag{5.27}$$

Since $\tilde{s} = O(x^{-3})$ is the product of Borel summable series (in $\mathbb{C} \setminus \mathbb{R}^+$), then, by Proposition 4.109 $e^{\tilde{s}}$ is Borel summable in $\mathbb{C} \setminus \mathbb{R}^+$. We note that $\tilde{y}_1 = 1 + o(1)$ and we cannot take the inverse Laplace transform of $\tilde{y}_1$ directly. It is convenient to make the substitution $\tilde{y}_k = x^k \tilde{\varphi}_k$. We get

$$\tilde{\varphi}_k' + (1 - k - 3\tilde{\varphi}_0^2 + kx^{-1})\tilde{\varphi}_k = 3\tilde{\varphi}_0 \sum_{j=1}^{k-1} \tilde{\varphi}_j \tilde{\varphi}_{k-j} + \sum_{j_1+j_2+j_3=k; j_i \geq 1} \tilde{\varphi}_{j_1} \tilde{\varphi}_{j_2} \tilde{\varphi}_{j_3} \tag{5.28}$$

or after Borel transform

$$-p\mathbf{\Phi} + (1 - \hat{k})\mathbf{\Phi} = -\hat{k} * \mathbf{\Phi} + 3Y_0^{*2} * \mathbf{\Phi} + 3Y_0 * \mathbf{\Phi} \boldsymbol{*} \mathbf{\Phi} + \mathbf{\Phi} \boldsymbol{*} \mathbf{\Phi} \boldsymbol{*} \mathbf{\Phi} \tag{5.29}$$

where $\mathbf{\Phi} = \{\Phi_j\}_{j \in \mathbb{N}}$, $(\hat{k}\mathbf{\Phi})_k = \Phi_k$ and $(F * \mathbf{G})_k := F * G_k$.

Here we reinterpret $\Phi_0, \Phi_1$ which have already been analyzed, as given, known functions (after rearrange (5.29) so that the terms containing $\varphi_1$ are explicitly written out). Consider the new equation on $L^1_{\mu,\nu;1} \subset L^1_{\mu,\nu}$, the subspace of sequences $\{\Phi_j\}_{j \in \mathbb{N}}$, $\Phi_1 = 0$ (and similar subspaces of other focusing algebras). They are focusing spaces too (check!).

**Proposition 5.30** *For any $\mu$, if $\nu$ is large enough, (5.29) is contractive. Thus (5.29) has a unique solution in this space. Similarly, it has a unique solution in $L^1_{\nu,\mu;1}(S), \mathcal{A}_{\nu,\mu;1}(S_K)$ for any $S$ and $S_K$ as in Proposition 5.20. Thus there is a $\nu$ large enough so that for all $k$*

$$\varphi_k(x) = \int_0^{\infty e^{-i \arg(x)}} e^{-xp} \Phi_k(p) dp \tag{5.31}$$

*exist for* $|x| > \nu$*. The functions* $\varphi_k(x) = \varphi_k(x)^+$ *are analytic in* $x$*, for* $\arg(x) \in (-\pi/2, 2\pi + \pi/2)$*. Similarly,* $\varphi_k(x) = \varphi_k(x)^-$ *are analytic in* $x$*, for* $\arg(x) \in (-2\pi - \pi/2, \pi/2)$*.*

*(ii) The function series*

$$\sum_{k=0}^{\infty} C_+^k e^{-kx} x^k \varphi_k^+(x) \tag{5.32}$$

*and*

$$\sum_{k=0}^{\infty} C_-^k e^{-kx} x^k \varphi_k^-(x) \tag{5.33}$$

*converge for sufficiently large* $\operatorname{Re} x$*,* $\arg(x) \in (-\pi/2, \pi/2)$ *and solve (4.54).*

**Note.** The solution cannot be written in the form (5.32) or (5.33) in a sector of opening more than $\pi$ centered on $\mathbb{R}^+$ because the exponentials would become large and convergence is not ensured anymore. This, generically, implies blow-up of the actual solutions, see §6.

**Exercise 5.34 (*)** *Prove Proposition 5.30.*

**Proposition 5.35** **Any** *Solution of (4.54) which is* $o(1)$ *as* $x \to +\infty$ *can be written in the form (5.32) or, equally well, in the form (5.32).*

**PROOF** Let $y_0 := y^+$ be the solution of (4.54) of the form (5.32) with $C = 0$. Let $y$ be another solution which is $o(1)$ as $x \to +\infty$ and let $\delta = y - y^+$. We have

$$\delta' = -\delta + 3y_0^2\delta + 3y_0\delta^2 + \delta^3 \tag{5.36}$$

or

$$\frac{\delta'}{\delta} = -1 + 3y_0^2 + 3y_0\delta + \delta^2 = -1 + o(1) \tag{5.37}$$

Thus (since we can integrate asymptotic relations),

$$\ln \delta = -x + o(x) \tag{5.38}$$

and thus

$$\delta = e^{-x+o(x)}$$

Returning to (5.37), we see that

$$\frac{\delta'}{\delta} = -1 + 3y_0^2 + 3y_0\delta + \delta^2 = -1 + O(1/x^2) \tag{5.39}$$

or

$$\delta = Ce^{-x}(1 + o(1)) \tag{5.40}$$

We then take $\delta = C_1 e^{-x+s}$ and obtain

$$s' = 3y_0^2 + 3y_0 e^{-x+s} + e^{-2x+2s} \tag{5.41}$$

where $s$ is small, or,

$$s = \int_{\infty}^{x} \left( 3y_0^2(t) + 3y_0(t)e^{-x+s(t)} + e^{-2t+2s(t)} \right) dt \tag{5.42}$$

Eq. (5.42) is contractive in the space of functions $s : [\nu, \infty) \mapsto \mathbb{C}$ in the sup norm. The solution of this equation is then unique. But $s = \ln(y_1 - y^+) + x$ where $y_1$ is the solution of the form (5.32) with $C = 1$ is already a solution of (5.42), so they must coincide. $\square$

### 5.3c Analytic structure along $\mathbb{R}^+$

The approach sketched in this section is simple, but of limited scope as it relies substantially on the ODE origin of the convolution equations.

A different, complete proof, that uses the differential equation only minimally is given in §5.11.

*

By Proposition 5.20, $Y = Y_0$ is analytic in any region of the form $B \cup S_K$. We now sketch a proof that $Y_0$ has analytic continuation along curves that do not pass through the integers.

For this purpose we use (5.33) and (5.32) in order to derive the behavior of $Y$. It is a way of exploiting what Écalle has discovered in more generality, *bridge equations.*

We start with exploring a relatively trivial, nongeneric possibility, namely that $y^+ = y^- =: y_0$. (This is not the case for our equation, though we will not prove it here; we still analyze this case since it may occur in other equations.)

$$y^{\pm} = \int_0^{\infty e^{\pm i\epsilon}} Y(p)e^{-px} dp = y_0 \tag{5.43}$$

we have $y \sim \tilde{y}_0$ in a sector of arbitrarily large opening. By inverse Laplace transform arguments, $Y$ is analytic in an arbitrarily large sector in $\mathbb{C} \setminus \{0\}$. On the other hand, we already know that $Y$ is analytic at the origin, and it is thus entire, of exponential order at most one. Then, $\tilde{y}_0$ converges.

**Exercise 5.44** *Complete the details in the argument above.*

We now consider the generic case $y^+ \neq y^-$. Then there exists $S \neq 0$ so that

$$y^+ = \int_0^{\infty e^{i\epsilon}} e^{-px} Y(p) dp = y^- + \sum_{k=1}^{\infty} S^k e^{-kx} x^k \varphi_k^-(x) \tag{5.45}$$

Thus

$$\int_{\infty e^{-i\epsilon}}^{\infty e^{+i\epsilon}} e^{-px} Y(p) dp = \int_1^{\infty} e^{-px}(Y^+(p) - Y^-(p)) dp = \sum_{k=1}^{\infty} S^k e^{-kx} x^k \varphi_k^-(x) \tag{5.46}$$

In particular, we have

$$\frac{1}{x}\int_1^\infty e^{-px}(Y^+(p)-Y^-(p))dp = Se^{-x}\int_0^{\infty e^{-i\epsilon}} e^{-px}\varphi_1(p)dp + O(x^2e^{-2x})$$
$$= S\int_1^{\infty e^{-i\epsilon}} e^{-px}\varphi_1(p-1)dp + O(x^2e^{-2x}) \quad (5.47)$$

Then, by Proposition 2.21, $\int_0^p Y^+ = \int_0^p Y^- + S\varphi_1(p-1)$ on $(1,2)$. (It can be checked that $\int Y$ has lateral limits on $(1,2)$, by looking at the convolution equation in a focusing space of functions continuous up to the boundary.)

Since $\varphi_1$ is continuous, this is means $\int_0^p Y^+ = S\varphi_1(p-1) + \int_0^p Y^-$ or $Y^+ = Y^- + SY_1(p-1)$, or yet, $Y^+(1+s) = Y^-(1+s) + SY_1(s)$ everywhere in the right half $s$ plane where $Y^- - Y_1$ is analytic, in particular in the fourth quadrant. Thus the analytic continuation of $Y$ from the upper half plane along a curve passing between 1 and 2 exists in the lower half plane; it equals the continuation of two functions along a straight line not crossing any singularities. The proof proceeds by induction, reducing the number of crossings at the expense of using more of the functions $Y_2, Y_3$, *etc.*.

This analysis can be adapted to general *differential* equations, and it allows for finding the resurgence structure (singularities in $p$) by constructing and solving Riemann-Hilbert problems, in the spirit above.

---

## 5.4 General setting

By relatively simple algebraic transformations a higher order differential equation can be transformed into a first order vectorial equation (differential system) and vice-versa [15]. The vectorial form has some technical advantages.

We consider the differential system

$$\mathbf{y}' = \mathbf{f}(x,\mathbf{y}) \qquad \mathbf{y}\in\mathbb{C}^n \quad (5.48)$$

under the following *assumptions:*

(a1) The function $\mathbf{f}$ is analytic at $(\infty, 0)$.

(a2) A condition slightly weaker than nonresonance (see §5.6a ) holds: for any half plane $\mathbb{H}$ in $\mathbb{C}$, the eigenvalues $\lambda_i$ of the linearization

$$\hat{\Lambda} := -\left(\frac{\partial f_i}{\partial y_j}(\infty,0)\right)_{i,j=1,2,\dots n} \quad (5.49)$$

lying in $\mathbb{H}$ are linearly independent over $\mathbb{Z}$. In particular, all eigenvalues are distinct and none of them is zero.

Writing out explicitly a few terms in the expansion of $\mathbf{f}$, relevant to leading order asymptotics, we get

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x, \mathbf{y}) \tag{5.50}$$

---

## 5.5 Normalization procedures: an example

Many equations that are not presented in the form (5.50) can be brought to this form by changes of variables. The key idea for so doing in a systematic way is to calculate the transseries solutions of the equation, find the transformations which bring the transseries to the normal form (5.82), and then apply the same transformations to the differential equation. The first part of the analysis need not be rigorous, as the conclusions are made rigorous in the steps that follow it.

We illustrate this on a simple equation

$$u' = u^3 - t \tag{5.51}$$

in the limit $t \to +\infty$. This is not of the form (5.50) because $g(u, t) = u^3 - t$ is not analytic in $t$ *at* $t = \infty$. This can be however remedied in the way described.

As we have already seen before, dominant balance for large $t$ requires writing the equation (5.51) in the form

$$u = (t + u')^{1/3} \tag{5.52}$$

and we have $u' \ll t$. Three branches of the cubic root are possible and are investigated similarly, but we aim here merely at illustration and choose the simplest. Iterating (5.52) in the usual way, we are lead to a formal series solution in the form

$$\tilde{u} = t^{1/3} + \frac{1}{9}t^{-4/3} + \dots = t^{1/3} \sum_{k=0}^{\infty} \frac{\tilde{u}_k}{t^{5k/3}} \tag{5.53}$$

To find the full transseries we now substitute $u = \tilde{u} + \delta$ in (5.51) and keep the dominant terms. We get

$$\frac{\delta'}{\delta} = \left(\frac{9}{5}t^{5/3} + \frac{2}{3}\ln t\right)'$$

from which it follows that

$$\delta = C t^{2/3} e^{\frac{9}{5} t^{5/3}} \tag{5.54}$$

Since exponents in a normalized transseries solution are linear in the (normalized) variable, the critical time is $t^{5/3}$. We take $x = (At)^{5/3}$; the formal power series (5.53) takes the form

$$\tilde{u} = x^{1/5} \sum_{k=0}^{\infty} \frac{\tilde{u}_k}{x^k} \tag{5.55}$$

But the desired form is $\sum_{k=0}^{\infty} \frac{b_k}{x^k}$. Thus the appropriate dependent variable is $h = B x^{1/5} u$. The choice of $A$ and $B$ is made so as to simplify the final analysis. We choose $A = -B^2/5, 15/B^5 = -1/9$ and we are led to the equation

$$h' + \frac{1}{5x} h + 3h^3 - \frac{1}{9} = 0 \tag{5.56}$$

which is analytic at infinity, as expected. The only remaining transformation is to subtract out a few terms out of $h$, to make the nonlinearity formally small. This is done by calculating, again by dominant balance, the first two terms in the $1/x$ power expansion of $h$, namely $1/3 - x^{-1}/15$ and subtracting them out of $h$, i.e., changing to the new dependent variable $y = h - 1/3 + x^{-1}/15$. This yields

$$y' \;=\; -y \;+\; \frac{1}{5x}\, y \;+\; g(y, x^{-1}) \tag{5.57}$$

where

$$g(y, x^{-1}) = -3(y^2 + y^3) + \frac{3y^2}{5x} - \frac{1}{15x^2} - \frac{y}{25x^2} + \frac{1}{3^2 5^3 x^3} \tag{5.58}$$

## 5.6 Further discussion of assumptions and normalization

Under the assumptions (a1) and (a2), $\hat{\Lambda}$ in (5.50) can be diagonalized by a linear change of the dependent variable $\mathbf{y}$. It can be checked that by a further substitution of the form $\mathbf{y}_1 = (I + x^{-1}\hat{V})\mathbf{y}$, the new matrix $\hat{A}$ can be arranged to be diagonal. No assumptions on $\hat{A}$ are needed in this second step. See also [54]. Thus, without loss of generality we can suppose that the system is already presented in *prepared* form, meaning:

(n1) $\hat{\Lambda} = \mathrm{diag}(\lambda_i)$ and
(n2) $\hat{A} = \mathrm{diag}(\alpha_i)$

For convenience, we rescale $x$ and reorder the components of $\mathbf{y}$ so that (n3) $\lambda_1 = 1$, and, with $\phi_i = \arg(\lambda_i)$, we have $\phi_i \le \phi_j$ if $i < j$. To simplify the notation, we formulate some of our results relative to $\lambda_1 = 1$; they can be easily adapted to any other eigenvalue.

A substitution of the form $\mathbf{y} = \mathbf{y}_1 x^{-N}$ for some $N \ge 0$ ensures that

(n4) $\mathrm{Re}\,(\alpha_j) > 0,\ j = 1, 2, \ldots, n$.

**Note 5.59** The case $\mathrm{Re}\,(\alpha_j) < 0$ treated in [24] and §5.11 is simpler but it cannot be arranged by simple transformations, while the general case (n4) is dealt with in [22]; in both papers the notation is notation is slightly different:

$$\hat{B} := -\hat{A} \text{ and } \quad \boldsymbol{\beta} := -\boldsymbol{\alpha} \tag{5.60}$$

Finally, through a transformation of the form $\mathbf{y} \leftrightarrow \mathbf{y} - \sum_{k=1}^{M} \mathbf{a}_k x^{-k}$ and $\mathbf{y} \leftrightarrow (1 + \hat{A}_1 x^{-1} + \cdots + \hat{A}_{M+1} x^{-M-1})\mathbf{y}$ we arrange that [2]

(n5) $\mathbf{f}_0 = O(x^{-M-1})$ and $\mathbf{g}(x, \mathbf{y}) = O(\mathbf{y}^2, x^{-M-1}\mathbf{y})$. We choose $M > 1 + \max_i \mathrm{Re}\,(\alpha_i)$ (cf. (n2)).

### 5.6a Nonresonance

(1) $\lambda_i$, $i = 1, ..., n_1$ are assumed $\mathbb{Z}$-linearly independent for any $d$. (2) Let $\theta \in [0, 2\pi)$ and $\tilde{\boldsymbol{\lambda}} = (\lambda_{i_1}, ..., \lambda_{i_p})$ where $\left|\arg \lambda_{i_j} - \theta\right| \in (-\pi/2, \pi/2)$ (those eigenvalues contained in the open half-plane $\mathbb{H}_\theta$ centered along $\mathrm{e}^{\mathrm{i}\theta}$). We require that for any $\theta$ the complex numbers in the *finite* set

$$N_{i;\mathbf{k}} := \{\tilde{\lambda}_i - \mathbf{k} \cdot \tilde{\boldsymbol{\lambda}} \in \mathbb{H}_\theta : \mathbf{k} \in \mathbb{N}^p,\ i = 1, ..., p\} \tag{5.61}$$

have *distinct* directions $d_{i;\mathbf{k}}$. The directions of $\arg(-N_{i;\mathbf{k}})$ are Stokes rays for the transseries of the ODE.

It can be easily seen that the set of $\boldsymbol{\lambda}$ which satisfy (1) and (2) has full measure; see also [22].

Without the nonresonance assumption, the main series $\tilde{\mathbf{y}}_0$ (but not the full transseries) is shown to be *multisummable* in [4].

## 5.7 Overview of results

Known results for this type of equations are informally summarized as follows.

*i All* $\tilde{\mathbf{y}}_{\mathbf{k}}$ *are* BE *summable* [3] *in a common half plane, of the form* $\mathbb{H}_0 = \{x : \mathrm{Re}\,(x) > x_0\}$.

---

[2] This latter transformation is omitted in [22].

[3] Due to special properties of this ODE setting, the process used here is simpler than, but equivalent to, BE summation.

*ii* *The Borel sums* $\mathbf{y}_{\mathbf{k}} = \mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}$ *are analytic in* $\mathbb{H}_0$.

*iii* *There exists a constant* $\mathbf{c}$ *independent of* $\mathbf{k}$ *so that* $\sup_{x\in\mathbb{H}_0} |\mathbf{y}_{\mathbf{k}}| \le \mathbf{c}^{\mathbf{k}}$. *Thus, the new series,*

$$\mathbf{y} = \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x) \tag{5.62}$$

*is convergent for any* $\mathbf{C}$ *for which the corresponding expansion (5.82) is a transseries, in a region given by the condition* $|C_i e^{-\lambda_i x} x^{\alpha_i}| < c_i^{-1}$ *(remember that* $C_i$ *is zero if* $|e^{-\lambda_i x}|$ *is not small).*

*iv* *The function* $\mathbf{y}$ *obtained in this way is a solution of the differential equation (5.50).*

*v* *Any solution of the differential equation (5.50) which tends to zero in some direction* $d$ *can be written in the form (5.62) for a unique* $\mathbf{C}$, *this constant depending usually on the sector where* $d$ *is (Stokes phenomenon).*

*vi* *The* BE *summation operator* $\mathcal{LB}$ *is the usual Borel summation in any direction* $d$ *of* $x$ *which is not a Stokes line. However* $\mathcal{LB}$ *is still an isomorphism, whether* $d$ *is a Stokes direction or not.*

## 5.8 Further notation

### 5.8a Regions in the $p$ plane

We use the convention $\mathbb{N} \ni 0$. Let

$$\mathcal{W} = \{p \in \mathbb{C} : p \neq k\lambda_i\,, \forall k \in \mathbb{N}, i = 1, 2, \ldots, n\} \tag{5.63}$$

The Stokes directions of $\tilde{\mathbf{y}}_0$ are $d_j = \{p : \arg(p) = -\phi_j\}, j = 1, 2, \ldots, n$. The rays $\{p : \arg(p) = \phi_j\}, j = 1, 2, \ldots, n$ are singular for $\mathbf{Y}_0$, and we simply speak of them as singular rays.

We construct the surface $\mathcal{R}$, consisting of homotopy classes [4] of smooth curves in $\mathcal{W}$ starting at the origin, moving away from it, and crossing at most one singular line, at most once (see Fig. 1):

---

[4] classes of curves that can be continuously deformed into each other without crossing points outside $\mathcal{W}$.

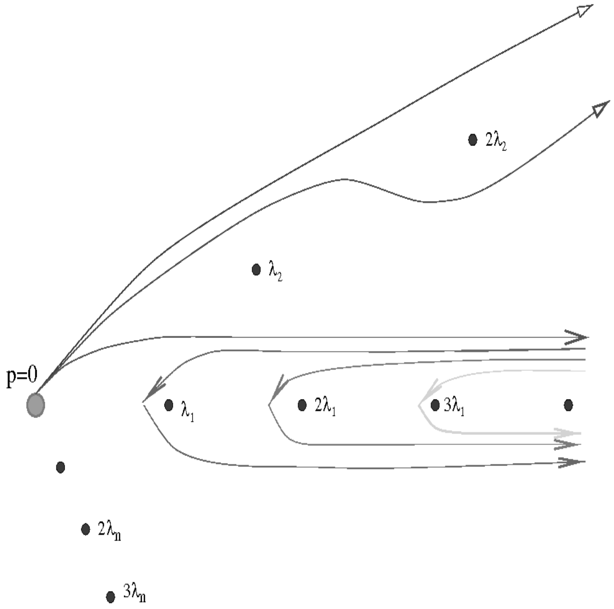

**FIGURE 5.1**: The paths near $\lambda_1$ relate to the medianized averages.

$$\mathcal{R} := \Big\{\gamma : (0,1) \mapsto \mathcal{W} : \ \gamma(0_+) = 0; \ \frac{\mathrm{d}}{\mathrm{d}t}|\gamma(t)| > 0; \ \arg(\gamma(t)) \text{ monotonic}\Big\}$$

(modulo homotopies) (5.64)

Define $\mathcal{R}_1 \subset \mathcal{R}$ by (5.64) with the supplementary restriction $\arg(\gamma) \in (\psi_n - 2\pi, \psi_2)$ where $\psi_n = \max\{-\pi/2, \phi_n - 2\pi\}$ and $\psi_2 = \min\{\pi/2, \phi_2\}$.

$\mathcal{R}_1$ may be viewed as the part of $\mathcal{R}$ above a sector containing $\mathbb{R}^+$.

Similarly we let $\mathcal{R}'_1 \subset \mathcal{R}_1$ with the restriction that the curves $\gamma$ do not cross any singular direction *other* than $\mathbb{R}^+$, and we let $\psi_\pm = \pm\max(\pm\arg\gamma)$ with $\gamma \in \mathcal{R}'_1$.

By symmetry (renumbering the directions) it suffices to analyze the singularity structure of $\mathbf{Y}_0$ in $\mathcal{R}_1$ only. However, (c1) breaks this symmetry for $\mathbf{k} \neq 0$ and the properties of these $\mathbf{Y}_\mathbf{k}$ will be analyzed along some other directions as well.

## 5.8b Ordering on $\mathbb{N}^{n_1}$

The notation is essentially the one already used for multiseries and transseries:

We write $\mathbf{k} \succeq \mathbf{k}'$ if $k_i \geq k'_i$ for all $i$ and $\mathbf{k} \succ \mathbf{k}'$ if $\mathbf{k} \succeq \mathbf{k}'$ and $\mathbf{k} \neq \mathbf{k}'$. The relation $\succ$ is a well ordering on $\mathbb{N}^{n_1}$. We let $\mathbf{e}_j$ be the unit vector in the $j^{\text{th}}$ direction in $\mathbb{N}^{n_1}$.

### 5.8c Analytic continuations between singularities

By $AC_\gamma(f)$ we denote the analytic continuation of $f$ along a curve $\gamma$.

For the analytic continuations near a singular line $d_{i;\mathbf{k}}$ the notation is similar to Écalle's:

$f^-$ is the branch of $f$ along a path $\gamma$ with $\arg(\gamma) = \phi_i - 0$, while $f^{-j+}$ denotes the branch along a path that crosses the singular line between $j\lambda_i$ and $(j+1)\lambda_i$, from right to left (see also [24]).

We write $\mathcal{P}f$ for $\int_0^p f(s)\mathrm{d}s$ and $\mathcal{P}_\gamma f$ if integration is along the curve $\gamma$.

---

## 5.9 Analytic properties of $\mathbf{Y_k}$ and resurgence

We let $m_i = 2 + \lfloor \operatorname{Re}\alpha_i \rfloor$ and $\alpha_i' = m_i - \alpha_i$.

**Theorem 5.65** *(i)* $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$ *is analytic in* $\mathcal{R}\cup\{0\}$.

*The singularities of* $\mathbf{Y}_0$ *(which are contained in the set* $\{l\lambda_j : l \in \mathbb{N}^+, j = 1, 2, \ldots, n\}$*) are described as follows. For* $l \in \mathbb{N}^+$ *and small* $z$*, using the notations explained in §5.8a we have*

$$\mathbf{Y}_0^{\pm}(z + l\lambda_j) = \pm\Big[(\pm S_j)^l (\ln z)^{0,1}\mathbf{Y}_{l\mathbf{e}_j}(z)\Big]^{(lm_j)} + \mathbf{B}_{lj}(z) =$$

$$\Big[z^{l\alpha_j'-1}(\ln z)^{0,1}\,\mathbf{A}_{lj}(z)\Big]^{(lm_j)} + \mathbf{B}_{lj}(z)\ (l = 1, 2, \ldots) \qquad (5.66)$$

*where the power of* $\ln z$ *is one iff* $l\alpha_j \in \mathbb{Z}$*, and* $\mathbf{A}_{lj}, \mathbf{B}_{lj}$ *are analytic for small* $z$*. The functions* $\mathbf{Y_k}$ *are, exceptionally, analytic at* $p = l\lambda_j$*,* $l \in \mathbb{N}^+$*, iff the Stokes constants vanish, i.e.*

$$S_j = r_j\Gamma(\alpha_j')\left(\mathbf{A}_{1,j}\right)_j(0) = 0 \qquad (5.67)$$

*where* $r_j = 1 - \mathrm{e}^{2\pi i(\alpha_j'-1)}$ *if* $l\alpha_j \notin \mathbb{Z}$ *and* $r_j = -2\pi i$ *otherwise.*

*

**Analyticity and resurgence of $\mathbf{Y_k}$, $\mathbf{k} \succ \mathbf{0}$.**

*(ii)* $\mathbf{Y_k} = \mathcal{B}\tilde{\mathbf{y}}_\mathbf{k}$, $|\mathbf{k}| > 1$*, are analytic in* $\mathcal{R}\backslash S_\mathbf{k}$*, where*

$$S_\mathbf{k} = \{-\mathbf{k}'\cdot\boldsymbol{\lambda} + \lambda_i : \mathbf{k}' \le \mathbf{k}, 1 \le i \le n\} \qquad (5.68)$$

*(*$S_\mathbf{k}$ *turns out to be a set of actual singularities, generically).*

*For* $l \in \mathbb{N}$ *and* $p$ *near* $l\lambda_j$*,* $j = 1, 2, \ldots, n$ *there exist* $\mathbf{A} = \mathbf{A}_{\mathbf{k}jl}$ *and* $\mathbf{B} = \mathbf{B}_{\mathbf{k}jl}$ *analytic at zero so that for small* $z$ *we have*

$$\mathbf{Y}_{\mathbf{k}}^{\pm}(z+l\lambda_j)=\pm\Big[(\pm S_j)^l\binom{k_j+l}{l}(\ln z)^{0,1}\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(z)\Big]^{(lm_j)}+\mathbf{B}_{\mathbf{k}lj}(z)=$$
$$\Big[z^{\mathbf{k}\cdot\boldsymbol{\alpha}'+l\alpha'_j-1}(\ln z)^{0,1}\,\mathbf{A}_{\mathbf{k}lj}(z)\Big]^{(lm_j)}+\mathbf{B}_{\mathbf{k}lj}(z)\ (l=0,1,2,\ldots)\tag{5.69}$$

*where the power of* $\ln z$ *is* $0$ *iff* $l=0$ *or* $\mathbf{k}\cdot\boldsymbol{\alpha}+l\alpha_j\notin\mathbb{Z}$ *and* $\mathbf{A}_{\mathbf{k}0j}=\mathbf{e}_j/\Gamma(\alpha'_j)$. *Near* $p\in\{\lambda_i-\mathbf{k}'\cdot\boldsymbol{\lambda}:0\prec\mathbf{k}'\le\mathbf{k}\}$, *(where* $\mathbf{Y}_0$ *is analytic)* $\mathbf{Y}_{\mathbf{k}}$, $\mathbf{k}\ne 0$ *have convergent Puiseux series.*

### 5.9 .1 Properties of the analytic continuation along singular rays

In the following formulas we make the convention $\mathbf{Y}_{\mathbf{k}}(p-j)=0$ for $p<j$.

**Theorem 5.70** *i) For all* $\mathbf{k}$ *and* $\mathrm{Re}\,(p)>j, \mathrm{Im}\,(p)>0$ *as well as in distributions (more precisely in* $\mathcal{D}'_{m,\nu}$*; see §5.13 and [22]) we have*

$$\mathbf{Y}_{\mathbf{k}}^{\pm j\mp}(p)-\mathbf{Y}_{\mathbf{k}}^{\pm(j-1)\mp}(p)=(\pm S_1)^j\binom{k_1+j}{j}\left(\mathbf{Y}_{\mathbf{k}+j\mathbf{e}_1}^{\mp}(p-j)\right)^{(mj)}\tag{5.71}$$

*and also,*

$$\mathbf{Y}_{\mathbf{k}}^{\pm}=\mathbf{Y}_{\mathbf{k}}^{\mp}+\sum_{j\ge 1}\binom{j+k_1}{k_1}(\pm S_1)^j(\mathbf{Y}_{\mathbf{k}+j\mathbf{e}_1}^{\mp}(p-j))^{(mj)}\tag{5.72}$$

**Remark 5.73 (Resurgence)** The fact that the singular part of $\mathbf{Y}_{\mathbf{k}}(p+l\lambda_j)$ in (5.66) and (5.69) is a *multiple* of $\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(p)$ is the effect of resurgence and provides a way of determining the $\mathbf{Y}_{\mathbf{k}}$ given $\mathbf{Y}_0$ provided the $S_j$ are nonzero. Since, generically, the $S_j$ are nonzero, given one formal solution, (generically) an $n$ parameter family of solutions can be constructed out of it, without using (5.50) in the process; the differential equation itself is then recoverable.

*

**Averaging.** We extend $\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$ along $d_{j,\mathbf{k}}$ by the weighted average of analytic continuations

$$\mathbf{Y}_{\mathbf{k}}^{\gamma}:=\mathbf{Y}_{\mathbf{k}}^{+}+\sum_{j=1}^{\infty}\gamma^j\left(\mathbf{Y}_{\mathbf{k}}^{-}-\mathbf{Y}_{\mathbf{k}}^{-(j-1)+}\right)\tag{5.74}$$

**Proposition 5.75** *Relation (5.74), with* $\mathrm{Re}\,\gamma=1/2$ *gives the most general reality preserving, linear operator mapping formal power series solutions of (5.50) to solutions of (5.88) in* $\mathcal{D}'_{m,\nu}$.

This follows from Proposition 23 in [22] and Theorem 5.70 below.

For the choice $\gamma = 1/2$ the expression (5.74) coincides with the one in which + and − are interchanged (Proposition 34 in [22]), and this yields a reality-preserving summation.

We call $\mathcal{B}_{\frac{1}{2}}\tilde{\mathbf{y}}_{\mathbf{k}} = \mathbf{Y}_{\mathbf{k}}^{ba}$ the balanced average of $\mathbf{Y}_{\mathbf{k}}$, and it is the restriction of the median average to generic systems of ODEs, becoming much simpler because of the extra structure inherited from the differential equation. It is *not* multiplicative in general, that is outside the ODE context, while the median average *is*.

**Remark 5.76** Clearly, if $\mathbf{Y}_{\mathbf{k}}$ is analytic along $d_{j,\mathbf{k}}$, then the terms in the infinite sum vanish and $\mathbf{Y}_{\mathbf{k}}^{\gamma} = \mathbf{Y}_{\mathbf{k}}$. For any $\mathbf{Y}_{\mathbf{k}}$ this is the case for all directions apart from the finitely many singular ones.

It follows from (5.74) and Theorem 5.77 below that the Laplace integral of $\mathbf{Y}_{\mathbf{k}}^{\gamma}$ along $\mathbb{R}^+$ can be deformed into contours as those depicted in Fig. 1, with weight $-(-\gamma)^k$ for a contour turning around $k\lambda_1$.

In addition to symmetry (the balanced average equals the half sum of the upper and lower continuations on $(0, 2\lambda_j)$), an asymptotic property uniquely picks $C = 1/2$. Namely, for $C = 1/2$ alone are the $\mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}$ always *summable to the least term*, see [22], [18].

### 5.9a Summability of the transseries

The following is an abbreviated form of a theorem in [22]. The statements are more precise there and more details are given about the functions, but to present them here would require introducing many new notations.

For clarity we again specialize to a sector in $\mathbb{H}$ in $x$ containing $\lambda_1 = 1$ in which (c1) holds (and for $p$ in the associated domain $\mathcal{R}'_1$), but $\lambda_1$ plays no special role as discussed in the introduction.

**Theorem 5.77** – *The limits in distributions of* $\mathbf{Y}_{\mathbf{k}}^{+n-}(p)$ *and* $\mathbf{Y}_{\mathbf{k}}^{-n+}(p)$ *on* $\mathbb{R}^+$ *exist for all $k$ and $n$.*

– *There is an $x_0$ large enough so that for* $\mathrm{Re}\,(xp/|p|) > x_0$ *all the Laplace transforms in distributions,* $\int_S e^{-xp}\mathbf{Y}_{\mathbf{k}}^{+n-}(p)dp$ *exist and are bounded by* $\mu^{\mathbf{k}}$ *for some* $\mu > 0$.

–*The series*

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^+ + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{\mathbf{k}\cdot\boldsymbol{\alpha}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^+ \tag{5.78}$$

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^- + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{\mathbf{k}\cdot\boldsymbol{\alpha}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^- \tag{5.79}$$

*are convergent and either of them provides the most general decaying solution of (5.50) along $S$ (so there is a multiplicity of representation).*

*–The Laplace transformed average*

$$\mathcal{L}\Big[\mathbf{Y}_{\mathbf{k}}^{+}+\sum_{j=1}^{\infty}\gamma^{j}\left(\mathbf{Y}_{\mathbf{k}}^{-}-\mathbf{Y}_{\mathbf{k}}^{-(j-1)+}\right)\Big] \tag{5.80}$$

*is the most general Borel sum of* $\tilde{\mathbf{y}}_{\mathbf{k}}$ *that commutes with complex conjugation and multiplication.*

Of special interest are the cases $\gamma = 1/2$, discussed below, and also $\gamma = 0, 1$ which give:

$$\mathbf{y} = \mathcal{L}\mathbf{Y}_0^{\pm} + \sum_{|\mathbf{k}|>0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda}x} x^{\mathbf{k}\cdot\boldsymbol{\alpha}} \mathcal{L}\mathbf{Y}_{\mathbf{k}}^{\pm} \tag{5.81}$$

*Formal solutions.* We we allow for complex valued transseries,

$$\mathbf{y} \;=\; \tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0;|\mathbf{k}|>0} C_1^{k_1}\cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\boldsymbol{\alpha}} \tilde{\mathbf{y}}_{\mathbf{k}} \qquad (x \;\to\; \infty, \arg(x)\;\text{fixed}) \tag{5.82}$$

under the assumption that $\mathrm{Re}\,(-\mathbf{k}\cdot\boldsymbol{\lambda}\; x) < 0$. That is, agreeing to omit the terms with $C_i = 0$, with $x$ in $d$, we should have

(c1) $\arg(x) + \phi_i \in (-\pi/2, \pi/2)$ for all $i$ such that $C_i \neq 0$. In other words, $C_i \neq 0$ implies that $\lambda_i$ lies in a half-plane centered on $\overline{d}$, the complex conjugate direction to $d$.

*

The series $\tilde{\mathbf{y}}_0$ is a formal solution of (5.50) while, for $\mathbf{k} \neq 0$, $\tilde{\mathbf{y}}_{\mathbf{k}}$ satisfy a hierarchy of linear differential equations [54] (see also § 5.12b for further details). In

$$\mathbf{y} = \mathcal{L}\mathcal{B}\tilde{\mathbf{y}}_0 + \sum_{\mathbf{k}\geq 0;|\mathbf{k}|>0} C_1^{k_1}\cdots C_n^{k_n} \mathrm{e}^{-(\mathbf{k}\cdot\boldsymbol{\lambda})x} x^{\mathbf{k}\cdot\boldsymbol{\alpha}} \mathcal{L}\mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}} \quad (x \to \infty, \arg(x)\;\text{fixed}) \tag{5.83}$$

given $\mathbf{y}$, the value of $C_i$ can only change (and it usually does) when $\arg(x) + \arg(\lambda_i - \mathbf{k}\cdot\boldsymbol{\lambda}) = 0$, $k_i \in \mathbb{N}\cup\{0\}$, i.e. when crossing one of the (finitely many by (c1)) Stokes lines.

---

## 5.10 Outline of the proofs

### 5.10a Summability of the transseries in nonsingular directions: a sketch

We have

$$\mathbf{g}(x,\mathbf{y}) = \sum_{|\mathbf{l}|\geq 1} \mathbf{g}_{\mathbf{l}}(x)\mathbf{y}^{\mathbf{l}} = \sum_{s\geq 0;|\mathbf{l}|\geq 1} \mathbf{g}_{s,\mathbf{l}}x^{-s}\mathbf{y}^{\mathbf{l}} \quad (|x|>x_0, |\mathbf{y}|<y_0) \tag{5.84}$$

where we denote

$$\mathbf{y}^{\mathbf{l}} = y_1^{l_1}\cdots y_n^{l_n},\ |\mathbf{l}| = l_1+\cdots+l_n;\quad \text{also } |\mathbf{y}| := \max\{|y_i| : i=1,...,n\} \tag{5.85}$$

By construction $\mathbf{g}_{s,\mathbf{l}} = 0$ if $|\mathbf{l}| = 1$ and $s \leq M$.

The formal inverse Laplace transform of $\mathbf{g}(x,\mathbf{y}(x))$ (formal since we have not yet shown that $\mathcal{L}^{-1}\mathbf{y}$ exists) is given by:

$$\mathcal{L}^{-1}\left(\sum_{|\mathbf{l}|\geq \mathbf{1}} \mathbf{y}(x)^{\mathbf{l}} \sum_{s\geq 0} \mathbf{g}_{s,\mathbf{l}}x^{-s}\right) = \sum_{|\mathbf{l}|\geq \mathbf{1}} \mathbf{G}_{\mathbf{l}} * \mathbf{Y}^{*\mathbf{l}} + \sum_{|\mathbf{l}|\geq \mathbf{2}} \mathbf{g}_{0,\mathbf{l}}\mathbf{Y}^{*\mathbf{l}} \tag{5.86}$$

$$= \mathcal{N}(\mathbf{Y}) = (\mathcal{L}^{\mathbf{-1}}\mathbf{g})(\mathbf{p},*\mathbf{Y}) = \mathbf{G}(\mathbf{p},*\mathbf{Y}) \tag{5.87}$$

where the last two equalities are mere suggestive notations. For instance, $\mathcal{L}^{-1}\mathbf{g}(\mathbf{1}/\mathbf{x},\mathbf{y})$, after series expansion in convolution powers of $\mathbf{Y}$ is essentially the Taylor series in $\mathbf{y}$ of $\mathbf{g}$, with multiplication replaced by convolution.

(Direct calculations, using expanded formulas for $\mathbf{G}(p,*\mathbf{Y})$, *etc*, are given in §5.11.)

Also, $\mathbf{G}_{\mathbf{l}}(p) = \sum_{s=1}^{\infty} \mathbf{g}_{s,\mathbf{l}}p^{s-1}/s!$ and $(\mathbf{G}_{\mathbf{l}} * \mathbf{Y}^{*\mathbf{l}})_j := (\mathbf{G}_{\mathbf{l}})_j * Y_1^{*l_1} * .. * Y_n^{*l_n}$. By (n5), $\mathbf{G}_{1,\mathbf{l}}^{(l)}(0) = 0$ if $|\mathbf{l}| = 1$ and $l \leq M$.

Thus the inverse Laplace transform of (5.50) is the convolution equation:

$$-p\mathbf{Y} = \mathbf{F}_0 - \hat{\Lambda}\mathbf{Y} + \hat{A}\mathcal{P}\mathbf{Y} + \mathbf{G}(p,*\mathbf{Y}) \tag{5.88}$$

Let $\mathbf{d}_{\mathbf{j}}(x) := \sum_{\mathbf{l}\geq\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \mathbf{g}_{\mathbf{l}}(x)\tilde{\mathbf{y}}_0^{\mathbf{l}-\mathbf{j}}$. Straightforward calculation (see Appendix § 5.12b ; cf. also [24]) shows that the components $\tilde{\mathbf{y}}_{\mathbf{k}}$ of the transseries satisfy the hierarchy of differential equations

$$\mathbf{y}_{\mathbf{k}}' + \left(\hat{\Lambda} - \frac{1}{x}\left(\hat{A} + \mathbf{k}\cdot\boldsymbol{\alpha}\right) - \mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{y}_k + \sum_{|\mathbf{j}|=1} \mathbf{d}_{\mathbf{j}}(x)(\mathbf{y}_{\mathbf{k}})^{\mathbf{j}} = \mathbf{t}_{\mathbf{k}} \tag{5.89}$$

where $\mathbf{t}_{\mathbf{k}} = \mathbf{t}_{\mathbf{k}}\left(\mathbf{y}_0, \{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right)$ is a *polynomial* in $\{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}$ and in $\{\mathbf{d}_{\mathbf{j}}\}_{\mathbf{j}\leq\mathbf{k}}$ (see (5.281)), with $\mathbf{t}(\mathbf{y}_0,\emptyset) = 0$; $\mathbf{t}_{\mathbf{k}}$ satisfies the homogeneity relation

$$\mathbf{t}_{\mathbf{k}}\left(\mathbf{y}_0, \left\{C^{\mathbf{k}'}\mathbf{y}_{\mathbf{k}'}\right\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right) = C^{\mathbf{k}}\mathbf{t}_{\mathbf{k}}\left(\mathbf{y}_0, \{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right) \tag{5.90}$$

Taking $\mathcal{L}^{-1}$ in (5.89) we get, with $\mathbf{D_j} = \sum_{\mathbf{l}\ge\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \left[\mathbf{G_l} * \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})} + \mathbf{g}_{0,\mathbf{l}} * \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})}\right]$,

$$\left(-p + \hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{Y_k} - \left(\hat{A} + \mathbf{k}\cdot\boldsymbol{\alpha}\right)\mathcal{P}\mathbf{Y_k} + \sum_{|\mathbf{j}|=1} \mathbf{D_j} * \mathbf{Y}_\mathbf{k}^{\mathbf{j}} = \mathbf{T_k} \tag{5.91}$$

where of course, for $|\mathbf{j}| = 1$, we have $\mathbf{Y}_\mathbf{k}^{*\mathbf{j}} = \mathbf{Y}_\mathbf{k}^{\mathbf{j}}$; now $\mathbf{T_k}$ is a *convolution* polynomial, cf. (5.205).

*

Since $\mathbf{g}$ is assumed analytic for small $|\mathbf{y}| < \epsilon$ we have

$$\mathbf{g}(1/x, \mathbf{y}) = \sum_{\mathbf{l}} \mathbf{g_l}\mathbf{y}^{\mathbf{l}} \tag{5.92}$$

and by Cauchy's formula and Fubini we have

$$\mathbf{g_l} = const. \oint \cdots \oint \frac{\mathbf{g}(1/x, \mathbf{s})}{s_1^{l_1+1}\cdots s_n^{l_n+1}} d\mathbf{s} \tag{5.93}$$

and therefore by (n5) pp. 164 we have

$$|\mathbf{g_l}| \le A_1^{|\mathbf{l}|}|x|^{-M-1} \tag{5.94}$$

for some $A_1$ and thus

$$|\mathbf{G_l}| \le A^{|\mathbf{l}|}|p|^M; \quad |\mathbf{g}_{0,\mathbf{l}}| \le A^{|\mathbf{l}|} \tag{5.95}$$

for some $A > 0$, where $\mathbf{G_l} = \mathcal{L}^{-1}\mathbf{g_l}$. Similarly,

$$\mathbf{g}(1/x, \mathbf{y}+\mathbf{h}) - \mathbf{g}(1/x, \mathbf{y}) = \sum_{\mathbf{j}\succ 0} \mathbf{g_l}(1/x; \mathbf{y})\mathbf{h}^{\mathbf{l}} \tag{5.96}$$

or

$$\mathcal{L}^{-1}\mathbf{g}(1/x, \mathbf{y}+\mathbf{h}) - \mathcal{L}^{-1}\mathbf{g}(1/x, \mathbf{y}) = \mathcal{L}^{-1}\left(\sum_{\mathbf{j}\succ 0} \mathbf{g_l}(1/x; \mathbf{y})\mathbf{h}^{\mathbf{l}}\right) = \sum_{\mathbf{j}\succ 0} \check{\mathbf{G}}_\mathbf{l} * \mathbf{H}^{*\mathbf{l}} \tag{5.97}$$

where

$$|\check{\mathbf{G}}_\mathbf{l}| \le B^{|\mathbf{l}|}|p|^M \tag{5.98}$$

It follows that in any of the focusing norms used we have

$$\|\mathbf{G}(p, *\mathbf{Y})\|_\nu \to 0 \quad \nu \to \infty \tag{5.99}$$

Similarly, we have

$$\mathbf{G}(p, *(\mathbf{Y}+\mathbf{H})) - \mathbf{G}(p, *\mathbf{Y}) = \mathbf{D}_1(p, *\mathbf{Y}) * \mathbf{H} + o(\|\mathbf{H}\|_\nu) \tag{5.100}$$

where

$$\mathbf{D}_1(p, *\mathbf{Y}) = \left(\mathcal{L}^{-1}\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right)(p, *\mathbf{Y}); \quad \|\mathbf{D}_1(p, *\mathbf{Y})\| \to 0 \text{ as } \nu \to \infty \tag{5.101}$$

We write eq. (5.88) in the form

$$(-p + \hat{\Lambda})\mathbf{Y} = \mathbf{F}_0 + \hat{A}(1 * \mathbf{Y}) + \mathbf{G}(p, *\mathbf{Y}) \tag{5.102}$$

Consider a region in $\mathbb{C}$ of the form

$$\mathcal{S} = \{p \in \mathbb{C} : |p| \le \epsilon \text{ or } \arg(p) \in (a, b)\} \tag{5.103}$$

not containing singular directions. Then the matrix $(-p+\hat{\Lambda})$ is invertible and

$$\mathbf{Y} = (-p + \hat{\Lambda})^{-1}\Big[\mathbf{F}_0 + \hat{A}(1 * \mathbf{Y}) + \mathbf{G}(p, *\mathbf{Y})\Big] \tag{5.104}$$

Let $\mathcal{A}(\mathcal{S})$ be any of the focusing algebras over $\mathcal{S}$.

**Proposition 5.105** *Eq. (5.104) is contractive in $\mathcal{S}$ and has a unique solution there.*

**PROOF** This follows immediately from (5.100) and (5.101). ☐

## 5.10b Higher terms of the transseries

The equations with $|\mathbf{k}| = 1$ are special: they are singular at $p = 0$. Indeed, with $\mathbf{k} = \mathbf{e}_m =: \mathbf{e}$ (the $m$-th unit vector, see pp.166) we have

$$\left(-p + \hat{\Lambda} - \mathbf{e}\cdot\boldsymbol{\lambda}\right)\mathbf{Y}_{\mathbf{e}} - \left(\hat{A} + \mathbf{e}\cdot\boldsymbol{\alpha}\right)\mathcal{P}\mathbf{Y}_{\mathbf{e}} + \sum_{|\mathbf{j}|=1}\mathbf{D}_{\mathbf{j}} * \mathbf{Y}_{\mathbf{e}}^{*\mathbf{j}} = 0 \tag{5.106}$$

where $\hat{\Lambda} - \mathbf{e}\cdot\boldsymbol{\lambda}$ is not invertible.

Suppose first $\mathbf{D}_{\mathbf{j}} = 0$. Then (5.106) is just a system of linear ODEs, written in integral form, at a *regular* singularity. The fact that the singularity is regular follows from the assumptions (a1)–(n5). The general theory of ODE applies and $\mathbf{Y}_{\mathbf{e}}$ has a convergent Frobenius expansion at zero.

In general, for small $p$, $\mathbf{D}_{\mathbf{j}} = O(p^2)$, because of the behavior of $\mathbf{Y}_0$. Then near the origin, the contribution of these terms is small. One writes

$$\left(-p + \hat{\Lambda} - \mathbf{e}\cdot\boldsymbol{\lambda}\right)\mathbf{Y}_{\mathbf{e}} - \left(\hat{A} + \mathbf{e}\cdot\boldsymbol{\alpha}\right)\mathcal{P}\mathbf{Y}_{\mathbf{e}} = \mathbf{R} \tag{5.107}$$

where

$$\mathbf{R} = -\sum_{|\mathbf{j}|=1}\mathbf{D}_{\mathbf{j}} * \mathbf{Y}_{\mathbf{e}}^{*\mathbf{j}} \tag{5.108}$$

and inverts the operator on the left side of (5.107). In the process of inversion one free constant, $C_m$, is generated (a free constant was to be expected, since

(5.106) is homogeneous). For any value of $C_m$ however, the ensuing integral equation is contractive.

For $|\mathbf{k}| > 1$, $\left(-p + \hat{\Lambda} - \mathbf{k} \cdot \boldsymbol{\lambda}\right)$ is invertible in $\mathcal{S}$, again by (a1)–(n5). The homogeneous part of the (*linear* equations) for $\mathbf{Y_k}$ is regular. But the inhomogeneous term is singular at $p = 0$, due to $\mathbf{Y_e}$.

The equations are treated in a space designed to accommodate these singularities. Contractivity of the system of equations for the $\mathbf{Y_k}$; $|\mathbf{k}| > 1$, as well as summability of the resulting transseries and the fact that it solves (5.50) are then shown essentially as in §5.3b .

The analysis of the convolution equations along singular directions is more delicate, but not so different in spirit from the analysis of the equation for $\mathbf{Y_e}$ at $p = 0$: near singular points, the convolution equations are well approximated by regularly singular ODEs.

However, new problems arise if $\operatorname{Re} \boldsymbol{\alpha} > 0$. Then the singularities of the functions $\mathbf{Y_k}$, alway located in $S_{\mathbf{k}}$, are nonintegrable, of strengths growing with the distance to the origin, as seen in (5.69). In [22] this is dealt with by working in $\mathcal{D}'_{m,\nu}$.

## *5.11 Detailed proofs, for $\operatorname{Re}(\alpha_1) < 0$ and a one-parameter transseries

This section essentially follows [24]. The simplifying assumptions are removed in [22].

We use the notation (5.60) and $\beta = \beta_1$. We have $\operatorname{Re} \beta > 0$. The substitution described above (n4) makes

$$\operatorname{Re}(\beta) \in (0, 1] \tag{5.109}$$

The *one*-parameter family of transseries formally small in a half-plane is given by

$$\tilde{\mathbf{y}} = \tilde{\mathbf{y}}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{\mathbf{y}}_k \tag{5.110}$$

where the series $\tilde{\mathbf{y}}_k$ are small; (5.89) becomes

$$\mathbf{y}_k' + \left(\hat{\Lambda} + \frac{1}{x}\hat{B} - k - \partial \mathbf{g}(x, \mathbf{y}_0)\right)\mathbf{y}_k = \sum_{|\mathbf{l}|>1} \mathbf{g}^{(\mathbf{l})}(x, \mathbf{y}_0)\mathbf{l}!^{-1} \sum_{\Sigma m=k} \prod_{i=1}^{n} \prod_{j=1}^{l_i} (\mathbf{y}_{m_{i,j}})_i \tag{5.111}$$

where $\mathbf{g}^{(\mathbf{l})} := \partial^{(\mathbf{l})}\mathbf{g}/\partial \mathbf{y}^{\mathbf{l}}$, $(\partial \mathbf{g})\mathbf{y}_k := \sum_{i=1}^{n} (\mathbf{y}_k)_i (\partial \mathbf{g}/\partial y_i)$, and $\sum_{\Sigma m=k}$ stands for the sum over all integers $m_{i,j} \geq 1$ with $1 \leq i \leq n, 1 \leq j \leq l_i$ such that $\sum_{i=1}^{n} \sum_{j=1}^{l_i} m_{i,j} = k$.

Since $m_{i,j} \geq 1$, we have $\sum m_{i,j} = k$ (fixed) and $\operatorname{card}\{m_{i,j}\} = |\mathbf{l}|$, the sums in (5.111) contain only a *finite* number of terms. We use the convention $\prod_{i\in\emptyset} \equiv 0$.

In the following, we choose the usual branch of the logarithm, positive for $x > 1$.

**Proposition 5.112** *i) The function* $\mathbf{Y}_0 := \mathcal{B}\tilde{\mathbf{y}}_0$ *is analytic in* $\mathcal{W}$ *and Laplace transformable along any direction in* $\mathcal{W}$*. In a neighborhood of* $p = 1$ *we have*

$$\mathbf{Y}_0(p) = \begin{cases} S_\beta(1-p)^{\beta-1}\mathbf{A}(p) + \mathbf{B}(p) \text{ for } \beta \neq 1 \\ S_\beta \ln(1-p)\mathbf{A}(p) + \mathbf{B}(p) \text{ for } \beta = 1 \end{cases} \tag{5.113}$$

*(see (5.109)), where* $\mathbf{A}$*,* $\mathbf{B}$ *are (*$\mathbb{C}^n$*-valued) analytic functions in a neighborhood of* $p = 1$*.*

*ii) The functions* $\mathbf{Y}_k := \mathcal{B}\tilde{\mathbf{y}}_k,\ k = 0, 1, 2, ..$ *are analytic in* $\mathcal{R}_1$*.*

*iii) For small* $p$ *we have*

$$\mathbf{Y}_0(p) = p\mathbf{A}_0(p); \quad \mathbf{Y}_k(p) = p^{k\beta-1}\mathbf{A}_k(p),\ k \in \mathbb{N} \tag{5.114}$$

*where* $\mathbf{A}_k$*,* $k \geq 0$*, are analytic functions in a neighborhood of* $p = 0$ *in* $\mathbb{C}$*.*

*iv) If* $S_\beta = 0$ *then* $\mathbf{Y}_k$*,* $k \geq 0$*, are analytic in* $\mathcal{W} \cup \mathbb{N}$*.*

*v) The analytic continuations of* $\mathbf{Y}_k$ *along paths in* $\mathcal{R}_1$ *are in* $L^1_{\text{loc}}(\mathbb{R}^+)$ *(their singularities along* $\mathbb{R}^+$ *are integrable* [5]*). The analytic continuations of the* $\mathbf{Y}_k$ *in* $\mathcal{R}_1$ *can be expressed in terms of each other through resurgence relations:*

$$S_\beta^k \mathbf{Y}_k \circ \tau_k = \left(\mathbf{Y}_0^- - \mathbf{Y}_0^{-^{k-1}+}\right), \qquad \text{on } (0,1); \quad (\tau_a := p \mapsto p - a) \tag{5.115}$$

*relating the higher index series in the transseries to the first series and*

$$\mathbf{Y}_k^{-^m+} = \mathbf{Y}_k^+ + \sum_{j=1}^{m} \binom{k+j}{k} S_\beta^j \mathbf{Y}_{k+j}^+ \circ \tau_j \tag{5.116}$$

$S_\beta$ is related to the Stokes constant [54] $S$ by

$$S_\beta = \begin{cases} \dfrac{iS}{2\sin(\pi(1-\beta))} \text{ for } \beta \neq 1 \\ \dfrac{iS}{2\pi} \text{ for } \beta = 1 \end{cases}$$

---

[5] For integrability, the condition $\operatorname{Re}\beta > 0$ is essential.

Let $\mathbf{Y}$ be one of the functions $\mathbf{Y}_k$ and define, on $\mathbb{R}^+ \cap \mathcal{R}_1$ the "balanced average" of $\mathbf{Y}$ [6]:

$$\mathbf{Y}^{ba} = \mathbf{Y}^+ + \sum_{k=1}^{\infty} 2^{-k} \left( \mathbf{Y}^- - \mathbf{Y}^{-^{k-1}+} \right) \mathcal{H} \circ \tau_k \tag{5.117}$$

($\mathcal{H}$ is Heaviside's function). For any value of $p$, only finitely many terms (5.117) are nonzero. Moreover, the balanced average preserves reality in the sense that if (5.50) is real and $\tilde{\mathbf{y}}_0$ is real then $\mathbf{Y}^{ba}$ is real on $\mathbb{R}^+ - \mathbb{N}$ (and in this case the formula can be symmetrized by taking 1/2 of the expression above plus 1/2 of the same expression with + and − interchanged). Equation (5.117) has the main features of medianization (cf. [31]), in particular (unlike individual analytic continuations, see Appendix 5.12a ) commutes with convolution (cf. Theorem 5.128). $\mathbf{Y}^{ba}$ is exponentially bounded at infinity for the functions we are dealing with.

Let again $\tilde{\mathbf{y}}$ be one of $\tilde{\mathbf{y}}_k$ and $\mathbf{Y} = \mathcal{B}\tilde{\mathbf{y}}$. We define:

$$\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}} := \mathcal{L}_\phi \,\mathbf{Y} = x \mapsto \int_0^{\infty e^{i\phi}} \mathbf{Y}(p) e^{-px} dp \ \text{ if } \phi \neq 0$$

$$\mathcal{L}_0 \mathcal{B}\tilde{\mathbf{y}} := \mathcal{L}_0 \,\mathbf{Y} = x \mapsto \int_0^{\infty} \mathbf{Y}^{ba}(p) e^{-px} dp \ \text{ if } \phi = 0 \tag{5.118}$$

(the first relation is the usual one, the second one defines summation along the Stokes line).

The connection between true and formal solutions of the differential equation is given in the following theorem:

**Theorem 5.119** *i) There is a large enough $\nu$ such that, for* $\mathrm{Re}\,(x) > \nu$ *the Laplace transforms $\mathcal{L}_\phi \mathbf{Y}_k$ exist for all $k \geq 0$ in*

$$\mathcal{W}_1 := \{p : p \notin \mathbb{N} \cup \{0\} \text{ and } \ \arg p \in (-\phi_-, \phi_+)\} \tag{5.120}$$

*where we denote, for simplicity, by $\phi^+$ the singular direction nearest to $\mathbb{R}^+$ in $\mathbb{H}$ (similarly for $\phi_-$). For $\phi \in (-\phi_-, \phi_+)$ and any $C$, the series*

$$\mathbf{y}(x) = (\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}}_0)(x) + \sum_{k=1}^{\infty} C^k e^{-kx} (\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}}_k)(x) \tag{5.121}$$

*is convergent for large enough $x$ in $\mathbb{H}$.*

*The function $\mathbf{y}$ in (5.121) is a solution of the differential equation (5.50).*

*Furthermore, for any $k \geq 0$ we have $\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}}_k \sim \tilde{\mathbf{y}}_k$ in $\mathbb{H}$ and $\mathcal{L}_\phi \mathcal{B}\tilde{\mathbf{y}}_k$ is a solution of the corresponding equation in (5.89).*

---

[6] As mentioned, it coincides with Écalle's medianization, but is simpler here due to the special features of ODEs.

*ii) Conversely, given* $\phi$, any *solution of (5.50) having* $\tilde{\mathbf{y}}_0$ *as an asymptotic series* $\mathbb{H}$ *can be written in the form (5.121), for a unique* $C$.

*iii) The constant* $C$*, associated in ii) with a given solution* $\mathbf{y}$ *of (5.50), depends on the angle* $\phi$*:*

$$C(\phi)=\begin{cases} C(0_+) \quad \textit{for } \phi>0 \\ C(0_+)-\frac{1}{2}S_\beta \quad \textit{for } \phi=0 \\ C(0_+)-S_\beta \quad \textit{for } \phi<0 \end{cases} \tag{5.122}$$

*(see also (5.113) ).*

When $\phi$ is not a singular direction, the description of the solutions is quite simple:

**Proposition 5.123** *i) With* $\phi$ *describing a ray in* $\mathcal{W}$*, the equation (5.88) has a unique solution in* $L^1_{\text{loc}}(\Phi)$*, namely* $\mathbf{Y}_0=\mathcal{B}\tilde{\mathbf{y}}_0$*.*

*ii) For any ray in* $\mathcal{W}_1$*, the system (5.88), (5.91) has the general solution solution* $C^k\mathbf{Y}_k=C^k\mathcal{B}\tilde{\mathbf{y}}_k,\ k\geq 0$*.*

The more interesting case $\phi=0$ is addressed in the following theorem:

**Theorem 5.124** *i) The general solution in* $L^1_{\text{loc}}(\mathbb{R}^+)$ *of (5.88) is*

$$\mathbf{Y}_C(p)=\sum_{k=0}^{\infty}C^k\mathbf{Y}_k^{ba}(p-k)\mathcal{H}(p-k) \tag{5.125}$$

*with* $C\in\mathbb{C}$ *arbitrary.*

*ii) Near* $p=1$, $\mathbf{Y}_C$ *is given by:*

$$\mathbf{Y}_C(p)=\begin{cases} S_\beta(1-p)^{\beta-1}\mathbf{A}(p)+\mathbf{B}(p) \textit{ for } p<1 \\ C(1-p)^{\beta-1}\mathbf{A}(p)+\mathbf{B}(p) \textit{ for } p>1 \end{cases} \quad (\beta\neq 1) \tag{5.126}$$

$$\mathbf{Y}_C(p)=\begin{cases} S_\beta\ln(1-p)\mathbf{A}(p)+\mathbf{B}(p) \textit{ for } p<1 \\ (S_\beta\ln(1-p)+C)\mathbf{A}(p)+\mathbf{B}(p) \textit{ for } p>1 \end{cases} \quad (\beta=1)$$

*where* $\mathbf{A}$ *and* $\mathbf{B}$ *are analytic in a neighborhood of* $p=1$*.*

*iii) With the choice* $\mathbf{Y}_0=\mathbf{Y}_0^{ba}$*, the general solution of (5.91) in* $L^1_{\text{loc}}(\mathbb{R}^+)$ *is* $C^k\mathbf{Y}_k^{ba}, k\in\mathbb{N}$*.*

Comparing (5.126) with (5.113) we see that if $S\neq 0$ (which is the generic case), then the general solution on $(0,2)$ of (5.88) is a linear combination of the upper and lower analytic continuations of $\mathcal{B}\tilde{\mathbf{y}}_0$:

$$\mathbf{Y}_C=\lambda_C\mathbf{Y}_0^+ +(1-\lambda_C)\mathbf{Y}_0^- \tag{5.127}$$

Finally we mention the following result, which shows that the balanced average, like medianization [31], commutes with convolution.

**Theorem 5.128** *If $f$ and $g$ are analytic in $\mathcal{R}_1$ then $f*g$ extends analytically in $\mathcal{R}_1$ and furthermore,*

$$(f*g)^{ba} = f^{ba} * g^{ba} \tag{5.129}$$

As a consequence of the linearity of the balanced averaging and its commutation with convolution, if $\tilde{\mathbf{t}}_{1,2}$ are the transseries of the solutions $\mathbf{f}_{1,2}$ of differential equations of the type considered here and if $\mathcal{LB}\tilde{\mathbf{t}}_{1,2} = \mathbf{f}_{1,2}$ then

$$\mathcal{LB}\left(a\tilde{\mathbf{t}}_1 + b\tilde{\mathbf{t}}_2\right) = a\mathbf{f}_1 + b\mathbf{f}_2 \tag{5.130}$$

Moreover, what is less obvious, we have for the componentwise product formula

$$\mathcal{LB}(\tilde{\mathbf{t}}_1\tilde{\mathbf{t}}_2) = \mathbf{f}_1\mathbf{f}_2 \tag{5.131}$$

## 5.11a Comments

We look more carefully at the behavior along singular directions. As mentioned, at singular points, the convolution equations are to leading order linear, regularly perturbed, ODEs. In nonlinear equations, one singularity is replicated periodically along its complex direction, via autoconvolution.

The next task is to find a Borel summation valid along the singular directions while preserving all properties of usual Borel summation. The formulas are valid in the context of ODEs, where they offer simplicity, as well as complete classification of well-behaved averages, but do not substitute for the general averages of Écalle. The latter have the expected properties regardless of the origin of the expansion, see [30].

We first obtain the general solution in $L^1_{\text{loc}}$ of the convolution system (5.91) in $\mathcal{W}$ and then, separately, on the Stokes line $\mathbb{R}^+$. We show that along a ray in $\mathcal{W}$, the solution is unique whereas along the ray $\mathbb{R}^+$ there is a one-parameter family of solutions of the system, branching off at $p=1$. We show that any $L^1_{\text{loc}}$ solution of the system is exponentially bounded at infinity (uniformly in $k$). Therefore the Laplace transforms exist and solve (5.50). Conversely, any solution of (5.50) with the required asymptotic properties is inverse Laplace transformable, therefore it has to be one of the previously obtained solutions of the equation corresponding to $k=0$. We then study the regularity properties of the solutions of the convolution equation by local analysis.

Having the complete description of the family of $L^1_{\text{loc}}$ solutions we compare different formulas for one given solution and obtain resurgence identities; resurgence, together with the local properties of the solutions are instrumental in finding the analytic properties of $\mathbf{Y}_k$ in $\mathcal{R}_1$.

## 5.11b The convolution equation away from singular rays

We denote by $L_{1,1}(\mathcal{E})$ the set of functions which are locally integrable along each ray in $\mathcal{E}$ (an intersection of usual $L_1$ spaces).

**Proposition 5.132** *There is a unique solution of (5.88) in* $L_{1,1}(\mathcal{W})$ *namely* $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$.

*This solution is analytic in* $\mathcal{W}$*, Laplace transformable along any ray* $\arg p = \phi$ *contained in* $\mathcal{W}$ *and* $\mathcal{L}_\phi \mathbf{Y}_0$ *is a solution of (5.50).*

For the proof we need a few more results.

**Remark 5.133** *There is a constant* $K > 0$ *(independent of* $p$ *and* $\mathbf{l}$*) so that for all* $p \in \mathbb{C}$ *and all* $\mathbf{l} \geq \mathbf{0}$

$$|\mathbf{G}_{\mathbf{l}}(p)| \leq K {c_0}^{|\mathbf{l}|} e^{c_0 |p|} \tag{5.134}$$

*if* $c_0 > \max\{x_0^{-1}, y_0^{-1}\}$.

**PROOF** From the analyticity assumption it follows that

$$|\mathbf{g}_{m,\mathbf{l}}| \leq \text{Const } {c_0}^{m+|\mathbf{l}|} \tag{5.135}$$

where the constant is independent on $m$ and $\mathbf{l}$.
Then,

$$|\mathbf{G}_{\mathbf{l}}(p)| \leq \text{Const } {c_0}^{|\mathbf{l}|+1} \frac{e^{c_0|p|} - 1}{c_0|p|} \leq \text{Const } {c_0}^{|\mathbf{l}|+1} e^{c_0|p|}$$

$\square$

Consider the ray segments

$$\Phi_D = \{\alpha e^{i\phi} : 0 \leq \alpha \leq D\} \tag{5.136}$$

the $L^1$ norm with exponential weight along $\Phi_D$

$$\|f\|_{\nu,\Phi} = \|f\|_\nu := \int_\Phi e^{-\nu|p|} |f(p)||dp| = \int_0^D e^{-\nu t} |f(te^{i\phi})| dt \tag{5.137}$$

and the space

$$L^1_\nu(\Phi_D) := \{f : \|f\|_\nu < \infty\}$$

(if $D < \infty$, $L^1_\nu(\Phi_D) = L^1_{\text{loc}}(\Phi_D)$).

Let $\mathcal{K} \in \mathbb{C}$ be a bounded domain, $\operatorname{diam}(\mathcal{K}) = D < \infty$. On the space of continuous functions on $\mathcal{K}$ we take the uniform norm with exponential weight:

$$\|f\|_u := D \sup_{p \in \mathcal{K}} \{|f(p)| e^{-\nu|p|}\} \tag{5.138}$$

(which is equivalent to the usual uniform norm).

Let $\mathcal{O} \subset \mathcal{D}$, $\mathcal{O} \ni 0$ be a *star-shaped, open set*, $\operatorname{diam}(\mathcal{O}) = D$ containing a ray segment $\Phi$. Let $\mathcal{A}$ be the space of analytic functions $f$ in $\mathcal{O}$ such that $f(0) = 0$, endowed with the norm (5.138).

**Proposition 5.139** *The spaces $L^1_\nu(\Phi_D)$ and $\mathcal{A}$ are focusing algebras, see §5.1, §5.2.*

**Corollary 5.140** *Let $f$ be continuous along $\Phi_D$, $D < \infty$ and $g \in L^1_\nu(\Phi_D)$. Given $\epsilon > 0$ there exists a large enough $\nu$ and $K = K(\epsilon, \Phi_D)$ so that for all $k$*

$$\|f * g^{*k}\|_u \le K\,\epsilon^k$$

By Proposition 5.139 we can choose $\nu = \nu(\epsilon, \Phi_D)$ so large that $\|g\|_\nu \le \epsilon$ (we make use of the focusing nature of the norm). Then, by Proposition 5.139 and Eq. (5.138) we have:

$$\left|\int_0^{pe^{i\phi}} f(pe^{i\phi} - s)g^{*k}(s)ds\right| \le D^{-1}e^{\nu|p|}\|f\|_u \int_0^{pe^{i\phi}} e^{-\nu|s|}|g^{*k}(s)||ds| \le$$
$$D^{-1}e^{\nu|p|}\|f\|_u\|g\|_\nu^k \le K\,\epsilon^k$$

□

**Remark 5.141** *By (5.134), for any $\nu > c_0$, and $\Phi_D \subset \mathbb{C}$, $D \le \infty$*

$$\|\mathbf{G_l}\|_\nu \le K{c_0}^{|\mathbf{l}|}\int_0^\infty |dp|e^{|p|(c_0-\nu)} = K\frac{{c_0}^{|\mathbf{l}|}}{\nu - c_0} \tag{5.142}$$

where we wrote

$$\mathbf{f} \in L^1_\nu(\Phi_D) \text{ iff } \ \||\mathbf{f}|\|_\nu \in L^1_\nu(\Phi_D) \tag{5.143}$$

(and similarly for other norms of vector functions).

**PROOF of Proposition 5.132** We first show existence and uniqueness in $L_{1,1}(\mathcal{W})$.

Then we show that for large enough $\nu$ there exists a unique solution of (5.88) in $L^1_\nu(\Phi_\infty)$. Since this solution is also in $L^1_{\text{loc}}(\Phi_\infty)$ it follows that our (unique) $L^1_{\text{loc}}$ solution is Laplace transformable. Analyticity is proven by usual fixed point methods in a space of analytic functions. ▯

**Proposition 5.144** *i) For $\Phi_D \in \mathcal{W}$ and large enough $\nu$, the operator*

$$\mathcal{N}_1 := \mathbf{Y}(p) \mapsto (\hat\Lambda - p)^{-1}\left(\mathbf{F}_0(p) - \hat B\int_0^p \mathbf{Y}(s)ds + \mathcal{N}(\mathbf{Y})(p)\right) \tag{5.145}$$

*is contractive in a small enough neighborhood of the origin with respect to $\|\|\|_u$ if $D < \infty$ and with respect to $\|\|\|_\nu$ for $D \le \infty$.*

*ii) For $D \leq \infty$ the operator $\mathcal{N}$ given formally in (5.86) is continuous in $L^1_{\text{loc}}(\Phi_D)$. The last sum in (5.86) converges uniformly on compact subsets of $\Phi_D$. $\mathcal{N}(L^1_{\text{loc}}(\Phi_D))$ is contained in the absolutely continuous functions on $\Phi_D$ [47]. Moreover, if $\mathbf{v}_n \to \mathbf{v}$ in $\|\|_\nu$ on $\Phi_D$, $D \leq \infty$, then for $\nu' \geq \nu$ large enough, $\mathcal{N}(\mathbf{v}_n)$ exist and converge in $\|\|_{\nu'}$ to $\mathbf{v}$.*

Here we make use of Remark 5.18 to obtain at the same time a number of needed properties of the solutions (analyticity, bounds at infinity, etc.).

The last statements in (ii) amount to saying that $\mathcal{N}$ is continuous in the topology of the inductive limit of the $L^1_\nu$.

**PROOF** Since $\hat{\Lambda}$ and $\hat{B}$ are constant matrices we have

$$\|\mathcal{N}_1(\mathbf{Y})\|_{u,\nu} \leq \text{Const}(\Phi)\left(\|\mathbf{F}_0\|_{u,\nu} + \|\mathbf{Y}\|_{u,\nu}\|1\|_\nu + \|\mathcal{N}(\mathbf{Y})\|_{u,\nu}\right) \tag{5.146}$$

As both $\|1\|_\nu$ and $\|\mathbf{F}_0\|_{u,\nu}$ are $O(\nu^{-1})$ for large $\nu$, the fact that $\mathcal{N}_1$ maps a small ball into itself follows from the following remark.

**Remark 5.147** *Let $\epsilon > 0$ be small enough. Then, there is a $K$ so that for large $\nu$ and all $\mathbf{v}$ such that $\|\mathbf{v}\|_{u,\nu} =: \delta < \epsilon$,*

$$\|\mathcal{N}(\mathbf{v})\|_{u,\nu} \leq K\left(\nu^{-1} + \|\mathbf{v}\|_{u,\nu}\right)\|\mathbf{v}\|_{u,\nu} \tag{5.148}$$

By (5.135) and (5.142), for large $\nu$ and some positive constants $C_1, ..., C_5$,

$$\begin{aligned}
\|\mathcal{N}(\mathbf{v})\|_{u,\nu} &\leq C_1\left(\sum_{|\mathbf{l}|\geq 1}\|\mathbf{G}_{\mathbf{l}}\|_\nu \|\mathbf{v}\|^{|\mathbf{l}|}_{u,\nu} + \sum_{|\mathbf{l}|\geq 2}\|\mathbf{g}_{0,\mathbf{l}}\|_\nu \|\mathbf{v}\|^{|\mathbf{l}|}_{u,\nu}\right) \\
&\leq \frac{C_2}{\nu}\left(\sum_{|\mathbf{l}|\geq 1}\frac{{c_0}^{|\mathbf{l}|}}{\nu - c_0}\delta^{|\mathbf{l}|} + \sum_{|\mathbf{l}|\geq 2}{c_0}^{|\mathbf{l}|}\delta^{|\mathbf{l}|}\right) \leq \left(C_2\sum_{m=1}^{\infty} + \sum_{m=2}^{\infty}\right){c_0}^m\delta^m \sum_{|\mathbf{l}|=m} 1 \\
&\leq \left(\frac{C_4}{\nu} + c_0\delta\right)\sum_{m=1}^{\infty}{c_0}^m\delta^m(m+4)^n \leq \left(\frac{C_4}{\nu} + c_0\delta\right)C_5\delta
\end{aligned} \tag{5.149}$$

□

To show that $\mathcal{N}_1$ is a contraction we need the following:

**Remark 5.150**

$$\|\mathbf{h}_{\mathbf{l}}\| := \|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}} - \mathbf{f}^{*\mathbf{l}}\| \leq |\mathbf{l}|\left(\|\mathbf{f}\| + \|\mathbf{h}\|\right)^{|\mathbf{l}|-1}\|\mathbf{h}\| \tag{5.151}$$

*where $\|\| = \|\|_u$ or $\|\|_\nu$.*

This estimate is useful when $\mathbf{h}$ is a "small perturbation". The proof of (5.151) is a simple induction on $\mathbf{l}$, with respect to the lexicographic ordering. For $|\mathbf{l}| = 1$, (5.151) is clear; assume (5.151) holds for all $\mathbf{l} < \mathbf{l}_1$ and that $\mathbf{l}_1$ differs from its predecessor $\mathbf{l}_0$ at the position k (we can take $k = 1$), i.e., $(\mathbf{l}_1)_1 = 1 + (\mathbf{l}_0)_1$. We have:

$$\begin{aligned}
&\|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}_1} - \mathbf{f}^{*\mathbf{l}_1}\| = \|(\mathbf{f}+\mathbf{h})^{*\mathbf{l}_0} * (\mathbf{f}_1 + \mathbf{h}_1) - \mathbf{f}^{*\mathbf{l}_1}\| = \\
&\quad \|(\mathbf{f}^{*\mathbf{l}_0} + \mathbf{h}_{\mathbf{l}_0}) * (f_1 + h_1) - \mathbf{f}^{*\mathbf{l}_1}\| = \|\mathbf{f}^{*\mathbf{l}_0} * h_1 + \mathbf{h}_{\mathbf{l}_0} * f_1 + \mathbf{h}_{\mathbf{l}_0} * h_1\| \le \\
&\qquad \|\mathbf{f}\|^{|\mathbf{l}_0|}\|\mathbf{h}\| + \|\mathbf{h}_{\mathbf{l}_0}\|\|\mathbf{f}\| + \|\mathbf{h}_{\mathbf{l}_0}\|\|\mathbf{h}\| \le \\
&\qquad \|\mathbf{h}\| \left(\|\mathbf{f}\|^{|\mathbf{l}_0|} + |\mathbf{l}_0|(\|\mathbf{f}\| + \|\mathbf{h}\|)^{|\mathbf{l}_0|}\right) \le \\
&\qquad\qquad \|\mathbf{h}\|(|\mathbf{l}_0| + 1)(\|\mathbf{f}\| + \|\mathbf{h}\|)^{|\mathbf{l}_0|} \qquad (5.152)
\end{aligned}$$

**Remark 5.153** *For small $\delta$ and large enough $\nu$, $\mathcal{N}_1$ defined in a ball of radius $\delta$ centered at zero, is contractive in the norms $\|\|_{u,\nu}$.*

By (5.146) and (5.148) we know that the ball is mapped into itself for large $\nu$. Let $\epsilon > 0$ be small and let $\mathbf{f}, \mathbf{h}$ be so that $\|\mathbf{f}\| < \delta - \epsilon, \|\mathbf{h}\| < \epsilon$. Using (5.151) and the notations (5.88) (5.146) and $\|\| = \|\|_{u,\nu}$ we obtain, for some positive constants $C_1, ..., C_4$ and large $\nu$,

$$\begin{aligned}
&\|\mathcal{N}_1(\mathbf{f}+\mathbf{h}) - \mathcal{N}_1(\mathbf{f})\| \le C_1 \| \left(\sum_{|\mathbf{l}|\ge\mathbf{2}} \mathbf{g}_{0,\mathbf{l}} \cdot + \sum_{|\mathbf{l}|\ge\mathbf{1}} \mathbf{G}_{\mathbf{l}} *\right) \left((\mathbf{f}+\mathbf{h})^{*\mathbf{l}} - \mathbf{f}^{*\mathbf{l}}\right) \| \le \\
&C_2\|\mathbf{h}\| \left(\sum_{|\mathbf{l}|\ge\mathbf{1}} \frac{{c_0}^{|\mathbf{l}|}}{\nu - c_0} |\mathbf{l}| \|\delta^{|\mathbf{l}|-1} + \sum_{|\mathbf{l}|\ge\mathbf{2}} |\mathbf{l}| {\mathbf{c_0}}^{|\mathbf{l}|} \delta^{|\mathbf{l}|-\mathbf{1}}\right) \le (C_3\nu^{-1} + C_4\delta)\|\mathbf{h}\|
\end{aligned} \qquad (5.154)$$

To finish the proof of Proposition 5.144 take $\mathbf{v} \in \mathcal{A}$. Given $\epsilon > 0$ we can choose $\nu$ large enough (by Proposition 5.139) to make $\|\mathbf{v}\|_u < \epsilon$. Then the sum in the formal definition of $\mathcal{N}$ is convergent in $\mathcal{A}$, by (5.149). Now, if $D < \infty$, then $L^1_{\mathrm{loc}}(\Phi_D) = L^1_\nu(\Phi_D)$ for any $\nu > 0$. If $\mathbf{v}_n \to \mathbf{v}$ in $L^1_\nu(\Phi_D)$, we choose $\epsilon$ small enough, then $\nu$ large so that $\|\mathbf{v}\|_\nu < \epsilon$, and finally $n_0$ large so that for $n > n_0$ $\|\mathbf{v}_n - \mathbf{v}\|_\nu < \epsilon$ (note that $\|\|_\nu$ decreases w.r. to $\nu$) thus $\|\mathbf{v}_n\|_\nu < 2\epsilon$ and continuity (in $L^1_\nu(\Phi_D)$ as well as in $L^1_{\mathrm{loc}}(\Phi_\infty) \equiv \cup_{k\in\Phi_\infty} L^1_\nu(0,k)$) follows from Remark 5.153. Continuity with respect to the topology of the inductive limit of the $L^1_\nu$ is proven in the same way. It is straightforward to show that $\mathcal{N}(L^1_{\mathrm{loc}}(\Phi)) \subset C_a(\Phi)$, where $C_a$ are absolutely continuous functions [47].

$\Box$ $P_{5.144}$

The fact that $\mathcal{L}_\phi \mathbf{Y}_0$ is a solution of (5.50) follows from Proposition 5.144 and the properties of $\mathcal{L}$ (see also the proof of Proposition 5.211).

Since $\mathbf{Y}_0(p)$ is analytic for small $p$, $(\mathcal{L}\mathbf{Y}_0)(x)$ has an asymptotic series for large $x$, which has to agree with $\tilde{\mathbf{y}}_0$ since $\mathcal{L}\mathbf{Y}_0$ solves (5.50). This shows that $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$.

$\square$ *P*5.132

**Remark 5.155** *For any $\delta$ there is a constant $K_2 = K_2(\delta, |p|)$ so that for all* $\mathbf{l}$ *we have*

$$|\mathbf{Y}_0^{*\mathbf{l}}(p)| \le K_2 \delta^{|\mathbf{l}|} \tag{5.156}$$

The estimates (5.156) follow immediately from analyticity and from Corollary 5.140.

$\square$

## 5.11c Behavior of $\mathbf{Y}_0(p)$ near $p = 1$.

The point $p = 1$ is a singular point of the convolution equation. The solution is generally singular too. Its behavior at the singularity is derived using the convolution equation alone.

*

Let $\mathbf{Y}_0$ be the unique solution in $L_{1,1}(\mathcal{W})$ of (5.88) and let $\epsilon > 0$ be small. Define

$$\mathbf{H}(p) := \begin{cases} \mathbf{Y}_0(p) & \text{for } p \in \mathcal{W}, |p| < 1-\epsilon \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{h}(1-p) := \mathbf{Y}_0(p) - \mathbf{H}(p) \tag{5.157}$$

In terms of $\mathbf{h}$, for real $z = 1 - p, z < \epsilon$, the equation (5.88) reads:

$$-(1-z)\mathbf{h}(z) = \mathbf{F}_1(z) - \hat{\Lambda}\mathbf{h}(z) + \hat{B}\int_\epsilon^z \mathbf{h}(s)ds + \mathcal{N}(\mathbf{H}+\mathbf{h}) \tag{5.158}$$

where

$$\mathbf{F}_1(1-s) := \mathbf{F}_0(s) - \hat{B}\int_0^{1-\epsilon} \mathbf{H}(s)ds$$

**Proposition 5.159** *i) For small $\epsilon$, $\mathbf{H}^{*\mathbf{l}}(1+z)$ extends to an analytic function in the disk $\mathbb{D}_\epsilon := \{z : |z| < \epsilon\}$. Furthermore, for any $\delta$ there is an $\epsilon$ and a constant $K_1 := K_1(\delta, \epsilon)$ so that for $z \in \mathbb{D}_\epsilon$ the analytic continuation satisfies the estimate*

$$|\mathbf{H}^{*\mathbf{l}}(1+z)| < K_1 \delta^l \tag{5.160}$$

**PROOF** The case $|\mathbf{l}| = 1$ is clear: $\mathbf{H}$ itself extends as the zero analytic function. We assume by induction on $|\mathbf{l}|$ that Proposition 5.159 is true for all $\mathbf{l}$, $|\mathbf{l}| \le l$ and show that it then holds for (e.g.) $H_1 * \mathbf{H}^{*\mathbf{l}}$, for all $\mathbf{l}$, $|\mathbf{l}| \le l$.

$\mathbf{H}$ is analytic in an $\epsilon$–neighborhood of $[0, 1-2\epsilon]$, and therefore so is $\mathbf{H}^{*\mathbf{l}}$. Taking first $z \in \mathbb{R}^+,\ z < \epsilon$, we have

$$\int_0^{1-z} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)ds = \int_0^{1-\epsilon} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)ds = \int_0^{1/2} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)ds + \int_{1/2}^{1-\epsilon} H_1(s)\mathbf{H}^{*\mathbf{l}}(1-z-s)ds \tag{5.161}$$

The integral on $[1/2, 1-\epsilon]$ is analytic for small $z$, since the argument of $\mathbf{H}^{*\mathbf{l}}$ varies in an $\epsilon$-neighborhood of $[0, 1/2]$; the integral on $[0, 1/2)$ equals

$$\int_{1/2-z}^{1-z} H_1(1-z-t)\mathbf{H}^{*\mathbf{l}}(t)dt = \left(\int_{1/2-z}^{1/2} + \int_{1/2}^{1-\epsilon} + \int_{1-\epsilon}^{1-z}\right) H_1(1-z-t)\mathbf{H}^{*\mathbf{l}}(t)dt \tag{5.162}$$

In (5.162) the integral on $[1/2-z, 1/2]$ is clearly analytic in $\mathbb{D}_\epsilon$, the second one is the integral of an analytic function of the parameter $z$ with respect to the absolutely continuous measure $\mathbf{H}^{*\mathbf{l}}dt$ whereas in the last integral, both $\mathbf{H}^{*\mathbf{l}}$ (by induction) and $H_1$ extend analytically in $\mathbb{D}_\epsilon$.

To prove now the induction step for the estimate (5.160), fix $\delta$ small and let:

$$\eta < \delta;\ M_1 := \max_{|p|<1/2+\epsilon} |\mathbf{H}(p)|;\ M_2(\epsilon) := \max_{0 \le x \le 1-\epsilon} |\mathbf{H}(p)|;\quad \epsilon < \frac{\delta}{4M_1} \tag{5.163}$$

Let $K_2 := K_2(\eta; \epsilon)$ be large enough so that (5.156) holds with $\eta$ in place of $\delta$ for *real* $x \in [0, 1-\epsilon]$ and *also* in an $\epsilon$ neighborhood in $\mathbb{C}$ of the interval $[0, 1/2 + 2\epsilon]$. We use (5.156) to estimate the second integral in the decomposition (5.161) and the first two integrals on the rhs of (5.162). For the last integral in (5.162) we use the induction hypothesis. If $K_1 > 2K_2(2M_1 + M_2)$, it follows that $|\mathbf{H}^{*\mathbf{l}} * H_1|$ is bounded by (the terms are in the order explained above):

$$M_2(\epsilon)K_2\eta^l + M_1K_2\eta^l + M_1K_2\eta^l + (2\epsilon)M_1K_1\delta^l < K_1\delta^{l+1} \tag{5.164}$$

$\square$

**Proposition 5.165** *The equation (5.158) can be written as*

$$-(1-z)\mathbf{h}(z) = \mathbf{F}(z) - \hat{\Lambda}\mathbf{h}(z) + \hat{B}\int_{\epsilon}^{z}\mathbf{h}(s)ds - \sum_{j=1}^{n}\int_{\epsilon}^{z} h_j(s)\mathbf{D}_j(s-z)ds \quad (5.166)$$

*where*

$$\mathbf{F}(z) := \mathcal{N}(\mathbf{H})(1-z) + \mathbf{F}_1(z) \quad (5.167)$$

$$\mathbf{D}_j = \sum_{|\mathbf{l}|\geq 1} l_j \mathbf{G}_{\mathbf{l}} * \mathbf{H}^{*\bar{\mathbf{l}}^j} + \sum_{|\mathbf{l}|\geq 2} l_j \mathbf{g}_{0,\mathbf{l}} \mathbf{H}^{*\bar{\mathbf{l}}^j};\ \bar{\mathbf{l}}^j := (l_1, l_2, ..(l_j - 1), ..., l_n) \quad (5.168)$$

*extend to analytic functions in* $\mathbb{D}_\epsilon$ *(cf. Proposition 5.159). Moreover, if* $\mathbf{H}$ *is a vector in* $L^1_\nu(\mathbb{R}^+)$ *then, for large* $\nu$, $\mathbf{D}_j \in L^1_\nu(\mathbb{R}^+)$ *and the functions* $\mathbf{F}(z)$ *and* $\mathbf{D}_j$ *extend to analytic functions in* $\mathbb{D}_\epsilon$*.*

**PROOF** Noting that $(\mathbf{Y}_0 - \mathbf{H})^{*2}(1-z) = 0$ for $\epsilon < 1/2$ and $z \in \mathbb{D}_\epsilon$ the result is easily obtained by re-expanding $\mathcal{N}(\mathbf{H} + \mathbf{h})$ since Proposition 5.159 guarantees the uniform convergence of the series thus obtained. The proof that $\mathbf{D}_j \in L^1_\nu$ for large $\nu$ is very similar to the proof of (5.154). The analyticity properties follow easily from Proposition 5.159, since the series involved in $\mathcal{N}(\mathbf{H})$ and $\mathbf{D}_j$ converge uniformly for $|z| < \epsilon$. □

Consider again the equation (5.166). Let $\hat{\Gamma} = \hat{\Lambda} - (1-z)\hat{1}$, where $\hat{1}$ is the identity matrix. By construction $\hat{\Gamma}$ and $\hat{B}$ are block-diagonal, their first block is one-dimensional: $\hat{\Gamma}_{11} = z$ and $\hat{B}_{11} = \beta$. We write this as $\hat{\Gamma} = z \oplus \hat{\Gamma}_c(z)$ and similarly, $\hat{B} = \beta \oplus \hat{B}_c$, where $\hat{\Gamma}_c$ and $\hat{B}_c$ are $(n-1)\times(n-1)$ matrices. $\hat{\Gamma}_c(z)$ and $\hat{\Gamma}_c^{-1}(z)$ are analytic in $\mathbb{D}_\epsilon$.

**Lemma 5.169** *The function* $\mathbf{Y}_0$ *given in Proposition 5.132 can be written in the form*

$$\mathbf{Y}_0(p) = (1-p)^{\beta-1}\mathbf{a}_1(p) + \mathbf{a}_2(p) \quad (\beta \neq 1)$$

$$\mathbf{Y}_0(p) = \ln(1-p)\mathbf{a}_1(p) + \mathbf{a}_2(p) \quad (\beta = 1) \quad (5.170)$$

*for* $p$ *in the region* $(\mathbb{D}_\epsilon + 1) \cap \mathcal{W}$ $(\mathbb{D}_\epsilon + 1 := \{1 + z : z \in \mathbb{D}_\epsilon\})$ *where* $\mathbf{a}_1$, $\mathbf{a}_2$ *are analytic functions in* $\mathbb{D}_\epsilon + 1$ *and* $(\mathbf{a}_1)_j = 0$ *for* $j > 1$.

*Proof.*

Let $\mathbf{Q}(z) := \int_\epsilon^z \mathbf{h}(s)ds$. By Proposition 5.132, $\mathbf{Q}$ is analytic in $\mathbb{D}_\epsilon \cap (1 - \mathcal{W})$. From (5.166) we obtain

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{Q}'(z) - (\beta \oplus \hat{B}_c)\mathbf{Q}(z) = \mathbf{F}(z) - \sum_{j=1}^{n} \int_{\epsilon}^{z} \mathbf{D}_j(s-z) Q_j'(s)ds \quad (5.171)$$

or, after integration by parts in the rhs of (5.171), ($\mathbf{D}_j(0) = 0$, cf. (5.168)),

$$(z \oplus \hat{\Gamma}_c(z))\mathbf{Q}'(z) - (\beta \oplus \hat{B}_c)\mathbf{Q}(z) = \mathbf{F}(z) + \sum_{j=1}^{n} \int_{\epsilon}^{z} \mathbf{D}_j'(s-z) Q_j(s)ds \quad (5.172)$$

With the notation $(Q_1, \mathbf{Q}_\perp) := (Q_1, Q_2, ..., Q_n)$ we write the system in the form

$$\begin{gathered}(z^{-\beta}Q_1(z))' = z^{-\beta-1}\left(F_1(z) + \sum_{j=1}^{n}\int_{\epsilon}^{z} D_{1j}'(s-z)Q_j(s)ds\right) \\ (e^{\hat{C}(z)}\mathbf{Q}_\perp)' = e^{\hat{C}(z)}\hat{\Gamma}_c(z)^{-1}\left(\mathbf{F}_\perp + \sum_{j=1}^{n}\int_{\epsilon}^{z} \mathbf{D}'_\perp(s-z)Q_j(s)ds\right) \\ \hat{C}(z) := -\int_0^z \hat{\Gamma}_c(s)^{-1}\hat{B}_c(s)ds \\ \mathbf{Q}(\epsilon) = 0 \quad (5.173)\end{gathered}$$

After integration we get:

$$\begin{gathered}Q_1(z) = R_1(z) + J_1(\mathbf{Q}) \\ \mathbf{Q}_\perp(z) = \mathbf{R}_\perp(z) + J_\perp(\mathbf{Q}) \quad (5.174)\end{gathered}$$

with

$$\begin{aligned}
&J_1(\mathbf{Q}) = z^{\beta}\int_{\epsilon}^{z} t^{-\beta-1}\sum_{j=1}^{n}\int_{\epsilon}^{t} Q_j(s)D'_{1j}(t-s)dsdt\\
&\quad J_{\perp}(\mathbf{Q})(z) := e^{-\hat{C}(z)}\int_{\epsilon}^{z} e^{\hat{C}(t)}\hat{\Gamma}_c(t)^{-1}\left(\sum_{j=1}^{n}\int_{\epsilon}^{z}\mathbf{D}'_{\perp}(s-z)Q_j(s)ds\right)dt\\
&\qquad\qquad \mathbf{R}_{\perp}(z) := e^{-\hat{C}(z)}\int_{\epsilon}^{z} e^{\hat{C}(t)}\hat{\Gamma}_c(t)^{-1}\mathbf{F}_{\perp}(t)dt\\
&R_1(z) = z^{\beta}\int_{\epsilon}^{z} t^{-\beta-1}F_1(t)dt \qquad (\beta \neq 1)\\
&R_1(z) = F_1(0) + F_1'(0)z\ln z + z\int_{\epsilon}^{z}\frac{F_1(s)-F_1(0)-sF_1'(0)}{s}ds \quad (\beta = 1)
\end{aligned} \tag{5.175}$$

Consider the following space of functions:

$$\begin{aligned}
\mathcal{Q}_{\beta} &= \Big\{\mathbf{Q} \text{ analytic in } \mathbb{D}_{\epsilon}\cap(\mathcal{W}-1) : \mathbf{Q} = z^{\beta}\mathbf{A}(z)+\mathbf{B}(z)\Big\} \text{ for } \beta\neq 1 \text{ and}\\
\mathcal{Q}_{1^+} &= \Big\{\mathbf{Q} \text{ analytic in } \mathbb{D}_{\epsilon}\cap(\mathcal{W}-1) : \mathbf{Q} = z\ln z\mathbf{A}(z)+\mathbf{B}(z)\Big\}
\end{aligned} \tag{5.176}$$

where $\mathbf{A}, \mathbf{B}$ are analytic in $\mathbb{D}_{\epsilon}$. (The decomposition of $\mathbf{Q}$ in (5.176) is unambiguous since $z^{\beta}$ and $z\ln z$ are not meromorphic in $\mathbb{D}_{\epsilon}$.)

The norm

$$\|\mathbf{Q}\| = \sup\{|\mathbf{A}(z)|, |\mathbf{B}(z)| : z\in\mathbb{D}_{\epsilon}\} \tag{5.177}$$

makes $\mathcal{Q}_{\beta}$ a Banach space.
For $A(z)$ analytic in $\mathbb{D}_{\epsilon}$ the following elementary identities are useful in what follows:

$$\begin{aligned}
&\int_{\epsilon}^{z} A(s)s^r ds = Const + z^{r+1}\int_0^1 A(zt)t^r dt = Const + z^{r+1}A_1(z)\\
&\quad\int_0^z s^r\ln s\, A(s)ds = z^{r+1}\ln z\int_0^1 A(zt)t^r dt + z^{r+1}\int_0^1 A(zt)t^r\ln t dt
\end{aligned} \tag{5.178}$$

where $A_1$ is analytic and the second equality is obtained by differentiating with respect to $r$ the first equality.

Using (5.178) it is straightforward to check that the rhs of (5.174) extends to a linear inhomogeneous operator on $\mathcal{Q}_\beta$ with image in $\mathcal{Q}_\beta$ and that the norm of $J$ is $O(\epsilon)$ for small $\epsilon$. For instance, one of the terms in $J$ for $\beta = 1$,

$$\begin{aligned} z\int_0^z t^{-2}\int_0^t s\ln s\, A(s)D'(t-s)ds = \\ z^2\ln z\int_0^1\int_0^1 \sigma A(z\tau\sigma)D'(z\tau - z\tau\sigma)d\sigma d\tau + \\ z^2\int_0^1 d\tau\int_0^1 d\sigma\sigma(\ln\tau + \ln\sigma)A(z\tau\sigma)D'(z\tau - z\tau\sigma) \end{aligned} \tag{5.179}$$

manifestly in $\mathcal{Q}_\beta$ if $A$ is analytic in $\mathbb{D}_\epsilon$. Comparing with (5.176), the extra power of $z$ accounts for a norm $O(\epsilon)$ for this term.

Therefore, in (5.173) $(1-J)$ is invertible and the solution $\mathbf{Q} \in \mathcal{Q}_\beta \subset \mathcal{L}(\mathcal{D})$. In view of the the uniqueness of $\mathbf{Y}_0$ (cf. Proposition 5.132), the rest of the proof of Lemma 5.169 is immediate.

## 5.11d General solution of (5.88) on $[0, 1+\epsilon]$

Let $\mathbf{Y}_0$ be the solution given by Proposition 5.132, take $\epsilon$ small enough and denote by $\mathcal{O}_\epsilon$ a neighborhood in $\mathbb{C}$ of width $\epsilon$ of the interval $[0, 1+\epsilon]$.

**Remark 5.180** . $\mathbf{Y}_0 \in L^1(\mathcal{O}_\epsilon)$. *As* $\phi \to \pm 0$, $\mathbf{Y}_0(pe^{i\phi}) \to \mathbf{Y}_0^{\pm}(p)$ *in the sense of* $L^1([0, 1+\epsilon])$ *and also in the sense of pointwise convergence for* $p \neq 1$, *where*

$$\mathbf{Y}_0^{\pm} := \begin{cases} \mathbf{Y}_0(p) & p < 1 \\ (1 - p \pm 0i)^{\beta-1}\mathbf{a}_1(p) + \mathbf{a}_2(p) & p > 1 \end{cases} \quad (\beta \neq 1)$$

$$\mathbf{Y}_0^{\pm} := \begin{cases} \mathbf{Y}_0(p) & p < 1 \\ \ln(1 - p \pm 0i)\mathbf{a}_1(p) + \mathbf{a}_2(p) p > 1 \end{cases} \quad (\beta = 1) \tag{5.181}$$

*Moreover,* $\mathbf{Y}_0^{\pm}$ *are* $L^1_{\text{loc}}$ *solutions of the convolution equation (5.88) on the interval* $[0, 1+\epsilon]$.

The proof is immediate from Lemma 5.169 and Proposition 5.144.

□

**Proposition 5.182** *For any* $\lambda \in \mathbb{C}$ *the combination* $\mathbf{Y}_\lambda = \lambda\mathbf{Y}_0^+ + (1-\lambda)\mathbf{Y}_0^-$ *is a solution of (5.88) on* $[0, 1+\epsilon]$.

*Proof.* For $p \in [0,1) \cup (1, 1+\epsilon]$ let $\mathbf{y}_\lambda(p) := \mathbf{Y}_\lambda - \mathbf{H}(p)$. Since $\mathbf{y}_\lambda^{*2} = 0$ the equation (5.88) is actually linear in $\mathbf{y}_\lambda$ (compare with (5.166)).

$\square$

*

Note: We consider the application $\mathcal{B}_\lambda := \tilde{\mathbf{y}}_0 \mapsto \mathbf{Y}_\lambda$ and require that it be compatible with complex conjugation of functions $\mathcal{B}_\lambda(\tilde{\mathbf{y}}_0^*) = (\mathcal{B}_\lambda(\tilde{\mathbf{y}}_0))^*$ where $F^*(z) := \overline{F(\overline{z})}$. We get Re $\lambda = 1/2$. It is natural to choose $\lambda = 1/2$ to make the linear combination a true average. This choice corresponds, on $[0, 1+\epsilon]$, to the balanced averaging (5.117).

*

**Remark 5.183** *For any $\delta > 0$ there is a constant $C(\delta)$ so that for large $\nu$*

$$\|(\mathbf{Y}_0^{ba})^{*\mathbf{l}}\|_u < C(\delta)\delta^{|\mathbf{l}|} \quad \forall \mathbf{l} \textit{ with } |\mathbf{l}| > 1 \tag{5.184}$$

*($\|\|_u$ is taken on the interval $[0, 1+\epsilon]$).*

Without loss of generality, assume that $l_1 > 1$. Using the notation (5.168) we get

$$\left\|\int_0^p (\mathbf{Y}_0)_1^{ba}(s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(p-s)ds\right\|_u \le$$
$$\left\|\int_0^{\frac{p}{2}} (\mathbf{Y}_0^{ba})_1(s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(p-s)ds\right\|_{u_2} + \left\|\int_0^{\frac{p}{2}} (\mathbf{Y}_0)_1(p-s)(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(s)ds\right\|_{u_2} \tag{5.185}$$

($\|\|_{u_2}$ refers to the interval $p \in [0, 1/2+\epsilon/2]$.) The first $u_2$ norm can be estimated directly using Corollary 5.140 whereas we majorize the second one by

$$\|(\mathbf{Y}_0^{ba})_1\|_\nu \|(\mathbf{Y}_0^{ba})^{*\bar{\mathbf{l}}^1}(x)\|_{u_2}$$

and apply Corollary 5.140 to it for $|\mathbf{l}| > 2$ (if $|\mathbf{l}| = 2$ simply observe that $(\mathbf{Y}_0^{ba})^{*\mathbf{l}}$ is analytic on $[0, 1/2+\epsilon/2]$).

$\square$

**Lemma 5.186** *The set of all solutions of (5.88) in $L^1_{\text{loc}}([0, 1+\epsilon])$ is parameterized by a complex constant $C$ and is given by*

$$\mathbf{Y}_0(p) = \begin{cases} \mathbf{Y}_0^{ba}(p) & \textit{for } p \in [0,1) \\ \mathbf{Y}_0^{ba}(p) + C(p-1)^{\beta-1}\mathbf{A}(p) & \textit{for } p \in (1, 1+\epsilon] \end{cases} \tag{5.187}$$

*for $\beta \neq 1$ or, for $\beta = 1$,*

$$\mathbf{Y}_0(p) = \begin{cases} \mathbf{Y}_0^{ba}(p) & \textit{for } p \in [0,1) \\ \mathbf{Y}_0^{ba}(p) + C(p-1)\mathbf{A}(p) & \textit{for } p \in (1, 1+\epsilon] \end{cases} \tag{5.187$'$}$$

*where* $\mathbf{A}$ *extend analytically in a neighborhood of* $p = 1$.

*Different values of* $C$ *correspond to different solutions.*

*This result remains true if* $\mathbf{Y}_0^{ba}$ *is replaced by any other combination* $\mathbf{Y}_\lambda := \lambda\mathbf{Y}_0^+ + (1-\lambda)\mathbf{Y}_0^-$, $\lambda \in \mathbb{C}$.

*Proof.*

We look for solutions of (5.88) in the form

$$\mathbf{Y}^{ba}(p) + \mathbf{h}(p-1) \tag{5.188}$$

From Lemma 5.169 it follows that $\mathbf{h}(p-1) = 0$ for $p < 1$. Note that

$$\mathcal{N}(\mathbf{Y}_0^{ba} \circ \tau_{-1} + \mathbf{h})(z) = \mathcal{N}(\mathbf{Y}_0^{ba})(1+z) + \sum_{j=1}^{n} \int_0^z h_j(s)\mathbf{D}_j(z-s)ds \tag{5.189}$$

where the $\mathbf{D}_j$ are given in (5.168), and by Remark 5.184 all infinite sums involved are uniformly convergent. For $z < \epsilon$ (5.88) translates into (compare with (5.166)):

$$-(1+z)\mathbf{h}(z) = -\hat{\Lambda}\mathbf{h}(z) - \hat{B}\int_0^z \mathbf{h}(s)ds + \sum_{j=1}^{n} \int_0^z h_j(s)\mathbf{D}_j(z-s)ds \tag{5.190}$$

Let

$$\mathbf{Q}(z) := \int_0^z \mathbf{h}(s)ds \tag{5.191}$$

As we are looking for solutions $\mathbf{h} \in L^1$, we have $\mathbf{Q} \in C_a[0,\epsilon]$, and $\mathbf{Q}(0) = 0$. Following the same steps as in the proof of Lemma 5.169 we get the system of equations:

$$(z^{-\beta}Q_1(z))' = z^{-\beta-1}\sum_{j=1}^{n}\int_0^z D'_{1j}(z-s)Q_j(s)ds$$

$$(e^{\hat{C}(z)}\mathbf{Q}_\perp)' = e^{\hat{C}(z)}\hat{\Gamma}_c(z)^{-1}\sum_{j=1}^{n}\int_0^z \mathbf{D}'_\perp(z-s)Q_j(s)ds$$

$$\hat{C}(z) := -\int_0^z \hat{\Gamma}_c(s)^{-1}\hat{B}_c(s)ds$$

$$\mathbf{Q}(0) = 0 \tag{5.192}$$

which by integration gives

$$(\hat{1} + J)\mathbf{Q}(z) = C\mathbf{R}(z) \tag{5.193}$$

where $C \in \mathbb{C}$ and

$$\begin{gathered}
(J(\mathbf{Q}))_1(z) = z^{\beta} \int_0^z t^{-\beta-1} \sum_{j=1}^{n} \int_0^t Q_j(s) D'_{1j}(t-s)\,ds\,dt \\
J(\mathbf{Q})_{\perp}(z) := e^{-\hat{C}(z)} \int_0^z e^{\hat{C}(t)} \hat{\Gamma}_c(t)^{-1} \left( \sum_{j=1}^{n} \int_0^z \mathbf{D}'_{\perp}(z-s) Q_j(s)\,ds \right) dt \\
\mathbf{R}_{\perp} = 0 \\
R_1(z) = z^{\beta}
\end{gathered} \tag{5.194}$$

First we note the presence of an arbitrary constant $C$ in (5.193) (Unlike in Lemma 5.169 when the initial condition, given at $z = \epsilon$ was determining the integration constant, now the initial condition $\mathbf{Q}(0) = 0$ is satisfied for all $C$).

For small $\epsilon$ the norm of the operator $J$ defined on $C_a[0, \epsilon]$ is $O(\epsilon)$, as in the proof of Lemma 5.169. Given $C$ the solution of the system (5.192) is unique and can be written as

$$\mathbf{Q} = C\mathbf{Q}_0; \quad \mathbf{Q}_0 := (\hat{1} + J)^{-1}\mathbf{R} \neq 0 \tag{5.195}$$

It remains to find the analytic structure of $\mathbf{Q}_0$. We now introduce the space

$$\mathcal{Q} = \left\{ \mathbf{Q} : [0, \epsilon) \mapsto \mathbb{C}^n : \mathbf{Q} = z^{\beta}\mathbf{A}(z) \right\} \tag{5.196}$$

where $\mathbf{A}(z)$ extends to an analytic function in $\mathbb{D}_{\epsilon}$. With the norm (5.177) (with $\mathbf{B} \equiv \mathbf{0}$), $\mathcal{Q}$ is a Banach space. As in the proof of Lemma 5.169 the operator $J$ extends naturally to $\mathcal{Q}$ where it has a norm $O(\epsilon)$ for small $\epsilon$. It follows immediately that

$$\mathbf{Q}_0 \in \mathcal{Q} \tag{5.197}$$

The formulas (5.187), (5.187') follow from (5.188) and (5.191).

□

**Remark 5.198** *If $S_{\beta} \neq 0$ (cf. Lemma 5.169) then the* general *solution of (5.88) is given by*

$$\mathbf{Y}_0(p) = (1 - \lambda)\mathbf{Y}_0^{+}(p) + \lambda\mathbf{Y}_0^{-}(p) \tag{5.199}$$

*with $\lambda \in \mathbb{C}$.*

Indeed, if $\mathbf{a}_1 \not\equiv 0$ (cf. Lemma 5.169) we get at least two distinct solutions of (5.193) (i.e., two distinct values of $C$) by taking different values of $\lambda$ in (5.199). The remark follows from (5.197) (5.196) and Lemma 5.186.

□

### 5.11e The solutions of (5.88) on $[0,\infty)$

In this section we show that the leading asymptotic behavior of $\mathbf{Y}_p$ as $p \to 1^+$ determines a unique solution of (5.88) in $L^1_{\text{loc}}(\mathbb{R}^+)$. Furthermore, any $L^1_{\text{loc}}$ solution of (5.88) is exponentially bounded at infinity and thus Laplace transformable. We also study some properties of these solutions and of their Laplace transforms.

Let $\check{\mathbf{Y}}$ be a solution of (5.88) on an interval $[0, 1+\epsilon]$, which we extend to $\mathbb{R}^+$ letting $\check{\mathbf{Y}}(\mathbf{p}) = \mathbf{0}$ for $p > 1+\epsilon$. For a large enough $\nu$, define

$$\mathcal{S}_{\check{\mathbf{Y}}} := \{\mathbf{f} \in L^1_{\text{loc}}([0,\infty)) : \mathbf{f}(p) = \check{\mathbf{Y}}(p) \text{ on } [0, 1+\epsilon]\} \tag{5.200}$$

and

$$\mathcal{S}_0 := \{\mathbf{f} \in L^1_{\text{loc}}([0,\infty)) : \mathbf{f}(p) = 0 \text{ on } [0, 1+\epsilon]\} \tag{5.201}$$

We extend $\check{\mathbf{Y}}$ to $\mathbb{R}^+$ by setting $\check{\mathbf{Y}}(p) = 0$ for $p > 1+\epsilon$. For $p \geq 1+\epsilon$ (5.88) reads:

$$-p(\check{\mathbf{Y}} + \boldsymbol{\delta}) = F_0 - \hat{\Lambda}(\check{\mathbf{Y}} + \boldsymbol{\delta}) - \hat{B}\int_0^p (\check{\mathbf{Y}} + \boldsymbol{\delta})(s)ds + \mathcal{N}(\check{\mathbf{Y}} + \boldsymbol{\delta}) \tag{5.202}$$

with $\boldsymbol{\delta} \in \mathcal{S}_0$, or

$$\boldsymbol{\delta} = -\check{\mathbf{Y}} + (\hat{\Lambda} - p)^{-1}\left(F_0 - \hat{B}\int_0^p (\check{\mathbf{Y}} + \boldsymbol{\delta})(s)ds + \mathcal{N}(\check{\mathbf{Y}} + \boldsymbol{\delta})\right) := \mathcal{M}(\boldsymbol{\delta}) \tag{5.203}$$

For small $\phi_0 > 0$ and $0 \leq \rho_1 < \rho_2 \leq \infty$, consider the truncated sectors

$$S^{\pm}_{(\rho_1,\rho_2)} := \{z : z = \rho e^{\pm i\phi}, \rho_1 < \rho < \rho_2;\ 0 \leq \phi < \phi_0\} \tag{5.204}$$

and the spaces of functions analytic in $S^{\pm}_{(\rho_1,\rho_2)}$ and continuous in its closure:

$$\mathcal{Q}^{\pm}_{\rho_1,\rho_2} = \left\{\mathbf{f} : \mathbf{f} \in C(\overline{S_{(\rho_1,\rho_2)}});\ \mathbf{f} \text{ analytic in } S^{\pm}_{(\rho_1,\rho_2)}\right\} \tag{5.205}$$

which are Banach spaces with respect to $\|\|_u$ on compact subsets of $\overline{S_{(\rho_1,\rho_2)}}$.

**Proposition 5.206** *i) Given $\check{\mathbf{Y}}$, the equation (5.203) has a unique solution in $L^1_{\text{loc}}[1+\epsilon,\infty)$. For large $\nu$, this solution is in $L^1_\nu([1+\epsilon,\infty))$ and thus Laplace transformable.*

*ii) Let* $\mathbf{Y}_0$ *be the solution defined in Proposition 5.132. Then*

$$\mathbf{Y}_0^{\pm}(p) := \lim_{\phi\to\pm 0} \mathbf{Y}_0(pe^{i\phi}) \in C(\mathbb{R}^+\backslash\{1\}) \cap L^1_{\rm loc}(\mathbb{R}^+) \tag{5.207}$$

*(and the limit exists pointwise on* $\mathbb{R}^+\backslash\{1\}$ *and in* $L^1_{\rm loc}(\mathbb{R}^+)$*.)*

*Furthermore,* $\mathbf{Y}_0^{\pm}$ *are particular solutions of (5.88) and*

$$\mathbf{Y}_0^{\pm}(p) = (1-p)^{\beta-1}\mathbf{a}^{\pm}(p) + \mathbf{a}_1^{\pm}(p) \quad (\beta \neq 1)$$

$$\mathbf{Y}_0^{\pm}(p) = \ln(1-p)\,\mathbf{a}^{\pm}(p) + \mathbf{a}_1^{\pm}(p) \quad (\beta = 1) \tag{5.208}$$

*where* $\mathbf{a}^{\pm}$ *and* $\mathbf{a}_1^{\pm}$ *are analytic near* $p=1$*.*

*Proof*

Note first that by Proposition 5.144, $\mathcal{M}$ (eq. (5.203)) is well defined on $\mathcal{S}_0$, (eq.(5.201)). Moreover, since $\check{\mathbf{Y}}$ is a solution of (5.88) on $[0, 1+\epsilon)$, we have, for $\boldsymbol{\delta}_0 \in \mathcal{S}_0$, $\mathcal{M}(\boldsymbol{\delta}) = 0$ a.e. on $[0, 1+\epsilon)$, i.e.,

$$\mathcal{M}(\mathcal{S}_0) \subset \mathcal{S}_0$$

**Remark 5.209** *For large* $\nu$*,* $\mathcal{M}$ *is a contraction in a small neighborhood of the origin in* $\|\|_{u,\nu}$*.*

Indeed, $\sup\{\|(\hat{\Lambda}-p)^{-1}\|_{\mathbb{C}^n\mapsto\mathbb{C}^n} : p \geq 1+\epsilon\} = O(\epsilon^{-1})$ so that

$$\|\mathcal{M}(\boldsymbol{\delta}_1) - \mathcal{M}(\boldsymbol{\delta}_2)\|_{u,\nu} \leq \frac{\rm Const}{\epsilon}\|\mathcal{N}(\boldsymbol{\delta}_1)) - \mathcal{N}(\boldsymbol{\delta}_2))\|_{u,\nu} \tag{5.210}$$

The rest follows from (5.154) —Proposition 5.144 and Proposition 5.139 applied to $\check{\mathbf{Y}}$.

□

The existence of a solution of (5.203) in $\mathcal{S}_0 \cap L^1_\nu([0,\infty))$ for large enough $\nu$ is now immediate.

Uniqueness in $L^1_{\rm loc}$ is tantamount to uniqueness in $L^1([1+\epsilon, K]) = L^1_\nu([1+\epsilon, K]$, for all $K-1-\epsilon \in \mathbb{R}^+$. Now, assuming $\mathcal{M}$ had two fixed points in $L^1_\nu([1+\epsilon, K])$, by Proposition 5.139, we can choose $\nu$ large enough so that these solutions have arbitrarily small norm, in contradiction with Remark 5.209.

*ii*). For $p<1, \mathbf{Y}_0^{\pm}(p) = \mathbf{Y}_0(p)$. For $p \in (1, 1+\epsilon)$ the result follows from Lemma 5.169. Noting that (in view of the estimate (5.149)) $\mathcal{M}(\mathcal{Q}^{\pm}{}_{1+\epsilon,\infty}) \subset \mathcal{V}^{\pm}{}_{1+\epsilon,\infty}$, the rest of the proof follows from the Remark 5.209 and Lemma 5.169.

□

### 5.11f General $L^1_{loc}$ solution of the convolution equation

**Proposition 5.211** *There is a one parameter family of solutions of equation (5.88) in* $L^1_{\text{loc}}[0,\infty)$*, branching off at* $p=1$ *and in a neighborhood of* $p=1$ *all solutions are of the form (5.187), (5.187′). The general solution of (5.88) is Laplace transformable for large* $\nu$ *and the Laplace transform is a solution of the original differential equation in the half-space* $\mathrm{Re}\,(x)>\nu$*.*

*Proof.* Let $\mathbf{Y}$ be any solution of (5.88). By Lemma 5.186 and Proposition 5.206, $\nu$ large implies that $\mathbf{Y}\in L^1_\nu([0,\infty))$ (thus $\mathcal{L}\mathbf{Y}$ exists), that $\|\mathbf{Y}\|_\nu$ is small and, in particular, that the sum defining $\mathcal{N}$ in (5.86) is convergent in $L^1_\nu(\mathbb{R}^+)$. We have

$$\mathcal{L}\sum_{|\mathbf{l}|\geq \mathbf{1}}\mathbf{G}_{\mathbf{l}}*\mathbf{Y}^{*\mathbf{l}}+\sum_{|\mathbf{l}|\geq 2}\mathbf{g}_{0,\mathbf{l}}\mathbf{Y}^{*\mathbf{l}}$$
$$=\sum_{|\mathbf{l}|\geq \mathbf{1}}(\mathcal{L}\mathbf{G}_{\mathbf{l}})(\mathcal{L}\mathbf{Y})^{\mathbf{l}}+\sum_{|\mathbf{l}|\geq 2}\mathbf{g}_{0,\mathbf{l}}\,(\mathcal{L}\mathbf{Y})^{\mathbf{l}}=\sum_{|\mathbf{l}|\geq \mathbf{1}}\mathbf{g}_{\mathbf{l}}\mathbf{y}^{\mathbf{l}}=\mathbf{g} \quad (5.212)$$

(and $\mathbf{g}(x,\mathbf{y}(x))$ is analytic for $\mathrm{Re}\,(x)>\nu$). The rest is straightforward.

□

**Corollary 5.213** *There is exactly a one parameter family of solutions of (5.50) having the asymptotic behavior described by* $\tilde{\mathbf{y}}_0$ *in the half-plane* $\mathrm{Re}\,(x)>0$.

*Proof.* Any solution with the properties stated in the corollary is inverse Laplace transformable and its inverse Laplace transform has to be one of the $L^1_{\text{loc}}$ solutions of the convolution equation (5.88). The rest of the proof follows from Proposition 5.211.

□

### 5.11g Equations and properties of $\mathbf{Y_k}$ and summation of the transseries

**Proposition 5.214** *Let* $\mathbf{Y}$ *be any* $L^1_{\text{loc}}(\mathbb{R}^+)$ *solution of (5.88). For large* $\nu$ *and some* $c>0$ *the coefficients* $\mathbf{d_m}$ *in (5.91) are bounded by*

$$|\mathbf{d_m}(p)|\leq e^{c_0 p}c^{|\mathbf{m}|}$$

Note that $\mathcal{L}^{-1}(\mathbf{g}^{(\mathbf{m})}(x,\mathbf{y})/\mathbf{m}!)$ is the coefficient of $\mathbf{Z}^{*\mathbf{m}}$ in the expansion of $\mathcal{N}(\mathbf{Y}+\mathbf{Z})$ in convolution powers of $Z$ (5.86):

$$\left(\left(\sum_{|\mathbf{l}|\ge\mathbf{2}}\mathbf{g}_{0,\mathbf{l}}\cdot+\sum_{|\mathbf{l}|\ge\mathbf{1}}\mathbf{G}_{\mathbf{l}}*\right)(\mathbf{Y}+\mathbf{Z})^{*\mathbf{l}}\right)_{\mathbf{Z}^{*\mathbf{m}}}=$$
$$\left(\left(\sum_{|\mathbf{l}|\ge\mathbf{2}}\mathbf{g}_{0,\mathbf{l}}\cdot+\sum_{|\mathbf{l}|\ge\mathbf{1}}\mathbf{G}_{\mathbf{l}}*\right)\sum_{0\le\mathbf{k}\le\mathbf{l}}\binom{\mathbf{l}}{\mathbf{k}}\mathbf{Z}^{*\mathbf{k}}\mathbf{Y}^{*(\mathbf{l}-\mathbf{k})}\right)_{\mathbf{Z}^{*\mathbf{m}}}=$$
$$\left(\sum_{|\mathbf{l}|\ge\mathbf{2}}\mathbf{g}_{0,\mathbf{l}}\cdot+\sum_{|\mathbf{l}|\ge\mathbf{1}}\mathbf{G}_{\mathbf{l}}*\right)\sum_{\mathbf{l}\ge\mathbf{m}}\binom{\mathbf{l}}{\mathbf{m}}\mathbf{G}_{\mathbf{l}}*\mathbf{Y}^{*(\mathbf{l}-\mathbf{m})} \qquad (5.215)$$

($\mathbf{m}$ is fixed) where $\mathbf{l}\ge\mathbf{m}$ means $l_i\ge m_i, i=1,...,n$ and $\binom{\mathbf{l}}{\mathbf{k}}:=\prod_{i=1}^n\binom{l_i}{k_i}$.

Let $\epsilon$ be small and $\nu$ large so that $\|\mathbf{Y}\|_\nu<\epsilon$. Then, for some constant $K$, we have (cf. (5.134))

$$\left|\left(\sum_{II}\mathbf{g}_{0,\mathbf{l}}\cdot+\sum_{I}\mathbf{G}_{\mathbf{l}}*\right)\binom{\mathbf{l}}{\mathbf{m}}\mathbf{G}_{\mathbf{l}}*\mathbf{Y}^{*(\mathbf{l}-\mathbf{m})}\right|\le\sum_{I}Ke^{c_0|p|}(c_0\epsilon)^{|\mathbf{l}-\mathbf{m}|}\binom{\mathbf{l}}{\mathbf{m}}=$$

$$\epsilon^{-|\mathbf{m}|}Ke^{c_0|p|}\prod_{i=1}^n\sum_{l_i\ge m_i}\binom{l_i}{m_i}(c_0\epsilon)^{l_i}=K\frac{e^{c_0|p|}{c_0}^{|\mathbf{m}|}}{(1-\epsilon c_0)^{|\mathbf{m}|+n}}\le e^{c_0|p|}c^{|\mathbf{m}|} \qquad (5.216)$$

(where $I(II,resp.)\equiv\{|\mathbf{l}|\ge 1(2,resp.);\mathbf{l}\ge\mathbf{m}\}$) for large enough $\nu$.

□

For $k=1$, $\mathbf{R}_1=0$ and equation (5.91) is (5.190) (with $p\leftrightarrow z$) but now on the whole line $\mathbb{R}^+$. For small $z$ the solution is given by (5.195) (note that $\mathbf{D}_1=\mathbf{d}_{(1,0,\dots,0)}$ and so on) and depends on the free constant $C$ (5.195). We choose a value for $C$ (the values of $\mathbf{Y}_1$ on $[0,\epsilon]$ are then determined) and we write the equation of $\mathbf{Y}_1$ for $p\ge\epsilon$ as

$$(\hat\Lambda-1-p)\mathbf{Y}_1(p)-\hat A\int_\epsilon^p\mathbf{Y}_1(s)ds-\sum_{j=1}^n\int_\epsilon^p(\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)ds$$
$$=\mathbf{R}(p):=\int_0^\epsilon\mathbf{Y}_1(s)ds+\sum_{j=1}^n\int_0^\epsilon(\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)ds \qquad (5.217)$$

($\mathbf{R}$ only depends on the values of $\mathbf{Y}_1(p)$ on $[0,\epsilon]$). We write

$$(1+J_1)\mathbf{Y}_1=\hat Q_1^{-1}\mathbf{R} \qquad (5.218)$$

with $Q_1 = 1 - \hat{\Lambda} + p$. The operator $J_1$ is defined by $(J_1\mathbf{Y}_1)(p) := 0$ for $p < \epsilon$, while, for $p > \epsilon$ we write

$$(J_1\mathbf{Y}_1)(p) := Q_1^{-1}\left(\hat{B}\int_\epsilon^p \mathbf{Y}_1(s)ds - \sum_{j=1}^n \int_\epsilon^p (\mathbf{Y}_1)_j(s)\mathbf{D}_j(p-s)ds\right)$$

By Proposition 5.165, Proposition 5.139 and the Banach algebra properties, cf. §5.1, and noting that $\sup_{p>\epsilon}\|Q_1^{-1}\| = O(\epsilon^{-1})$, we find that $(1 + J_1)$ is invertible as an operator in $L^1_\nu$ since:

$$\|J_1\|_{L^1_\nu \mapsto L^1_\nu} < \sup_{p>\epsilon}\|\hat{Q}_1^{-1}\|\left(\|\hat{B}\|\|1\|_\nu + n\max_{1\le j\le n}\|\mathbf{D_j}\|_\nu\right) \to 0 \text{ as } \nu \to \infty \tag{5.219}$$

Given $C$, $\mathbf{Y}_1$ is therefore uniquely determined from (5.218) as an $L^1_\nu(\mathbb{R}^+)$ function.

The analytic structure of $\mathbf{Y}_1$ for small $z$ is contained in in (5.187), (5.187'). As a result,

$$\mathcal{L}(\mathbf{Y}_1)(x) \sim C\sum_{k=0}^\infty \frac{\Gamma(k+\beta)}{x^{k+\beta}}\mathbf{a}_k \tag{5.220}$$

where $\sum_{k=0}^\infty \mathbf{a}_k z^k$ is the series of $\mathbf{a}(z)$ near $z = 0$.

Correspondingly, we write (5.91) as

$$(1 + J_k)\mathbf{Y}_k = \hat{Q}_k^{-1}\mathbf{R}_k \tag{5.221}$$

with $\hat{Q}_k := (-\hat{\Lambda} + p + k)$ and

$$(J_k\mathbf{h})(p) := \hat{Q}_k^{-1}\left(\hat{B}\int_0^p \mathbf{h}(s)ds - \sum_{j=1}^n \int_0^p h_j(s)\mathbf{D}_j(p-s)ds\right) \tag{5.222}$$

$$\|J_k\|_{L^1_\nu \mapsto L^1_\nu} < \sup_{p\ge 0}\|\hat{Q}_k^{-1}\|\left(\|\hat{B}\|\|1\|_\nu + n\max_{1\le j\le n}\|\mathbf{D_j}\|_\nu\right) \tag{5.223}$$

Since $\sup_{p\ge 0}\|\hat{Q}_k^{-1}\| \to 0$ as $k \to \infty$ we have

$$\sup_{k\ge 1}\left\{\|J_k\|_{L^1_\nu \mapsto L^1_\nu}\right\} \to 0 \text{ as } \nu \to \infty \tag{5.224}$$

Thus,

**Proposition 5.225** *For large $\nu$, $(1 + J_k), k \ge 1$ are simultaneously invertible in $L^1_\nu$, (cf. 5.224). Given $\mathbf{Y}_0$ and $C$, $\mathbf{Y}_k, k \ge 1$ are uniquely determined and moreover, for $k \ge 2$, the following estimate holds*

$$\|\mathbf{Y}_k\|_\nu \le \frac{\sup_{p\ge 0}\|\hat{Q}_k^{-1}\|}{1-\sup_{k\ge 1}\|J_k\|_{L^1_\nu\mapsto L^1_\nu}}\|\mathbf{R}_k\|_\nu := K\|\mathbf{R}_k\|_\nu \tag{5.226}$$

□

(Note: There is a *one-parameter* only freedom in $\mathbf{Y}_k$: a change in $\mathbf{Y}_0$ can be compensated by a corresponding change in $C$.)

Because of the condition $\sum m = k$ in the definition of $\mathbf{R}_\mathrm{k}$, we get, by an easy induction, the homogeneity relation with respect to the free constant $C$,

$$\mathbf{Y}_k^{[C]} = C^k\mathbf{Y}_k^{[C=1]} =: C^k\mathbf{Y}_k \tag{5.227}$$

**Proposition 5.228** *For any $\delta > 0$ there is a large enough $\nu$, so that*

$$\|\mathbf{Y}_k\|_\nu < \delta^k,\ \ k = 0, 1, ... \tag{5.229}$$

*Each $\mathbf{Y}_k$ is Laplace transformable and $\mathbf{y}_k = \mathcal{L}(\mathbf{Y}_k)$ solve (5.89).*

*Proof*

We first show inductively that the $\mathbf{Y}_k$ are bounded. Choose $r$ small enough and $\nu$ large so that $\|\mathbf{Y}_0\|_\nu < r$. Note that in the expression of $\mathbf{R}_k$, only $\mathbf{Y}_i$ with $i < k$ appear. We show by induction that $\|\mathbf{Y}_k\|_\nu < r$ for all $k$. Using (5.226), (5.91) the explanation to (5.89) and Proposition 5.214 we get

$$\|\mathbf{Y}_k\|_\nu < K\|\mathbf{R}_k\|_\nu \le \sum_{|\mathbf{l}|>1} {c_0}^{|\mathbf{l}|} r^k \sum_{\Sigma m=k} 1 \le r^k \left(\sum_{l>1}\binom{l}{k}{c_0}^l\right)^n \le (r(1{+}c_0)^n)^k < r \tag{5.230}$$

if $r$ is small which completes this induction step. But now if we look again at (5.230) we see that in fact $\|\mathbf{Y}_k\|_\nu \le (r(1 + c_0)^n)^k$. Choosing $r$ small enough, (and to that end, $\nu$ large enough) the first part of Proposition 5.228 follows. Laplace transformability as well as the fact that $\mathbf{y}_k$ solve (5.89) follow immediately from (5.229) (observe again that, given $k$, there are only finitely many terms in the sum in $\mathbf{R}_k$).

□

Therefore,

**Remark 5.231** *The series*

$$\sum_{k=0}^{\infty} C^k(\mathbf{Y}_k\cdot\mathcal{H})\circ\tau_k \tag{5.232}$$

*is convergent in $L^1_\nu$ for large $\nu$ and thus the sum is Laplace transformable. By Proposition 5.229 we have*

$$\mathcal{L}\sum_{k=0}^{\infty} C^k (\mathbf{Y}_k \mathcal{H}) \circ \tau_k = \sum_{k=0}^{\infty} C^k e^{-kx} \mathcal{L}\mathbf{Y}_k \tag{5.233}$$

*is uniformly convergent for large $x$ (together with its derivatives with respect to $x$). Thus (by construction) (5.233) is a solution of (5.50).*

$\square$

(Alternatively, we could have checked in a straightforward way that the series (5.232), truncated to order $N$ is a solution of the convolution equation (5.88) on the interval $p \in [0, N)$ and in view of the $L^1_\nu(\mathbb{R}^+)$ (or even $L^1_{\mathrm{loc}}$) convergence it has to be one of the general solutions of the convolution equation and therefore provide a solution to (5.50).)

*Proof of Proposition 5.112, ii)*

We now show (5.114). This is done from the system (5.91) by induction on $k$. For $k = 0$ and $k = 1$ the result follows from Proposition 5.132 and Proposition 5.180. For the induction step we consider the operator $J_k$ (5.222) on the space

$$\mathcal{Q}_k = \left\{ \mathbf{Q} : [0, \epsilon) \mapsto \mathbb{C} : \mathbf{Q}(z) = z^{k\beta - 1} \mathbf{A}_k(z) \right\} \tag{5.234}$$

where $\mathbf{A}_k$ extends as an analytic function in a neighborhood $\mathbb{D}_\epsilon$ of $z = 0$. Endowed with the norm

$$\|\mathbf{Q}\| = \sup_{z \in \mathbb{D}_\epsilon} |\mathbf{A}_k(z)|$$

$\mathcal{Q}_k$ is a Banach space.

**Remark 5.235** *For $k \in \mathbb{N}$ the operators $J_k$ in (5.222) extend continuously to $\mathcal{Q}_k$ and their norm is $O(\epsilon)$. The functions $\mathbf{R}_k$, $k \in \mathbb{N}$ (cf. (5.221), (5.91)), belong to $\mathcal{Q}_k$. Thus for $k \in \mathbb{N}$, $\mathbf{Y}_k \in \mathcal{Q}_k$.*

If $A, B$ are analytic then for $z < \epsilon$

$$\int_0^z ds\, s^{k\beta - 1} A(s) B(z - s) = z^{k\beta} \int_0^1 dt\, t^r A(zt) B(z(1 - t)) \tag{5.236}$$

is in $\mathcal{Q}_{\mathrm{k}}$ with norm $O(\epsilon)$ and the assertion about $J_k$ follows easily. Therefore $\mathbf{Y}_k \in \mathcal{Q}_{\mathrm{k}}$ if $\mathbf{R}_k \in \mathcal{Q}_{\mathrm{k}}$. We prove both of these properties by induction and (by the homogeneity of $\mathbf{R}_k$ and the fact that $\mathbf{R}_k$ depends only on $\mathbf{Y}_m, m < k$) this amounts to checking that if $\mathbf{Y}_m \in \mathcal{Q}_m$ and $\mathbf{Y}_n \in \mathcal{Q}_n$ then

$$\mathbf{Y}_m * \mathbf{Y}_n \in \mathcal{Q}_{m+n}$$

as a result of the identity

$$\int_0^z ds\, s^r A(s)(z-s)^q B(z-s) = z^{r+q+1} \int_0^1 dt\, t^r (1-t)^q A(zt) B(z-zt)$$

$\square$

It is now easy to see that $\mathcal{L}_\phi \mathcal{B} \tilde{\mathbf{y}}_k \sim \tilde{\mathbf{y}}_k$ (cf. Theorem 5.119). Indeed, note that in view of Remark 5.235 and Proposition 5.228, $\mathcal{L}(\mathbf{Y}_k)$ have asymptotic power series that can be differentiated for large $x$ in the positive half plane. Since $\mathcal{L}(\mathbf{Y}_k)$ are actual solutions of the system (5.89) their asymptotic series are formal solutions of (5.89) and by the uniqueness of the formal solution of (5.89) once $C$ is given, the property follows.

In the next subsection, we prove that the general solution of the system (5.89) can be obtained by means of Borel transform of formal series and analytic continuation.

We define $\mathbf{Y}^+$ to be the function defined in Proposition 5.206, extended in $\mathcal{W} \cap \mathbb{C}^+$ by the unique solution of (5.88) $\mathbf{Y}_0$ provided by Proposition 5.132. (We define $\mathbf{Y}^-$ correspondingly.)

By Proposition 5.206 *ii*), $\mathbf{Y}^\pm$ are solutions of (5.88) on $[0,\infty)$ (cf. (5.205)). By Lemma 5.186 any solution on $[0,\infty)$ can be obtained from, say, $\mathbf{Y}^+$ by choosing $C$ and then solving uniquely (5.203) on $[1+\epsilon,\infty)$ (Proposition 5.206). We now show that the solutions of (5.218), (5.221) are continuous boundary values of functions analytic in a region bounded by $\mathbb{R}^+$.

**Remark 5.237** *The function* $\mathbf{D}(s)$ *defined in (5.168) by substituting* $\mathbf{H} = \mathbf{Y}^\pm$, *is in* $\mathcal{Q}^\pm_{0,\infty}$ *(cf. (5.205)).*

By Proposition 5.206, *ii*) it is easy to check that if $\mathbf{H}$ is any function in $\mathcal{Q}^+_{0,A}$ then $\mathbf{Y}^+ * \mathbf{Q} \in \mathcal{T}^+_{0,A}$. Thus, with $\mathbf{H} = \mathbf{Y}^+$, all the terms in the infinite sum in (5.168) are in $\mathcal{Q}^+_{0,A}$. For fixed $A > 0$, taking $\nu$ large enough, the norm $\rho_\nu$ of $\mathbf{Y}^+$ in $L^1_\nu$ can be made arbitrarily small uniformly in all rays in $S^+_{0,A}$ (5.205) (Proposition 5.206). Then by Corollary 5.140 and Proposition 5.206 *ii*), the uniform norm of each term in the series

$$\mathbf{D}_j = \sum_{|\mathbf{l}|\ge 1} l_j \mathbf{G}_{\mathbf{l}} * \left(\mathbf{Y}^\pm\right)^{*\bar{\mathbf{l}}^j} + \sum_{|\mathbf{l}|\ge 2} l_j \mathbf{g}_{0,\mathbf{l}} \left(\mathbf{Y}^\pm\right)^{*\bar{\mathbf{l}}^j} \ ; \ \bar{\mathbf{l}}^j := (l_1, l_2, ..(l_j - 1), ..., l_n) \tag{5.238}$$

can be estimated by $Const\, \rho_\nu^{|\mathbf{l}-1|} c^{|\mathbf{l}|}$ and thus the series converges uniformly in $\mathcal{Q}^+{}_{0,\infty}$, for large $\nu$. $\square$

**Lemma 5.239** *i) The system (5.91) with* $\mathbf{Y}_0 = \mathbf{Y}^+$ *(or* $\mathbf{Y}^-$*) and given* $C$ *(say* $C = 1$*) has a unique solution in* $L^1_{\rm loc}(\mathbb{R}^+)$, *namely* $\mathbf{Y}^+_k$, *(*$\mathbf{Y}^-_k$, *resp.),* $k \in \mathbb{N}$. *Furthermore, for large* $\nu$ *and all* $k$, $\mathbf{Y}^+_k \in \mathcal{Q}^+_{0,\infty}$ *(*$\mathbf{Y}^-_k \in \mathcal{Q}^-_{0,\infty}$*) (cf. (5.205)).*

*ii) The general solution of the equation (5.88) in* $L^1_{\rm loc}(\mathbb{R}^+)$ *can be written in either of the forms:*

$$\mathbf{Y}^+ + \sum_{k=1}^{\infty} C^k(\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k \quad \textit{or} \quad \mathbf{Y}^- + \sum_{k=1}^{\infty} C^k(\mathbf{Y}_k^- \cdot \mathcal{H}) \circ \tau_k \tag{5.240}$$

**PROOF**

i) The first part follows from the same arguments as Proposition 5.225. For the last statement it is easy to see (cf. (5.236)) that $J_k\mathcal{Q}_{0,\infty}^+ \subset \mathcal{Q}_{0,\infty}^+$ the inequalities (5.223), (5.224) hold for $\|\|_{\mathcal{Q}_{0,\mathrm{A}}\mapsto\mathcal{Q}_{0,\mathrm{A}}}$ ($A$ arbitrary) replacing $\|\|_{L^1_\nu\mapsto L^1_\nu}$ (cf. §5.1).

*ii*) We already know that $\mathbf{Y}^+$ solves (5.91) for $k=0$. For $k>0$ by *i*) $C^k\mathbf{Y}_k \in \mathcal{Q}_{0,\infty}$ and so, by continuity, the boundary values of $\mathbf{Y}_k^+$ on $\mathbb{R}^+$ solve the system (5.91) on $\mathbb{R}^+$ in $L^1_{\mathrm{loc}}$. The rest of *ii)* follows from Lemma 5.186, Proposition 5.206 and the arbitrariness of $C$ in (5.240) (cf. also (5.195).

$\Box$

### 5.11h Analytic structure, resurgence, averaging

Having the general structure of the solutions of (5.88) given in Proposition 5.123 and in Lemma 5.239 we can obtain various analytic identities. The function $\mathbf{Y}_0^\pm := \mathbf{Y}^\pm$ has been defined in the previous section.

**Proposition 5.241** *For $m \geq 0$,*

$$\mathbf{Y}_m^- = \mathbf{Y}_m^+ + \sum_{k=1}^{\infty} \binom{m+k}{m} S_\beta^k(\mathbf{Y}_{m+k}^+ \cdot \mathcal{H}) \circ \tau_k \tag{5.242}$$

*Proof.*

$\mathbf{Y}_0^-(p)$ is a particular solution of (5.88). It follows from Lemma 5.239 that the following identity holds on $\mathbb{R}^+$:

$$\mathbf{Y}_0^- = \mathbf{Y}_0^+ + \sum_{k=1}^{\infty} S_\beta^k(\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k \tag{5.243}$$

since, by (5.126) and (5.113), (5.243) holds for $p \in (0,2)$.

By Lemma 5.239 for any $C_+$ there is a $C_-$ so that

$$\mathbf{Y}_0^+ + \sum_{k=1}^{\infty} C_+^k(\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = \mathbf{Y}_0^- + \sum_{k=1}^{\infty} C_-^k(\mathbf{Y}_k^- \cdot \mathcal{H}) \circ \tau_k \tag{5.244}$$

To find the relation between $C_+$ and $C_-$ we take $p \in (1,2)$; we get, comparing with (5.243):

$$\mathbf{Y}_0^+(p) + C_+\mathbf{Y}_1(p-1) = \mathbf{Y}_0^-(p) + C_-\mathbf{Y}_1(p-1) \Rightarrow C_+ = C_- + S_\beta \tag{5.245}$$

whence, for any $C \in \mathbb{C}$,

$$\mathbf{Y}_0^+ + \sum_{k=1}^{\infty} (C + S_\beta)^k (\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = \mathbf{Y}_0^- + \sum_{k=1}^{\infty} C^k (\mathbf{Y}_k^- \cdot \mathcal{H}) \circ \tau_k \tag{5.246}$$

Differentiating $m$ times with respect to $C$ and taking $C = 0$ we get

$$\sum_{k=m}^{\infty} \frac{k!}{(k-m)!} S_\beta^{k-m} (\mathbf{Y}_k^+ \cdot \mathcal{H}) \circ \tau_k = m! (\mathbf{Y}_m^- \cdot \mathcal{H}) \circ \tau_m$$

from which we obtain (5.242) by rearranging the terms and applying $\tau_{-m}$.

$\square$

**Proposition 5.247** *The functions* $\mathbf{Y}_k$, $k \geq 0$, *are analytic in* $\mathcal{R}_1$.

**PROOF**

Starting with (5.243), if we take $p \in (1, 2)$ and obtain:

$$\mathbf{Y}_0^-(p) = \mathbf{Y}_0^+(p) + S_\beta \mathbf{Y}_1(p-1) \tag{5.248}$$

By Proposition 5.206 and Lemma 5.239 the lhs of (5.248) is analytic in a lower half plane neighborhood of $(\varepsilon, 1-\varepsilon)$, $(\forall \varepsilon \in (0,1))$ and continuous in the closure of such a neighborhood. The rhs is analytic in an upper half plane neighborhood of $(\varepsilon, 1-\varepsilon)$, $(\forall \varepsilon \in (0,1))$ and continuous in the closure of such a neighborhood. Thus, $\mathbf{Y}_0^-(p)$ can be analytically continued along a path crossing the interval $(1, 2)$ from below, i.e., $\mathbf{Y}_0^{-+}$ exists and is analytic.

Now, in (5.243), let $p \in (2, 3)$. Then,

$$\begin{aligned} S_\beta^2 \mathbf{Y}_2(p-2) &= \mathbf{Y}_0(p)^- - \mathbf{Y}(p)^+ - S_\beta \mathbf{Y}_1(p-1)^+ = \\ &\mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^+ - \mathbf{Y}_0(p)^{-+} + \mathbf{Y}_0(p)^+ = \mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^{-+} \end{aligned} \tag{5.249}$$

and, in general, taking $p \in (k, k+1)$ we get

$$S_\beta^k \mathbf{Y}_k(p-k) = \mathbf{Y}_0(p)^- - \mathbf{Y}_0(p)^{-^{k-1}+} \tag{5.250}$$

Using (5.250) inductively, the same arguments that we used for $p \in (0,1)$ show that $\mathbf{Y}_0^{-^k}(p)$ can be continued analytically in the upper half plane. Thus, we have

**Remark 5.251** *The function* $\mathbf{Y}_0$ *is analytic in* $\mathcal{R}_1$. *In fact, for* $p \in (j, j+1)$, $k \in \mathbb{N}$,

$$\mathbf{Y}_0^{-^j+}(p) = \mathbf{Y}_0^+(p) + \sum_{k=1}^{j} S_\beta^k \mathbf{Y}_k^+(p-k) \mathcal{H}(p-k) \tag{5.252}$$

The relation (5.252) follows from (5.250) and (5.243).

▯

Note: Unlike (5.243), the sum in (5.252) contains finitely many terms. For instance we have:

$$\mathbf{Y}_0^{-+}(p) = \mathbf{Y}_0^{+}(p) + \mathcal{H}(p-1)\mathbf{Y}_1^{+}(p-1). \quad (\forall p \in \mathbb{R}^+) \tag{5.253}$$

Analyticity of $\mathbf{Y}_m$, $m \geq 1$ is shown inductively on $m$, using (5.242) and following exactly the same course of proof as for $k = 0$.

□

**Remark 5.254** *If $S_\beta = 0$ then $\mathbf{Y}_k$ are analytic in $\mathcal{W}_1 \cup \mathbb{N}$.*

Indeed, this follows from (5.243) (5.242) and Lemma 5.239, *i*).

□

On the other hand, if $S_\beta \neq 0$, then all $\mathbf{Y}_k$ are analytic continuations of the Borel transform of $\mathbf{Y}_0$ (cf. (5.249)) -an instance of resurgence.
Moreover, we can now calculate $\mathbf{Y}_0^{ba}$. By definition, (see the discussion before Remark 5.183) on the interval $(0, 2)$,

$$\mathbf{Y}_0^{ba} = \frac{1}{2}(\mathbf{Y}_0^{+} + \mathbf{Y}_0^{-}) = \mathbf{Y}_0^{+} + \frac{1}{2}S_\beta(\mathbf{Y}_1\,\mathcal{H}) \circ \tau_1 \tag{5.255}$$

Now we are looking for a solution of (5.88) which satisfies the condition (5.255). By comparing with Lemma 5.239, which gives the general form of the solutions of (5.88), we get, now on the whole positive axis,

$$\mathbf{Y}_0^{ba} = \mathbf{Y}_0^{+} + \sum_{k=1}^{\infty} \frac{1}{2^k} S_\beta^k(\mathbf{Y}_k^{+}\mathcal{H}) \circ \tau_k \ (\text{on } \mathbb{R}^+) \tag{5.256}$$

which we can rewrite using (5.250):

$$\mathbf{Y}_0^{ba} = \mathbf{Y}_0^{+} + \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\mathbf{Y}_0^{-^k} - \mathbf{Y}_0^{-^{k-1}+}\right)(\mathcal{H} \circ \tau_k) \tag{5.257}$$

**Proposition 5.258** *Let $\psi_1(p), \psi_2(p)$ be analytic in $\mathcal{R}_1$, and such that for any path $\gamma = t \mapsto t\exp(i\phi(t))$ in $\mathcal{R}_1$,*

$$|\psi_{1,2}(\gamma(t))| < f_\gamma(t) \in L^1_{\text{loc}}(\mathbb{R}^+) \tag{5.259}$$

*Assume further that for some large enough $\nu, M$ and any path $\gamma$ in $\mathcal{R}_1$ we have*

$$\int_\gamma |\psi_{1,2}|(s)e^{-\nu|s|}|ds| < M \tag{5.260}$$

*Then the analytic continuation* $AC_\gamma(\psi_1 * \psi_2)$ *along a path* $\gamma$ *in* $\mathcal{R}_1$*, of their convolution product* $\psi_1 * \psi_2$ *(defined for small* $p$ *by (2.20)) exists, is locally integrable and satisfies (5.259) and, for the same* $\nu$ *and some* $\gamma$*-independent* $M' > 0$*,*

$$\int_\gamma |\psi_1 * \psi_2|(s) e^{-\nu|s|} |ds| < M' \tag{5.261}$$

*Proof.* Since

$$2\psi_1 * \psi_2 = (\psi_1 + \psi_2) * (\psi_1 + \psi_2) - \psi_1 * \psi_1 - \psi_2 * \psi_2 \tag{5.262}$$

it is enough to take $\psi_1 = \psi_2 = \psi$. For $p \in \mathbb{R}^+ \backslash \mathbb{N}$ we write:

$$\psi^- = \psi^+ + \sum_{k=1}^{\infty} (\mathcal{H} \cdot \psi_k^+) \circ \tau_k \tag{5.263}$$

The functions $\psi_k$ are *defined* inductively (the superscripts "+,(-)" mean, as before, the analytic continuations in $\mathcal{R}_1$ going below(above) the real axis). In the same way (5.250) was obtained we get by induction:

$$\psi_k = (\psi^- - \psi^{-^{k-1}+}) \circ \tau_{-k} \tag{5.264}$$

where the equality holds on $\mathbb{R}^+ \backslash \mathbb{N}$ and $+, -$ mean the upper and lower continuations. For any $p$ only finitely many terms in the sum in (5.263) are nonzero. The sum is also convergent in $\|\|_\nu$ (by dominated convergence; note that, by assumption, the functions $\psi^{--\cdots-\pm}$ belong to the same $L^1_\nu$).

If $t \mapsto \gamma(t)$ in $\mathcal{R}_1$, is a straight line, other than $\mathbb{R}^+$, then:

$$AC_\gamma((\psi * \psi)) = AC_\gamma(\psi) *_\gamma AC_\gamma(\psi) \text{ if } \arg(\gamma(t))\text{=const} \neq 0 \tag{5.265}$$

(Since $\psi$ is analytic along such a line). The notation $*_\gamma$ means (2.20) with $p = \gamma(t)$.

Note though that, suggestive as it might be, (5.265) is *incorrect* if the condition stated there is not satisfied and $\gamma$ is a path that crosses the real line (see the Appendix, Section 5.12a )!

We get from (5.265), (5.263)

$$(\psi * \psi)^- = \psi^- * \psi^- = \psi^+ * \psi^+ + \sum_{k=1}^{\infty} \left( \mathcal{H} \sum_{m=0}^{k} \psi_m^+ * \psi_{k-m}^+ \right) \circ \tau_k =$$

$$(\psi * \psi)^+ + \sum_{k=1}^{\infty} \left( \mathcal{H} \sum_{m=0}^{k} (\psi_m * \psi_{k-m})^+ \right) \circ \tau_k \tag{5.266}$$

and now the analyticity of $\psi * \psi$ in $\mathcal{R}_1$ follows: on the interval $p \in (m, m+1)$ we have from (5.264)

$$\big(\psi * \psi\big)^{-^j}(p) = \big(\psi * \psi\big)^{-}(p) = \big(\psi^{*2}\big)^{+}(p) + \sum_{k=1}^{j}\sum_{m=0}^{k}\big(\psi_m * \psi_{k-m}\big)^{+}(p-k) \tag{5.267}$$

Again, formula (5.267) is useful for analytically continuing $\big(\psi * \psi\big)^{-^j}$ along a path as the one depicted in Fig. 5.1. By dominated convergence, $(\psi * \psi)^{\pm} \in \mathcal{V}^{\pm}_{(0,\infty)}$, (5.205). By (5.264), $\psi_m$ are analytic in $\mathcal{R}_1^+ := \mathcal{R}_1 \cap \{p : \mathrm{Im}\,(p) > 0\}$ and thus by (5.265) the rhs. of (5.267) can be continued analytically in $\mathcal{R}_1^+$. The same is then true for $(\psi * \psi)^-$. The function $(\psi * \psi)$ can be extended analytically along paths that cross the real line from below. Likewise, $(\psi * \psi)^+$ can be continued analytically in the lower half plane so that $(\psi * \psi)$ is analytic in $\mathcal{R}_1$.

Combining (5.267), (5.265) and (5.262) we get a similar formula for the analytic continuation of the convolution product of two functions, $f, g$ satisfying the assumptions of Proposition 5.258

$$(f * g)^{-^j +} = f^+ * g^+ + \sum_{k=1}^{j}\left(\mathcal{H}\sum_{m=0}^{k} f_m^+ * g_{k-m}^+\right) \circ \tau_k \tag{5.268}$$

Note that (5.268) corresponds to (5.263) and in those notations we have:

$$\big(f * g\big)_k = \sum_{m=0}^{k} f_m * g_{k-m} \tag{5.269}$$

Integrability as well as (5.261) follow from (5.264), (5.267) and Remark 5.139. $\Box_{P5.258}$

By (5.117) and (5.264),

$$\psi^{ba} = \psi^+ + \sum_{k=1}^{\infty}\frac{1}{2^k}(\psi_k^+\mathcal{H}) \circ \tau_k$$

so that

$$\psi^{ba} * \psi^{ba} = \left(\psi^+ + \sum_{k=1}^{\infty}\frac{1}{2^k}(\mathcal{H}\circ\tau_k)(\psi_k^+ \circ \tau_k)\right)^{*2} =$$

$$\psi^+ * \psi^+ + \sum_{k=1}^{\infty}\frac{1}{2^k}\mathcal{H}\circ\tau_k\sum_{m=0}^{k}(\psi_m^+ \circ \tau_m) * (\psi_{k-m}^+ \circ \tau_{k-m}) \circ \tau_k =$$

$$\psi^+ * \psi^+ + \sum_{k=1}^{\infty}\frac{1}{2^k}\mathcal{H}\circ\tau_k\sum_{m=0}^{k}(\psi_m * \psi_{k-m})^+ \circ \tau_k = (\psi^{*2})^{ba} \tag{5.270}$$

To finish the proof of Theorem 5.119 note that on any finite interval the sum in (5.117) has only a finite number of terms and by (5.270) balanced averaging commutes with any finite sum of the type

$$\sum_{k_1,\ldots,k_n} c_{k_1..k_n} f_{k_1} * \ldots * f_{k_n} \tag{5.271}$$

and then, by continuity, with any sum of the form (5.271), with a finite or infinite number of terms, provided it converges in $L^1_{\rm loc}$. Averaging thus commutes with all the operations involved in the equations (5.221). By uniqueness therefore, if $\mathbf{Y}_0 = \mathbf{Y}^{ba}$ then $\mathbf{Y}_k = \mathbf{Y}_k^{ba}$ for all $k$. Preservation of reality is immediate since (5.88), (5.91) are real if (5.50) is real, therefore $\mathbf{Y}_0^{ba}$ is real-valued on $\mathbb{R}^+\backslash\mathbb{N}$ (since it is real-valued on $[0,1)\cup(1,2)$) and so are, inductively, all $\mathbf{Y}_k$.

---

## 5.12 Appendix

### 5.12a $AC(f*g)$ versus $AC(f)*AC(g)$

Typically, the analytic continuation along curve in $\mathcal{W}_1$ which is not homotopic to a straight line does not commute with convolution.

**Remark 5.272** *Let $\psi$ be a function satisfying the conditions stated in Proposition 5.258 and assume that $p=1$ is a branch point of $\psi$. Then,*

$$(\psi*\psi)^{-+} \neq \psi^{-+} * \psi^{-+} \tag{5.273}$$

*Proof*
Indeed, by (5.268) and (5.264)

$$\begin{gathered}(\psi*\psi)^{-+} = \psi^+ * \psi^+ + 2[(\psi^+ * \psi_1^+)\mathcal{H}] \circ \tau_1 \neq \psi^{-+} * \psi^{-+} = \\ [\psi^+ + (\mathcal{H}\psi_1^+)\circ\tau_1]^{*2} = \psi^+ * \psi^+ + 2[(\psi^+ * \psi_1^+)\mathcal{H}] \circ \tau_1 + [\mathcal{H}(\psi_1^+ * \psi_1^+)] \circ \tau_2\end{gathered} \tag{5.274}$$

since in view of (5.264), in our assumptions, $\psi_1 \not\equiv 0$ and thus $\psi_1 * \psi_1 \not\equiv 0$.
□

There is also the following intuitive reasoning leading to the same conclusion. For a generic system of the form (5.50), $p=1$ is a branch point of $\mathbf{Y}_0$ and so $\mathbf{Y}_0^- \neq \mathbf{Y}_0^{-+}$. On the other hand, if $AC_{-+}$ commuted with convolution, then $\mathcal{L}(\mathbf{Y}_0^{-+})$ would provide a solution of (5.50). By Lemma 5.239, $\mathcal{L}(\mathbf{Y}_0^-)$ is a different solution (since $\mathbf{Y}_0^- \neq \mathbf{Y}_0^{-+}$). As $\mathbf{Y}_0^-$ and $\mathbf{Y}_0^{-+}$ coincide up to $p=2$ we have $\mathcal{L}(\mathbf{Y}_0^{-+}) - \mathcal{L}(\mathbf{Y}_0^-) = e^{-2x(1+o(1))}$ as $x \to +\infty$. By Theorem 5.119 however, no two solutions of (5.50) can differ by less than $e^{-x(1+o(1))}$ without actually being equal (also, heuristically, this can be checked using formal perturbation theory), contradiction.

## 5.12b Derivation of the equations for the transseries for general ODEs.

Consider first the scalar equation

$$y' = f_0(x) - \lambda y - x^{-1}By + g(x,y) = -y + x^{-1}By + \sum_{k=1}^{\infty} g_k(x) y^k \tag{5.275}$$

For $x \to +\infty$ we take

$$y = \sum_{k=0}^{\infty} y_k \mathrm{e}^{-kx} \tag{5.276}$$

where $y_k$ can be formal series $x^{-s_k}\sum_{n=0}^{\infty} a_{kn} x^{-n}$, with $a_{k,0} \neq 0$, or actual functions with the condition that (5.276) converges uniformly. Let $y_0$ be the first term in (5.276) and $\delta = y - y_0$. We have

$$\begin{aligned} y^k - y_0^k - k y_0^{k-1}\delta &= \sum_{j=2}^{k} \binom{k}{j} y_0^{k-j}\delta^j = \sum_{j=2}^{k} \binom{k}{j} y_0^{k-j} \sum_{i_1,\dots,i_j=1}^{\infty} \prod_{s=1}^{j} \left( y_{i_s} \mathrm{e}^{-i_s x} \right) \\ &= \sum_{m=1}^{\infty} \mathrm{e}^{-mx} \sum_{j=2}^{k} \binom{k}{j} y_0^{k-j} \sum_{(i_s)}^{(m;j)} \prod_{s=1}^{j} y_{i_s} \end{aligned} \tag{5.277}$$

where $\sum_{(i_s)}^{(m;j)}$ means the sum over all positive integers $i_1, i_2, \dots, i_j$ satisfying $i_1 + i_2 + \cdots + i_j = m$. Let $d_1 = \sum_{k\geq 1} k g_k y_0^{k-1}$. Introducing $y = y_0 + \delta$ in (5.275) and equating the coefficients of $\mathrm{e}^{-lx}$ we get, by separating the terms containing $y_l$ for $l \geq 1$ and interchanging the $j,k$ orders of summation,

$$\begin{aligned} y_l' + (\lambda(1-l) + x^{-1}B - d_1(x))y_l &= \sum_{j=2}^{\infty} \sum_{(i_s)}^{(l;j)} \prod_{s=1}^{j} y_{i_s} \sum_{k\geq\{2,j\}} \binom{k}{j} g_k y_0^{k-j} \\ = \sum_{j=2}^{l} \sum_{(i_s)}^{(l;j)} \prod_{s=1}^{j} y_{i_s} \sum_{k\geq\{2,j\}} \binom{k}{j} g_k y_0^{k-j} &=: \sum_{j=2}^{l} d_j(x) \sum_{(i_s)}^{(l;j)} \prod_{s=1}^{j} y_{i_s} \end{aligned} \tag{5.278}$$

where for the middle equality we note that the infinite sum terminates because $i_s \geq 1$ and $\sum_{s=1}^{j} i_s = l$.

For a vectorial equation like (5.50) we first write

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} - x^{-1}\hat{B}\mathbf{y} + \sum_{\mathbf{k}\succ 0} \mathbf{g}_{\mathbf{k}}(x)\mathbf{y}^{\mathbf{k}} \tag{5.279}$$

with $\mathbf{y}^{\mathbf{k}} := \prod_{i=1}^{n_1}(\mathbf{y})_i^{k_i}$. As with (5.278), we introduce the transseries (5.110) in (5.279) and equate the coefficients of $\exp(-\mathbf{k}\cdot\boldsymbol{\lambda}x)$. Let $\mathbf{v}_{\mathbf{k}} = x^{-\mathbf{k}\cdot\mathbf{m}}\mathbf{y}_{\mathbf{k}}$ and

$$\mathbf{d}_{\mathbf{j}}(x) = \sum_{\mathbf{l}\geq\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \mathbf{g}_{\mathbf{l}}(x)\mathbf{v}_0^{\mathbf{l}-\mathbf{j}} \tag{5.280}$$

Noting that, by assumption, $\mathbf{k}\cdot\boldsymbol{\lambda} = \mathbf{k}'\cdot\boldsymbol{\lambda} \Leftrightarrow \mathbf{k} = \mathbf{k}'$ we obtain, for $\mathbf{k}\in\mathbb{N}^{n_1}$, $\mathbf{k}\succ 0$

$$\mathbf{v}_{\mathbf{k}}' + \left(\hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\hat{I} + x^{-1}\hat{B}\right)\mathbf{v}_{\mathbf{k}} + \sum_{|\mathbf{j}|=1}\mathbf{d}_{\mathbf{j}}(x)(\mathbf{v}_{\mathbf{k}})^{\mathbf{j}}$$
$$= \sum_{\substack{\mathbf{j}\leq\mathbf{k}\\ |\mathbf{j}|\geq 2}} \mathbf{d}_{\mathbf{j}}(x) \sum_{(\mathbf{i}_{mp}:\mathbf{k})} \prod_{m=1}^{n} \prod_{p=1}^{j_m} \left(\mathbf{v}_{\mathbf{i}_{mp}}\right)_m = \mathbf{t}_{\mathbf{k}}(\mathbf{v}) \tag{5.281}$$

where $\binom{\mathbf{l}}{j} = \prod_{j=1}^{n}\binom{l_i}{j_i}$, $(\mathbf{v})_m$ means the component $m$ of $\mathbf{v}$, and $\sum_{(\mathbf{i}_{mp}:\mathbf{k})}$ stands for the sum over all vectors $\mathbf{i}_{mp}\in\mathbb{N}^n$, with $p\leq j_m, m\leq n$, so that $\mathbf{i}_{mp}\succ 0$ and $\sum_{m=1}^{n}\sum_{p=1}^{j_m}\mathbf{i}_{mp} = \mathbf{k}$. We use the convention $\prod_{\emptyset} = 1, \sum_{\emptyset} = 0$. With $m_i = 1 - \lfloor \mathrm{Re}\,\beta_i \rfloor$ we obtain for $\mathbf{y}_{\mathbf{k}}$

$$\mathbf{y}_{\mathbf{k}}' + \left(\hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\hat{I} + x^{-1}(\hat{B} + \mathbf{k}\cdot\mathbf{m})\right)\mathbf{y}_{\mathbf{k}} + \sum_{|\mathbf{j}|=1}\mathbf{d}_{\mathbf{j}}(x)(\mathbf{y}_{\mathbf{k}})^{\mathbf{j}} = \mathbf{t}_{\mathbf{k}}(\mathbf{y}) \tag{5.282}$$

There are clearly finitely many terms in $\mathbf{t}_{\mathbf{k}}(\mathbf{y})$. To find a (not too unrealistic) upper bound for this number of terms, we compare with $\sum_{(\mathbf{i}_{mp})'}$ which stands for the same as $\sum_{(\mathbf{i}_{mp})}$ except with $\mathbf{i}\geq 0$ instead of $\mathbf{i}\succ 0$. Noting that $\binom{k+s-1}{s-1} = \sum_{a_1+\ldots+a_s=k} 1$ is the number of ways $k$ can be written as a sum of $s$ integers, we have

$$\sum_{(\mathbf{i}_{mp})} 1 \leq \sum_{(\mathbf{i}_{mp})'} 1 = \prod_{l=1}^{n_1} \sum_{(\mathbf{i}_{mp})_l} 1 = \prod_{l=1}^{n_1} \binom{k_l + |\mathbf{j}| - 1}{|\mathbf{j}| - 1} \leq \binom{|\mathbf{k}| + |\mathbf{j}| - 1}{|\mathbf{j}| - 1}^{n_1} \tag{5.283}$$

**Remark 5.284** *Equation (5.281) can be written in the form (5.90)*

*Proof.* The fact that only predecessors of $\mathbf{k}$ are involved in $\mathbf{t}(\mathbf{y}_0,\cdot)$ and the homogeneity property of $\mathbf{t}(\mathbf{y}_0,\cdot)$ follow immediately by combining the conditions $\sum\mathbf{i}_{mp} = \mathbf{k}$ and $\mathbf{i}_{mp}\succ 0$. □

The formal inverse Laplace transform of (5.282) is then

$$\left(-p+\hat{\Lambda}-\mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{Y}_{\mathbf{k}}+\left(\hat{B}+\mathbf{k}\cdot\mathbf{m}\right)\mathcal{P}\mathbf{Y}_{\mathbf{k}}+\sum_{|\mathbf{j}|=1}\mathbf{D}_{\mathbf{j}}*(\mathbf{Y}_{\mathbf{k}})^{\mathbf{j}}=\mathbf{T}_{\mathbf{k}}(\mathbf{Y}) \tag{5.285}$$

with

$$\mathbf{T}_{\mathbf{k}}(\mathbf{Y})=\mathbf{T}\left(\mathbf{Y}_0,\{\mathbf{Y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right)=\sum_{\mathbf{j}\le k;\ |\mathbf{j}|>1}\mathbf{D}_{\mathbf{j}}(p)*\sum_{(\mathbf{i}_{mp};\mathbf{k})}{}^{*}\prod_{m=1}^{n_1}{}^{*}\prod_{p=1}^{j_m}\left(\mathbf{Y}_{\mathbf{i}_{mp}}\right)_m \tag{5.286}$$

and

$$\mathbf{D}_{\mathbf{j}}=\sum_{\mathbf{l}\ge\mathbf{m}}\binom{\mathbf{l}}{\mathbf{m}}\mathbf{G}_{\mathbf{l}}*\mathbf{Y}_0^{*(\mathbf{l}-\mathbf{m})}+\sum_{\mathbf{l}\ge\mathbf{m};|\mathbf{l}|\ge 2}\binom{\mathbf{l}}{\mathbf{m}}\mathbf{g}_{0,\mathbf{l}}\mathbf{Y}_0^{*(\mathbf{l}-\mathbf{m})} \tag{5.287}$$

## 5.12c Appendix: formal diagonalization

Consider again the equation

$$\mathbf{y}'=\mathbf{f}_0(x)-\hat{\Lambda}\mathbf{y}+\frac{1}{x}\hat{A}\mathbf{y}+\mathbf{g}(x,\mathbf{y}) \tag{5.288}$$

If $\hat{\Lambda}$ is diagonalizable, then it can be easily diagonalized in (5.288) by the substitution $\mathbf{y}=\hat{C}\mathbf{y}^{[1]}$, where $=\hat{C}^{-1}\hat{\Lambda}\hat{C}$ is diagonal.

So we can assume that $\hat{\Lambda}$ is already diagonal. Now, a transformation of the form $\mathbf{y}=(I+x^{-1}\hat{V})\mathbf{y}^{[1]}$ brings (5.288), up to terms of order $\mathbf{y}/x^2$, to an equation of the type

$$\mathbf{y}'=\mathbf{f}_0(x)-\hat{\Lambda}\mathbf{y}+\frac{1}{x}\left(\hat{A}+\hat{V}\hat{\Lambda}-\hat{V}\hat{\Lambda}\right)\mathbf{y}+\mathbf{g}(x,\mathbf{y}) \tag{5.289}$$

Now we regard the map

$$\hat{\hat{\Lambda}}\hat{V}:=\mapsto\hat{V}\hat{\Lambda}-\hat{V}\hat{\Lambda}$$

as a linear map on the space of matrices $\hat{V}$, or, which is the same, on $\mathbb{C}^{2n}$. The equation

$$\hat{\hat{\Lambda}}\hat{V}=\hat{X} \tag{5.290}$$

has a unique solution *iff* $\hat{X}$ is not in the kernel of $\hat{\hat{\Lambda}}$, which by definition, consists in all matrices such that $\hat{\hat{\Lambda}}\hat{Y}=0$, or, in other words, all matrices which commute with $\hat{\Lambda}$. Since the eigenvalues of $\hat{\Lambda}$ are distinct, it is easy to check that $\hat{\hat{\Lambda}}\hat{Y} = 0$ implies $\hat{Y}$ is diagonal. So, we can change the *off-diagonal* elements of $\hat{A}$ at will, in particular we can choose them to be zero. By further transformations $\mathbf{y} = (I + x^{-j}\hat{V})\mathbf{y}^{[1]}$, $j = 2...m$, we can diagonalize the coefficients of $x^{-2}\mathbf{y}, ..., x^{-m}\mathbf{y}$.

So, we can assume all coefficients of $x^{-j}\mathbf{y}$ up to any fixed $m$ are diagonal. To show that we can actually assume the coefficients of $x^{-j}\mathbf{y}$, $j = 2...m$, to be zero it is then enough to show that this is possible for a scalar equation

$$y' = f_0(x) - \Lambda y + \frac{1}{x}Ay + (A_2x^{-2} + \cdots + A_mx^{-m})y + \mathbf{g}(x,y) \quad (5.291)$$

As usual, by subtracting terms, we can assume $f_0(x) = O(x^{-M})$ for any choice of $M$, so for the purpose of this argument, we can see that we can safely assume $f_0$ is absent.

$$y' = -\Lambda y + \frac{1}{x}Ay + (A_2x^{-2} + \cdots + A_mx^{-m})y + \mathbf{g}(x,y) \quad (5.292)$$

Now, taking $y = (1 + c_1/x + c_2/x^2 + \cdots + c_m/x^m)y^{[1]}$ where for suitable $c_i$ (check!).

## *5.13 Appendix: The $C^*$–algebra of staircase distributions, $\mathcal{D}'_{m,\nu}$

Let $\mathcal{D}$ be the space of test functions (compactly supported $C^\infty$ functions on $(0,\infty)$) and $\mathcal{D}(0,x)$ be the test functions on $(0,x)$.

We say that $f \in \mathcal{D}'$ is a **staircase distribution** if for any $k = 0, 1, 2, ...$ there is an $L^1$ function on $[0, k+1]$ so that $f = F_k^{(km)}$ (in the sense of distributions) when restricted to $\mathcal{D}(0, k+1)$ or

$$F_k := \mathcal{P}^{mk} f \in L_1(0, k+1) \quad (5.293)$$

(since $f \in L^1_{\rm loc}[0, 1-\epsilon]$ and $\mathcal{P}f$ is well defined, [22]). With this choice we have

**Remark 5.294** This space is natural to singular convolution equations arising in ODEs. The solutions are generically singular when $p = n\lambda$ where $\lambda$ is an eigenvalue of $\hat{\Lambda}$ and $n \in \mathbb{N}$. If the singularity at $\lambda$ is nonintegrable, so are generically the singularities at multiples of it, and the strength of the singularity (such as, the order of the pole) grows linearly in $n$.

$$F_{k+1} = \mathcal{P}^m F_k \text{ on } [0,k] \text{ and } F_k^{(j)}(0) = 0 \text{ for } j \le mk - 1 \quad (5.295)$$

We denote these distributions by $\mathcal{D}'_m$ ($\mathcal{D}'_m(0,k)$ respectively, when restricted to $\mathcal{D}(0,k)$) and observe that $\bigcup_{m>0}\mathcal{D}'_m \supset S'$, the distributions of slow growth. The inclusion is strict since any element of $S'$ is of finite order.

Let $f \in L^1$. Taking $F = \mathcal{P}^j f \in C^j$ we have, by integration by parts and noting that the boundary terms vanish,

$$(F * F)(p) = \int_0^p F(s)F(p-s)\mathrm{d}s = \int_0^p F^{(j)}(s)\mathcal{P}^j F(p-s) \qquad (5.296)$$

so that $F * F \in C^{2j}$ and

$$(F * F)^{(2j)} = f * f \qquad (5.297)$$

This motivates the following definition: for $f, \tilde{f} \in \mathcal{D}'_m$ let

$$f * \tilde{f} := (F_k * \tilde{F}_k)^{(2km)} \quad \text{in } \mathcal{D}'(0, k+1) \qquad (5.298)$$

We first check that the definition is consistent in the sense that

$$(F_{k+1} * F_{k+1})^{(2m(k+1))} = (F_k * F_k)^{(2mk)}$$

on $\mathcal{D}(0, k+1)$. For $p < k+1$ integrating by parts and using (5.295) we obtain

$$\frac{\mathrm{d}^{2m(k+1)}}{\mathrm{d}p^{2m(k+1)}} \int_0^p F_k(s)\mathcal{P}^{2m}\tilde{F}_k(p-s)\mathrm{d}s = \frac{\mathrm{d}^{2mk}}{\mathrm{d}p^{2mk}} \int_0^p F_k(s)\tilde{F}_k(p-s)\mathrm{d}s \quad (5.299)$$

The same argument shows that the definition is compatible with the embedding of $\mathcal{D}'_m$ in $\mathcal{D}'_{m'}$ with $m' > m$. Convolution is commutative and associative: with $f, g, h \in \mathcal{D}'_m$ and identifying $(f * g)$ and $h$ by the natural inclusion with elements in $\mathcal{D}'_{2m}$ we obtain $(f * g) * h = ((F * G) * H)^{(4mk)} = f * (g * h)$.

**Note 5.300** *The construction is over $\mathbb{R}^+$; the delta distribution at zero for instance is not obtained in this way.*

The following staircase decomposition exists in $\mathcal{D}'_m$.

**Lemma 5.301** . *For each $f \in \mathcal{D}'_m$ there is a* unique *sequence $\{\Delta_i\}_{i=0,1,\dots}$ such that $\Delta_i \in L^1(\mathbb{R}^+)$, $\Delta_i = \Delta_i \chi_{[i,i+1]}$ and*

$$f = \sum_{i=0}^{\infty} \Delta_i^{(mi)} \qquad (5.302)$$

*Also (cf. (5.295)),*

$$F_i = \sum_{j \le i} \mathcal{P}^{m(i-j)} \Delta_i \; \text{ on } [0, i+1) \qquad (5.303)$$

Note that the infinite sum is $\mathcal{D}'-$convergent since for a given test function only a finite number of distributions are nonzero.

*Proof*

We start by showing (5.303). For $i = 0$ we take $\Delta_0 = F_0\chi[0,1]$ (where $F_0\chi[0,1] := \phi \mapsto \int_0^1 F_0(s)\phi(s)\mathrm{d}s$). Assuming (5.303) holds for $i < n$ we simply note that

$$\begin{aligned}\Delta_n &:= \chi_{[0,n+1]}\left(F_n - \sum_{j\le n-1}\mathcal{P}^{m(n-j)}\Delta_j\right)\\ &= \chi_{[0,n+1]}\Big(F_n - \mathcal{P}^m(F_{n-1}\chi_{[0,n]})\Big) = \chi_{[n,n+1]}\Big(F_n - \mathcal{P}^m(F_{n-1}\chi_{[0,n]})\Big)\end{aligned} \tag{5.304}$$

(with $\chi_{[n,\infty]}F_n$ defined in the same way as $F_0\chi[0,1]$ above) has, by the induction hypothesis and (5.295) the required properties. Relation (5.302) is immediate. It remains to show uniqueness. Assuming (5.302) holds for the sequences $\Delta_i, \tilde{\Delta}_i$ and restricting $f$ to $\mathcal{D}(0,1)$ we see that $\Delta_0 = \tilde{\Delta}_0$. Assuming $\Delta_i = \tilde{\Delta}_i$ for $i < n$ we then have $\Delta_n^{(mn)} = \tilde{\Delta}_n^{(mn)}$ on $\mathcal{D}(0,n+1)$. It follows ([22]) that $\Delta_n(x) = \tilde{\Delta}_n(x) + P(x)$ on $[0,n+1)$ where $P$ is a polynomial (of degree $< mn$). Since by definition $\Delta_n(x) = \tilde{\Delta}_n(x) = 0$ for $x < n$ we have $\Delta_n = \tilde{\Delta}_n(x)$. □

The expression (5.298) hints to decrease in regularity, but this is not the case. In fact, we check that the regularity of convolution is not worse than that of its arguments.

**Remark 5.305**

$$(\cdot * \cdot) : \mathcal{D}_n \mapsto \mathcal{D}_n \tag{5.306}$$

Since

$$\chi_{[a,b]} * \chi_{[a',b']} = \Big(\chi_{[a,b]} * \chi_{[a',b']}\Big)\chi_{[a+a',b+b']} \tag{5.307}$$

we have

$$F * \tilde{F} = \sum_{j+k\le\lfloor p\rfloor}\mathcal{P}^{m(i-j)}\Delta_j * \mathcal{P}^{m(i-k)}\tilde{\Delta}_k = \sum_{j+k\le\lfloor p\rfloor}\Delta_j * \mathcal{P}^{m(2i-j-k)}\tilde{\Delta}_k \tag{5.308}$$

which is manifestly in $C^{2mi-m(j+k)}[0,p) \subset C^{2mi-m\lfloor p\rfloor}[0,p)$.

□

### 5.13 .1 Norms on $\mathcal{D}'_m$

For $f \in \mathcal{D}'_m$ define

$$\|f\|_{\nu,m} := c_m \sum_{i=0}^{\infty} \nu^{im} \|\Delta_i\|_{L^1_\nu} \tag{5.309}$$

(the constant $c_m$, immaterial for the moment, is defined in (5.322). When no confusion is possible we will simply write $\|f\|_\nu$ for $\|f\|_{\nu,m}$ and $\|\Delta\|_\nu$ for $\|\Delta_i\|_{L^1_\nu}$ (no other norm is used for the $\Delta$'s). Let ${\mathcal{D}'}_{m,\nu}$ be the distributions in $\mathcal{D}'_m$ such that $\|f||_\nu < \infty$.

**Remark 5.310** *$\|\cdot\|_\nu$ is a norm on ${\mathcal{D}'}_{m,\nu}$.*

If $\|f\|_\nu = 0$ for all $i$, then $\Delta_i = 0$ whence $f = 0$. In view of Lemma 5.301 we have $\|0\|_\nu = 0$. All the other properties are immediate.

**Remark 5.311** *${\mathcal{D}'}_{m,\nu}$ is a Banach space. The topology given by $\|\cdot\|_\nu$ on ${\mathcal{D}'}_{m,\nu}$ is stronger than the topology inherited from $\mathcal{D}'$.*

*Proof.* If we let $\mathcal{D}'_{m,\nu}(k, k+1)$ be the subset of ${\mathcal{D}'}_{m,\nu}$ where all $\Delta_i = 0$ except for $i = k$, with the norm (5.309), we have

$${\mathcal{D}'}_{m,\nu} = \bigoplus_{k=0}^{\infty} \mathcal{D}'_{m,\nu}(k, k+1) \tag{5.312}$$

and we only need to check completeness of each $\mathcal{D}'_{m,\nu}(k, k+1)$ which is immediate: on $L^1[k, k+1]$, $\|\cdot\|_\nu$ is equivalent to the usual $L^1$ norm and thus if $f_n \in \mathcal{D}'_{m,\nu}(k, k+1)$ is a Cauchy sequence then $\Delta_{k,n} \overset{L_\nu}{\to} \Delta_k$ (whence weak convergence) and $f_n \overset{\mathcal{D}'_{m,\nu}(k,k+1)}{\to} f$ where $f = \Delta_k^{(mk)}$. □

**Lemma 5.313** *The space ${\mathcal{D}'}_{m,\nu}$ is a $C^*$ algebra with respect to convolution.*

*Proof.* Let $f, \tilde{f} \in \mathcal{D}'_{m,\nu}$ with

$$f = \sum_{i=0}^{\infty} \Delta_i^{(mi)} \quad , \quad \tilde{f} = \sum_{i=0}^{\infty} \tilde{\Delta}_i^{(mi)}$$

Then

$$f * \tilde{f} = \sum_{i,j=0}^{\infty} \Delta_i^{(mi)} * \tilde{\Delta}_j^{(mj)} = \sum_{i,j=0}^{\infty} \left(\Delta_i * \tilde{\Delta}_j\right)^{m(i+j)} \tag{5.314}$$

and the support of $\Delta_i * \tilde{\Delta}_j$ is in $[i+j, i+j+2]$ i.e. $\Delta_i * \tilde{\Delta}_j = \chi_{[i+j,i+j+2]} \Delta_i * \tilde{\Delta}_j$.

We first evaluate the norm in $\mathcal{D}'_{m,\nu}$ of the terms $\left(\Delta_i * \tilde{\Delta}_j\right)^{m(i+j)}$.

**I. Decomposition formula.** Let $f = F^{(mk)} \in \mathcal{D}'(\mathbb{R}_+)$, where $F \in L^1(\mathbb{R}_+)$, and $F$ is supported in $[k, k+2]$ i.e., $F = \chi_{[k,k+2]}F$ $(k \geq 0)$. Then $f \in \mathcal{D}'_m$ and the decomposition of $f$ (cf. (5.302)) has the terms:

$$\Delta_0 = \Delta_1 = ... = \Delta_{k-1} = 0 \quad , \quad \Delta_k = \chi_{[k,k+1]}F \tag{5.315}$$

and

$$\Delta_{k+n} = \chi_{[k+n,k+n+1]}G_n, \quad \text{where } G_n = \mathcal{P}^m\left(\chi_{[k+n,\infty)}G_{n-1}\right), \quad G_0 = F \tag{5.316}$$

*Proof of the decomposition formula.* We use first line of (2.98) of the paper

$$\Delta_j = \chi_{[j,j+1]}\left(F_j - \sum_{i=0}^{j-1}\mathcal{P}^{m(j-i)}\Delta_i\right) \tag{5.317}$$

where, in our case, $F_k = F$, $F_{k+1} = \mathcal{P}^m F$, ..., $F_{k+n} = \mathcal{P}^{mn}F$, ....

The relations (5.315) follow directly from (5.317). Formula (5.316) is shown by induction on $n$. For $n = 1$ we have

$$\Delta_{k+1} = \chi_{[k+1,k+2]}\left(\mathcal{P}^m F - \mathcal{P}^m\Delta_k\right)$$

$$= \chi_{[k+1,k+2]}\mathcal{P}^m\left(\chi_{[k,\infty)}F - \chi_{[k,k+1]}F\right) = \chi_{[k+1,k+2]}\mathcal{P}^m\left(\chi_{[k+1,\infty)}F\right)$$

Assume (5.316) holds for $\Delta_{k+j}$, $j \leq n-1$. Using (5.317), with $\chi = \chi_{[k+n,k+n+1]}$ we have

$$\Delta_{k+n} = \chi\left(\mathcal{P}^{mn}F - \sum_{i=k}^{n-1}\mathcal{P}^{m(n-i)}\Delta_i\right) = \chi\mathcal{P}^m\left(G_{n-1} - \Delta_{n-1}\right)$$

$$= \chi\mathcal{P}^m\left(\chi_{[k+n-1,\infty)}G_{n-1} - \chi_{[k+n-1,k+n]}G_{n-1}\right) = \chi\mathcal{P}^m\left(\chi_{[k+n,\infty)}G_{n-1}\right) \quad \Box$$

**II. Estimating $\Delta_{k+n}$.** For $f$ as in **I**, we have

$$||\Delta_{k+1}||_\nu \leq \nu^{-m}||F||_\nu \quad , \quad ||\Delta_{k+2}||_\nu \leq \nu^{-2m}||F||_\nu \tag{5.318}$$

and, for $n \geq 3$

$$||\Delta_{k+n}||_\nu \leq e^{2\nu-n\nu}(n-1)^{nm-1}\frac{1}{(nm-1)!}||F||_\nu \tag{5.319}$$

*Proof of estimates of $\Delta_{k+n}$.*

(A) Case $n = 1$.

$$\begin{aligned}
||\Delta_{k+1}||_\nu &\le \int_{k+1}^{k+2} dt\, e^{-\nu t}\mathcal{P}^m\left(\chi_{[k+1,\infty)}|F|\right)(t)\\
&= \int_{k+1}^{k+2} dt\, e^{-\nu t}\int_{k+1}^{t} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-1}} ds_m |F(s_m)|\\
&\le \int_{k+1}^{k+2} ds_m |F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_2}^{\infty} ds_1 \int_{s_1}^{\infty} dt\, e^{-\nu t}\\
&\qquad\qquad = \int_{k+1}^{k+2} ds_m |F(s_m)| e^{-\nu s_m}\nu^{-m} \le \nu^{-m}||F||_\nu \qquad (5.320)
\end{aligned}$$

(B) Case $n = 2$:

$$||\Delta_{k+1}||_\nu \le \int_{k+2}^{k+3} dt\, e^{-\nu t}\mathcal{P}^m\left(\chi_{[k+2,\infty)}\mathcal{P}^m\left(\chi_{[k+1,\infty)}|F|\right)\right)$$

$$= \int_{k+2}^{k+3} dt\, e^{-\nu t}\int_{k+2}^{t} dt_1 \int_{k+2}^{t_1} dt_2 \ldots \int_{k+2}^{t_{m-1}} dt_m \times \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \ldots \int_{k+1}^{s_{m-1}} ds_m |F(s_m)|$$

$$\le \int_{k+2}^{k+3} ds_m |F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_2}^{\infty} ds_1 \int_{\max\{s_1,k+2\}}^{\infty} dt_m \times \int_{t_m}^{\infty} dt_{m-1} \ldots \int_{t_1}^{\infty} dt\, e^{-\nu t}$$

$$= \int_{k+2}^{k+3} ds_m |F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_2}^{\infty} ds_1 e^{-\nu \max\{s_1,k+2\}}\nu^{-m-1}$$

$$\le \int_{k+2}^{k+3} ds_m |F(s_m)| \int_{s_m}^{\infty} ds_{m-1} \ldots \int_{s_3}^{\infty} ds_2 e^{-\nu s_2}\nu^{-m-2} = \int_{k+2}^{k+3} ds_m |F(s_m)| e^{-\nu s_m}\nu^{-2m}$$

(C) Case $n \ge 3$. We first estimate $G_2, \ldots, G_n$:

$$|G_2(t)| \le \mathcal{P}^m\left(\chi_{[k+2,\infty)}\mathcal{P}^m\left(\chi_{[k+1,\infty)}|F|\right)\right)(t)$$

$$= \int_{k+2}^{t} dt_1 \int_{k+2}^{t_1} dt_2 \dots \int_{k+2}^{t_{m-1}} dt_m \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \dots \int_{k+1}^{s_{m-1}} ds_m |F(s_m)|$$

and using the inequality

$$|F(s_m)| = |F(s_m)|\chi_{[k,k+2]}(s_m) \le |F(s_m)|e^{-\nu s_m} e^{\nu(k+2)}$$

we get

$$|G_2(t)| \le e^{\nu(k+2)} ||F||_\nu \int_{k+1}^{t} dt_1 \int_{k+1}^{t_1} dt_2 \dots, \\ \times \int_{k+1}^{t_{m-1}} dt_m \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \dots \int_{k+1}^{s_{m-2}} ds_{m-1}$$

$$|G_2(t)| \le e^{\nu(k+2)} ||F||_\nu \int_{k+1}^{t} dt_1 \int_{k+1}^{t_1} dt_2 \dots \int_{k+1}^{t_{m-1}} dt_m \\ \times \int_{k+1}^{t_m} ds_1 \int_{k+1}^{s_1} ds_2 \dots, \int_{k+1}^{s_{m-2}} ds_{m-1}$$

$$= e^{\nu(k+2)} ||F||_\nu (t-k-1)^{2m-1} \frac{1}{(2m-1)!}$$

The estimate of $G_2$ is used for bounding $G_3$:

$$|G_3(t)| \le \mathcal{P}^m \left(\chi_{[k+3,\infty)}|G_2|\right) \le \mathcal{P}^m \left(\chi_{[k+1,\infty)}|G_2|\right)$$

$$\le e^{\nu(k+2)} ||F||_\nu (t-k-1)^{3m-1} \frac{1}{(3m-1)!}$$

and similarly (by induction)

$$|G_n(t)| \le e^{\nu(k+2)} ||F||_\nu (t-k-1)^{nm-1} \frac{1}{(nm-1)!}$$

Then

$$||\Delta_{k+n}||_\nu \le e^{\nu(k+2)} ||F||_\nu \frac{1}{(nm-1)!} \int_{k+n}^{k+n+1} dt\, e^{-\nu t} (t-k-1)^{nm-1}$$

and, for $\nu \ge m$ the integrand is decreasing, and the inequality (5.319) follows. **III. Final Estimate**. Let $\nu_0 > m$ be fixed. For $f$ as in **I**, we have for any $\nu > \nu_0$,

$$||f|| \le c_m \nu^{km} ||F||_\nu \tag{5.321}$$

for some $c_m$, if $\nu > \nu_0 > m$.
*Proof of Final Estimate*

$$||f|| = \sum_{n\ge 0} \nu^{km+kn}||\Delta_{k+n}||_\nu \le \nu^{km}||F||_\nu \left[3 + \sum_{n\ge 3} \nu^{nm} e^{2\nu - n\nu} \frac{(n-1)^{nm-1}}{(nm-1)!}\right]$$

and, using $n-1 \le (mn-1)/m$ and a crude Stirling estimate we obtain

$$||f|| \le \nu^{km}||F||_\nu \left[3 + me^{2\nu-1} \sum_{n\ge 3} \left(e^{m-\nu}\nu^m/m^m\right)^n\right] \le c_m \nu^{km}||F||_\nu \quad (5.322)$$

Thus (5.321) is proven for $\nu > \nu_0 > m$.
**End of the proof**. From (5.314) and (5.321) we get

$$||f * \tilde{f}|| \le \sum_{i,j=0}^{\infty} || \left(\Delta_i * \tilde{\Delta}_j\right)^{m(i+j)} ||$$

$$\le \sum_{i,j=0}^{\infty} c_m^2 \nu^{m(i+j)}||\Delta_i * \tilde{\Delta}_j||_\nu \le c_m^2 \sum_{i,j=0}^{\infty} \nu^{m(i+j)}||\Delta_i||_\nu \, ||\tilde{\Delta}_j||_\nu = c_m^2 ||f|| \, ||\tilde{f}||$$

□

**Remark 5.323** *Let $f \in \mathcal{D}'_{m,\nu}$ for some $\nu > \nu_0$ where $\nu_0^m = \mathrm{e}^{\nu_0}$. Then $f \in \mathcal{D}'_{m,\nu'}$ for all $\nu' > \nu$ and furthermore,*

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \quad (5.324)$$

*Proof.* We have

$$\nu^{mk} \int_k^{k+1} |\Delta_k(s)|\mathrm{e}^{-\nu s}\mathrm{d}s = (\nu^m \mathrm{e}^{-\nu})^k \int_0^1 |\Delta_k(s+k)|\mathrm{e}^{-\nu s}\mathrm{d}s \quad (5.325)$$

which is decreasing in $\nu$. The rest follows from the monotone convergence theorem. □

### 5.13 .2 Embedding of $L^1_\nu$ in $\mathcal{D}'_m$

**Lemma 5.326** *i) Let $f \in L^1_{\nu_0}$ (cf. Remark 5.323). Then $f \in \mathcal{D}'_{m,\nu}$ for all $\nu > \nu_0$.*

*ii) $\mathcal{D}(\mathbb{R}^+ \backslash \mathbb{N}) \cap L^1_\nu(\mathbb{R}^+)$ is dense in $\mathcal{D}_{m,\nu}$ with respect to the norm $\|\|_\nu$.*

*Proof.*

Note that if for some $\nu_0$ we have $f \in L^1_{\nu_0}(\mathbb{R}^+)$ then

$$\int_0^p |f(s)|\mathrm{d}s \leq \mathrm{e}^{\nu_0 p}\int_0^p |f(s)|\mathrm{e}^{-\nu_0 s}\mathrm{d}s \leq \mathrm{e}^{\nu_0 p}\|f\|_{\nu_0} \tag{5.327}$$

to which, application of $\mathcal{P}^{k-1}$ yields

$$\mathcal{P}^k|f| \leq \nu_0^{-k+1}\mathrm{e}^{\nu_0 p}\|f\|_{\nu_0} \tag{5.328}$$

Also, $\mathcal{P}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \leq {\nu_0}^{-1}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p}$ so that

$$\mathcal{P}^m\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \leq {\nu_0}^{-m}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \tag{5.329}$$

so that, by (5.304) (where now $F_n$ and $\chi_{[n,\infty]}F_n$ are in $L^1_{\mathrm{loc}}(0,n+1)$) we have for $n>1$,

$$|\Delta_n| \leq \|f\|_{\nu_0}\mathrm{e}^{\nu_0 p}{\nu_0}^{1-mn}\chi_{[n,n+1]} \tag{5.330}$$

Let now $\nu$ be large enough. We have

$$\sum_{n=2}^{\infty}\nu^{mn}\int_0^{\infty}|\Delta_n|\mathrm{e}^{-\nu p}\mathrm{d}p \leq \nu_0\|f\|_{\nu_0}\sum_{n=2}^{\infty}\int_n^{n+1}\mathrm{e}^{-(\nu-\nu_0)p}\left(\frac{\nu}{\nu_0}\right)^p\mathrm{d}p$$
$$= \frac{\mathrm{e}^{-2(\nu-\nu_0-\ln(\nu/\nu_0))}}{\nu-\nu_0-\ln(\nu/\nu_0)}\nu_0\|f\|_{\nu_0} \tag{5.331}$$

For $n=0$ we simply have $\|\Delta_0\| \leq \|f\|$, while for $n=1$ we write

$$\|\Delta_1\|_\nu \leq \|1^{*(m-1)} * |f|\|_\nu \leq \nu^{m-1}\|f\|_\nu \tag{5.332}$$

Combining the estimates above, the proof of (i) is complete. To show (ii), let $f \in \mathcal{D}'_{m,\nu}$ and let $k_\epsilon$ be such that $c_m\sum_{i=k_\epsilon}^{\infty}\nu^{im}\|\Delta_i\|_\nu < \epsilon$. For each $i \leq k_\epsilon$ we take a function $\delta_i$ in $\mathcal{D}(i,i+1)$ such that $\|\delta_i - \Delta_i\|_\nu < \epsilon 2^{-i}$. Then $\|f - \sum_{i=0}^{k_\epsilon}\delta_i^{(mi)}\|_{m,\nu} < 2\epsilon$. □

*Proof* of continuity of $f(p) \mapsto pf(p)$. If $f(p) = \sum_{k=0}^{\infty}\Delta_k^{(mk)}$ then $pf = \sum_{k=0}^{\infty}(p\Delta_k)^{(mk)} - \sum_{k=0}^{\infty}mk\mathcal{P}(\Delta_k^{(mk)}) = \sum_{k=0}^{\infty}(p\Delta_k^{(mk)}) - 1*\sum_{k=0}^{\infty}(mk\Delta_k)^{(mk)}$. The rest is obvious from continuity of convolution, the embedding shown above and the definition of the norms.

# Chapter 6

## Asymptotic and transasymptotic matching; formation of singularities

Transasymptotic matching stands for matching at the level of transseries. Matching can be exact, in that a BE summable transseries, valid in one region, is matched to another BE summable transseries, valid in an adjacent region, or asymptotic, when a transseries is matched to a classical asymptotic expansion. An example of exact matching is (5.121), with the connection formula (5.122), valid for systems of ODEs. In this case the two transseries exactly represent one function, and the process is very similar to analytic continuation; it is a process of continuation through transseries.

The collection of matched transseries represents exactly the function on the union of their domain of validity. For *linear* ODEs, matching is global, in that it is valid in a full (ramified, since the solution might not be single valued) neighborhood of infinity. In this case, by the results in §5 we see that for any $\phi$ we have

$$\mathbf{y} = \mathcal{L}_\phi \mathbf{Y}_0^+ + \sum_{|\mathbf{k}|=1} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\mathbf{k}\cdot\boldsymbol{\lambda} x} x^{\mathbf{k}\cdot\boldsymbol{\alpha}} \mathcal{L}_\phi \mathbf{Y}_{\mathbf{k}}^+ \tag{6.1}$$

(where if $\phi$ corresponds to a Stokes line, $\mathcal{L}_\phi$ is understood as the balanced average) and transitions of the form (5.122) occur at Stokes rays, where the constant that changes is the one corresponding to $\lambda_i$, where $\lambda_i x$ is real and positive. The representation is uniform in a ramified neighborhood of infinity.

We emphasized linear, since solutions of nonlinear ODEs usually develop **infinitely many singularities** as we shall see, and even natural boundaries in a neighborhood of infinity, and in the latter case transasymptotic matching often ends there (though at times it suggests pseudo-analytic continuation formulas.) The information contained in the transseries suffices to determine, very accurately for large values of the variable, the position and often the type of singularities. We first look at a number of simple examples, which should provide the main ideas for a more general analysis, found in [23] together with rigorous proofs.

A simple example of transasymptotic matching showing formation of singularities, for linear *difference* equations, is seen in §4.4e .

### 6.0a Transseries and singularities. Discussion

For nonlinear systems, a solution described by a transseries in some sector, usually forms *quasiperiodic arrays of singularities* on the edges of *formal* validity of the transseries. (Note that the change seen in (5.122) lies well *within* the domain of validity of the transseries.)

Assume $\mathbf{y}' = \mathbf{f}(1/x, \mathbf{y})$ is a nonlinear system, with an irregular singularity at infinity, and which is amenable to the normal form studied in §5. Assume $t = x^r$ is the critical time needed for normalization and $\mathbf{y}_0$ is a solution which decays along a Stokes line, take it to be $\mathbb{R}^+$. Then this solution generically develops arrays of singularities near the line $x^r \in i\mathbb{R}$; the singularity position is, to leading order, periodic in $x^r$. The precise location is a function of the constant $\mathbf{C}$ in the transseries, that is on the size of exponentially small terms on the Stokes line.

These actual singularities are reflections of the Borel plane singularities. Say, the equation is of the form

$$y'' = \lambda^2 y + A(1/x, y) \tag{6.2}$$

with $\lambda > 0$, $A(z_1, z_2)$ analytic at 0, nonlinear in $z_2$ and of order $O(z_1^2, z_2^2)$ for small $\mathbf{z}$. Written as a system, (6.2) satisfies the assumptions in §5. Then, there is a one parameter family of solutions $y(x; C)$ which decay in $\mathbb{H}$ and these, by § 5, and in the notations there, can be written in the form

$$y_0^+ + \sum_{k=1}^{\infty} C^k e^{-\lambda k x} y_k^+(x)$$

where $y_k$ are Borel summed series. When $\lambda x \sim i|x| + \ln C$, the exponentials become $O(1)$ and the sum in (6.2) usually diverges. Then, see Proposition 6.19, $y(x, C)$ is singular at all points in an array asymptotically given by

$$x_n = \lambda^{-1}(2n\pi i + \ln C) + c_1 + o(1) \quad (n \to +\infty) \tag{6.3}$$

and it is analytic inbetween the points in the array, where $c_1$ depends only on the equation. This is a “first” array of singularities and to the left of it others arrays can be found similarly.

Note, in comparison, that the singularities of $\mathcal{L}^{-1} y_0$ are located at $p_n = n\lambda, n \in \mathbb{Z} \setminus \{0\}$.

See also (6.65), valid for the Painlevé equation $P_1$ and Fig. 6.3.

## 6.1 Transseries reexpansion and singularities. Abel's equation.

We examine Abel's equation (5.51); its normal form is (5.58). We write the decaying formal asymptotic series solution as

$$y \sim \sum_{j=2}^{\infty} \frac{a_{j,0}}{x^j} \equiv \tilde{y}_0(x) \tag{6.4}$$

where $a_{j,0}$ can be determined algorithmically, and their values are immaterial for now. If $y_0$ is a particular solution to (5.57) with asymptotic series $\tilde{y}_0$ then, $y_0$ and $y_0 + \delta$ will have the same asymptotic series if $\delta = o(x^{-n})$ for any $n$, i.e, if $\delta$ is a term beyond all orders for the asymptotic series $\tilde{y}_0$. Furthermore, $\delta$ satisfies

$$\delta' = -\delta + \frac{1}{5x}\,\delta \tag{6.5}$$

which has the solution $\delta \sim Cx^{1/5}e^{-x}$, where $C$ is an arbitrary constant. The full transseries solution is obtained as usual by substituting

$$y = y_0 + \sum_{k=1}^{\infty} C^k\, x^{k/5} e^{-kx}\, y_k \tag{6.6}$$

in (5.57) and equating coefficients of $e^{-kx}$ to determine a set of differential equations for $y_k$, in which we look for solutions which are not exponentially growing in $\mathbb{H}$; the only such solutions are of the form

$$y_k(x) \sim \sum_{j=0}^{\infty} \frac{a_{j,k}}{x^j} \equiv \tilde{y}_k(x) \tag{6.7}$$

Arbitrariness only appears in the choice of $a_{0,1}$; all other coefficients are determined recursively. Since $C$ is arbitrary, there is no loss of generality in setting $a_{0,1} = 1$. We rewrite the transseries (6.6) in the form

$$\tilde{y}_0(x) + \sum_{k=1}^{\infty} C^k \xi^k\, \tilde{y}_k(x) \tag{6.8}$$

with $\xi = x^{1/5}e^{-x}$. By Theorem 5.65, (6.8) is Borel summable and

$$y = y_0(x) + \sum_{k=1}^{\infty} C^k \xi^k\, y_k(x) \tag{6.9}$$

where

$$y_k(x) = \mathcal{L}Y_k = \int_C e^{-px} Y_k(p)\,dp = \mathcal{L}\mathcal{B}\tilde{y}_k \tag{6.10}$$

and

$$Y_k(p) = \mathcal{B}[\tilde{y}_k] \tag{6.11}$$

By the definition of Borel summation, the contours in the Laplace transforms in (6.10) are taken so that $-px$ is real and negative. Thus, analytic continuation in $x$ in the upper half plane involves simultaneous analytic continuation in $p$ in the lower half plane. We note, again using Theorem 5.65 that $Y_k$ are analytic $\mathbb{C}\setminus\mathbb{R}^+$. Then, $y_k$ are analytic in $x$ (and bounded by some $c^k$) in a sector with angles $(-\pi/2, 5\pi/2)$.

Convergence of the series (6.8) depends in an essential way on the size of **effective variable** $\xi$. The solution $y(x)$ is analytic in a sector in $\mathbb{H}$ of any angle $< \pi$. But $\xi$ becomes large in the left half plane. The series is not expected to converge there.

The key to understanding the behavior of $y(x)$ for $x$ beyond its analyticity region is to look carefully at the borderline region, where (6.9) barely converges, and see what expansion is adequate there and beyond. Convergence is marginal along curves so that $\xi$ is small enough, but as $|x| \to \infty$, is nevertheless larger than all *negative* powers of $x$. In this case, any term in the transseries of the form $\xi^k a_{0,k}$ is larger than any other term of the form $\xi^l\ a_{j,l}x^{-j}$, if $k, l \geq 0$ and $j > 0$. Then though the transseries is still valid, and its summation converges, the terms are disordered: smaller terms are followed by both smaller and larger terms.

The natural thing to do is to properly reorder the terms. This will give the expansion in a form that is suited for this marginal region, and as it turns out, beyond it as well.

In the aforementioned domain, the largest terms are those containing no inverse power of $x$, namely

$$y(x) \sim \sum_{k\geq 0} \xi^k a_{0,k} \equiv F_0(\xi) \tag{6.12}$$

Next in line, insofar as orders of magnitudes are concerned, are the terms containing only the first power of $x^{-1}$ and any power of $\xi$, followed by the group of terms containing $x^{-2}$ and any power of $\xi$ and so on. The result is

$$y(x) \sim \sum_{j=0}^{\infty} x^{-j} \sum_{k=0}^{\infty} \xi^k a_{j,k} \equiv \sum_{j=0}^{\infty} \frac{F_j(\xi)}{x^j} \tag{6.13}$$

This is a new type of expansion.

It is intuitively clear that the region of validity of (6.13), while overlapping as expected with the transseries region, goes beyond it. This is because unless $\xi$ approaches some singular value of $F_j$, $F_j$ is much smaller than $x$. By the same token, we can read, with high accuracy, the location of the singularities of $y$ from this expansion. All this will be phrased rigorously in §6.5.

The new expansion (6.13) will usually break down further on, in which case we do exactly the same, namely push the expansion close to its boundary of validity, rearrange the terms there and obtaining a new expansion. This works until true singularities of $y$ are reached.

The expansion (6.13) has a two-scale structure, with scales $\xi$ and $x$, with the $\xi$-series of each $F_j$ *analytic* in $\xi$ for small $\xi$. This may seem paradoxical, as it suggests that we have started with a series with zero radius of convergence and ended up, by mere rearrangement, with a convergent one. This is not the case. The new series still diverges factorially, because the $F_k$ *as a function of* $k$ grow factorially.

## 6.2 Determining the $\xi$ reexpansion in practice

In §6.1 we have found (6.13) by rearranging the series by hand. This procedure is quite cumbersome; there is a better way to obtain (6.13).

Namely, now that we know how the expansion should look like, we can substitute (6.13) in the original differential equation and identify the terms order by order in $1/x$, thinking of $\xi$ as an independent variable. In view of the simple changes of coordinates involved, we can make this substitution in (5.56), which is simpler.

We obtain

$$9\xi F_0' = (3F_0)^3 - 1; \qquad F_0'(0) = 1;\; F_0(0) = 1/3 \tag{6.14}$$

while for $k \geq 1$ we have

$$-\xi F_k' + 9F_0^2 F_k = \left(k - 1 - \frac{\xi}{5}\right) F_{k-1}' + \sum_{\substack{j_1+j_2+j_2+j_3=k \\ j_i \neq 0}} F_{j_1} F_{j_2} F_{j_3} \tag{6.15}$$

The condition $F_0'(0) = 1$ comes from the fact that the coefficient of $\xi = Ce^{-x}x^{1/5}$ in the transseries is one, while $F_0(0) = h(\infty)$. Of course, the equation for $F_0$ can be solved in closed form. First we treat it abstractly. If we take $F_0 = 1/3 + \xi G$, then it can be written in integral form as

$$G = 1 + 3\xi \int_0^\xi (G^2(s) + G^3(s))ds \tag{6.16}$$

which is contractive in the ball of radius say 2 in the sup norm of functions analytic in $\xi$ for $|\xi| < \epsilon$, for small enough $\epsilon$. Thus $F_0$ is analytic in $\xi$ small, that is, the series (6.12) converges.

We see that the equations for $F_k$ are linear.

**Exercise 6.17** Show that for $k = 1$ we have a one parameter family of solutions which are analytic at $\xi = 0$, of the form $-1/15 + c\xi + \cdots$. There is a choice of $c_1$ so that the equation for $F_2$ has a one-parameter family of solutions analytic at $\xi = 0$, parameterized by $c_2$, and by induction there is a choice of $c_k$ so that the equation for $F_{k+1}$ has a one-parameter family of solutions parameterized by $c_{k+1}$ and so on.

**Remark 6.18** *With this choice of constants, clearly, $F_j$ is singular only if $F_0$ is singular.*

## 6.3 Conditions for formation of singularities

**Proposition 6.19** *Let $\lambda_1 = 1$ and $\xi = x^{\alpha}e^{-x}$. Assume $F_0$ is not entire (this is generic [1]). Say the unit disk $\mathbb{D}_1$ is the maximal disk containing $\xi = 0$ where $F_0$ is analytic, [2] and assume $\xi_0 \in \partial\mathbb{D}_1$ is a singular point of $F_0$ with the property that $F_0$ admits analytic continuation in a neighborhood of $\xi_0$. Then $y$ is singular at infinitely many points, asymptotically given by*

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_0 + o(1) \ (n \to \infty) \tag{6.20}$$

**Remark 6.21** We note that asymptotically $y$ is a function of $\xi = Ce^{-x}x^{\alpha}$. This means that

1. The boundary of analyticity of $F_0$, the unit circle, translates in a boundary of analyticity of $y$ *near the imaginary line* (the image of $\partial\mathbb{D}_1$ under ln, as $-x + \alpha \ln x = 2n\pi i + \ln \xi_0 - \ln C + o(1)$).
2. The singularities are repeated nearly periodically, since the $y$ depends on $x$ through the $2\pi i$- periodic variable $e^{-x}$.

We need the following result which is in some sense a converse of Morera's theorem.

**Lemma 6.22** *Let $B_r = \{\xi : |\xi| < r\}$ and assume that $f(\xi)$ is analytic on the universal covering of $B_r \backslash \{0\}$. Assume further that for any circle around zero $\mathcal{C} \subset B_r \backslash \{0\}$ and any $g(\xi)$ analytic in $B_r$ we have $\oint_{\mathcal{C}} f(\xi)g(\xi)d\xi = 0$. Then $f$ is in fact analytic in $B_r$.*

**PROOF** Let $a \in B_r \backslash \{0\}$. It follows that $\int_a^{\xi} f(s)ds$ is single-valued in $B_r \backslash \{0\}$. Thus $f$ is single-valued and, by Morera's theorem, analytic in $B_r \backslash \{0\}$. Since by assumption $\oint_{\mathcal{C}} f(\xi)\xi^n d\xi = 0$ for all $n \geq 0$, there are no negative powers of $\xi$ in the Laurent series of $f(\xi)$ about zero: $f$ extends as an analytic function at zero. ☐

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[1] After suitable changes of variables, see comments after Theorem 6.58.

[2] By Theorem 6.58 $F_0$ is always analytic at zero.

**PROOF of Proposition 6.19**

By lemma 6.22 there is a circle $\mathcal{C}$ around $\xi_s$ and a function $g(\xi)$ analytic in $B_r(\xi - \xi_s)$ so that $\oint_{\mathcal{C}} F_0(\xi)g(\xi)d\xi = 1$. In a neighborhood of $x_n \in X$ the function $f(x) = \mathrm{e}^{-x}x^{\alpha_1}$ is conformal and for large $x_n$

$$\oint_{f^{-1}(\mathcal{C})} \mathbf{y}(x)\frac{g(f(x))}{f(x)}dx$$
$$= -\oint_{\mathcal{C}} (1 + O(x_n^{-1}))(F_0(\xi) + O(x_n^{-1}))g(\xi)d\xi = 1 + O(x_n^{-1}) \neq 0 \quad (6.23)$$

It follows from lemma 4.158 that for large enough $x_n$ $y(x)$ is not analytic inside $\mathcal{C}$ either. Since the radius of $\mathcal{C}$ can be taken $o(1)$ the result follows.

□

**Remark 6.24** Proposition 6.19 clearly extends to the case where $F_0$ and $y$ are vectors.

**Exercise 6.25 (*)** Let $X > 0$ be large and $\epsilon > 0$ be small. The expansion (6.12) is asymptotic along any curve of the form in Fig. 6.2, if with the properties

- $|x| > X$ along the curve, the length of the curve is $O(X^m)$ and no singularity of $F_0$ is approached at a distance less than $\epsilon$.

For example, a contractive mapping integral equation can be written for the remainder

$$y(x) - \sum_{j=0}^{N} \frac{F_j(\xi)}{x^j} \quad (6.26)$$

for $N$ conveniently large.

---

## 6.4 Abel's equation, continued

**Proposition 6.27** *(i) The solutions $u = u(z; C)$ of 5.51 which have algebraic behavior in the right half plane have the asymptotic expansion*

$$u(z) \sim z^{1/3}\left(1 + \frac{1}{9}z^{-5/3} + \sum_{k=0}^{\infty} \frac{F_k\left(C\xi(z)\right)}{z^{5k/3}}\right)$$
$$(\text{as } z \to \infty; \quad z \in \mathcal{R}_{z;K,\epsilon}) \quad (6.28)$$

*where*

$$\xi(z) = x(z)^{1/5}\mathrm{e}^{-x(z)}, \ \textit{and } x(z) = -\frac{9}{5}z^{5/3} \tag{6.29}$$

*(ii) In the "steep ascent" strips* $\arg(\xi) \in (a_1, a_2)$, $|a_2 - a_1| < \pi$ *starting in* $\mathcal{A}_K$ *(see p. 228) and crossing the boundary of* $\mathcal{A}_K$*, the function* $u$ *has at most one singularity, when* $\xi(z) = \xi_0$ *or* $\xi_1$*, and* $u(z) = z^{1/3}\mathrm{e}^{\pm 2\pi i/3}(1 + o(1))$ *as* $z \to \infty$ *(the sign is determined by* $\arg(\xi)$*).*

*(iii) Up to obvious changes of variables, the singularities of* $u(z; C)$*, for* $C \neq 0$*, are located within* $o(1)$ *of the singularities of* $F_0$*, which are described in §6.4a .*

Let $f = F_0 - 1/3$. The equation for $f(\xi)$ is, cf. (6.48),

$$\xi f' = f(1 + 3f + 3f^2); \qquad f'(0) = 1 \tag{6.30}$$

so that

$$\xi = \xi_0 f(\xi)(f(\xi) + \omega_0)^{-\theta}(f(\xi) + \overline{\omega_0})^{-\overline{\theta}} \tag{6.31}$$

with $\xi_0 = 3^{-1/2}\exp(-\frac{1}{6}\pi\sqrt{3})$, $\omega_0 = \frac{1}{2} + \frac{i\sqrt{3}}{6}$ and $\theta = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. and, cf. (6.49),

$$\xi F_k' = (3f+1)^2 F_k + R_k(f, ..., F_{k-1})$$
$$(\text{ for } k \geq 1 \text{ and where } R_1 = \frac{3}{5}f^3) \tag{6.32}$$

The functions $F_k$, $k \geq 1$ can also be obtained in closed form, order by order.

**Remark 6.33** By Theorem 6.58 below, the relation $y \sim \tilde{y}$ holds in the sector

$$S_{\delta_1} = \{x \in \mathbb{C} : \arg(x) \geq -\frac{\pi}{2} + \delta, \ |Cx^{1/5}e^{-x}| < \delta_1\}$$

for some $\delta_1 > 0$ and any small $\delta > 0$. Theorem 6.58 also ensures that $y \sim \tilde{y}$ holds in fact on a larger region, surrounding singularities of $F_0$ (and thus of $y$). To apply this result we need the surface of analyticity of $F_0$ and an estimate for the location of its singularities. We postpone its formulation which needs more notation, until the study of particular examples motivates that.

**Lemma 6.34** *(i) The function* $F_0$ *is analytic on the universal covering* $\mathcal{R}_\Xi$ $\mathbb{C} \setminus \Xi$ [3] *where*

$$\Xi = \{\xi_p = (-1)^{p_1}\xi_0 \exp(p_2\pi\sqrt{3}) : p_{1,2} \in \mathbb{Z}\} \tag{6.35}$$

---

[3] This consists in classes of curves in $\mathbb{C} \setminus \Xi$, where two curves are not considered distinct if they can be continuously deformed into each-other without crossing $\Xi$.

*and its singularities are algebraic of order* $-1/2$*, located at points lying above* $\Xi$*. Fig. 6.4a sketches the Riemann surface associated to* $F_0$*.*

*(ii) (The first Riemann sheet.) The function* $F_0$ *is analytic in* $\mathbb{C}\backslash\Big((-\infty,\xi_0]\cup[\xi_1,\infty)\Big)$*.*

## 6.4a Singularities of $F_0$

The rhs of (6.14) is analytic except at $F_0=\infty$, thus $F_0$ is analytic except at points where $F_0\to\infty$. From (6.31) it follows that $\lim_{F_0\to\infty}\xi\in\Xi$ and (i) follows straightforwardly; in particular, as $\xi\to\xi_p\in\Xi$ we have $(\xi-\xi_p)^{1/2}F_0(\xi)\to\sqrt{-\xi_p/6}$.

(ii) We now examine on which sheets in $\mathcal{R}_\Xi$ these singularities are located, and start with a study of the first Riemann sheet (where $F_0(\xi)=\xi+O(\xi^2)$ for small $\xi$). Finding which of the points $\xi_p$ are singularities of $F_0$ on the first sheet can be rephrased in the following way. On which constant phase (equivalently, steepest ascent/descent) paths of $\xi(F_0)$, which extend to $|F_0|=\infty$ in the plane $F_0$, is $\xi(F_0)$ uniformly bounded?

Constant phase paths are governed by the equation $\mathrm{Im}\,(\mathrm{d}\ln\xi)=0$. Thus, denoting $F_0=X+iY$, since $\xi'/\xi=\left(F_0+3F_0^2+3F_0^3\right)^{-1}$ one is led to the *real* differential equation $\mathrm{Im}\,(\xi'/\xi)\mathrm{d}X+\mathrm{Re}\,(\xi'/\xi)\mathrm{d}Y=0$ (cf. §3.6a ), or

$$Y(1+6X+9X^2-3Y^2)\mathrm{d}X \\ -(X+3X^2-3Y^2+3X^3-9XY^2)\mathrm{d}Y=0 \quad (6.36)$$

We are interested in the field lines of (6.36) which extend to infinity. Noting that the singularities of the field are $(0,0)$ (unstable node, in a natural parameterization) and $P_\pm=(-1/2,\pm\sqrt{3}/6)$ (stable foci, corresponding to $-\overline{\omega_0}$ and $-\omega_0$), the phase portrait is easy to draw (see Fig. 6.4a ) and there are only two curves starting at $(0,0)$ so that $|F_0|\to\infty$, $\xi$ bounded, namely $\pm\mathbb{R}^+$, along which $\xi\to\xi_0$ and $\xi\to\xi_1$, respectively.

Fig. 6.4a encodes the structure of singularities of $F_0$ on $\mathcal{R}_\Xi$ in the following way. A given class $\gamma\in\mathcal{R}_\Xi$ can be represented by a curve composed of rays and arcs of circle. In Fig. 6.4a , in the $F_0$-plane, this corresponds to a curve $\gamma'$ composed of constant phase (dark gray) lines or constant modulus (light gray) lines. Curves in $\mathcal{R}_\Xi$ terminating at singularities of $F_0$ correspond in Fig 2. to curves so that $|F_0|\to\infty$ (the four dark gray separatrices $S_1,...,S_4$). Thus to calculate, on a particular Riemann sheet of $\mathcal{R}_\Xi$, where $F_0$ is singular, one needs to find the limit of $\xi$ in (6.31), as $F_0\to\infty$ along along $\gamma'$ followed by $S_i$. This is straightforward, since the branch of the complex powers $\theta,\overline{\theta}$, is calculated easily from the index of $\gamma'$ with respect to $P_\pm$.

Remark 6.33 can now be applied on relatively compact subdomains of $\mathcal{R}_\Xi$ and used to determine a uniform asymptotic representation $y\sim\tilde{y}$ in domains surrounding singularities of $y(x)$, and to obtain their asymptotic location. Going back to the original variables, similar information on $u(z)$ follows. Let

$$\mathcal{D} = \{|\xi| < K \mid \xi \notin (-\infty, \xi_1) \cup (\xi_0, +\infty) \ , \ |\xi - \xi_0| > \epsilon, |\xi - \xi_1| > \epsilon, \} \quad (6.37)$$

(for any small $\epsilon > 0$ and large positive $K$) the corresponding domain in the $z$-plane is shown in Fig. 6.3.

In general, we fix $\epsilon > 0$ small, and some $K > 0$ and define $\mathcal{A}_K = \{z : \arg z \in \left(\frac{3}{10}\pi - 0, \frac{9}{10}\pi + 0\right) , \ |\xi(z)| < K\}$ and let $\mathcal{R}_{K,\Xi}$ be the universal covering of $\Xi \cap \mathcal{A}_K$ and $\mathcal{R}_{z;K,\epsilon}$ the corresponding Riemann surface in the $z$ plane, with $\epsilon-$ neighborhoods of the points projecting on $z(x(\Xi))$ deleted.

Applying Remark 6.33 to (5.57) it follows that for $n \to \infty$, a given solution $y$ is singular at points $\tilde{x}_{p,n}$ such that $\xi(\tilde{x}_{p,n})/\xi_p = 1 + o(1)$ ($|\tilde{x}_{p,n}|$ large).

Now, $y$ can only be singular if $|y| \to \infty$ (otherwise the rhs of (5.57) is analytic). If $\tilde{x}_{p,n}$ is a point where $y$ is unbounded, with $\delta = x - \tilde{x}_{p,n}$ and $v = 1/y$ we have

$$\frac{\mathrm{d}\delta}{\mathrm{d}v} = vF_s(v, \delta) \quad (6.38)$$

where $F_s$ is analytic near $(0, 0)$. It is easy to see that this differential equation has a unique solution with $\delta(0) = 0$ and that $\delta'(0) = 0$ as well.

The result is then that the singularities of $u$ are also algebraic of order $-1/2$.

**Proposition 6.39** *If $z_0$ is a singularity of $u(z;C)$ then in a neighborhood of $z_0$ we have*

$$u = \pm\sqrt{-1/2}(z - z_0)^{-1/2} A_0((z - z_0)^{1/2}) \quad (6.40)$$

*where $A_0$ is analytic at zero and $A_0(0) = 1$.*

**Notes.** 1. The local behavior near a singularity could have been guessed by local Painlevé analysis and the method of dominant balance, with the standard ansatz near a singularity, $u \sim Const.(z - z_0)^p$. The results however are **global**: Proposition 6.27 gives the behavior of *a fixed* solution at infinitely many singularities, and gives the **position** of these singularities as soon as $C_1$ (or the position of only one of these singularities) is known (and in addition show that the power behavior ansatz is correct in this case).

2. By the substitution $y = v/(1 + v)$ in (5.57) we get

$$v' = -v - 27\,\frac{v^3}{1 + v} - 10\,v^2 + \frac{1}{5t}v + g^{[1]}(t^{-1}, v) \quad (6.41)$$

The singularities of $v$ are at the points where $v(t) = -1$.

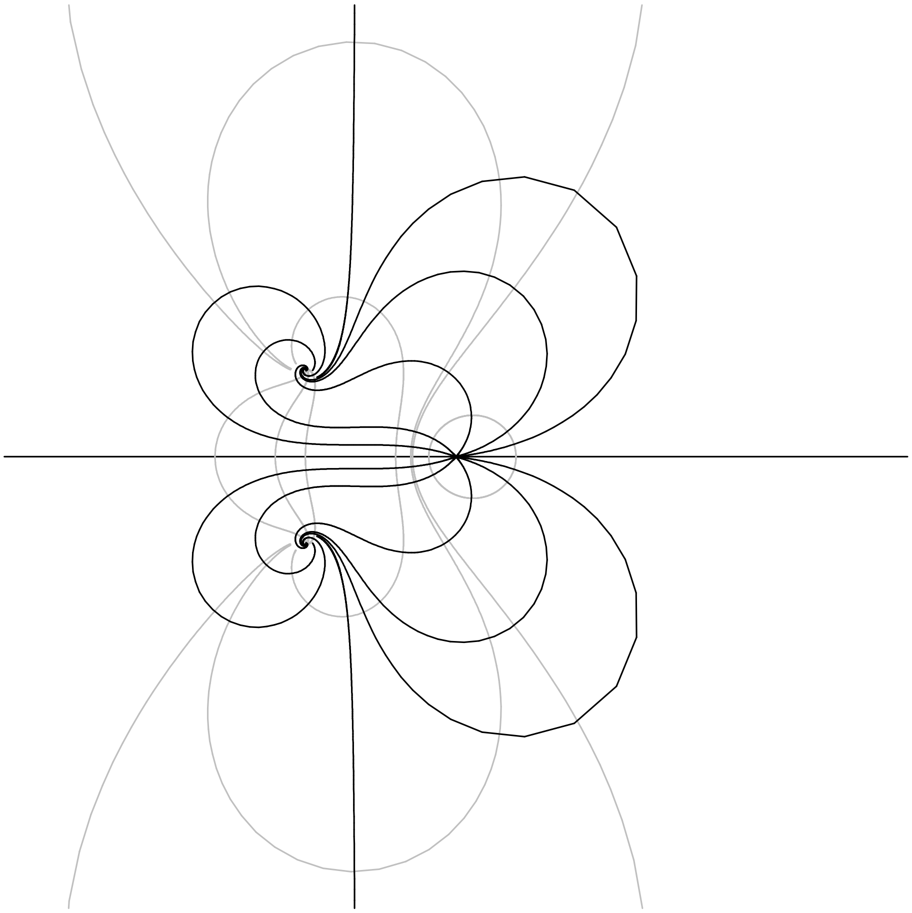

**FIGURE 6.1**: The dark lines represent the phase portrait of (6.36), as well as the lines of steepest variation of $|\xi(u)|$. The light gray lines correspond to the orthogonal field, and to the lines $|\xi(u)| = const$.

3. **It is not always the case** that the singularities of $y$ must be of the same *type* as the singularities of $F_0$. The position, as we argued is asymptotically the same, but near singularities the expansion (6.13) becomes invalid and it must either be re-matched to an expansion valid near singularities or, again, we can rely on the differential equation to see what these singularities are.

Further examples and discussions follow, in §6.6a and §6.6b .

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## 6.5 General case

The setting is that of § 5. The region where the formal or summed transseries is valid is

$$S_t = \left\{ x \in \mathbb{C}\,;\ \text{if } C_j \neq 0 \text{ then } x^{\alpha_j} e^{-\lambda_j x} = o(1), j = 1, ..., n \right\} \tag{6.42}$$

This sector might be the whole $\mathbb{C}$ if all $C_j = 0$; otherwise it lies between two antistokes lines, and has opening at most $\pi$.

If we have normalized the equation in such a way that $\lambda_1 = 1$, and $\lambda_m$ is the eigenvalue in the fourth quadrant (if there is such an eigenvalue) with the most *negative* angle, then in the upper half plane, $S_t$ will be controlled, roughly, by the condition $\operatorname{Re}(\lambda_m x) > 0$ $(*)$. If we examine the first quadrant, it is now convenient to rotate again the independent variable so that the first eigenvalue for which (*) fails is $\lambda_1$. Since originally no exponentials associated with $\lambda_j$ belonging to the second or third quadrant were allowed, then after this new rotation there will be no eigenvalue in the fourth quadrant, and the region of validity *in the first quadrant* would be, roughly, up to the imaginary line.

### 6.5a Notation

We recall that

$$\mathbf{y}(x) = \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{k}\cdot\boldsymbol{\alpha}} \mathbf{y}_{\mathbf{k}}(x) = \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{k}\cdot\boldsymbol{\alpha}} \mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}(x) \equiv \mathcal{LB}\tilde{\mathbf{y}}(x) \tag{6.43}$$

for some constants $\mathbf{C} \in \mathbb{C}^n$, where $M_j = \lfloor \operatorname{Re}\alpha_j \rfloor + 1$ ($\lfloor\cdot\rfloor$ is the integer part), and

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) = \sum_{j=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};j}}{x^j} \tag{6.44}$$

All terms in (6.43) with $\mathbf{k}$ not a multiple of $\mathbf{e}_1 = (1, 0, ..., 0)$ are subdominant (small). Thus, for $x$ near $i\mathbb{R}^+$ we only need to look at

$$\mathbf{y}^{[1]}(x) = \sum_{k\geq 0} C_1^k \mathrm{e}^{-kx} x^{k\alpha_1} \mathbf{y}_{k\mathbf{e}_1}(x) \tag{6.45}$$

As in the first order example analyzed before, the region of convergence of (6.45) (thus of (6.43)) is then determined by the effective variable $\xi = C_1 \mathrm{e}^{-x} x^{\alpha_1}$ (since $\mathbf{y}_{k\mathbf{e}_1} \sim \tilde{\mathbf{y}}_{k\mathbf{e}_1;0}/x^{k(\alpha_1 - M_1)}$). As in the simple examples at the beginning of the chapter, the leading behavior of $\mathbf{y}^{[1]}$ is expected to be

$$\mathbf{y}^{[1]}(x) \sim \sum_{k\geq 0} (C_1 \mathrm{e}^{-x} x^{\alpha_1})^k \tilde{\mathbf{s}}_{k\mathbf{e}_1;0} \equiv \mathbf{F}_0(\xi) \tag{6.46}$$

(cf. (6.44)); moreover, taking into account all terms in $\tilde{\mathbf{s}}_{k\mathbf{e}_1}$ we get

$$\mathbf{y}^{[1]}(x) \sim \sum_{r=0}^{\infty} x^{-r} \sum_{k=0}^{\infty} \xi^k \tilde{\mathbf{y}}_{k\mathbf{e}_1;r} \equiv \sum_{j=0}^{\infty} \frac{\mathbf{F}_j(\xi)}{x^j} \tag{6.47}$$

### 6.5b The recursive system for $\mathbf{F}_m$

The functions $\mathbf{F}_m$ are determined recursively, from their differential equation. Formally the calculation is the following.

The series $\tilde{\mathbf{F}} = \sum_{m\geq 0} x^{-m} \mathbf{F}_m(\xi)$ is a formal solution of (5.288); substitution in the equation and identification of coefficients of $x^{-m}$ yields the recursive system

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_0 = \xi^{-1}\left(\hat{\Lambda}\mathbf{F}_0 - \mathbf{g}(0, \mathbf{F}_0)\right) \tag{6.48}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_m + \hat{N}\mathbf{F}_m = \alpha_1 \frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_{m-1} + \mathbf{R}_{m-1} \qquad \text{for } m \geq 1 \tag{6.49}$$

where $\hat{N}$ is the matrix

$$\xi^{-1}(\partial_{\mathbf{y}}\mathbf{g}(0, \mathbf{F}_0) - \hat{\Lambda}) \tag{6.50}$$

and the function $\mathbf{R}_{m-1}(\xi)$ depends only on the $\mathbf{F}_k$ with $k < m$:

$$\xi \mathbf{R}_{m-1} = -\left[(m-1)I + \hat{A}\right]\mathbf{F}_{m-1} - \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbf{g}\left(z; \sum_{j=0}^{m-1} z^j \mathbf{F}_j\right)\Bigg|_{z=0} \tag{6.51}$$

For more detail see [23] Section 4.3.

To leading order we have $\mathbf{y} \sim \mathbf{F}_0$ where $\mathbf{F}_0$ satisfies the autonomous (after a substitution $\xi = e^{\zeta}$) equation

$$\mathbf{F}_0' = \hat{\Lambda}\mathbf{F}_0 - \mathbf{g}(0, \mathbf{F}_0)$$

which can be solved in closed form for first order equations ($n = 1$) (the equation for $F_0$ is separable, and for $k \geq 1$ the equations are linear), as well as in other interesting cases (see e.g. §6.6b ).

To determine the $\mathbf{F}_m$'s associated to $\mathbf{y}$ we first note that these functions are analytic at $\xi = 0$ (cf. Theorem 6.58). Denoting by $F_{m,j}$, $j = 1, .., n$ the components of $\mathbf{F}_m$, a simple calculation shows that (6.48) has a unique analytic solution satisfying $F_{0,1}(\xi) = \xi + O(\xi^2)$ and $F_{0,j}(\xi) = O(\xi^2)$ for $j = 2, ..., n$. For $m = 1$, there is a one parameter family of solutions of (6.49) having a Taylor series at $\xi = 0$, and they have the form $F_{1,1}(\xi) = c_1\xi + O(\xi^2)$ and $F_{1,j}(\xi) = O(\xi^2)$ for $j = 2, ..., n$. The parameter $c_1$ is determined from the condition that (6.49) has an analytic solution for $m = 2$. For this value of $c_1$ there is a one parameter family of solutions $\mathbf{F}_2$ analytic at $\xi = 0$ and this new parameter is determined by analyzing the equation of $\mathbf{F}_3$. The procedure can be continued to any order in $m$, in the same way; in particular, the constant $c_m$ is only determined at step $m+1$ from the condition of analyticity of $\mathbf{F}_{m+1}$.

### 6.5c General results and properties of the $\mathbf{F}_m$

We describe in detail the results but omit many proofs, given in [23] which roughly follow the lines sketched in §6.1, but are rather lengthy.

Let $d$ be a direction in the $x$-plane which is not not an antistokes line. Consider a solution $\mathbf{y}(x)$ of (5.288) satisfying the assumptions in §5.6a . We define

$$S_a = S_a\left(\mathbf{y}(x); \epsilon\right) = S_\epsilon^+ \cup S_\epsilon^- \tag{6.52}$$

where

$$S_\epsilon^{\pm} = \Big\{x\,;\ |x| > R\ ,\ \arg(x) \in [-\frac{\pi}{2} \mp \epsilon, \frac{\pi}{2} \mp \epsilon] \text{ and } \\ \left|C_j^- e^{-\lambda_j x} x^{\alpha_j}\right| < \delta^{-1} \text{ for } j = 1, ..., n\Big\} \tag{6.53}$$

We use the representation of $\mathbf{y}$ as summation of its transseries $\tilde{\mathbf{y}}(x)$ (5.62) in the direction $d$. Let

$$p_{j;\mathbf{k}} = \lambda_j - \mathbf{k} \cdot \boldsymbol{\lambda}\ ,\quad j = 1, ..., n_1\ ,\quad \mathbf{k} \in \mathbb{Z}_+^{n_1} \tag{6.54}$$

For simplicity we *assume*, what is generically the case, that no $\overline{p_{j;\mathbf{k}}}$ lies on the antistokes lines bounding $S_t$.

We *assume* that not all parameters $C_j$ are zero, say $C_1 \neq 0$. Then $S_t$ is bounded by two antistokes lines and its opening is at most $\pi$.

We arrange that

(a) $\arg(\lambda_1) < \arg(\lambda_2) < ... < \arg(\lambda_{n_1})$

and, by construction,

(b) $\text{Im}\,\lambda_k \geq 0$.

The solution $\mathbf{y}(x)$ is then analytic in a region $S_a$.

The locations of singularities of $\mathbf{y}(x)$ depend on the constant $C_1$ (constant which may change when we cross the Stokes line $\mathbb{R}^+$). We need its value in the sector between $\mathbb{R}^+$ and $i\mathbb{R}_+$, the next Stokes line.

Fix some small, positive $\delta$ and $c$. Denote

$$\xi = \xi(x) = C_1 \mathrm{e}^{-x} x^{\alpha_1} \tag{6.55}$$

and

$$\mathcal{E} = \Big\{ x\,;\, \arg(x) \in \Big[ -\frac{\pi}{2} + \delta, \frac{\pi}{2} + \delta \Big] \text{ and } \\ \mathrm{Re}\,(\lambda_j x/|x|) > c \text{ for all } j \text{ with } 2 \le j \le n_1 \Big\} \tag{6.56}$$

Also let

$$\mathcal{S}_{\delta_1} = \{ x \in \mathcal{E}\,;\, |\xi(x)| < \delta_1 \} \tag{6.57}$$

The sector $\mathcal{E}$ contains $S_t$, except for a thin sector at the lower edge of $S_t$ (excluded by the conditions $\mathrm{Re}\,(\lambda_j x/|x|) > c$ for $2 \le j \le n_1$, or, if $n_1 = 1$, by the condition $\arg(x) \ge -\frac{\pi}{2} + \delta$), and may extend beyond $i\mathbb{R}_+$ since there is no condition on $\mathrm{Re}\,(\lambda_1 x)$—hence $\mathrm{Re}\,(\lambda_1 x) = \mathrm{Re}\,(x)$ may change sign in $\mathcal{E}$ and $\mathcal{S}_{\delta_1}$.

Figure 6.2 is drawn for $n_1 = 1$; $\mathcal{E}$ contains the gray regions and extends beyond the curved boundary.

**Theorem 6.58** *(i) The functions $\mathbf{F}_m(\xi)$; $m \ge 1$, are analytic in $\mathcal{D}$ (see (6.37); note that by construction $\mathbf{F}_0$ is analytic in $\mathcal{D}$) and for some positive $B, K$ we have*

$$|F_m(\xi)| \le K m! B^m, \quad \xi \in \mathcal{D} \tag{6.59}$$

*(ii) For large enough $R$, the solution $\mathbf{y}(x)$ is analytic in $\mathcal{D}_x := \{x : \xi(x) \in \mathcal{D}\}$ and has the asymptotic representation*

$$\mathbf{y}(x) \sim \sum_{m=0}^{\infty} x^{-m} \mathbf{F}_m(\xi(x)) \quad (x \in \mathcal{D}_x,\ |x| \to \infty) \tag{6.60}$$

*In fact, the following Gevrey-like estimates hold*

$$\left| \mathbf{y}(x) - \sum_{j=0}^{m-1} x^{-j} \mathbf{F}_j(\xi(x)) \right| \le K_2 m! B_2^m |x|^{-m} \quad (m \in \mathbb{N}^+,\ x \in \mathcal{D}_x) \tag{6.61}$$

*(iii) Assume $\mathbf{F}_0$ has an isolated singularity at $\xi_s \in \Xi$ and that the projection of $\mathcal{D}$ on $\mathbb{C}$ contains a punctured neighborhood of (or an annulus of inner radius $r$ around) $\xi_s$.*

*Then, if $C_1 \neq 0$, $\mathbf{y}(x)$ is singular at a distance at most $o(1)$ $(r + o(1)$, respectively) of $x_n \in \xi^{-1}(\{\xi_s\}) \cap \mathcal{D}_x$, as $x_n \to \infty$.*
*The collection $\{x_n\}_{n\in\mathbb{N}}$ forms a nearly periodic array*

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1) \tag{6.62}$$

*as $n \to \infty$.*

**Remarks.** 1. The singularities $x_n$ satisfy $C_1 e^{-x_n} x_n^{\alpha_1} = \xi_s(1 + o(1))$ (for $n \to \infty$). Therefore, the singularity array lies slightly to the left of the antistokes line $i\mathbb{R}_+$ if $\mathrm{Re}\,(\alpha_1) < 0$ (this case is depicted in Fig. 6.2) and slightly to the right of $i\mathbb{R}_+$ if $\mathrm{Re}\,(\alpha_1) > 0$.

2. To find singularities, the system (5.288) find the normalization that gives an $\alpha_1$ is as **small** as possible, undoing the transformations described in (n4) on p. 164, which serve the different purpose of unifying the treatment of generic ODEs. Enlarging $\alpha$, always possible, yields an $\mathbf{F}_0$ which is entire, and manifest singularity information is lost. See also the comments on p. 238.

3. By (6.61) a truncation of the two-scale series (6.60) at an $m$ dependent on $x$ ($m \sim |x|/B$) is seen to produce exponential accuracy $o(\mathrm{e}^{-|x/B|})$, see e.g. [5].

4. Theorem 6.58 can also be used to determine precisely the nature of the singularities of $\mathbf{y}(x)$. In effect, for any $n$, the representation (6.60) provides $o(\mathrm{e}^{-K|x_n|})$ estimates on $\mathbf{y}$ down to an $o(\mathrm{e}^{-K|x_n|})$ distance of an actual singularity $x_n$. In most instances this is more than sufficient to match to a suitable local integral equation, contractive in a tiny neighborhood of $x_n$, providing rigorous control of the singularity. See also §6.6.

## 6.6 Further examples

### 6.6a The Painlevé equation $\mathrm{P_I}$.

Proposition 6.63 below shows, in (i), how the constant $C$ beyond all orders is associated to a truncated solution $y(z)$ of $\mathrm{P_I}$ for $\arg(z) = \pi$ (formula (6.64)) and gives the position of one array of poles $z_n$ of the solution associated to $C$ (formula (6.65)), and in (ii) provides uniform asymptotic expansion to all orders of this solution in a sector centered on $\arg(z) = \pi$ and one array of poles (except for small neighborhoods of these poles) in formula (6.67).

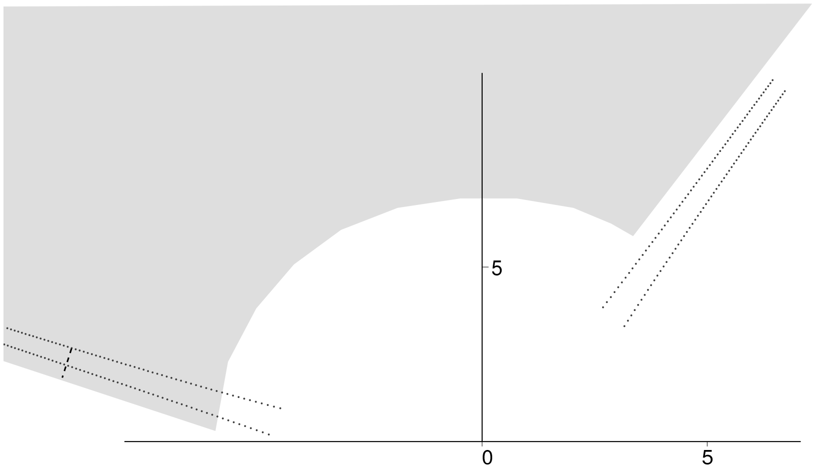

**FIGURE 6.2**: Singularities on the boundary of $S_t$ for (5.51). The gray region lies in the projection on $\mathbb{C}$ of the Riemann surface where (6.28) holds. The short dotted line is a generic cut delimiting a first Riemann sheet.

**Proposition 6.63** *(i) Let $y$ be a solution of (4.81) such that $y(z) \sim \sqrt{-z/6}$ for large $z$ with $\arg(z) = \pi$. For any $\phi \in (\pi, \pi + \frac{2}{5}\pi)$ the following limit determines the constant $C$ (which does not depend on $\phi$ in this range) in the transseries $\tilde{y}$ of $y$:*

$$\lim_{\substack{|z|\to\infty \\ \arg(z)=\phi}} \xi(z)^{-1} \left( \sqrt{\frac{6}{-z}}\, y(z) - \sum_{k \le |x(z)|} \frac{\tilde{y}_{0;k}}{z^{5k/2}} \right) = C \tag{6.64}$$

*(Note that the constants $\tilde{y}_{0;k}$ do* not *depend on $C$). With this definition, if $C \neq 0$, the function $y$ has poles near the antistokes line $\arg(z) = \pi + \frac{2}{5}\pi$ at all points $z_n$, where, for large $n$*

$$z_n = -\frac{(60\pi i)^{4/5}}{24} \left( n^{\frac{4}{5}} + i L_n n^{-\frac{1}{5}} + \left( \frac{L_n^2}{8} - \frac{L_n}{4\pi} + \frac{109}{600\pi^2} \right) n^{-\frac{6}{5}} \right) + O\left( \frac{(\ln n)^3}{n^{\frac{11}{5}}} \right) \tag{6.65}$$

*with $L_n = \frac{1}{5\pi} \ln\left(\frac{\pi i C^2}{72} n\right)$, or, more compactly,*

$$\xi(z_n) = 12 + \frac{327}{(-24z_n)^{5/4}} + O(z_n^{-5/2}) \quad (z_n \to \infty) \tag{6.66}$$

*(ii) Let* $\epsilon \in \mathbb{R}^+$ *and define*

$$\mathcal{Z} = \{z : \arg(z) > \frac{3}{5}\pi; |\xi(z)| < 1/\epsilon; \ |\xi(z) - 12| > \epsilon\}$$

*(the region starts at the antistokes line* $\arg(z) = \frac{3}{5}\pi$ *and extends slightly beyond the next antistokes line,* $\arg(z) = \frac{7}{5}\pi$*). If* $y \sim \sqrt{-z/6}$ *as* $|z| \to \infty$, $\arg(z) = \pi$, *then for* $z \in \mathcal{Z}$ *we have*

$$y \sim \sqrt{\frac{-z}{6}} \left(1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} + \sum_{k=0}^{\infty} \frac{30^k H_k(\xi)}{(-24z)^{5k/4}}\right) \qquad (|z| \to \infty, \ z \in \mathcal{Z}) \tag{6.67}$$

*The functions* $H_k$ *are rational, and* $H_0(\xi) = \xi(\xi/12 - 1)^{-2}$*. The expansion (6.67) holds uniformly in the sector* $\pi^{-1}\arg(z) \in (3/5, 7/5)$ *and also on one of its sides, where* $H_0$ *becomes dominant, down to an* $o(1)$ *distance of the actual poles of* $y$ *if* $z$ *is large.*

**Proof**. We prove the corresponding statements for the normal form (4.85). One returns to the variables of (4.81) by simple substitutions, which we omit.

Most of Proposition 6.63 is a direct consequence of Theorems 1 and 2. For the one-parameter family of solutions which are small in $\mathbb{H}$ we then have

$$h \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi(x)) \tag{6.68}$$

where $\xi(x) = x^{-1/2}e^{-x}$ (thus $\alpha = -1/2$).
As in the first example we find $H_k$ by substituting (6.68) in (4.85).

The equation of $H_0$ is

$$\xi^2 H_0'' + \xi H_0' = H_0 + \frac{1}{2}H_0^2$$

The general solution of this equation is a Weierstrass elliptic function of $\ln\xi$, as expected from the general knowledge of the asymptotic behavior of the Painlevé solutions (see [36]). For our special initial condition, $H_0$ analytic at zero and $H_0(\xi) = \xi(1 + o(1))$, the solution is a degenerate elliptic function, namely,

$$H_0(\xi) = \frac{\xi}{(\xi/12 - 1)^2}$$

**Important remark.** *One of the two free constants in the general solution $H_1$ is determined by the condition of analyticity at zero of $H_1$ (this constant multiplies terms in $\ln \xi$). It is interesting to note that the remaining constant is only determined in the* next *step, when solving the equation for $H_2$! This pattern is typical (see §6.5b ).*

Continuing this procedure we obtain successively:

$$H_1 = \left(216\,\xi + 210\,\xi^2 + 3\,\xi^3 - \frac{1}{60}\,\xi^4\right)(\xi - 12)^{-3} \tag{6.69}$$

$$H_2 = \left(1458\xi + 5238\xi^2 - \frac{99}{8}\xi^3 - \frac{211}{30}\xi^4 + \frac{13}{288}\xi^5 + \frac{\xi^6}{21600}\right)(\xi - 12)^{-4} \tag{6.70}$$

We omit the straightforward but quite lengthy inductive proof that all $H_k$ are rational functions of $\xi$. The reason the calculation is tedious is that this property holds for (4.85) but *not* for its generic perturbations, and the last potential obstruction to rationality, successfully overcome by (4.85), is at $k = 6$. On the positive side, these calculations are algorithmic and are very easy to carry out with the aid of a symbolic language program.

In the same way as in Example 1 one can show that the corresponding singularities of $h$ are double poles: all the terms of the corresponding asymptotic expansion of $1/h$ are *analytic* near the singularity of $h$! All this is again straightforward, and lengthy because of the potential obstruction at $k = 6$.

Let $\xi_s$ correspond to a zero of $1/h$. To leading order, $\xi_s = 12$, by Theorem 6.58 (iii). To find the next order in the expansion of $\xi_s$ one substitutes $\xi_s = 12 + A/x + O(x^{-2})$, to obtain

$$1/h(\xi_s) = \frac{(A - 109/10)^2}{12^3 x^2} + O(1/x^3)$$

whence $A = 109/10$ (because $1/h$ is analytic at $\xi_s$) and we have

$$\xi_s = 12 + \frac{109}{10x} + O(x^{-2}) \tag{6.71}$$

Given a solution $h$, its constant $C$ in $\xi$ for which (6.68) holds can be calculated from asymptotic information in any direction above the real line by near least term truncation, namely

$$C = \lim_{\substack{x \to \infty \\ \arg(x) = \phi}} \exp(x) x^{1/2} \left( h(x) - \sum_{k \le |x|} \frac{\tilde{h}_{0,k}}{x^k} \right) \tag{6.72}$$

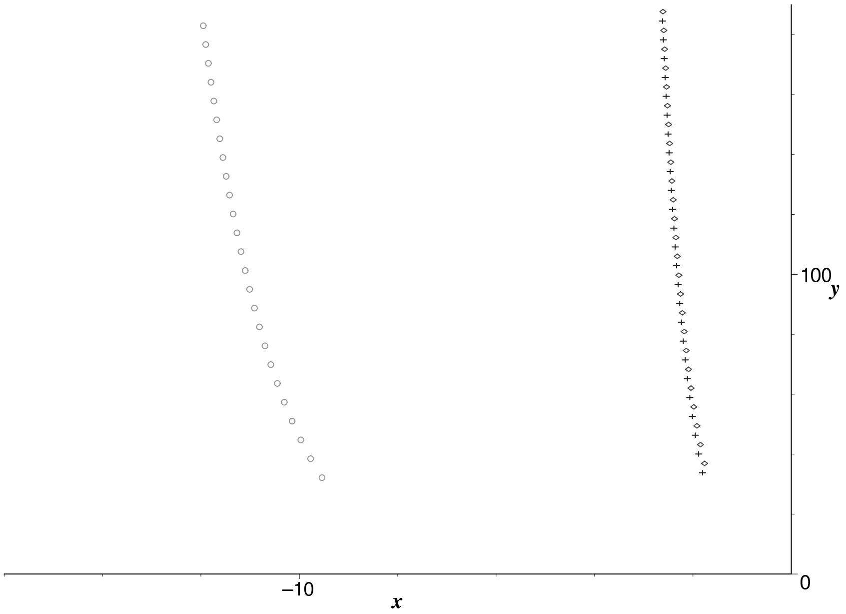

**FIGURE 6.3**: Poles of (4.85) for $C = -12$ ($\diamond$) and $C = 12$ ($+$), calculated via (6.71). The light circles are on the second line of poles for to $C = -12$.

(this is a particular case of much more general formulas [18] where $\sum_{k>0} \tilde{h}_{0,k} x^{-k}$ is the common asymptotic series of all solutions of (4.85) which are small in $\mathbb{H}$.

$\square$

**General comments.** The expansion scales, $x$ and $x^{-1/2}\mathrm{e}^{-x}$ are crucial. Only for this choice one obtains an expansion which is valid both in $S_t$ and near poles of (4.85). For instance, the more general second scale $x^a \mathrm{e}^{-x}$ introduces logarithmic singularities in $H_j$, except when $a \in -\frac{1}{2} + \mathbb{Z}$. With these logarithmic terms, the two scale expansion would only be valid in an $O(1)$ region in $x$, what is sometimes called a "patch at infinity", instead of more than a sector. Also, $a \in -\frac{1}{2} - \mathbb{N}$ introduces obligatory singularities at $\xi = 0$ precluding the validity of the expansion in $S_t$. The case $a \in -\frac{1}{2} + \mathbb{N}$ produces instead an expansion valid in $S_t$ but not near poles. Indeed, the substitution $h(x) = g(x)/x^n,\ n \in \mathbb{N}$ has the effect of changing $\alpha$ to $\alpha + n$ in the normal form. This in turn amounts to restricting the analysis to a region far away from the poles, and then all $H_j$ will be entire. In general we need thus to make (by substitutions in (5.288)) $a = \alpha$ minimal compatible with the assumptions (a1) and (a2), as this ensures the widest region of analysis.

### 6.6b The Painlevé equation $\mathbf{P}_{\mathrm{II}}$

This equation reads:

$$y'' = 2y^3 + xy + \gamma \tag{6.73}$$

(Incidentally, this example also shows that for a given equation distinct solution manifolds associated to distinct asymptotic behaviors may lead to different normalizations.) After the change of variables

$$x = (3t/2)^{2/3}; \quad y(x) = x^{-1}(t\,h(t) - \gamma)$$

one obtains the normal form equation

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\gamma^2 + 1}{9t^2}\right) h - \frac{8}{9}h^3 + \frac{8\gamma}{3t}h^2 + \frac{8(\gamma^3 - \gamma)}{9t^3} = 0 \tag{6.74}$$

and

$$\lambda_1 = 1, \quad \alpha_1 = -1/2; \; \xi = \frac{e^{-t}}{\sqrt{t}}; \; \; \xi^2 F_0'' + \xi F_0' = F_0 + \frac{8}{9}F_0^3$$

The initial condition is (always): $F_0$ analytic at 0 and $F_0'(0) = 1$. This implies

$$F_0(\xi) = \frac{\xi}{1 - \xi^2/9}$$

Distinct normalizations (and sets of solutions) are provided by

$$x = (At)^{2/3}; \;\; y(x) = (At)^{1/3}\left(w(t) - B + \frac{\gamma}{2At}\right)$$

if $A^2 = -9/8, B^2 = -1/2$. In this case,

$$\begin{aligned} w'' + \frac{w'}{t} + w\left(1 + \frac{3B\gamma}{tA} - \frac{1 - 6\gamma^2}{9t^2}\right) w \\ - \left(3B - \frac{3\gamma}{2tA}\right) w^2 + w^3 + \frac{1}{9t^2}\left(B(1 + 6\gamma^2) - t^{-1}\gamma(\gamma^2 - 4)\right) \end{aligned} \tag{6.75}$$

so that

$$\lambda_1 = 1, \alpha_1 = -\frac{1}{2} - \frac{3}{2}\frac{B\gamma}{A}$$

implying

$$\xi^2 F_0'' + \xi F_0' - F_0 = 3BF_0^2 - F_0^3$$

and, with the same initial condition as above, we now have

$$F_0 = \frac{2\xi(1 + B\xi)}{\xi^2 + 2}$$

The first normalization applies for the manifold of solutions such that $y \sim -\frac{\gamma}{x}$ (for $\gamma = 0$ $y$ is exponentially small and behaves like an Airy function) while the second one corresponds to $y \sim -B - \frac{\gamma}{2}x^{-3/2}$. The analysis can be completed as in the other examples.

# Chapter 7

# Other classes of problems

## 7.1 Difference equations

### 7.1a Setting

Let us now look at difference systems of equations which can be brought to the form

$$\mathbf{x}(n+1) = \hat{\Lambda}\left(I + \frac{1}{n}\hat{A}\right)\mathbf{x}(n) + \mathbf{g}(n, \mathbf{x}(n)) \tag{7.1}$$

where $\hat{\Lambda}$ and $\hat{A}$ are constant coefficient matrices, $\mathbf{g}$ is convergently given for small $\mathbf{x}$ by

$$\mathbf{g}(n, \mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{g}_{\mathbf{k}}(n)\mathbf{x}^{\mathbf{k}} \tag{7.2}$$

with $\mathbf{g}_{\mathbf{k}}(n)$ analytic in $n$ at infinity and

$$\mathbf{g}_{\mathbf{k}}(n) = O(n^{-2}) \text{ as } n \to \infty, \text{ if } \sum_{j=1}^{m} k_j \le 1 \tag{7.3}$$

under nonresonance conditions: Let $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ and $\mathbf{a} = (a_1, ..., a_n)$ where $e^{-\mu_k}$ are the eigenvalues of $\hat{\Lambda}$ and the $a_k$ are the eigenvalues of $\hat{A}$. Then the nonresonance condition is

$$(\mathbf{k}\cdot\boldsymbol{\mu} = 0 \mod 2\pi i \text{ with } \mathbf{k} \in \mathbb{Z}^{m_1}) \Leftrightarrow \mathbf{k} = 0. \tag{7.4}$$

The theory of these equations is remarkably similar to that of differential equations. We consider the solutions of (7.1) which are small as $n$ becomes large.

### 7.1b Transseries for difference equations

Braaksma [13] showed that the recurrences (7.1) possess $l$-parameter transseries solutions of the form

$$\tilde{\mathbf{x}}(n) := \sum_{\mathbf{k}\in\mathbb{N}^m} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} t^{\mathbf{k}\cdot\mathbf{a}} \tilde{\mathbf{x}}_{\mathbf{k}}(n) \tag{7.5}$$

where $\tilde{\mathbf{x}}_{\mathbf{k}}(n)$ are formal power series in powers of $n^{-1}$ and $l \le m$ is chosen such that, after reordering the indices, we have $\mathrm{Re}\,(\mu_j) > 0$ for $1 \le j \le l$.

It is shown in [13] that

$$\mathbf{X}_{\mathbf{k}} = \mathcal{B}\tilde{\mathbf{x}_{\mathbf{k}}} \text{ are analytic in a sectorial neighborhood } \mathcal{S} \text{ of } \mathbb{R}^+, \text{ and}$$

$$\sup_{p\in\mathcal{S},\mathbf{k}\in\mathbb{N}^m} \left| A^{|\mathbf{k}|} e^{-\nu|p|} \mathbf{X}_{\mathbf{k}} \right| < \infty \tag{7.6}$$

Furthermore, the functions $\mathbf{x}_{\mathbf{k}}$ defined by

$$\mathbf{x}_{\mathbf{k}}(n) = \int_0^\infty e^{-np} \mathbf{X}_{\mathbf{k}}(p)dp \tag{7.7}$$

are asymptotic to the series $\tilde{\mathbf{x}}_{\mathbf{k}}$ i.e.

$$\mathbf{x}_{\mathbf{k}}(n) \;\sim\; \tilde{\mathbf{x}}_{\mathbf{k}}(n) \quad (n \to +\infty) \tag{7.8}$$

and in any direction different from a Stokes one,

$$\mathbf{x}(n) = \sum_{\mathbf{k}\in\mathbb{N}^l} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} n^{\mathbf{k}\cdot\mathbf{a}} \mathbf{x}_{\mathbf{k}}(n) \tag{7.9}$$

is a solution of (7.1), if $n > y_0$, $t_0$ large enough.

There is a freedom of composition with periodic functions. For example, the general solution of $x_{n+1} = x_n$ is an arbitrary 1-periodic function. This freedom permeates both the formal and analytic theory. It can be ruled out by disallowing purely oscillatory terms in the transseries.

### 7.1c Application: Extension of solutions of difference equations to the complex $n$ plane

If the formal series solution, say in $1/n$, of a difference equation is Borel summable, then the expression (7.7) of the Borel sum allows for continuation in the complex domain. Since Borel summation preserves relations, the continuation in $\mathbb{C}$ will as well. This extends to transseries. Furthermore, this continuation is unique, in the following sense: The values of $\mathbf{x}$ on the integers uniquely determine $\mathbf{x}$. It is then easy to check that condition (7.6) implies that the sum

$$\mathbf{x}(t) = \sum_{\mathbf{k}\in\mathbb{N}^{n_0}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} t} t^{\mathbf{k}\cdot\mathbf{a}} \mathbf{x}_{\mathbf{k}}(t) \tag{7.10}$$

is convergent in the half plane $\mathbb{H} = \{t : \mathrm{Re}\,(t) > t_0\}$, for $t_0$ large enough.

**Definition 7.11** *Define the continuation of* $\mathbf{x}_{\mathbf{k}}(n)$ *in the half plane* $\{t : \mathrm{Re}\,(t) > t_0\}$ *by* $\mathbf{x}(t)$ *by (7.10).*

***THEOREM 7.1***

*The following uniqueness property holds. If in the assumptions (7.6)–(7.9) we have* $\mathbf{x}(n) = 0$ *for all except possibly finitely many* $n \in \mathbb{N}$*, then* $\mathbf{x}(t) = 0$ *for all* $t \in \mathbb{C}$, $\mathrm{Re}\,(t) > t_0$.

For a proof, see [17]. In particular, as it is easy to see, the formal expansion of $\ln\Gamma(n)$ turns out to be exactly the extension of $\ln\Gamma$ in $\mathbb{C}$, see (4.61).

This extension, in turn, and transasymptotic matching, can be used to check the Painlevé property in difference equations, and determine their integrability properties.

### 7.1d Extension of the Painlevé criterion to difference equations.

The function $\mathbf{x}$ is analytic in $\mathbb{H}$ and has, in general, nontrivial singularities in $\mathbb{C}\backslash\mathbb{H}$. In particular, Painlevé's test of integrability, absence of movable non-isolated singularities, extends then to difference equations. The representation (7.10) and Theorem 7.1 make the following definition natural. As in the case of differential equations, fixed singularities are singular points whose location is the same for all solutions; they define a common Riemann surface. Other singularities (i.e., whose location depends on initial data) are called *movable*.

**Definition 7.12** *A difference equation has the Painlevé property if its solutions are analyzable and their analytic continuations on a Riemann surface* common to all solutions*, have only* isolated *singularities.*

For instance, the Gamma function satisfies the Painlevé criterion, as seen in (4.127). But the solution of an equation as simple as the logistic equation $x_{n+1} = ax_n(1 - x_n)$ fails the criterion, except in the known integrable cases $a = -2, 0, 2, 4$, [17].

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## 7.2 PDEs

Borel summability has been developed substantially in PDE settings as well. It comes as a particularly useful tool in nonlinear PDEs, since, unlike in ODEs, existence and uniqueness of solutions are not known in general. Wherever applicable, Borel summation provides actual solutions by simply summing formal ones most often much more easily accessible.

These formal solutions exist only if the initial conditions are smooth enough. This cannot be assumed in general, and in this context it is useful to reinterpret Borel summation as a regularization tool. When solutions corresponding to analytic initial data are Borel summable, it means that the Borel transformed equation, which has the Borel sums as solutions must be more regular. Indeed, Borel transforms are by definition analytic, and thus the transformed equation has analytic solutions if the data is analytic, a sign of better regularity of the equation altogether.

### 7.2a Example: regularizing the heat equation

$$f_{xx} - f_t = 0 \tag{7.13}$$

Since (7.13) is parabolic, power series solutions

$$f = \sum_{k=0}^{\infty} t^k F_k(x) = \sum_{k=0}^{\infty} \frac{F_0^{(2k)}}{k!} t^k \tag{7.14}$$

are divergent even if $F_0$ is analytic (but not entire). Nevertheless, under suitable assumptions, Borel summability results of such formal solutions have been shown by Lutz, Miyake, and Schäfke [40] and more general results of multisummability of linear PDEs have been obtained by Balser [5].

The heat equation can be regularized by a suitable Borel transform . The divergence implied, under analyticity assumptions, by (7.14) is $F_k = O(k!)$ which indicates Borel summation with respect to $t^{-1}$. Indeed, the substitution

$$t = 1/\tau; \quad f(t,x) = t^{-1/2} g(\tau, x) \tag{7.15}$$

yields

$$g_{xx} + \tau^2 g_\tau + \frac{1}{2}\tau g = 0$$

which becomes after formal inverse Laplace transform (Borel transform) in $\tau$,

$$p\hat{g}_{pp} + \frac{3}{2}\hat{g}_p + \hat{g}_{xx} = 0 \tag{7.16}$$

which is brought, by the substitution $\hat{g}(p,x) = p^{-\frac{1}{2}} u(x, 2p^{\frac{1}{2}})$; $y = 2p^{\frac{1}{2}}$, to the wave equation, which is hyperbolic, thus *regular*

$$u_{xx} - u_{yy} = 0. \tag{7.17}$$

Existence and uniqueness of solutions to regular equations is guaranteed by Cauchy-Kowalevsky theory. For this simple equation the general solution is certainly available in explicit form: $u = f_1(x-y) + f_2(x+y)$ with $f_1, f_2$ arbitrary twice differentiable functions. Since the solution of (7.17) is related to a solution of (7.13) through (7.15), to ensure that we do get a solution it is easy to check that we need to choose $f_1 = f_2 =: u$ (up to an irrelevant additive constant which can be absorbed into $u$) which yields,

$$f(t,x) = t^{-\frac{1}{2}} \int_0^\infty y^{-\frac{1}{2}} \left[ u\left(x + 2\,y^{\frac{1}{2}}\right) + u\left(x - 2\,y^{\frac{1}{2}}\right)\right] \exp\left(-\frac{y}{t}\right) dy \qquad (7.18)$$

which, after splitting the integral and making the substitutions $x \pm 2\,y^{\frac{1}{2}} = s$ is transformed into the usual Heat kernel solution,

$$f(t,x) = t^{-\frac{1}{2}} \int_{-\infty}^\infty u(s) \exp\left(-\frac{(x-s)^2}{4t}\right) ds \qquad (7.19)$$

### 7.2b Higher order nonlinear systems of evolution PDEs

For partial differential equations with analytic coefficients which can be transformed to equations in which the differentiation order in a distinguished variable, say time, is no less than the one with respect to the other variable(s), under some other natural assumptions, Cauchy-Kowalevsky theory (C-K) applies and gives existence and uniqueness of the initial value problem. A number of evolution equations do not satisfy these assumptions and even if formal power series solutions exist their radius of convergence is zero. The paper [20] provides a C-K type theory in such cases, providing existence, uniqueness and regularity of the solutions. Roughly, convergence is replaced by Borel summability, although the theory is more general.

Unlike in C-K, solutions of nonlinear evolution equations develop singularities which can be more readily studied from the local behavior near $t = 0$, and this is useful in determining and proving spontaneous blow-up. This is somewhat similar to the mechanism discussed in §6.

We describe some of the results in [20]. The proofs can be found in the paper. Roughly, the approach is similar to the ODE one. However, here the dual equation is a partial differential-convolution equation. It would superficially look like complicating the problem even further, but the built in regularity of the new equation makes its study in fact much easier than the one of the original equation. In [20], to simplify the algebra, and in fact reduce to an almost ODE-like equation, we make use of Écalle acceleration (cf. §8.2) although the type of divergence would not require it.

In the following, $\partial_{\mathbf{x}}^{\mathbf{j}} \equiv \partial_{x_1}^{j_1}\partial_{x_2}^{j_2}...\partial_{x_d}^{j_d}$, $|\mathbf{j}| = j_1+j_2+...+j_d$, $\mathbf{x}$ is in a poly-sector $\mathcal{S} = \{\mathbf{x} : |\arg x_i| < \frac{\pi}{2} + \phi; |\mathbf{x}| > a\}$ in $\mathbb{C}^d$ where $\phi < \frac{\pi}{2n}$, $\mathbf{g}\left(\mathbf{x}, t, \{\mathbf{y}_{\mathbf{j}}\}_{|\mathbf{j}|=0}^{n-1}\right)$ is a function analytic in $\{\mathbf{y}_{\mathbf{j}}\}_{|\mathbf{j}|=0}^{n-1}$ near $\mathbf{0}$ vanishing as $|\mathbf{x}| \to \infty$. The results in hold for $n$-th order nonlinear *quasilinear* partial differential equations of the form

$$\mathbf{u}_t + \mathcal{P}(\partial_{\mathbf{x}}^{\mathbf{j}})\mathbf{u} + \mathbf{g}\left(\mathbf{x}, t, \{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\}\right) = 0 \qquad (7.20)$$

where $\mathbf{u} \in \mathbb{C}^m$, for large $|\mathbf{x}|$ in $S$. Generically, the constant coefficient operator $\mathcal{P}(\partial_{\mathbf{x}})$ in the linearization of $\mathbf{g}(\infty, t, \cdot)$ is diagonalizable. It is then taken to be diagonal, with eigenvalues $\mathcal{P}_j$. $\mathcal{P}$ is subject to the requirement that for all $j \le m$ and $\mathbf{p} \ne 0$ in $\mathbb{C}^d$ with $|\arg p_i| \le \phi$ we have

$$\operatorname{Re}\mathcal{P}_j^{[n]}(-\mathbf{p}) > 0 \tag{7.21}$$

where $\mathcal{P}^{[n]}(\partial_{\mathbf{x}})$ is the principal symbol of $\mathcal{P}(\partial_{\mathbf{x}})$. Then the following holds. (The precise conditions and results are given in [20].)

**Theorem 7.22 (large $|\mathbf{x}|$ existence)** *Under the assumptions above, for any $T > 0$ (7.20) has a unique solution $\mathbf{u}$ that for $t \in [0, T]$ is $O(|\mathbf{x}|^{-1})$ and analytic in $\mathcal{S}$.*

Determining asymptotic properties of solutions of PDEs is substantially more difficult than the corresponding question for ODEs. Borel-Laplace techniques however provide a very efficient way to overcome this difficulty. The paper shows that formal series solutions are actually Borel summable, a fortiori asymptotic, to actual solutions.

**Condition 7.23** *The functions $\mathbf{b}_{\mathbf{q},\mathbf{k}}(\mathbf{x}, t)$ and $\mathbf{r}(\mathbf{x}, t)$ are analytic in $(x_1^{-\frac{1}{N_1}}, ..., x_d^{-\frac{1}{N_d}})$ for large $|\mathbf{x}|$ and some $N \in \mathbb{N}$.*

**Theorem 7.24** *If Condition 7.23 and the assumptions of Theorem 7.22 are satisfied, then the unique solution $\mathbf{f}$ found there can be written as*

$$\mathbf{f}(\mathbf{x}, t) = \int_{\mathbb{R}^{+d}} e^{-\mathbf{p}\cdot\mathbf{x}^{\frac{n}{n-1}}} \mathbf{F}_{1^+}(\mathbf{p}, t)d\mathbf{p} \tag{7.25}$$

*where $\mathbf{F}_{1^+}$ is (a) analytic at zero in $(p_1^{\frac{1}{nN_1}}, ..., p_d^{\frac{1}{nN_d}})$; (b) analytic in $\mathbf{p} \ne \mathbf{0}$ in the poly-sector $|\arg p_i| < \frac{n}{n-1}\phi + \frac{\pi}{2(n-1)}$, $i \le d$; and (c) exponentially bounded in the latter poly-sector.*

Existence and asymptoticity of the formal power series follow as a corollary, using Watson's lemma.

The analysis has been extended recently to the Navier-Stokes system in $\mathbb{R}^3$, see [25].

# Chapter 8

## Other important tools and developments

### 8.1 Resurgence, bridge equations, alien calculus, moulds

This is a powerful set of tools discovered by Écalle, which provide detailed analytic information on Borel transforms, linear and nonlinear Stokes phenomena and general summation rules along singular directions [30]. The recent article [48] provides a largely self-contained introduction.

### 8.2 Multisummability

Nongenerically, exponentials of different powers of $x$, such as $e^{-x^2}$ and $e^x$, may occur in the solutions of the same equation. If such is the case, no single critical time will work, and the formal solutions are effectively mixtures of solutions belonging to different Borel planes. Summation with respect to any single variable will result in superexponential growth and/or divergent expansions at the origin. For some reason, in applications, the need for full multisummability rarely occurs. More often a nongeneric equation can be effectively split into lower order, pure-type equations. In general though, *acceleration and multisummability* were introduced by Écalle, see [30] and [31], adequately deal with mixed divergences in wide settings. In PDEs however it is often helpful to use acceleration operators since they can further simplify or regularize the problem.

We only sketch the general procedure, and refer the interested reader to [30], [31], [5] and [13] for a detailed analysis.

Multisummation consists in Borel transform with respect to the lowest power of $x$ in the exponents of the transseries (resulting in oversummation of some components of the mixture, and superexponential growth) and a sequence of transformations called accelerations (which mirror in Borel space the passage from one power in the exponent to the immediately larger one) followed by a final Laplace transform in the highest power of $x$.

More precisely ([31]):

$$\mathcal{L}_{k_1} \circ \mathcal{A}_{k_2/k_1} \circ \cdots \circ \mathcal{A}_{k_q/k_{q-1}} \mathcal{S}\mathcal{B}_{k_q} \tag{8.1}$$

where $(\mathcal{L}_k f)(x) = (\mathcal{L}f)(x^k)$, $\mathcal{B}_k$ is the formal inverse of $\mathcal{L}_k$, $\alpha_i \in (0,1)$ and the acceleration operator $\mathcal{A}_\alpha$ is formal the image, in Borel space, of the change of variable from $x^\alpha$ to $x$ and is defined as

$$\mathcal{A}_\alpha \phi = \int_0^\infty C_\alpha(\cdot, s)\phi(s)ds \tag{8.2}$$

and where, for $\alpha \in (0,1)$, the kernel $C_\alpha$ is defined as

$$C_\alpha(\zeta_1, \zeta_2) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\zeta_2 z - \zeta_1 z^\alpha} dz \tag{8.3}$$

where we adapted the notations in [5] to the fact that the formal variable is large. In our example, $q = 2, k_2 = 1, k_1 = 2$.

In [13], W. Balser, B.L.J. Braaksma, J-P. Ramis, and Y. Sibuya proved of multisummability of series solutions of general nonlinear meromorphic ODEs in the spirit of Écalle's theory.

**Note 8.4** (i) Multisummability of type (8.1) can be equivalently characterized by decomposition of the series into terms which are ordinarily summable after changes of independent variable of the form $x \to x^\alpha$. This is shown in [5] where it is used to give an alternative proof of multisummability of series solutions of meromorphic ODEs, closer to the cohomological point of view of Ramis, see [41, 44, 45].

(ii) More general multisummability is described by Écalle [31], allowing, among others, for stronger than power-like acceleration. This is relevant to more general transseries equations.

## 8.3 Hyperasymptotics

Once a series has been summed to the least term, with exponential accuracy, there is no reason to stop there. Often asymptotic expansions for the difference between the function and the truncated series can be obtained too.

This new expansion can also be summed to the least term. The procedure can be continued in principle *ad infinitum.* In practice, after a few iterations, the order of truncation becomes one and no further improvement is obtained in this way.

The overall gain in accuracy however is significant in applications.

One can alternatively choose to truncate the successive series far beyond their least term, in a prescribed way which improves even the exponential order of the error (at the expense of making the calculation substantially more laborious).

This is an entry point into hyperasymptotics, a theory in its own right, and numerous papers address it. The concept and method were discovered by Berry [9], with ideas going back to Dingle [26] and further developed by Berry, Delabaere, Howls and Olde Daalhuis, see e.g. [7], [42].

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