Why certain integrals are "impossible".

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Outline

1. Introduction.
2. Elementary fields and functions.
3. Liouville's Theorem.
4. An example.

Probability

- Central Limit Theorem
- $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du$
- For probability applications, we need $\Phi(\infty) = 1$.
- This is not proved by finding a formula for $\Phi(x)$ (by finding an explicit antiderivative of $e^{-u^2/2}$) and taking the limit as $x \longrightarrow \infty$.

Number Theory

- Prime Number Theorem
- $\pi(x) = \#\{n \leq x \mid n \text{ is prime }\}$
- $Li(x) = \displaystyle\int_2^x \frac{1}{\ln(t)}\, dt$
- $\pi(x) \sim Li(x)$ as $x \to \infty$
- This is not proved by finding an explicit antiderivative of $\frac{1}{\ln(t)}$.
- If $u = \ln(t)$, then $\int \frac{1}{\ln(t)}\, dt = \int \frac{e^u}{u}\, du$.

Elementary formulas

- The indefinite integrals $\int e^{-u^2}\,du$ and $\int \frac{e^u}{u}\,du$ do not have elementary formulas.
- How does one prove such claims?
- First have to give a precise definition of "elementary formula".
- After all $\int e^{-u^2}\,du = \int_a^u e^{-x^2}\,dx + C$ for any constants a and C by FTC.

History

- Newton was perfectly happy to solve an integral by a power series.

- Leibniz preferred integration in "finite terms" and allowed transcendental functions like logarithms.

Elementary function

- An **elementary function** (roughly) should be a function of one variable built out of polynomials, exponentials, logarithms, trigonometric functions, and inverse trigonometric functions, by using the operations of addition, multiplication, division, root extraction, and composition.

- **Example**: $\dfrac{\sin^{-1}(x^3-1)}{\sqrt{\ln x + \cos(x/x^2+1)}}$

A simplification

- We will use $\mathbb{C}$-valued functions of the **real** variable x, i.e., our constants will be complex numbers.
- All trigonometric functions and inverse-trigonometric functions can be written in terms of complex exponentials and logarithms.
- $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$, $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$
- $\tan^{-1}(x) = \dfrac{1}{2i}(\ln(\dfrac{x-i}{x+i}) - i\pi)$
- $\sin^{-1}(x) = \tan^{-1}(\dfrac{x}{\sqrt{1-x^2}})$, $\cos^{-1}(x) = \tan^{-1}(\dfrac{\sqrt{1-x^2}}{x})$

Meromorphic functions

- A **meromorphic function** is a function defined on an open interval I of the real numbers whose values are complex numbers or ∞ with the property that sufficiently close to any x_0 in I the function is given by a convergent Laurent series in $x - x_0$.
- Rational functions are meromorphic on $\mathbf{R}$.
- Given a meromorphic function f, both e^f and $\ln f$ are meromorphic (one may have to restrict the domain of f).

Fields of meromorphic functions

- Let $\mathbb{C}(x)$ denote the field of rational functions. Notice that this field is closed under differentiation.
- Any elementary function (under our rough definition) should be in some "extension" of $\mathbb{C}(x)$.

Fields of meromorphic functions

- If $f_1, \ldots, f_n$ are meromorphic functions, let $\mathbb{C}(f_1, \ldots, f_n)$ denote the set of all meromorphic functions h of the form

$$h = \frac{p(f_1, \ldots, f_n)}{q(f_1, \ldots, f_n)}$$

  for some n-variable polynomials $p, q \neq 0$ and $q(f_1, \ldots, f_n)$ is not identically zero.

- This definition captures the operations of addition, multiplication, and division.

- It is not hard to show that the set $\mathbb{C}(f_1, \ldots, f_n)$ is a field and that this field is closed under differentiation.

- **Example:** $K = \mathbb{C}(x, \sin x, \cos x) = \mathbb{C}(x, e^{ix})$.

Elementary fields

- A field K is an **elementary field** if $K = \mathbb{C}(x, f_1, \ldots, f_n)$ and each f_j is
  - an exponential or logarithm of an element of $K_{j-1} = \mathbb{C}(x, f_1, \ldots, f_{j-1})$
  - or f_j is **algebraic** over K_{j-1}, that is f_j is a solution to an equation $g_l t^l + \cdots + g_1 t + g_0 = 0$ where $g_0, g_1, \ldots, g_l \in K_{j-1}$
- An elementary field is built from the the field of rational functions in finitely many steps by adjoining an exponential, a logarithm, or a solution to a polynomial.
- Composition is captured by adjoining exponentials or logarithms. Root extraction is captured by the adjunction of algebraic solutions.
- Elementary fields are closed under differentiation.

Elementary functions

- A meromorphic function f is an **elementary function** if it lies in some elementary field.

- **Example:** $f(x) = \sqrt[3]{\ln x + \cos(\frac{x}{x^2+i})}$ is an elementary function

$$\mathbb{C}(x) \subset \mathbb{C}(x, \ln x) \subset \mathbb{C}(x, \ln x, e^{i(\frac{x}{x^2+i})}) \subset \mathbb{C}(x, \ln x, e^{i(\frac{x}{x^2+i})}, f)$$

Elementary integration

- A meromorphic function f can be **integrated in elementary terms** if $f = g'$ for some elementary function g.

- Recall an elementary field is closed under differentiation so if f can be integrated in elementary terms, then necessarily f is also elementary.

Differential Galois theory

- We can rephrase our problem: Given an elementary function f, when does the differential equation $\frac{dy}{dx} - f = 0$ have an elementary solution?

- The answer is in the affirmative precisely when we can find a tower of fields with special properties.

- Consider the analogy with ordinary Galois theory.

Liouville's Thereom

- **Theorem (Liouville, 1835):** Let f be an elementary function and let K be any elementary field containing f. If f can be integrated in elementary terms then there exist nonzero $c_1, \ldots, c_n \in \mathbb{C}$, nonzero $g_1, \ldots, g_n \in K$, and an element $h \in K$ such that
$$f = \sum c_j \frac{g_j'}{g_j} + h'.$$

- If $f = \sum c_j \frac{g_j'}{g_j} + h'$, then $g = \sum c_j \ln(g_j) + h$ is an elementary antiderivative of f.

- The theorem is proved by induction on the length of a tower of fields constructing $K(g)$ where g is an antiderivative of f.

An important corollary

- **Corollary:** Let f and g be in $\mathbb{C}(x)$ with $f \neq 0$ and g nonconstant. If $f(x)e^{g(x)}$ can be integrated in elementary terms then there is a function $R(x)$ in $\mathbb{C}(x)$ such that $R'(x) + g'(x)R(x) = f(x)$.

- If $R(x) \in \mathbb{C}(x)$ satisfies $R'(x) + g'(x)R(x) = f(x)$, then $R(x)e^g(x)$ is an antiderivative of $f(x)e^{g(x)}$.

- We can apply this corollary to show that e^{-x^2} and e^x/x have no elementary antiderivatives.

Proof for e^{-x^2}

- Taking $f = 1$ and $g = -x^2$ in the Corollary, we must show the differential equation

$$R'(x) - 2xR(x) = 1 \quad (*)$$

has no solution for $R(x) \in \mathbb{C}(x)$.

- ODE's shows the general solution of $(*)$ is $R(x) = e^{x^2}(\int e^{-x^2}\, dx + c)$ for any $c \in \mathbb{C}$ ... but this doesn't help!

Proof for e^{-x^2}

- Suppose that $R(x) \in \mathbb{C}(x)$ is a solution to $(*)$.
- R cannot be a constant or a polynomial in x (by degree considerations).
- Write $R(x) = \frac{p(x)}{q(x)}$ for some nonzero relatively prime polynomials $p(x), q(x)$ with $q(x)$ nonconstant.
- Let $z_0 \in \mathbb{C}$ be a root of $q(x)$ of multiplicity $\mu \geq 1$. Then $p(z_0) \neq 0$ and $p(x)/q(x) = h(x)/(x - z_0)^{\mu}$ with $h(x) \in \mathbb{C}(x)$ having numerator and denominator that are non-vanishing at z_0.

Proof for e^{-x^2}

- The quotient rule yields

$$(\frac{p(x)}{q(x)})' = \frac{-h(x)}{\mu(x-z_0)^{\mu+1}} + \frac{h'(x)}{(x-z_0)^{\mu}}$$

- As $z \longrightarrow z_0$ in $\mathbb{C}$ the absolute value of $(p(x))/q(x))'|_{x=z}$ blows up like $A/|z-z_0|^{\mu+1}$ with $A = |h(z_0)/\mu| \neq 0$.
- $|-2z \cdot (p(z)/q(z))|$ has growth bounded by a constant multiple of $1/|z-z_0|^{\mu}$ as $z \longrightarrow z_0$.
- Therefore

$$|((\frac{p(x)}{q(x)})' - 2x \cdot (\frac{p(x)}{q(x)}))|_{x=z}| \sim \frac{A}{|z-z_0|^{\mu+1}}$$

  as $z \longrightarrow z_0$.
- This contradicts the identity $R'(x) - 2xR(x) = 1$.

References

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