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Elementary Functions

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Consider the three functions

$$f(x) = \cos\left(\ln\left(2\pi x + \sqrt{4\pi x^2 + 1}\right)\right)\left(e^{x^2} + e^{-x^2}\right) + \sin\left(\ln\left(2\pi x + \sqrt{4\pi x^2 + 1}\right)\right)\left(e^{x^2} - e^{-x^2}\right),$$

$$g(x) = \begin{cases} x, & x \le 2 \\ x^2, & x > 2 \end{cases},$$

$$h(x) = \begin{cases} x + 2, & x \le 2 \\ x^2, & x > 2 \end{cases}.$$

Are they all elementary functions? If not, then which do we call elementary functions? How can we use this knowledge to our advantage?

The fact that we are considering elementary functions might point to $g(x)$ and $h(x)$ as elementary functions since they are the simplest. Sherlock Holmes might see these functions and say to his companion, “these are elementary, my dear Watson.” However, this is incorrect; in actuality, $f(x)$ is an elementary functions and $g(x)$ is not an elementary function. Sadly, $h(x)$ perplexes us as we cannot conclude anything about

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its status as an elementary function. Despite what the name suggests, an elementary function is not necessarily "simple" and a "simple" function is not necessarily an elementary function. Unfortunately, the distinction between a function that is easy on the eyes and an elementary function is rarely discussed today and was seldom discussed in the past. This distinction is important to make, powerful algorithms can be created for solving problems that involve elementary functions and even those that do not. They provide us with a rare case of a set of mathematical objects which is both broad and has substantive implications. We challenge the reader, after reading this, to open an undergraduate textbook on engineering or physics to a random page and nearly all of the functions which appear are likely to be elementary functions.

Background

Throughout history, the need to solve scientific problems prompted growth in mathematical knowledge. As mathematicians created special functions to solve these problems, they sought to distinguish the functions which had been familiar for centuries.[1] Joseph Liouville (1809–1882) was among the first to give a primitive "definition" of elementary functions in showing that certain integrals are not solvable in terms of elementary functions [**8**], but his work was predated by other mathematicians such as Marie-Jean Marquis de Condorcet (1743–1794) [**3**], Pierre-Simon de Laplace (1749–1827) [**4**], and Niels Henrik Abel (1802–1829) [**9**, p. 352–369]. These facile definitions allowed for mathematicians to demonstrate certain results on integration, but they set a harmful precedent. The lack of rigor present in these early definitions of an elementary function continues into the modern day.

Most twentieth-century calculus textbooks have "elementary function" in the index but none provide a proper definition. Instead, they provide a description in order to be palatable and appeal to a wide readership. We later show via examples, that there are many ways in which they could benefit from being more stringent on particular details in their descriptions. Differential and Integral Calculus by Richard Courant devotes a section to exposing students to the various types of elementary functions beginning with polynomials and rational functions, then algebraic functions, and finally transcendental functions [**2**].[2] A more recent attempt is found in Calculus by Tom M. Apostol where he says that an elementary function can "be obtained from polynomials, exponentials, logarithms, trigonometric or inverse trigonometric functions in a finite number of steps by using the operations of addition, subtraction, multiplication, division, or composition" [**1**, p. 367]. Unfortunately, his "definition" is found toward the end of the first volume in the chapter on Taylor series rather than in the first chapter where functions are introduced, and it is incorrect which could lead to false conclusions by readers about what constitutes an elementary function. The descriptions given by Courant and Apostol allow for the misidentification of an arbitrary piecewise function as an elementary function. More often than not, twentieth-century calculus texts have ambiguities that can lead to misidentifications of elementary function.

This trend continues in contemporary calculus textbooks. Essential Calculus by James Stewart bears similarity to Apostol's textbook. Stewart explains that elementary functions "are the polynomials, rational functions, power functions (x^a), exponential functions (a^x), logarithmic functions, trigonometric and inverse trigonometric

[1]The strict definition of special functions is an interesting question which we do not answer here. For now, we leave it to readers to consider how we might provide an actual definition for a special function given our definition of an elementary function.

[2]This raises the question: exactly what do we mean by algebraic and transcendental functions?

functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition" [15, p. 339]. Stewart's description again allows for the misidentification of piecewise functions as elementary functions. Referencing elementary functions by name without giving a precise description is an unfortunate and common trend that can also be seen in Silverman [14] as well as Thomas and Finney [6].

Calculus serves as the introductory course to more rigorous college-level mathematics for most students. Further exposition on elementary functions in calculus textbooks and courses could aid students in solving complicated problems. Providing a rigorous definition allows for the creation of the theory of elementary functions. This theory leads to the creation of methods for solving challenging problems in mathematical analysis and its applications without necessarily drowning in the minutiae of those details presented here. When it comes to elementary functions, textbooks often expect readers to be satisfied with loose descriptions of elementary functions which only hinder their ability to solve problems with mathematical analysis. Basic definitions and theorems for functions given in these courses are only practical for trivial examples. Then, upon finishing, students are often left with the impression that they can only solve simple problems. Once elementary functions are properly described, we are able to free students from the constraints imposed by these definitions and theorems so that they can solve virtually any difficult problem that they will encounter.

It should be noted that elementary functions are defined on a level beyond that of the typical undergraduate in differential algebra. J.F. Ritt [10] was among the first to write on the subject and greatly expanded on the early work by Liouville. Elementary functions as defined in differential algebra are a more general class than the elementary functions of a single real variable that we define.[3] This definition is also dependent upon prior knowledge of abstract algebra, which suggests that while elementary functions are referenced as early as calculus they may not be defined until much later. Furthermore, while many useful theorems have been obtained from this definition (e.g. Liouville's aforementioned results on integration), its dependence upon abstract algebra distances it from the immediate application of elementary functions in calculus courses. For instance, every calculus student learns how to determine the maximal domain of an elementary function as a subset of the the set of real numbers. However, in differential algebra domains are often assumed to be subsets of the set of complex numbers and are often not given direct attention. Another potential difficulty is the typical use of exponential functions and logarithmic functions of a complex variable to define trigonometric and inverse trigonometric functions.[4] This is impossible to do when a function of a real variable is not defined through a function of a complex variable. In light of these difficulties for undergraduate students and the usefulness of elementary functions, a digestible definition or at least a proper description appropriate for calculus students seems to be a necessity.

As mentioned above, the rigorous study of elementary functions allows for the creation of algorithms that can be used to solve most types of problems in calculus in the most arbitrary case. If an algorithm cannot be created, then it still gives great strength in solving problems. For example, any problem of differentiation is reduced to memorizing a handful of derivatives of elementary functions and their operations. Then, finding the order of evaluation for the elementary function, reversing this order, and taking the corresponding derivatives of each elementary function and operation creates

[3]However, after defining elementary functions and giving several examples of their use, we also provide ways in which our definition may be generalized.

[4]For example, $\sin x = \left(e^{ix} - e^{-ix}\right)/2i$.

an algorithm for differentiation of an arbitrary elementary function. Later in the paper, continuity is treated in a similar manner in an example. Almost all topics in calculus can be treated in this manner with a few exceptions, e.g. the range of a function of two or more variables. Even in cases where we have a piecewise function, which is not necessarily an elementary function, a surprising theorem at the end of the paper shows that there are cases in which piecewise functions are elementary functions. However, before elementary functions may be enjoyed, they must be defined!

Rigorous definition of an elementary function of a real variable

We build the elementary functions from eight fundamental elementary functions that serve as the building blocks of elementary functions. Three fundamental elementary operations on functions bind the fundamental elementary functions. From there, we are able to combine these fundamental elementary functions with three fundamental elementary operations. These two preliminary definitions then allow us to define all elementary functions from basic building blocks.

Definition (Fundamental elementary functions). The following eight functions are referred to as the fundamental elementary functions of a real variable:

$$f_1(x) = c, \quad c \in \mathbb{R}, \text{ with domain } D_1 \subseteq \mathbb{R} \tag{1}$$

$$f_2(x) = x, \quad \text{with domain } D_2 \subseteq \mathbb{R} \tag{2}$$

$$f_3(x) = \frac{1}{x}, \quad \text{with domain } D_3 \subseteq \mathbb{R} \setminus \{0\} \tag{3}$$

$$f_4(x) = \sqrt[n]{x}, \quad n \in \mathbb{N}, \quad \begin{matrix} \text{if } \frac{n}{2} \in \mathbb{N} \text{ then with domain } D_4 \subseteq [0, +\infty) \\ \text{if } \frac{n+1}{2} \in \mathbb{N} \text{ then with domain } D_4 \subseteq \mathbb{R} \end{matrix} \tag{4}$$

$$f_5(x) = \sin x, \quad \text{with domain } D_5 \subseteq \mathbb{R} \tag{5}$$

$$f_6(x) = e^x, \quad \text{with domain } D_6 \subseteq \mathbb{R} \tag{6}$$

$$f_7(x) = \ln x, \quad \text{with domain } D_7 \subseteq (0, +\infty) \tag{7}$$

$$f_8(x) = \arccos x, \quad \text{with domain } D_8 \subseteq [-1, 1] \tag{8}$$

Definition (Fundamental elementary operations). For any two functions (of a real variable) $f(x)$ and $g(x)$ with domains D_f, D_g and ranges R_f, R_g, respectively, the following operations are called the fundamental elementary operations on elementary functions:

1. Addition: For all $x \in D = D_f \cap D_g \neq \emptyset$, $f(x)$ and $g(x)$ are both defined and have values a and b (where $a \in R_f$ and $b \in R_g$). Thus, for all $x \in D$ there is a unique corresponding real number $c = a + b$. Hence, we define a new function

$$(f + g)(x) = f(x) + g(x)$$

where the domain of $f + g$ is $D = D_f \cap D_g$. This function is called the sum of $f(x)$ and $g(x)$.

2. Multiplication: In a similar fashion to addition, we define

$$(fg)(x) = f(x) \cdot g(x)$$

where the domain of fg is $D = D_f \cap D_g$. This function is called the product of $f(x)$ and $g(x)$.

3. Composition of Functions: In order to define the composition $g \circ f$, it is necessary that the domain of f is such that the range of f is a subset of the domain of g. If and only if this condition is satisfied can we then find $(g \circ f)(x)$ for all $x \in D_f$. Thus, we define $g \circ f$ to be the composite function of $f(x)$ and $g(x)$, if and only if

$$\begin{cases} R_f \subseteq D_g \\ (g \circ f)(x) = g(f(x)), \quad x \in D_f \end{cases}.$$

Definition (Elementary functions). A function $F(x)$ with domain D is called an elementary function, if it can be obtained from one and the same set of fundamental elementary functions using a finite number of fundamental elementary operations in one and the same way for all $x \in D$.[5]

In other words, any elementary function may be written with only one formula that consists of the allowed functions and operations for all values in the domain used finitely many times. This is why the definition of an elementary function specifies that it employs a finite number of fundamental elementary operations in one and the same way for all $x \in D$. Of course, defining them in this way excludes functional series and piecewise functions from necessarily being elementary functions. However, if the functional series or piecewise function may be written as specified in the above definition, then it is an elementary function. As an example, the series $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$ with domain may be written as e^x so that it is an elementary function.

It follows directly from the definition that elementary functions can also be obtained by performing the fundamental elementary operations on any elementary functions. There are many other operations and elementary functions which can then be obtained. In the following section, we define other classes of elementary functions which are not fundamental elementary functions and other operations on elementary functions which are not fundamental elementary operations. Then we no longer worry about using only the fundamental elementary functions and operations, and instead have an extensive list of functions which are elementary functions and operations on these elementary functions. We can then consider finitely many of these elementary functions obtained from finitely many of these operations in one and the same way as an elementary function. This gives the breadth of elementary functions.

Basic results

The following is a list of a set of functions and operations on elementary functions (denoted $f(x)$ and $g(x)$) that also yield elementary functions along with a sketch of the proof:

[5]Not using a certain fundamental elementary operation is also considered to be using a finite number of this operation.

- Subtraction
  - We have $f(x) - g(x) = f(x) + (-1)(g(x))$ is an elementary function as it consists of multiplication and addition of elementary functions.
- Division
  - We have $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ is an elementary function as it consists of multiplication and composition of elementary functions.
- Polynomial functions
  - This follows from multiplication and addition of elementary functions carried on finitely many times.
- Rational functions
  - By definition, a rational function may be expressed as a quotient of two polynomials. Hence, rational functions are elementary functions from division of elementary functions and the fact that polynomial functions are elementary functions.
- Rational powers of elementary functions
  - We have $(f(x))^{\frac{m}{n}} = \sqrt[n]{(f(x))^m}$ is an elementary function using multiplication and composition of elementary functions where $m > 0$ or using division, multiplication, and composition of elementary functions where $m < 0$.
- Trigonometric functions
  - We can define $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ which consists of addition and composition of elementary functions. Using $\cos x$ and the ever-expanding list of elementary functions, we have that $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$, $\tan x = \frac{\sin x}{\cos x}$, and $\cot x = \frac{\cos x}{\sin x}$ are elementary functions.
- Inverse trigonometric functions
  - First, we define $\arcsin x = \frac{\pi}{2} - \arccos x$ which is clearly an elementary function. To obtain $\text{arccot} x$, let $\text{arccot} x = t$ so that $x = \cot t$ where $x \in (-\infty, +\infty)$ and $t \in (0, \pi)$. Through manipulation of various trigonometric identities, we can show that $t = \arccos \frac{x}{\sqrt{1+x^2}}$ so that $\text{arccot} x = \arccos \frac{x}{\sqrt{1+x^2}}$ is an elementary function. Then, $\arctan x = \frac{\pi}{2} - \text{arccot} x$ is also an elementary function.
- Functions of the form $\log_{f(x)} g(x)$
  - We define $\log_{f(x)} g(x) = \frac{\ln g(x)}{\ln f(x)}$ for all x in the domain such that $f(x) \neq 1$, $f(x) > 0$, and $g(x) > 0$. Hence, all logarithmic functions are elementary functions.
- Functions of the form $f(x)^{g(x)}$
  - We define $f(x)^{g(x)} = e^{g(x) \cdot \ln f(x)}$ for values of x such that $f(x) > 0$.

This list simplifies the way to obtain elementary functions. From the definition of an elementary function, applying finitely many elementary operations (including those listed above) to elementary functions results in an elementary function. This allows for the quick identification of elementary functions and has shown that they are a common group. Another basic result shows that elementary functions are very useful.

The following statements are theorems that are very often proven in textbooks, e.g. Lang [7], Rosenlicht [11], and Rudin [13], and thus will only be stated without proofs.

1. All eight, $f_1(x) - f_8(x)$, fundamental elementary functions are continuous everywhere in their domains except at the isolated points and are discontinuous at the isolated points and outside of the domain.[6]
2. The sum of two continuous functions is also continuous everywhere in its domain except at the isolated points and is discontinuous at the isolated points and outside of the domain.
3. The product of two continuous functions is also continuous everywhere in its domain except at the isolated points and is discontinuous at the isolated points and outside of the domain.
4. The composition of two continuous functions is also continuous everywhere in its domain except at the isolated points and is discontinuous at the isolated points and outside of the domain.

From these four theorems, one can conclude that all elementary functions are continuous in their domains, except at the isolated points at which they are discontinuous.[7] In some real analysis books, continuity is defined in a way that all functions are continuous at their isolated points as well. This gives that elementary functions are continuous everywhere in their domains. This is a powerful and useful result, as it allows for tedious $\epsilon - \delta$ or limit based arguments for continuity to be avoided when "finding the domain" of elementary functions. These definitions remain important for finding where piecewise functions are continuous or for proving fundamental results for continuity.

A surprising theorem

Thus far, it seems that the restriction that elementary functions may be written in one formula will preclude countless useful functions from being elementary functions. Perhaps the simplest case of this is the absolute value function, $|x|$. This is shown in the lemma below.

Lemma 1. *The absolute value function* $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ *is an elementary function.*

Proof. While the absolute value function is typically written as a piecewise function as shown above. We may also write $|x| = \sqrt{x^2}$. This clearly holds for all real values of x. Since $\sqrt{x^2}$ is a rational power of an elementary function, it is also an elementary function. ■

The Lemma above suggests that it is not always the case that piecewise functions are not elementary functions. In fact, we can extend this to a much more powerful theorem.

Theorem 1. *For a function $f(x)$ defined as*

$$f(x) = \begin{cases} g(x), & x < a \\ h(x), & x > a \end{cases}$$

[6] For a function $f(x)$ defined in a domain D, a point a is called an *isolated point* of D, if and only if $a \in D$ and there exists $\epsilon > 0$ such that $x \notin D$ for all $x \in (a - \epsilon, a + \epsilon) \setminus \{a\}$.

[7] As an example of an elementary function with an isolated point in it's domain, consider $f(x) = \sqrt{x} + \sqrt{x(x-1)}$ with domain $D = \{0\} \cup [1, +\infty)$.

where $g(x)$ is an elementary function in $D_g = (-\infty, a)$ and $h(x)$ is an elementary function in $D_h = (a, +\infty)$. The function $f(x)$ with domain $D_f = \mathbb{R} \setminus \{a\}$ is an elementary function.

Proof. Note that

$$\frac{|x| + x}{2x} = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

and

$$\frac{-|x| + x}{2x} = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

Applying a shift in x, the function $f(x)$ could be written as

$$f(x) = g(x)\frac{x - a - |x - a|}{2(x - a)} + h(x)\frac{x - a + |x - a|}{2(x - a)}$$

with domain $D_f = \mathbb{R} \setminus \{a\}$. Since the formula to calculate the value of $f(x)$ at any point in the domain D_f of $f(x)$ consists of only elementary functions and elementary operations, $f(x)$ is an elementary function. ■

By induction, it can be shown that a function of the form

$$f(x) = \begin{cases} g_1(x), & x < a_1 \\ g_2(x), & a_1 < x < a_2 \\ \vdots & \\ g_n(x), & a_{n-1} < x < a_n \\ g_{n+1}(x), & x > a_n \end{cases}$$

is also an elementary function.

It is interesting to consider whether or not the function $f(x)$ is an elementary function if it is of the form

$$f(x) = \begin{cases} g(x), & x \le a \\ h(x), & x > a \end{cases}$$

where we require that $\lim_{x \to a-0} g(x) = \lim_{x \to a+0} h(x)$ so that $f(x)$ is continuous at a. Unfortunately, we do not know if there is some way in which $f(x)$ can be written in terms of one formula so that it is an elementary function.[8]

We have shown that piecewise functions which exclude this "transition point" may be treated as elementary functions. While they can be written in one formula, this formula is not useful outside of proving a theorem and we may treat piecewise functions by observing their constituent formulas. In addition, a piecewise function that includes

[8] Like the question of defining special functions, we leave this for readers to consider!

the transition point may be treated by first excluding this point and treating the function as an elementary function. Then, reincorporate this point and apply definitions to observe what happens at the transition point.

Concluding example

Rather than leaving the reader with a summary of the main points of elementary functions, we feel it is best to leave them with an example demonstrating their newfound knowledge of elementary functions.

Example 1. Considering the function given with the formula $f(x) = \sqrt{x} + \sqrt{x(x-1)}$, the maximal possible domain is $D = \{0\} \cup [1, +\infty)$. Since $f(x)$ is an elementary function, it is continuous everywhere in its domain except at 0 which is an isolated point.

Example 2. Consider the function

$$g(x) = \begin{cases} \sin x^2 + \sqrt{x} & , \ x \in [0, +\infty] \\ |x| & , \ x \in (-\infty, 0) \end{cases}.$$

Where is this function continuous?

The domain of this function is $D = \mathbb{R}$ but the formula that we use depends on whether x is in $D_1 = [0, +\infty)$ or $D_2 = (-\infty, 0)$. The transition point for this function is 0 which is contained in D_1. Unfortunately, this transition point is in the domain so that it cannot be concluded that $g(x)$ is an elementary function. In order to use Theorem 1 this transition point must be excluded by considering

$$h(x) = \begin{cases} \sin x^2 + \sqrt{x} & , \ x \in (0, +\infty) \\ |x| & , \ x \in (-\infty, 0) \end{cases}.$$

By Theorem 1, $h(x)$ is an elementary function. Therefore, for $x \in (-\infty, 0) \cup (0, +\infty)$, $g(x)$ is an elementary function and is therefore continuous in this set. For $x = 0$, the definition of continuity must be applied. Thus, the problem of continuity has been reduced to applying the definition at only one point in the domain rather than at an arbitrary point in the domain.

For simplicity, we use the definition of continuity with limits rather than with $\epsilon - \delta$. For $x = 0$, we have

$$\lim_{x \to -0} g(x) = \lim_{x \to -0} |x| = 0$$

and

$$\lim_{x \to +0} g(x) = \lim_{x \to +0} \left(\sin x^2 + \sqrt{x}\right) = 0.$$

Since $g(0) = 0$, we have

$$f(0) = \lim_{x \to -0} g(x) = \lim_{x \to +0} g(x).$$

Hence, $g(x)$ is continuous at $x = 0$. Therefore, while $g(x)$ is not an elementary function, it is continuous everywhere in its domain. The beauty in this approach is that students are able to justify the continuity of $f(x)$ without resorting to applying the definition to an arbitrary point of the domain.

We conclude here with an example demonstrating the easy justification for continuity of a piecewise function given through the use of elementary functions. However, there are many other ways in which these techniques may be applied. Differentiation can be achieved by remembering how to differentiate several elementary functions. Taylor series expansions of elementary functions can be done by introducing Taylor series expansions of several fundamental elementary functions and several fundamental operations on elementary functions and their corresponding power series. Unfortunately, integration of elementary functions does not always provide an elementary function. One can consult texts by K.O. Geddes et al. [**5**, pp. 523–529] or Maxwell Rosenlicht [**12**] for a modern treatment of this result. However, integration techniques can be summarized by recalling a handful of simple integration formulas and categorizing them based on elementary functions. Our definition of elementary functions is for functions of a single real variable but can be extended to functions of several real variables as well. It can also be extended to functions of a complex variable but the notion of an analytic function is more useful for these functions. This provides further breadth that allows for the use of elementary functions in numerous subjects including calculus, real analysis, and differential equations.

Summary. The definition of elementary functions was never given proper attention historically and continues to be neglected in contemporary calculus textbooks and courses. Eight fundamental elementary functions and three fundamental elementary operations are defined. Then, an elementary function is defined using these. Several powerful results are given for elementary functions. Algorithms and formulas may be made allowing students to solve complicated problems easily if they recognize and understand elementary functions. Examples of these algorithms and their usage are provided.

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