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A TALE OF TWO LIOUVILLE CLOSURES

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An H -field is a type of ordered valued differential field with a natural interaction between ordering, valuation, and derivation. The main examples are Hardy fields and fields of transseries. Aschenbrenner and van den Dries (2002) proved that every H -field K has either exactly one or exactly two Liouville closures up to isomorphism over K , but the precise dividing line between these two cases was unknown. We prove here that this dividing line is determined by λ -freeness, a property of H -fields that prevents certain deviant behavior. In particular, we show that under certain types of extensions related to adjoining integrals and exponential integrals, the property of λ -freeness is preserved. In the proofs we introduce a new technique for studying H -fields, the *yardstick argument* which involves the rate of growth of pseudoconvergence.

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1. Introduction

Consider the classical ordinary differential equation

$$(*) \quad y' + fy = g$$

where f and g are sufficiently nice real-valued functions. To solve $(*)$, we first perform an *exponential integration* to obtain the so-called *integrating factor*

$$\mu = \exp \int f.$$

Then we perform an *integration* to obtain a solution $y = \mu^{-1} \int (g\mu)$. In this paper, we wish to consider integration and exponential integration in the context of H -fields. H -fields and all other terms used in this introduction will be properly defined in the body of this paper.

H -fields are a certain kind of ordered valued differential field introduced in [Aschenbrenner and van den Dries 2002] and include all *Hardy fields* containing \mathbb{R} ; Hardy fields are ordered differential fields of germs of real-valued functions defined on half-lines $(a, +\infty)$, (e.g., see [Bourbaki 1951, Chapitre V] or [Rosenlicht 1983a; 1983b]). Other examples include fields of *transseries* such as the *field of logarithmic-exponential transseries* \mathbb{T} and the *field of logarithmic transseries* \mathbb{T}_{\log} (e.g., see [Écalle 1992; van der Hoeven 2006; ADH 2017]). Our primary reference for the theory of H -fields, and all other things considered in this paper, is the work “Asymptotic differential algebra and model theory of transseries”, by Matthias Aschenbrenner, Lou van den Dries and Joris van der Hoeven, which we refer to as [ADH 2017].

A real closed H -field in which every equation of the form $(*)$ has a nonzero solution, with f and g ranging over K , is said to be *Liouville closed*. If K is an H -field, then a minimal Liouville closed H -field extension of K is called a *Liouville closure* of K . The main result of [Aschenbrenner and van den Dries 2002] is that for any H -field K , exactly one of the following occurs:

- (I) K has exactly one Liouville closure up to isomorphism over K .
- (II) K has exactly two Liouville closures up to isomorphism over K .

There are three distinct types of H -fields: an H -field K either is *grounded*, *has a gap*, or has *asymptotic integration*. According to that work, grounded H -fields fall into case (I) and H -fields with a gap fall into case (II). If an H -field has asymptotic integration, then it is either in case (I) or (II). However, the precise dividing line between (I) and (II) for asymptotic integration was not known.

The main result of this paper ([Theorem 12.1](#)) shows that this dividing line is exactly the property of λ -freeness. We prove that if an H -field is λ -free, then it is in case (I), and if an H -field has asymptotic integration and is not λ -free,

then it is in case (II). This follows by combining known facts about λ -freeness from [ADH 2017] with our new technical results which show that λ -freeness is preserved under certain adjunctions of integrals and exponential integrals. In order to “defend” the λ -freeness of an H -field in these types of extensions, we introduce the *yardstick argument*, which concerns the “rate of pseudoconvergence” when adjoining integrals and exponential integrals.

We use many definitions and cite many results from [ADH 2017]. As a general rule, any result taken directly from that reference is titled ADH instead of Lemma, Theorem, etc. In citing results in this way we do not imply that they are originally due to the authors of [ADH 2017]; for instance, ADH 4.1 is actually a classical fact of valuation theory due to Kaplansky. Furthermore, in citations we omit qualifiers when no confusion should arise, writing, for example, [Gehret 2017a, 3.2] instead of [Gehret 2017a, Lemma 3.2].

In Section 2, we introduce the notion of a subset S of an ordered abelian group Γ being *jammed*. A set S being jammed corresponds to the elements near the top of S becoming closer and closer together at an unreasonably fast rate. Being jammed is an exotic property which we later wish to avoid.

In Section 3, we recall the basic theory of asymptotic couples and introduce and study the *yardstick property* of subsets of asymptotic couples. Asymptotic couples are pairs (Γ, ψ) where Γ is an ordered abelian group and $\psi : \Gamma \setminus \{0\} \rightarrow \Gamma$ is a map which satisfies, among other things, a valuation-theoretic version of l’Hôpital’s rule. Asymptotic couples often arise as the value groups of H -fields, where the map ψ comes from the logarithmic derivative operation $f \mapsto f'/f$ for $f \neq 0$. Roughly speaking, a set S has the yardstick property if for any element $\gamma \in S$, there is a larger element $\gamma + \varepsilon(\gamma) \in S$ for a certain “yardstick” $\varepsilon(\gamma) > 0$ which depends on γ and which we can explicitly describe. In contrast to the notion of being jammed, the yardstick property is a desirable tame property. In Section 3 we show, among other things, that the two properties are incompatible, except in a single degenerate case. Asymptotic couples were introduced by Rosenlicht [1979; 1980; 1981] in order to study the value group of a differential field with a so-called *differential valuation*, what we call here a *differential-valued field*. For more on asymptotic couples, including the extension theory of asymptotic couples and some model-theoretic results concerning the asymptotic couples of \mathbb{T} and \mathbb{T}_{\log} , see [Aschenbrenner and van den Dries 2000; Aschenbrenner 2003; Gehret 2017b; 2017a] and [ADH 2017, §6.5, §9.2, §9.8 and §9.9].

In Section 4 we recall definitions concerning pseudocauchy sequences in valued fields and some of the elementary facts concerning pseudocauchy sequences. The main result of Section 4 is Lemma 4.4 which is a rational version of Kaplansky’s lemma (ADH 4.1). We assume the reader is familiar with basic valuation theory,

including notions such as henselianity. As a general reference, see [ADH 2017, Chapters 2 and 3] or [Engler and Prestel 2005].

In Section 5 we give the definitions and relevant properties of *differential fields*, *valued differential fields*, *asymptotic fields*, *pre-differential-valued fields*, *differential-valued fields*, *pre- H -fields* and *H -fields*. These are the types of fields we will be concerned with in the later sections. Nearly everything from this section is from [ADH 2017] except for Lemmas 5.1 and 5.4 which are needed in our proof of Theorem 12.1.

In Section 6 we give a survey of the property of λ -freeness, citing many definitions and results from [ADH 2017, §11.5 and §11.6]. Many of these results we cite, and later use, involve situations where λ -freeness is preserved in certain valued differential field extensions. The main result of this section is Proposition 6.19 which shows that a rather general type of field extension preserves λ -freeness. Proposition 6.19 is related to the yardstick property of Section 3.

In Section 7, Section 8, and Section 9, we show that under various circumstances, if a pre-differential-valued field or a pre- H -field K is λ -free, and we adjoin an integral or an exponential integral to K for an element in K that does not already have an integral or exponential integral, then the resulting field extension will also be λ -free. The arguments in all three sections mirror one another and the main results, Propositions 7.2, 8.3, and 9.3 are all instances of Proposition 6.19.

In Sections 10 and 11 we give two minor applications of the results of Sections 7, 8, and 9. In Section 10 we show that λ -freeness is preserved when passing to the *differential-valued hull* of a λ -free pre-differential-field K (Theorem 10.2). In Section 11 we show that for λ -free differential-valued fields K , the minimum henselian, integration-closed extension $K(\int)$ of K is also λ -free (Theorem 11.2).

In Section 12 we prove the main result of this paper, Theorem 12.1. Combining it with the results in Section 10, we also obtain a generalization to the setting of pre- H -fields (Corollary 12.3). Finally, we provide proofs of claims made in [Aschenbrenner and van den Dries 2002; 2005] (Corollary 12.6 and Remark 12.7).

Conventions. Throughout, m and n range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers. By “ordered set” we mean “totally ordered set”.

Let S be an ordered set. Below, the ordering on S will be denoted by \leq , and a subset of S is viewed as ordered by the induced ordering. We put $S_\infty := S \cup \{\infty\}$, $\infty \notin S$, with the ordering on S extended to a (total) ordering on S_∞ by $S < \infty$. Suppose that B is a subset of S . We put $S^{>B} := \{s \in S : s > b \text{ for every } b \in B\}$ and we denote $S^{>\{a\}}$ as just $S^{>a}$; similarly for \geq , $<$, and \leq instead of $>$. For $a, b \in S$ we put

$$[a, b] := \{x \in S : a \leq x \leq b\}.$$

A subset C of S is said to be *convex* in S if for all $a, b \in C$ we have $[a, b] \subseteq C$. A subset A of S is said to be a *downward closed* in S , if for all $a \in A$ and $s \in S$ we have $s < a \implies s \in A$. For $A \subseteq S$ we put

$$A^\downarrow := \{s \in S : s \leq a \text{ for some } a \in A\},$$

which is the smallest downward closed subset of S containing A .

A *well-indexed sequence* is a sequence (a_ρ) whose terms a_ρ are indexed by the elements ρ of an infinite well-ordered set without a greatest element.

Suppose that G is an ordered abelian group. Then we set $G^\neq := G \setminus \{0\}$. Also, $G^< := G^{<0}$; similarly for \geq, \leq , and $>$ instead of $<$. We define $|g| := \max\{g, -g\}$ for $g \in G$. For $a \in G$, the *archimedean class* of a is defined by

$$[a] := \{g \in G : |a| \leq n|g| \text{ and } |g| \leq n|a| \text{ for some } n \geq 1\}.$$

The archimedean classes partition G . Each archimedean class $[a]$ with $a \neq 0$ is the disjoint union of the two convex sets $[a] \cap G^<$ and $[a] \cap G^>$. We order the set $[G] := \{[a] : a \in G\}$ of archimedean classes by

$$[a] < [b] :\iff n|a| < |b| \text{ for all } n \geq 1.$$

We have $[0] < [a]$ for all $a \in G^\neq$, and

$$[a] \leq [b] \iff |a| \leq n|b| \text{ for some } n \geq 1.$$

We shall consider G to be an ordered subgroup of its *divisible hull* $\mathbb{Q}G$. The divisible hull of G is the divisible abelian group $\mathbb{Q}G := \mathbb{Q} \otimes_{\mathbb{Z}} G$ equipped with the unique ordering which makes it an ordered abelian group containing G as an ordered subgroup.

2. Ordered abelian groups

In this section Γ is an ordered abelian group, $S \subseteq \Gamma$, $\alpha \in \Gamma$ and $n \geq 1$. We define:

$$\alpha + nS := \{\alpha + n\gamma : \gamma \in S\}.$$

A set of the form $\alpha + nS$ is called an *affine transform* of S . Many qualitative properties of a set $S \subseteq \Gamma$ are preserved when passing to an affine transform, for instance:

Lemma 2.1. S has a supremum in $\mathbb{Q}\Gamma$ if and only if $\alpha + nS$ does.

Definition 2.2. We say that S is *jammed* (in Γ) if $S \neq \emptyset$ does not have a greatest element and for every nontrivial convex subgroup $\Delta \neq \{0\}$ of Γ , there is $\gamma_0 \in S$ such that for every $\gamma_1 \in S^{>\gamma_0}$, $\gamma_1 - \gamma_0 \in \Delta$.

Example 2.3. Suppose $\Gamma \neq \{0\}$ is such that $\Gamma^>$ does not have a least element. Then $S := \Gamma^{<\beta}$ is jammed for every $\beta \in \Gamma$. In particular, $\Gamma^<$ is jammed.

Most $\Gamma \neq \{0\}$ we will deal with are either divisible or else $[\Gamma^\neq]$ does not have a least element and so [Example 2.3](#) will provide a large collection of jammed subsets for such Γ . Of course, not all jammed sets are of the form $S^\downarrow = \Gamma^{<\beta}$.

Whether or not S is jammed in Γ depends on the archimedean classes of Γ in the following way:

Lemma 2.4. *Let Γ_1 be an ordered abelian group extension of Γ such that $[\Gamma^\neq]$ is coinital in $[\Gamma_1^\neq]$. Then S is jammed in Γ if and only if S is jammed in Γ_1 .*

Being jammed is also preserved by affine transforms:

Lemma 2.5. *S is jammed if and only if $\alpha + nS$ is jammed.*

Proof. (\implies) Let Δ be a nontrivial convex subgroup of Γ . Let $\gamma_0 \in S$ be such that for every $\gamma_1 \in S^{>\gamma_0}$, $\gamma_1 - \gamma_0 \in \Delta$. Consider the element $\delta_0 := \alpha + n\gamma_0 \in \alpha + nS$. Let $\delta_1 \in (\alpha + nS)^{>\delta_0}$. Then $\delta_1 = \alpha + n\gamma_1$ for some $\gamma_1 \in S^{>\gamma_0}$ and $\delta_1 - \delta_0 = n(\gamma_1 - \gamma_0) \in \Delta$. We conclude that $\alpha + nS$ is jammed.

(\impliedby) Let Δ be a nontrivial convex subgroup of Γ . Let $\delta_0 = \alpha + n\gamma_0 \in \alpha + nS$ be such that $\delta_1 - \delta_0 \in \Delta$ for all $\delta_1 \in (\alpha + nS)^{>\delta_0}$. Then for $\gamma_1 \in S^{>\gamma_0}$ we have $\delta_1 := \alpha + n\gamma_1 \in (\alpha + nS)^{>\delta_0}$ and so $\delta_1 - \delta_0 = n(\gamma_1 - \gamma_0) \in \Delta$. As Δ is convex, it follows that $\gamma_1 - \gamma_0 \in \Delta$. We conclude that S is jammed. \square

Whether or not S is jammed depends only on the downward closure S^\downarrow of S :

Lemma 2.6. *S is jammed if and only if S^\downarrow is jammed.*

3. Asymptotic couples

An *asymptotic couple* is a pair (Γ, ψ) where Γ is an ordered abelian group and $\psi : \Gamma^\neq \rightarrow \Gamma$ satisfies for all $\alpha, \beta \in \Gamma^\neq$,

$$(AC1) \quad \alpha + \beta \neq 0 \implies \psi(\alpha + \beta) \geq \min(\psi(\alpha), \psi(\beta));$$

$$(AC2) \quad \psi(k\alpha) = \psi(\alpha) \text{ for all } k \in \mathbb{Z}^\neq, \text{ in particular, } \psi(-\alpha) = \psi(\alpha);$$

$$(AC3) \quad \alpha > 0 \implies \alpha + \psi(\alpha) > \psi(\beta).$$

If in addition for all $\alpha, \beta \in \Gamma$,

$$(HC) \quad 0 < \alpha \leq \beta \implies \psi(\alpha) \geq \psi(\beta),$$

then (Γ, ψ) is said to be of *H-type*, or to be an *H-asymptotic couple*.

By convention we extend ψ to all of Γ by setting $\psi(0) := \infty$. Then

$$\psi(\alpha + \beta) \geq \min(\psi(\alpha), \psi(\beta))$$

holds for all $\alpha, \beta \in \Gamma$, and we construe $\psi : \Gamma \rightarrow \Gamma_\infty$ as a (non-surjective) valuation on the abelian group Γ . If (Γ, ψ) is of *H-type*, then this valuation is convex in the sense of [\[ADH 2017, §2.4\]](#).

For $\alpha \in \Gamma^\neq$ we shall also use the following notation:

$$\alpha^\dagger := \psi(\alpha), \quad \alpha' := \alpha + \psi(\alpha).$$

The following subsets of Γ play special roles:

$$\begin{aligned} (\Gamma^\neq)' &:= \{\gamma' : \gamma \in \Gamma^\neq\}, & (\Gamma^>)' &:= \{\gamma' : \gamma \in \Gamma^>\}, \\ \Psi &:= \psi(\Gamma^\neq) = \{\gamma^\dagger : \gamma \in \Gamma^\neq\} = \{\gamma^\dagger : \gamma \in \Gamma^>\}. \end{aligned}$$

Note that by (AC3) we have $\Psi < (\Gamma^>)'$. It is also the case that $(\Gamma^<)' < (\Gamma^>)'$:

ADH 3.1. *The map $\gamma \mapsto \gamma' = \gamma + \psi(\gamma) : \Gamma^\neq \rightarrow \Gamma$ is strictly increasing. In particular:*

- (1) $(\Gamma^<)' < (\Gamma^>)'$, and
- (2) for $\beta \in \Gamma$ there is at most one $\alpha \in \Gamma^\neq$ such that $\alpha' = \beta$.

Proof. This follows from [ADH 2017, 6.5.4(iii)]. □

ADH 3.2 [ADH 2017, 9.2.4]. *There is at most one β such that*

$$\Psi < \beta < (\Gamma^>)'.$$

If Ψ has a largest element, there is no such β .

Definition 3.3. Let (Γ, ψ) be an asymptotic couple. If $\Gamma = (\Gamma^\neq)'$, then we say that (Γ, ψ) has *asymptotic integration*. If there is $\beta \in \Gamma$ as in ADH 3.2, then we say that β is a *gap* in (Γ, ψ) and that (Γ, ψ) *has a gap*. Finally, we call (Γ, ψ) *grounded* if Ψ has a largest element, and *ungrounded* otherwise.

The notions of asymptotic integration, gaps and being grounded form an important trichotomy for H -asymptotic couples:

ADH 3.4 [ADH 2017, 9.2.16]. *Let (Γ, ψ) be an H -asymptotic couple. Then exactly one of the following is true:*

- (1) (Γ, ψ) has a gap, in particular, $\Gamma \setminus (\Gamma^\neq)' = \{\beta\}$ where β is a gap in Γ ;
- (2) (Γ, ψ) is grounded, in particular, $\Gamma \setminus (\Gamma^\neq)' = \{\max \Psi\}$;
- (3) (Γ, ψ) has asymptotic integration.

Remark 3.5. Gaps in H -asymptotic couples are the fundamental source of deviant behavior we wish to avoid. If β is a gap in an H -asymptotic couple (Γ, ψ) , then there is no $\alpha \in \Gamma$ such that $\alpha' = \beta$, or in other words, β cannot be asymptotically integrated. This presents us with an *irreversible choice*: if we wish to adjoin to (Γ, ψ) an asymptotic integral for β , then we have to choose once and for all if that asymptotic integral will be positive or negative. This phenomenon is referred to as the *fork in the road* and is the primary cause of H -fields having two nonisomorphic Liouville closures, as we shall see in Section 12 below. Gaps also prove to be a

main obstruction in the model theory of asymptotic couples. For more on this, see [Aschenbrenner and van den Dries 2000] and [Gehret 2017a].

Definition 3.6 (The Divisible Hull). Given an asymptotic couple (Γ, ψ) , ψ extends uniquely to a map $(\mathbb{Q}\Gamma)^\neq \rightarrow \mathbb{Q}\Gamma$, also denoted by ψ , such that $(\mathbb{Q}\Gamma, \psi)$ is an asymptotic couple. We call $(\mathbb{Q}\Gamma, \psi)$ the *divisible hull* of (Γ, ψ) . Here are some basic facts about the divisible hull:

- (1) $\psi((\mathbb{Q}\Gamma)^\neq) = \Psi = \psi(\Gamma^\neq)$;
- (2) if (Γ, ψ) is of H -type, then so is $(\mathbb{Q}\Gamma, \psi)$;
- (3) if (Γ, ψ) is grounded, then so is $(\mathbb{Q}\Gamma, \psi)$;
- (4) if $\beta \in \Gamma$ is a gap in (Γ, ψ) , then it is a gap in $(\mathbb{Q}\Gamma, \psi)$;
- (5) $(\Gamma^\neq)' = ((\mathbb{Q}\Gamma)^\neq)' \cap \Gamma$.

For proofs of these facts, see [ADH 2017, §6.5 and 9.2.8]. We say (Γ, ψ) has *rational asymptotic integration* if $(\mathbb{Q}\Gamma, \psi)$ has asymptotic integration.

In the rest of this section (Γ, ψ) is an H -asymptotic couple with asymptotic integration and we let α, β, γ range over Γ .

Definition 3.7. For $\alpha \in \Gamma$ we let $\int \alpha$ denote the unique element $\beta \in \Gamma^\neq$ such that $\beta' = \alpha$ and we call $\beta = \int \alpha$ the *integral* of α . This gives us a function $\int : \Gamma \rightarrow \Gamma^\neq$ which is the inverse of $\gamma \mapsto \gamma' : \Gamma^\neq \rightarrow \Gamma$. We define the *successor function* $s : \Gamma \rightarrow \Psi$ by $\alpha \mapsto \psi(\int \alpha)$. Finally, we define the *contraction map* $\chi : \Gamma^\neq \rightarrow \Gamma^<$ by $\alpha \mapsto \int \psi(\alpha)$. We extend χ to a function $\Gamma \rightarrow \Gamma^\leq$ by setting $\chi(0) := 0$.

The successor function gets its name from its behavior on $\psi(\Gamma_{\log}^\neq)$ in [Example 3.8](#) below (see [Gehret 2017a]). The contraction map gets its name from the way it contracts archimedean classes in the sense of [Lemma 3.9\(5\)](#) below. Contraction maps originate from the study of precontraction groups and ordered exponential fields (see [Kuhlmann 1994; 1995; 2000]).

Example 3.8 (The asymptotic couple of \mathbb{T}_{\log}). Define the abelian group $\Gamma_{\log} := \bigoplus_n \mathbb{R}e_n$, equipped with the unique ordering such that $e_n > 0$ for all n , and $[e_m] > [e_n]$ whenever $m < n$. It is convenient to think of an element $\sum r_i e_i$ of Γ_{\log} as the vector (r_0, r_1, r_2, \dots) . Next, we define the map $\psi : \Gamma_{\log}^\neq \rightarrow \Gamma_{\log}$ by

$$\underbrace{(0, \dots, 0)}_n, \underbrace{(r_n, r_{n+1}, \dots)}_{\neq 0} \mapsto \underbrace{(1, \dots, 1)}_{n+1}, (0, 0, \dots).$$

It is easy to verify that (Γ_{\log}, ψ) is an H -asymptotic couple with rational asymptotic integration. Furthermore, the functions \int , s , and χ are given by the following formulas:

(1) (**integral**) For $\alpha = (r_0, r_1, r_2, \dots) \in \Gamma_{\log}$, take the unique n such that $r_n \neq 1$ and $r_m = 1$ for $m < n$. Then the formula for $\alpha \mapsto \int \alpha$ is given as follows:

$$\alpha = (\underbrace{1, \dots, 1}_n, \underbrace{r_n}_{\neq 1}, r_{n+1}, r_{n+2}, \dots) \mapsto \int \alpha = (\underbrace{0, \dots, 0}_n, r_n - 1, r_{n+1}, r_{n+2}, \dots) : \Gamma_{\log} \rightarrow \Gamma_{\log}^{\neq}.$$

(2) (**successor**) For $\alpha = (r_0, r_1, r_2, \dots) \in \Gamma_{\log}$, take the unique n such that $r_n \neq 1$ and $r_m = 1$ for $m < n$. Then the formula for $\alpha \mapsto s(\alpha)$ is given as follows:

$$\alpha = (\underbrace{1, \dots, 1}_n, \underbrace{r_n}_{\neq 1}, r_{n+1}, r_{n+1}, \dots) \mapsto s(\alpha) = (\underbrace{1, \dots, 1}_{n+1}, 0, 0, \dots) : \Gamma_{\log} \rightarrow \Psi_{\log} \subseteq \Gamma_{\log}.$$

(3) (**contraction**) If $\alpha = 0$, then $\chi(\alpha) = 0$. Otherwise, for $\alpha = (r_0, r_1, r_2, \dots) \in \Gamma_{\log}^{\neq}$, take the unique n such that $r_n \neq 0$ and $r_m = 0$ for $m < n$. Then the formula for $\alpha \mapsto \chi(\alpha)$ is given as follows:

$$\alpha = (\underbrace{0, \dots, 0}_n, \underbrace{r_n}_{\neq 0}, r_{n+1}, \dots) \mapsto \chi(\alpha) = (\underbrace{0, \dots, 0}_{n+1}, -1, 0, 0, \dots) : \Gamma_{\log} \rightarrow \Gamma_{\log}^{\leq}.$$

For more on this example, see [Gehret 2017b; 2017a].

Lemma 3.9. For all $\alpha, \beta \in \Gamma$ and $\gamma \in \Gamma^{\neq}$:

- (1) (**integral identity**) $\int \alpha = \alpha - s\alpha$.
- (2) (**successor identity**) If $s\alpha < s\beta$, then $\psi(\beta - \alpha) = s\alpha$.
- (3) (**fixed point identity**) $\beta = \psi(\alpha - \beta)$ if and only if $\beta = s\alpha$.
- (4) $s\alpha < s^2\alpha$.
- (5) $[\chi(\gamma)] < [\gamma]$.
- (6) $\alpha \neq \beta \implies [\chi(\alpha) - \chi(\beta)] < [\alpha - \beta]$.
- (7) $\alpha < \beta \implies \alpha - \chi(\alpha) < \beta - \chi(\beta)$.

Proof. For (1)–(4) we direct the reader to [Gehret 2017a]. (1) is Lemma 3.2 there, (2) is Lemma 3.4 there, (3) is Lemma 3.7 there, (4) is Lemma 3.3 there, and (5) and (6) follow easily from [ADH 2017, 9.2.18 (iii,iv)]. (7) follows from (6). \square

Lemma 3.10. Suppose $\alpha \in (\Gamma^{<})'$ and $n \geq 1$. Then $\alpha + (n+1)(s\alpha - \alpha) \in (\Gamma^{>})'$.

Proof. Suppose $\alpha \in (\Gamma^{<})'$. Then we have

$$\begin{aligned} \alpha + (n+1)(s\alpha - \alpha) &= s\alpha + ns\alpha - n\alpha \\ &= \psi(\int \alpha) + n\psi(\int \alpha) - n(\int \alpha)' \\ &= \psi(\int \alpha) + n\psi(\int \alpha) - n(\int \alpha) - n\psi(\int \alpha) \\ &= \psi(\int \alpha) - n\int \alpha \\ &= (-n\int \alpha)' \in (\Gamma^{>})'. \end{aligned}$$

The last part follows because $\alpha \in (\Gamma^{<})'$ if and only if $\int \alpha \in \Gamma^{<}$ if and only if $-n \int \alpha \in \Gamma^{>}$ if and only if $(-n \int \alpha)' \in (\Gamma^{>})'$. \square

Lemma 3.11. *The sets Ψ and Ψ^\downarrow are jammed.*

Proof. By [Lemma 2.6](#), it suffices to show that $\Psi^\downarrow = (\Gamma^{<})'$ is jammed. By asymptotic integration and [ADH 3.4](#), $(\Gamma^{<})'$ is nonempty and does not have a largest element. Let Δ be a nontrivial convex subgroup of Γ . Take $\delta \in \Delta^{>}$ and set $\gamma_0 := (-\delta)' \in (\Gamma^{<})'$. Then

$$\gamma_0 + 2\delta = \gamma_0 + 2(-\int(-\delta)') = \gamma_0 + 2(-\int\gamma_0) = \gamma_0 + 2(s\gamma_0 - \gamma_0),$$

where the last equality follows from [Lemma 3.9\(1\)](#). Thus $\gamma_0 + 2\delta \in (\Gamma^{>})'$ by [Lemma 3.10](#). In particular, for every $\gamma_1 \in ((\Gamma^{<})')^{>\gamma_0}$, $\gamma_1 - \gamma_0 < 2\delta \in \Delta$. We conclude that $(\Gamma^{<})'$ is jammed. \square

Calculation 3.12. Suppose $\gamma \neq 0$. Then

$$\int(\gamma' - \int s\gamma') = \gamma + (s\gamma^\dagger - \gamma^\dagger) = \gamma - \chi(\gamma).$$

Proof. We begin by showing that

$$(A) \quad s(\gamma + s\gamma^\dagger) = \gamma^\dagger.$$

By (2) and (4) of [Lemma 3.9](#) we have

$$\psi(-\gamma) = \gamma^\dagger < s\gamma^\dagger = \psi(\gamma^\dagger - s\gamma^\dagger),$$

which implies $\psi(\gamma^\dagger - \gamma - s\gamma^\dagger) = \gamma^\dagger$. Now (A) follows by [Lemma 3.9\(3\)](#).

We now proceed with our main calculation. The first and second equalities below come from [Lemma 3.9\(1\)](#); the third from the definitions of s and $'$, and the last from (A).

$$\begin{aligned} \int(\gamma' - \int s\gamma') &= (\gamma' - \int s\gamma') - s(\gamma' - \int s\gamma') \\ &= (\gamma' - s\gamma' + s^2\gamma') - s(\gamma' - s\gamma' + s^2\gamma') \\ &= (\gamma + \gamma^\dagger - \gamma^\dagger + s\gamma^\dagger) - s(\gamma + \gamma^\dagger - \gamma^\dagger + s\gamma^\dagger) \\ &= \gamma + s\gamma^\dagger - s(\gamma + s\gamma^\dagger) \\ &= \gamma + (s\gamma^\dagger - \gamma^\dagger) \end{aligned}$$

Finally, note that $-\chi(\gamma) = s\gamma^\dagger - \gamma^\dagger$ follows from applying [Lemma 3.9\(1\)](#) to γ^\dagger and the definition of χ . \square

Lemma 3.13. *Let $\gamma \in (\Gamma^{>})'$. Then*

$$\int \gamma > -\int s\gamma = -\chi \int \gamma > 0.$$

Furthermore, if $\gamma_0, \gamma_1 \in (\Gamma^{>})'$, then $\gamma_0 \leq \gamma_1$ implies $-\int s\gamma_0 \leq -\int s\gamma_1$.

Proof. We have $s\gamma \in (\Gamma^<)'$ which implies that $-\int s\gamma > 0$, which gives the second part of the first inequality. For the first part we note that

$$\int \gamma > -\int s\gamma \iff \int \gamma + \int s\gamma > 0 \iff \int \gamma + \chi \int \gamma > 0,$$

this last equivalence being true because $\int \gamma > 0$ and $[\chi \int \gamma] < [\int \gamma]$ by [Lemma 3.9\(5\)](#).

For the second inequality, we have

$$\begin{aligned} \gamma_0 \leq \gamma_1 &\implies s\gamma_0 \geq s\gamma_1 && \text{since } \gamma_0, \gamma_1 \in (\Gamma^>)' \\ &\iff \int s\gamma_0 \geq \int s\gamma_1 && \text{by ADH 3.1} \\ &\iff -\int s\gamma_0 \leq -\int s\gamma_1. && \square \end{aligned}$$

Definition 3.14. Let S be a nonempty convex subset of Γ without a greatest element. We say that S has the *yardstick property* if there is $\beta \in S$ such that for every $\gamma \in S^{>\beta}$, $\gamma - \chi(\gamma) \in S$.

Note that if S is a nonempty convex subset of Γ without a greatest element, then S has the yardstick property if and only if S^\downarrow has the yardstick property. The following is immediate from [Lemma 3.9\(7\)](#):

Lemma 3.15. *Suppose S is a nonempty convex subset of Γ without a greatest element with the yardstick property. Then for every $\gamma \in S$, $\gamma - \chi(\gamma) \in S$.*

Remark 3.16. The yardstick property says that if you have an element $\gamma \in S$, then you can travel up the set S to a larger element $\gamma - \chi(\gamma)$ in a “measurable” way, i.e., you can increase upwards at least a distance of $-\chi(\gamma)$ and still remain in S . Similar to the property *jammed* from [Section 2](#), this is a qualitative property concerning the top of the set S . Unlike *jammed*, the yardstick property requires the asymptotic couple structure of (Γ, ψ) , and the contraction map χ in particular.

The yardstick property and being jammed are incompatible properties, except in the following case:

Lemma 3.17. *Let S be a nonempty convex subset of Γ without a greatest element with the yardstick property. Then S is jammed if and only if $S^\downarrow = \Gamma^<$.*

Proof. If $S = \Gamma^<$, then S is jammed. Now suppose that $S \neq \Gamma^<$. We must show that S is not jammed. In the first case, suppose $S \cap \Gamma^> \neq \emptyset$ and take $\gamma \in S \cap \Gamma^>$. Let Δ be a nontrivial convex subgroup of Γ such that $[\Delta] < [\chi(\gamma)]$. Now let $\gamma_0, \gamma_1 \in S$ be such that $\gamma < \gamma_0 < \gamma_0 - \chi(\gamma_0) < \gamma_1$. Note that

$$\gamma_1 - \gamma_0 > -\chi(\gamma_0) \geq -\chi(\gamma) > \Delta,$$

and we conclude that S is not jammed since $\gamma_0 > \gamma$ was arbitrary.

Next, suppose there is β such that $S < \beta < 0$. Let Δ be a nontrivial convex subgroup of Γ such that $[\beta] > [\chi(\beta)] > [\Delta]$. Let $\gamma \in S$ be arbitrary. Then $\gamma - \chi(\gamma) \in S$. Note that

$$(\gamma - \chi(\gamma)) - \gamma = -\chi(\gamma) \geq -\chi(\beta) > \Delta.$$

We conclude that S is not jammed since γ was arbitrary. \square

The following technical variant of the yardstick property will come in handy in Sections 7, 8, and 9:

Definition 3.18. Let $S \subseteq \Gamma$ be a nonempty convex set without a greatest element such that either $S \subseteq (\Gamma^{>})'$ or $S \subseteq (\Gamma^{<})'$. We say that S has the *derived yardstick property* if there is $\beta \in S$ such that for every $\gamma \in S^{>\beta}$,

$$\gamma - \int s\gamma \in S^{>\beta}.$$

Proposition 3.19. *Suppose $S \subseteq \Gamma$ is a nonempty convex set without a greatest element such that either $S \subseteq (\Gamma^{>})'$ or $S \subseteq (\Gamma^{<})'$ and S has the derived yardstick property. Then $\int S := \{\int s : s \in S\} \subseteq \Gamma$ is nonempty, convex, does not have a greatest element, and has the yardstick property.*

Proof. By ADH 3.1, $\int S$ is nonempty, convex, and does not have a greatest element. Let $\beta \in S$ be such that for every $\gamma \in S^{>\beta}$, $\gamma - \int s\gamma \in S$. Now take $\gamma \in (\int S)^{>\int\beta}$. Then $\gamma' \in S^{>\beta}$, so $\gamma' - \int s\gamma' \in S^{>\beta}$. Thus

$$\int(\gamma' - \int s\gamma') \in (\int S)^{>\int\beta}.$$

By Calculation 3.12,

$$\gamma - \chi(\gamma) \in (\int S)^{>\int\beta}.$$

We conclude that $\int S$ has the yardstick property. \square

Example 3.20. (The yardstick property in (Γ_{\log}, ψ)) To get a feel for what the yardstick property says, suppose $S \subseteq \Gamma_{\log}$ is nonempty, downward closed, and has the yardstick property. Then, given an element $\alpha \neq 0$ in S we may write

$$\alpha = (\underbrace{0, \dots, 0}_n, \underbrace{r_n}_{\neq 0}, r_{n+1}, \dots),$$

and then the yardstick property says that the following larger element is also in S :

$$\alpha - \chi(\alpha) = (\underbrace{0, \dots, 0}_n, r_n, r_{n+1}) - (\underbrace{0, \dots, 0}_{n+1}, -1, 0, 0, \dots) = (\underbrace{0, \dots, 0}_n, r_n, r_{n+1} + 1, \dots) \in S.$$

In fact, by iterating the yardstick property, we find that for *any* m , the following element is in S :

$$(\underbrace{0, \dots, 0}_n, r_n, r_{n+1} + m, \dots) \in S$$

Thus if Δ is the convex subgroup generated by $-\chi(\alpha)$, it follows that $\alpha + \Delta \subseteq S$.

4. Valued fields

In this section K is a valued field. Let \mathcal{O}_K denote its valuation ring, \mathfrak{o}_K the maximal ideal of \mathcal{O}_K , $v : K^\times \rightarrow \Gamma_K := v(K^\times)$ its valuation with value group Γ_K , and $\text{res} : \mathcal{O}_K \rightarrow \mathbf{k}_K := \mathcal{O}_K/\mathfrak{o}_K$ its residue map with residue field \mathbf{k}_K , which we may also denote as $\text{res}(K)$. We will suppress the subscript K when the valued field K is clear from context. By convention we extend v to a map $v : K \rightarrow \Gamma_\infty$ by setting $v(0) := \infty$.

Given $f, g \in K$ we have the following relations:

$$\begin{aligned} f \preceq g &: \iff vf \geq vg & (f \text{ is dominated by } g) \\ f < g &: \iff vf > vg & (f \text{ is strictly dominated by } g) \\ f \asymp g &: \iff vf = vg & (f \text{ is asymptotic to } g) \end{aligned}$$

For $f, g \in K^\times$, we have the additional relation:

$$f \sim g : \iff v(f - g) > vf \quad (f \text{ and } g \text{ are equivalent})$$

Both \asymp and \sim are equivalence relations on K and K^\times , respectively. We shall also use the following notation:

$$\begin{aligned} K^{<1} &:= \{f \in K : f < 1\} = \mathfrak{o}_K \\ K^{\preceq 1} &:= \{f \in K : f \preceq 1\} = \mathcal{O}_K \\ K^{>1} &:= \{f \in K : f > 1\} = K \setminus \mathcal{O}_K \end{aligned}$$

Pseudocauchy sequences and a Kaplansky lemma. Let (a_ρ) be a well-indexed sequence in K , and $a \in K$. Then (a_ρ) is said to *pseudoconverge to a* (written $a_\rho \rightsquigarrow a$) if for some index ρ_0 we have $a - a_\sigma < a - a_\rho$ whenever $\sigma > \rho > \rho_0$. In this case we also say that a is a *pseudolimit of (a_ρ)* . We say that (a_ρ) is a *pseudocauchy sequence in K* (or *pc-sequence in K*) if for some index ρ_0 we have

$$\tau > \sigma > \rho > \rho_0 \implies a_\tau - a_\sigma < a_\sigma - a_\rho.$$

If $a_\rho \rightsquigarrow a$, then (a_ρ) is necessarily a pc-sequence in K . A pc-sequence (a_ρ) is *divergent in K* if (a_ρ) does not have a pseudolimit in K .

Suppose that (a_ρ) is a pc-sequence in K and $a \in K$ is such that $a_\rho \rightsquigarrow a$. Also let $\gamma_\rho := v(a - a_\rho) \in \Gamma_\infty$, which is eventually in Γ and strictly increasing as a function of ρ . Recall *Kaplansky's Lemma*:

ADH 4.1 [ADH 2017, Prop. 3.2.1]. *Suppose $P \in K[X] \setminus K$. Then $P(a_\rho) \rightsquigarrow P(a)$. Furthermore, there are $\alpha \in \Gamma$ and $i \geq 1$ such that eventually $v(P(a_\rho) - P(a)) = \alpha + i\gamma_\rho$.*

Note that [ADH 4.1](#) concerns *polynomials* $P \in K[X]$. Below we give a version for rational functions, but first a few remarks.

Roughly speaking, we think of the eventual nature of the sequence (γ_ρ) as a “rate of convergence” for the pseudoconvergence $a_\rho \rightsquigarrow a$. [ADH 4.1](#) tells us that the rate of convergence for $P(a_\rho) \rightsquigarrow P(a)$ is very similar to that of $a_\rho \rightsquigarrow a$. Indeed, $(\alpha + i\gamma_\rho)$ is just an affine transform of (γ_ρ) in Γ . We want to show that applying rational functions to (a_ρ) will also have this property. Before we can do this, we need to recall a few more facts from valuation theory.

Suppose that (a_ρ) is a pc-sequence in K . A main consequence of [ADH 4.1](#) is that (a_ρ) falls into one of two categories:

- (1) (a_ρ) is of *algebraic type over K* if for *some* nonconstant $P \in K[X]$, $v(P(a_\rho))$ is eventually strictly increasing (equivalently, $P(a_\rho) \rightsquigarrow 0$).
- (2) (a_ρ) is of *transcendental type over K* if for *all* nonconstant $P \in K[X]$, $v(P(a_\rho))$ is eventually constant (equivalently, $P(a_\rho) \not\rightsquigarrow 0$).

Suppose (a_ρ) is a pc-sequence of transcendental type over K . Then (a_ρ) is divergent in K . Moreover, if $a_\rho \rightsquigarrow b$ with b in a valued field extension of K , then b will necessarily be transcendental over K .

Now suppose that (a_ρ) is a pc-sequence in K . Take ρ_0 as in the definition of “pseudocauchy sequence” and define $\gamma_\rho := v(a_{\rho'} - a_\rho) \in \Gamma$ for $\rho' > \rho > \rho_0$; this depends only on ρ and the sequence $(\gamma_\rho)_{\rho > \rho_0}$ is strictly increasing. We define the *width* of (a_ρ) to be the following upward closed subset of Γ_∞ :

$$\text{width}(a_\rho) = \{\gamma \in \Gamma_\infty : \gamma > \gamma_\rho \text{ for all } \rho > \rho_0\}.$$

The width of (a_ρ) is independent of the choice of ρ_0 . The following follows from various results in [\[ADH 2017, Chapters 2 and 3\]](#):

ADH 4.2. *Let (a_ρ) be a divergent pc-sequence in K and let b be an element of a valued field extension of K such that $a_\rho \rightsquigarrow b$. Then for $\sigma_\rho := v(b - a_\rho) \in \Gamma_\infty$, eventually $\sigma_\rho = \gamma_\rho$ and*

$$\text{width}(a_\rho) = \Gamma_\infty^{>v(b-K)} \quad \text{and} \quad v(b-K) = \Gamma_\infty^{<\text{width}(a_\rho)}$$

where $v(b-K) = \{v(b-a) : a \in K\} \subseteq \Gamma$.

Remark 4.3. Let b be an element of an immediate valued field extension of K . If $b \notin K$, then $v(b-K) \subseteq \Gamma$ is a nonempty downward closed subset of Γ without a greatest element. We think of $v(b-K)$ as encoding how well elements from K can approximate b . Below we will consider various qualitative properties of such a set $v(b-K)$ and consider what these properties say about the element b itself.

We say that pc-sequences (a_ρ) and (b_σ) in K are *equivalent* if they satisfy any of the following equivalent conditions:

- (1) (a_ρ) and (b_σ) have the same pseudolimits in every valued field extension of K ;
- (2) (a_ρ) and (b_σ) have the same width, and have a common pseudolimit in some valued field extension of K ;
- (3) there are arbitrarily large ρ and σ such that for all $\rho' > \rho$ and $\sigma' > \sigma$ we have $a_{\rho'} - b_{\sigma'} < a_{\rho'} - a_\rho$, and there are arbitrarily large ρ and σ such that for all $\rho' > \rho$ and $\sigma' > \sigma$ we have $a_{\rho'} - b_{\sigma'} < b_{\sigma'} - b_\sigma$.

See [ADH 2017, 2.2.17] for details of this equivalence.

Now we assume that L is an immediate extension of K , $a \in L \setminus K$, and (a_ρ) is a pc-sequence in K of transcendental type over K such that $a_\rho \rightsquigarrow a$.

Lemma 4.4. *Let $R(X) \in K(X) \setminus K$. Then there is an index ρ_0 such that, for $\rho > \rho_0$,*

- (1) $R(a_\rho) \in K$ (that is, $R(a_\rho) \neq \infty$);
- (2) $R(a_\rho) \rightsquigarrow R(a)$;
- (3) $v(R(a_\rho) - R(a)) = \alpha + i\gamma_\rho$, eventually, for some $\alpha \in \Gamma$ and $i \geq 1$;
- (4) $(\alpha + i\gamma_\rho)$ is eventually cofinal in $v(R(a) - K)$, with α and i as in (3);
- (5) $(R(a_\rho))$ is a divergent pc-sequence in K ; and
- (6) $v(R(a) - K) = (\alpha + i v(a - K))^\downarrow$, with α and i as in (3).

Proof. Let $R(X) = P(X)/Q(X)$ with $P, Q \in K[X]^\neq$. It is clear there exists ρ_0 such that $R(a_\rho) \in K$ for all $\rho > \rho_0$. Fix such a ρ_0 and assume $\rho > \rho_0$ for the rest of this proof.

We first consider the case that $R(X) = P(X) \in K[X] \setminus K$ is a polynomial. Then (2) and (3) follow from ADH 4.1. We will prove (5) and then (4) and (6) will follow. Assume towards a contradiction that there is $b \in K$ such that $R(a_\rho) \rightsquigarrow b$. Then $R(a_\rho) - b \rightsquigarrow 0$, so (a_ρ) is of algebraic type in view of $R(X) - b \in K[X] \setminus K$. This contradicts the assumption that (a_ρ) is a pc-sequence of transcendental type.

Next consider the case that $R(X) \in K(X) \setminus K[X]$. In particular, $Q(X) \in K[X] \setminus K$ and $Q \nmid P$. Then note that

$$\begin{aligned} v\left(\frac{P(a_\rho)}{Q(a_\rho)} - \frac{P(a)}{Q(a)}\right) &= v\left(\frac{P(a_\rho)Q(a) - P(a)Q(a_\rho)}{Q(a_\rho)Q(a)}\right) \\ &= v(P(a_\rho)Q(a) - P(a)Q(a_\rho)) - v(Q(a_\rho)) - v(Q(a)). \end{aligned}$$

The quantity $v(Q(a_\rho))$ is eventually constant since (a_ρ) is of transcendental type. Next, set $S(X) := P(X)Q(a) - P(a)Q(X) \in K(a)[X]$. Note that eventually $S(a_\rho) \neq 0$ and thus $S \neq 0$ (otherwise, the polynomial $Q(X) - (Q/P)(a)P(X)$ would be identically zero since it would have infinitely many distinct zeros, which would imply $Q \mid P$). Furthermore, $S(a) = 0$, which shows that $S \in K(a)[X] \setminus K(a)$. By ADH 4.1, it follows that $S(a_\rho) \rightsquigarrow S(a) = 0$. In particular, $v(S(a_\rho))$ is eventually

strictly increasing and there are $\alpha \in \Gamma$ and $i \geq 1$ such that eventually $v(S(a_\rho)) = \alpha + i\gamma_\rho$. This shows (2) and (3).

Finally, we will prove (5) and then (4) and (6) will follow. Assume towards a contradiction that $R(a_\rho) \rightsquigarrow b$ with $b \in K$. Then

$$v\left(\frac{P(a_\rho)}{Q(a_\rho)} - b\right) = v(P(a_\rho) - bQ(a_\rho)) - v(Q(a_\rho))$$

is eventually strictly increasing. Therefore so is $v(P(a_\rho) - bQ(a_\rho))$, since $v(Q(a_\rho))$ is eventually constant. This implies that (a_ρ) is of algebraic type, a contradiction. \square

5. Differential fields, differential-valued fields and H -fields

Differential fields. A *differential field* is a field K of characteristic zero equipped with a derivation ∂ on K , i.e., an additive map $\partial : K \rightarrow K$ which satisfies the Leibniz identity: $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in K$. For such K we identify \mathbb{Q} with a subfield of K in the usual way.

Let K be a differential field. For $a \in K$, we will often denote $a' := \partial(a)$, and for $a \in K^\times$ we will denote the *logarithmic derivative* of a as $a^\dagger := a'/a = \partial(a)/a$. For $a, b \in K^\times$, note that $(ab)^\dagger = a^\dagger + b^\dagger$, in particular, $(a^k)^\dagger = ka^\dagger$ for $k \in \mathbb{Z}$. The set $\{a \in K : a' = 0\} \subseteq K$ is a subfield of K and is called the *field of constants* of K , and denoted by C_K (or just C if K is clear from the context). If $c \in C$, then $(ca)' = ca'$ for $a \in K$. If $a, b \in K^\times$, then $a^\dagger = b^\dagger$ if and only if $a = bc$ for some $c \in C^\times$.

The following is routine:

Lemma 5.1. *Let K be a differential field. Suppose that $y_0, y_1, \ell \in K$ are such that $y_0, y_1 \notin C$ and $y_i'' = \ell y_i'$ for $i = 0, 1$. Then there are $c_0, c_1 \in C$ such that $c_0 \neq 0$ and $y_1 = c_0 y_0 + c_1$.*

In this paper we are primarily concerned with algebraic extensions and simple transcendental extensions of differential fields. In these cases, we have:

ADH 5.2 [ADH 2017, 1.9.2]. *Suppose K is a differential field and L is an algebraic field extension of K . Then ∂ extends uniquely to a derivation on L .*

ADH 5.3 [ADH 2017, 1.9.4]. *Suppose K is a differential field with field extension $L = K(x)$ where $x = (x_i)_{i \in I}$ is a family in L that is algebraically independent over K . Then there is for each family $(y_i)_{i \in I}$ in L a unique extension of ∂ to a derivation on L with $\partial(x_i) = y_i$ for all $i \in I$.*

If K is a differential field and $s \in K \setminus \partial(K)$, then ADH 5.3 allows us to *adjoin an integral for s* : let $K(x)$ be a field extension of K such that x is transcendental over K . Then by ADH 5.3 there is a unique derivation on $K(x)$ extending ∂ such that $x' = s$. Likewise, if $s \in K \setminus (K^\times)^\dagger$, then we can *adjoin an exponential integral for s* : take $K(x)$ as before and by ADH 5.3 there is a unique derivation on $K(x)$

extending ∂ such that $x' = sx$, and thus $x^\dagger = s$, i.e., “ $x = \exp(\int s)$ ”. Adjoining integrals and exponential integrals are basic examples of *Liouville extensions*:

A *Liouville extension* of K is a differential field extension L of K such that C_L is algebraic over C and for each $a \in L$ there are $t_1, \dots, t_n \in L$ with $a \in K(t_1, \dots, t_n)$ and for $i = 1, \dots, n$,

- (1) t_i is algebraic over $K(t_1, \dots, t_{i-1})$, or
- (2) $t_i' \in K(t_1, \dots, t_{i-1})$, or
- (3) $t_i \neq 0$ and $t_i^\dagger \in K(t_1, \dots, t_{i-1})$.

Valued differential fields. A *valued differential field* is a differential field K equipped with a valuation ring $\mathcal{O} \supseteq \mathbb{Q}$ of K . In particular, all valued differential fields have $\text{char } k = 0$.

An *asymptotic differential field*, or just *asymptotic field*, is a valued differential field K such that for all $f, g \in K^\times$ with $f, g < 1$,

$$(A) \quad f < g \iff f' < g'.$$

If K is an asymptotic field, then $C \subseteq \mathcal{O}$ and thus $v(C^\times) = \{0\}$. The following consequence of [Lemma 5.1](#) will be used in [Section 12](#) to obtain our main result:

Lemma 5.4. *Let K be an asymptotic field. Suppose that $y_0, y_1, \ell \in K$ are such that $y_0, y_1 \notin C$ and $y_i'' = \ell y_i'$ for $i = 0, 1$. Then $y_0 > 1$ if and only if $y_1 > 1$.*

The value group of an asymptotic field always has a natural asymptotic couple structure associated to it:

ADH 5.5 [[ADH 2017](#), 9.1.3]. *Let K be a valued differential field. The following are equivalent:*

- (1) K is an asymptotic field;
- (2) there is an asymptotic couple (Γ, ψ) with underlying ordered abelian group $\Gamma = v(K^\times)$ such that for all $g \in K^\times$ with $g \neq 1$ we have $\psi(vg) = v(g^\dagger)$.

If K is an asymptotic field, we call (Γ, ψ) as defined in [ADH 5.5\(2\)](#), the *asymptotic couple of K* .

Convention 5.6. Let L be an expansion of an asymptotic field, and P a property that an asymptotic couple may or may not have. Then “ L has property P ” means “the asymptotic couple of L has property P ”. For instance, when we say L is “of H -type”, equivalently “is H -asymptotic”, we mean that the asymptotic couple (Γ_L, ψ_L) of L is H -type. Likewise for the properties “asymptotic integration”, “grounded”, etc.

We say that an asymptotic field K is *pre-differential-valued*, or *pre-d-valued*, if the following holds:

$$(PDV) \quad \text{for all } f, g \in K^\times, \quad f \asymp 1, g \prec 1 \implies f' \prec g^\dagger.$$

Every ungrounded asymptotic field is pre-d-valued by [ADH 2017, 10.1.3].

Finally, we say that a pre-d-valued field K is *differential-valued*, or *d-valued*, if it satisfies one of the following three equivalent conditions:

- (1) $\mathcal{O} = C + \mathfrak{o}$.
- (2) $\{\text{res}(a) : a \in C\} = \mathfrak{k}$.
- (3) for all $f \asymp 1$ in K there exists $c \in C$ with $f \sim c$.

Suppose K is a pre-d-valued field of H -type. Define the \mathcal{O} -submodule

$$I(K) := \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\}$$

of K . We say that K has *integration* if $K = \partial K$, has *exponential integration* if $K = (K^\times)^\dagger$, has *small integration* if $I(K) = \partial \mathfrak{o}$, and has *small exponential integration* if $I(K) = (1 + \mathfrak{o})^\dagger$.

Lemma 5.7. *Let K be a pre-d-valued field of H -type with small integration. Then K is d-valued.*

Proof. Take $f \in K$ such that $f \asymp 1$. Then $f' \in I(K) = \partial \mathfrak{o}$, so we have $\varepsilon \in \mathfrak{o}$ such that $f' = \varepsilon'$. Hence $f - \varepsilon = c$ with $c \in C^\times$ and thus $f \sim c$. \square

Ordered valued differential fields. A *pre- H -field* is an ordered pre-d-valued field K of H -type whose ordering, valuation, and derivation interact as follows:

- (PH1) the valuation ring \mathcal{O} is convex with respect to the ordering;
- (PH2) for all $f \in K$, if $f > \mathcal{O}$, then $f' > 0$.

An *H -field* is a pre- H -field K that is also d-valued. Any ordered differential field with the trivial valuation is a pre- H -field.

Example 5.8. Consider the field $L = \mathbb{R}(x)$ with x transcendental over \mathbb{R} , equipped with the unique derivation which has constant field \mathbb{R} and $x' = 1$. Furthermore, equip L with the trivial valuation and the unique field ordering determined by requiring $x > \mathbb{R}$. It follows that L is a pre- H -field with residue field isomorphic to $\mathbb{R}(x)$. However, L is not an H -field. Indeed, the residue field is not even algebraic over the image of the constant field \mathbb{R} under the residue map.

Example 5.9. Consider the Hardy field \mathbb{Q} . Using [Rosenlicht 1983a, Theorem 2] twice, we can extend to the Hardy field $\mathbb{Q}(x)$ where $x' = 1$, and further extend to the Hardy field $K = \mathbb{Q}(x, \arctan(x))$ where $(\arctan(x))' = 1/(1+x^2)$. Each of these three Hardy fields are pre- H -fields (see [ADH 2017, §10.5]); however, \mathbb{Q}

Hardy field	value group	residue field	constant field	H -field?
\mathbb{Q}	$\{0\}$	\mathbb{Q}	\mathbb{Q}	Yes
$\mathbb{Q}(x)$	$\mathbb{Z}v(x)$	\mathbb{Q}	\mathbb{Q}	Yes
$K = \mathbb{Q}(x, \arctan(x))$	(I) $\mathbb{Z}v(x)$	(I) $\mathbb{Q}(\pi)$	(II) \mathbb{Q}	No

and $\mathbb{Q}(x)$ are H -fields, whereas K is *not* an H -field: the constant field of K is \mathbb{Q} whereas the residue field of K is $\mathbb{Q}(\pi)$. Note that in this example the residue field $\mathbb{Q}(\pi)$ is also not algebraic over the image of the constant field \mathbb{Q} . For details of these Hardy field extensions and justification of the claims about K , see the table and the following discussion:

- (I) Note that $\lim_{x \rightarrow \infty} \arctan(x) = \pi/2$, hence $\arctan(x) \lesssim 1$ and the residue field $\text{res}(K)$ of K contains $\mathbb{Q}(\pi)$. Recall that by the Lindemann–Weierstrass theorem [Lindemann 1882], π is transcendental over \mathbb{Q} , so $\text{res}(\arctan(x)) = \pi/2$ is transcendental over $\text{res}(\mathbb{Q}(x)) = \mathbb{Q}$. It follows that $\arctan(x)$ is transcendental over $\mathbb{Q}(x)$ (otherwise $\text{res}(K)$ would be algebraic over $\text{res}(\mathbb{Q}(x)) = \mathbb{Q}$). By [ADH 2017, 3.1.31], it follows that $\Gamma_K = \Gamma_{\mathbb{Q}(x)} = \mathbb{Z}v(x)$, and

$$\text{res}(K) = \text{res}(\mathbb{Q}(x))(\text{res}(\arctan(x))) = \mathbb{Q}(\pi/2) = \mathbb{Q}(\pi).$$

- (II) As K is a pre- H -field, it follows that the constant field is necessarily a subfield of the residue field $\mathbb{Q}(\pi)$. A routine brute force verification shows that $1/(1+x^2) \notin \partial(\mathbb{Q}(x))$. Thus the differential ring $\mathbb{Q}(x)[\arctan(x)]$ is simple by [ADH 2017, 4.6.10] (see the same work for definitions of *differential ring* and *simple differential ring*). Furthermore, as $\mathbb{Q}(x)[\arctan(x)]$ is finitely generated as a $\mathbb{Q}(x)$ -algebra, it follows that C_K is algebraic over \mathbb{Q} by [ADH 2017, 4.6.12]. However, \mathbb{Q} is algebraically closed in $\mathbb{Q}(\pi)$ (because π is transcendental over \mathbb{Q}) and so $C_K = \mathbb{Q}$.

Algebraic extensions. *In this subsection K is an asymptotic field.* We fix an algebraic field extension L of K . By ADH 5.2 we equip L with the unique derivation extending the derivation ∂ of K . By Chevalley’s Extension Theorem [ADH 2017, 3.1.15] we equip L with a valuation extending the valuation of K . Thus L is a valued differential field extension of K . We record here several properties that are preserved in this algebraic extension:

ADH 5.10. *The valued differential field L is an asymptotic field [ADH 2017, 9.5.3]. Also:*

- (1) *If K is of H -type, then so is L .*
- (2) *If K is pre-d-valued, then so is L [ADH 2017, 10.1.22].*
- (3) *K is grounded if and only if L is grounded.*

(1) and (3) of [ADH 5.10](#) follow from the corresponding facts about the divisible hull of an asymptotic couple; see [Definition 3.6](#).

Furthermore, assume that K is equipped with an ordering making it a pre- H -field, and $L|K$ is an algebraic extension of ordered differential fields.

ADH 5.11. *There is a unique convex valuation ring of L extending the valuation ring of K [[ADH 2017](#), 3.5.18]. Equipped with this valuation ring, L is a pre- H -field extension of K [[ADH 2017](#), 10.5.4]. Furthermore, if K is an H -field and $L = K^{\text{rc}}$, a real closure of K , then L is also an H -field [[ADH 2017](#), 10.5.6].*

6. λ -freeness

In this section K is an ungrounded H -asymptotic field with $\Gamma \neq \{0\}$.

Logarithmic sequences and λ -sequences.

Definition 6.1. A logarithmic sequence (in K) is a well-indexed sequence (ℓ_ρ) in $K^{>1}$ such that

- (1) $\ell'_{\rho+1} \asymp \ell'_\rho$, i.e., $v(\ell_{\rho+1}) = \chi(v\ell_\rho)$, for all ρ ;
- (2) $\ell_{\rho'} < \ell_\rho$ whenever $\rho' > \rho$;
- (3) (ℓ_ρ) is coinital in $K^{>1}$: for each $f \in K^{>1}$ there is an index ρ with $\ell_\rho \preccurlyeq f$.

Such sequences exist and can be constructed by transfinite recursion.

Definition 6.2. A λ -sequence (in K) is a sequence of the form $(\lambda_\rho) = (-(\ell_\rho^{\dagger\dagger}))$ where (ℓ_ρ) is a logarithmic sequence in K .

ADH 6.3 [[ADH 2017](#), 11.5.2]. *Every λ -sequence is a pc-sequence of width $\{\gamma \in \Gamma_\infty : \gamma > \Psi\}$.*

ADH 6.4 [[ADH 2017](#), 11.5.3]. *All λ -sequences are equivalent as pc-sequences.*

For the rest of this section we will fix in K a distinguished logarithmic sequence (ℓ_ρ) along with its corresponding λ -sequence (λ_ρ) . Nothing that we will discuss depends on the choice of this λ -sequence.

λ -freeness.

ADH 6.5 [[ADH 2017](#), 11.6.1]. *The following conditions on K are equivalent:*

- (1) (λ_ρ) has no pseudolimit in K ;
- (2) for all $s \in K$ there is $g \in K^{>1}$ such that $s - g^{\dagger\dagger} \succcurlyeq g^\dagger$.

Definition 6.6. If L is an H -asymptotic field, we say that L is λ -free (or has λ -freeness) if it is ungrounded with $\Gamma_L \neq \{0\}$, and it satisfies condition (2) in [ADH 6.5](#).

The following is immediate from the definition of λ -freeness and is a remark made after [[ADH 2017](#), 11.6.4]:

ADH 6.7. *Suppose L is an H -asymptotic extension of K such that Ψ is cofinal in Ψ_L . If L is λ -free, then so is K .*

ADH 6.8 [ADH 2017, 11.6.4]. *If K is a directed union of grounded asymptotic subfields, then K is λ -free.*

Lemma 6.9. *If K is a directed union of λ -free asymptotic subfields, then K is λ -free.*

Proof. This follows easily from the (2) characterization of λ -freeness. \square

Algebraic extensions. Ultimately, we will show that λ -freeness is preserved under arbitrary Liouville extensions of H -fields. For the time being, we have the following results concerning λ -freeness for algebraic extensions:

ADH 6.10 [ADH 2017, 11.6.7]. *If K is λ -free, then so is its henselization K^h .*

ADH 6.11 [ADH 2017, 11.6.8]. *K is λ -free if and only if the algebraic closure K^a of K is λ -free.*

Lemma 6.12. *Suppose K is equipped with an ordering making it a pre- H -field. If K is λ -free, then so is its real closure K^{rc} .*

Proof. This follows from ADH 6.11 and ADH 6.7, using the fact that $\Psi_{K^{rc}} = \Psi$. \square

Big exponential integration. The “big” exponential integral extensions considered here complement the Liouville extensions considered in Section 7, Section 8, and Section 9 below. In particular, we fix an element $s \in K$ that does not have an exponential integral in K , i.e., $s \notin (K^\times)^\dagger$, and we assume that s is *bounded away* from the logarithmic derivatives in K in the sense that

$$S := \{v(s - a^\dagger) : a \in K^\times\} \subseteq \Psi^\downarrow.$$

Then under the following circumstances, λ -freeness is preserved when adjoining an exponential integral for such an s :

ADH 6.13 [ADH 2017, 11.6.12]. *Let K be λ -free and Γ be divisible, and let $f^\dagger = s$, where $f \neq 0$ lies in an H -asymptotic field extension of K . Suppose*

- (1) *S does not have a largest element, or*
- (2) *S has a largest element and $[\gamma + vf] \notin [\Gamma]$ for some $\gamma \in \Gamma$.*

Then $K(f)$ is λ -free.

ADH 6.14 [ADH 2017, 10.5.20 and 11.6.13]. *Suppose K is equipped with an ordering making it a real closed H -field such that $s < 0$. Let $L = K(f)$ be a field extension of K such that f is transcendental over K , equipped with the unique derivation extending the derivation of K such that $f^\dagger = s$. Then there is a unique pair consisting of a valuation of $L = K(f)$ and a field ordering on L making it a*

pre- H -field extension of K with $f > 0$. With this valuation and ordering L is an H -field and Ψ is cofinal in Ψ_L . Furthermore, if K is λ -free, then so is L .

Gap creators. Let $s \in K$. We say that s creates a gap over K if vf is a gap in $K(f)$, for some element $f \neq 0$ in some H -asymptotic field extension of K with $f^\dagger = s$.

ADH 6.15 [ADH 2017, 11.6.1 and 11.6.8]. *If K is λ -free, then K has rational asymptotic integration, and no element of K creates a gap over K .*

Remark 6.16. ADH 6.15 suggests that one way to view λ -freeness is as a gap prevention property. How good is λ -freeness as a gap prevention property? Already the above results show that it is impossible to create a gap from algebraic extensions and certain exponential integral extensions of a λ -free field. However, we can do a little bit better than that: by our results Propositions 7.2, 8.3, and 9.3 below, it follows that λ -freeness is also safely preserved (and so gaps are prevented) when passing to much more general Liouville extensions of a λ -free field.

On the other hand, *not* being λ -free does not bode well for preventing a gap:

ADH 6.17. *Suppose K has asymptotic integration, Γ is divisible, and $\lambda_\rho \rightsquigarrow \lambda \in K$. Then $s = -\lambda$ creates a gap over K . Furthermore, for every H -asymptotic extension $K(f)$ of K such that $f^\dagger = s$, vf is a gap in $K(f)$.*

Proof. The first claim is [ADH 2017, 11.5.14] and the second claim is a remark after that. \square

The following will be our main method of producing gaps in Liouville extensions of H -fields in Section 12 below:

ADH 6.18. *Suppose that K is equipped with an ordering making it a real closed H -field with asymptotic integration, and $\lambda_\rho \rightsquigarrow \lambda \in K$. Let $L = K(f)$ be a field extension of K with f transcendental over K equipped with the unique derivation extending the derivation of K such that $f^\dagger = -\lambda$. Then there is a unique pair consisting of a valuation of L and a field ordering on L making it an H -field extension of K with $f > 0$. With this valuation and ordering, vf is a gap in L .*

Proof. By [ADH 2017, 11.5.13] we can apply 10.5.20 of the same work with either $-\lambda$ or λ playing the role of s , whichever one is negative. Either way, a positive exponential integral f of $-\lambda$ will be adjoined, as it is the reciprocal of a positive exponential integral of λ . Also $L = K(f)$. By ADH 6.17, vf is a gap in L . \square

The yardstick argument. Assume that $L = K(y)$ is an immediate H -asymptotic extension of K where y is transcendental over K . In particular, $v(y - K)$ is a nonempty downward closed subset of Γ without a greatest element.

Proposition 6.19. *Assume K is henselian and λ -free, and $v(y - K) \subseteq \Gamma$ has the yardstick property. Then $L = K(y)$ is λ -free.*

Proof. Assume towards a contradiction that L is not λ -free. Take $\lambda \in L \setminus K$ such that $\lambda_\rho \rightsquigarrow \lambda$. By [ADH 6.3](#), [ADH 4.2](#), and [Lemma 3.11](#), $v(\lambda - K) = \Psi^\downarrow$ is jammed. Furthermore, $v(\lambda - K)$ does not have a supremum in $\mathbb{Q}\Gamma$ because K is λ -free and hence has rational asymptotic integration. By the henselian assumption and [Lemma 4.4](#), there are $\alpha \in \Gamma$ and $n \geq 1$ such that $v(\lambda - K) = (\alpha + nv(y - K))^\downarrow$. Thus by [Lemmas 2.6](#) and [2.5](#), $v(y - K)$ is jammed as well. Since $v(y - K)$ also has the yardstick property, by [Lemma 3.17](#) it follows that $v(y - K) = \Gamma^{<}$. However, since $v(\lambda - K)$ does not have a supremum in $\mathbb{Q}\Gamma$, by [Lemma 2.1](#), neither does $v(y - K)$, a contradiction. \square

7. Small exponential integration

In this section K is a henselian pre-d-valued field of H-type and we fix an element $s \in K \setminus (K^\times)^\dagger$ such that $v(s) \in (\Gamma^{>})'$. In particular, K does not have small exponential integration. Take a field extension $L = K(y)$ with y transcendental over K , equipped with the unique derivation extending the derivation of K such that $(1 + y)^\dagger = y'/(1 + y) = s$.

ADH 7.1 [[ADH 2017](#), 10.4.3 and 10.5.18]. *There is a unique valuation of L that makes it an H-asymptotic extension of K with $y \not\approx 1$. With this valuation L is pre-d-valued, and is an immediate extension of K with $y < 1$. Furthermore, if K is equipped with an ordering making it a pre-H-field, then there is a unique ordering on L making it a pre-H-field extension of K .*

For the rest of this section equip L with this valuation. The main result of this section is the following:

Proposition 7.2. *If K is λ -free, then so is $L = K(y)$.*

The proof of [Proposition 7.2](#) is delayed until the end of the section. The following nonempty set will be of importance in our analysis:

$$S := \left\{ v \left(s - \frac{\varepsilon'}{1 + \varepsilon} \right) : \varepsilon \in K^{<1} \right\} \subseteq (\Gamma^{>})' \subseteq \Gamma_\infty.$$

ADH 7.3. *The set S does not have a largest element.*

Proof. This is Claim 1 in the proof of [[ADH 2017](#), 10.4.3]. \square

Lemma 7.4. *S is a downward closed subset of $(\Gamma^{>})'$; in particular, S is convex.*

Proof. Let $\varepsilon_1 < 1$ in K and $\alpha, \beta \in (\Gamma^{>})'$ be such that

$$\alpha < v \left(s - \frac{\varepsilon'_1}{1 + \varepsilon_1} \right) = \beta.$$

Let $\delta < 1$ in K be such that $v(\delta') = \alpha$ and set $\varepsilon_0 := \delta + \varepsilon_1 + \delta\varepsilon_1$. Note that

$$\begin{aligned} \frac{\varepsilon'_1}{1+\varepsilon_1} - \frac{\varepsilon'_0}{1+\varepsilon_0} &= \frac{\varepsilon'_1}{1+\varepsilon_1} - (1+\delta+\varepsilon_1+\delta\varepsilon_1)^\dagger \\ &= \frac{\varepsilon'_1}{1+\varepsilon_1} - ((1+\delta)(1+\varepsilon_1))^\dagger = \frac{\varepsilon'_1}{1+\varepsilon_1} - \frac{\delta'}{1+\delta} - \frac{\varepsilon'_1}{1+\varepsilon_1} = -\frac{\delta'}{1+\delta} \end{aligned}$$

and thus

$$v\left(\frac{\varepsilon'_1}{1+\varepsilon_1} - \frac{\varepsilon'_0}{1+\varepsilon_0}\right) = v\left(\frac{\delta'}{1+\delta}\right) = \alpha.$$

Finally,

$$v\left(s - \frac{\varepsilon'_0}{1+\varepsilon_0}\right) = v\left(\left(s - \frac{\varepsilon'_1}{1+\varepsilon_1}\right) + \left(\frac{\varepsilon'_1}{1+\varepsilon_1} - \frac{\varepsilon'_0}{1+\varepsilon_0}\right)\right) = \min(\beta, \alpha) = \alpha \in S. \quad \square$$

The next lemma shows that S is a transform of the positive portion of $v(y - K)$.

Lemma 7.5. $(v(y - K)^{>0})' = S$, and equivalently $v(y - K)^{>0} = \int S$.

Proof. (\subseteq) Let $\varepsilon \in K$ be such that $v(y - \varepsilon) > 0$. Then necessarily $\varepsilon < 1$ since $y < 1$ and so it suffices to prove that $(v(y - \varepsilon))' = v(y' - \varepsilon') \in S$. By (PDV) it follows that $(y - \varepsilon)' \succ \varepsilon'(y - \varepsilon)$. Thus

$$\begin{aligned} s - \frac{\varepsilon'}{1+\varepsilon} &= \frac{y'}{1+y} - \frac{\varepsilon'}{1+\varepsilon} = \frac{y'(1+\varepsilon) - \varepsilon'(1+y)}{(1+y)(1+\varepsilon)} = \frac{(1+\varepsilon)(y-\varepsilon)' - \varepsilon'(y-\varepsilon)}{(1+y)(1+\varepsilon)} \\ &\asymp (1+\varepsilon)(y-\varepsilon)' - \varepsilon'(y-\varepsilon) \asymp y' - \varepsilon'. \end{aligned}$$

We conclude that $v(y' - \varepsilon') = (v(y - \varepsilon))' \in S$.

For the (\supseteq) direction, suppose that $\alpha = v(s - \varepsilon'/(1+\varepsilon)) \in S$ where $\varepsilon \in K^{<1}$. Then the calculation in reverse shows that $\alpha = v(y' - \varepsilon') = (v(y - \varepsilon))' \in (v(y - K)^{>0})'$. \square

The next lemma gives us a “definable yardstick” that we can use for going up the set S . If K has small integration, then we can obtain a longer yardstick in the sense of Lemma 3.13, however the shorter yardstick will be good enough for our purposes.

Lemma 7.6. *Suppose $\gamma \in S$. Then $\gamma < \gamma - \int s\gamma \in S$. If $I(K) = \partial\circ$, then $\gamma < \gamma + \int \gamma \in S$. Thus S has the derived yardstick property and so $v(y - K)^{>0}$ and $v(y - K)$ both have the yardstick property.*

Proof. Let $\gamma \in S$ and take $\varepsilon < 1$ in K such that $\gamma = v(s - \varepsilon'/(1+\varepsilon))$. Next take $b < 1$ in K such that $v(b') = (v(b))' = \gamma$ (and so $v(b) = \int \gamma$). Take $u \in K$ with $s - \varepsilon'/(1+\varepsilon) = ub'$, so $u \asymp 1$. Next let $\delta < 1$ be such that $(1+\varepsilon)(1+ub) = 1+\delta$.

Now note that

$$\begin{aligned} s - \frac{\delta'}{1+\delta} &= s - ((1+\varepsilon)(1+ub))^\dagger = s - \frac{\varepsilon'}{1+\varepsilon} - \frac{(ub)'}{1+ub} \\ &= ub' - \frac{(ub)'}{1+ub} = \frac{u^2bb' - u'b}{1+ub}. \end{aligned}$$

However, since $\Psi \ni s^2\gamma < v(u') \in \Gamma^{>\Psi}$, we have

$$v(u'b) = v(u'b'(b^\dagger)^{-1}) = v(u') - \psi \int \gamma + \gamma > s^2\gamma - s\gamma + \gamma = -\int s\gamma + \gamma,$$

the last step following from [Lemma 3.9\(1\)](#). Thus by [Lemma 3.13](#), we have

$$v\left(s - \frac{\delta'}{1+\delta}\right) \geq \min(v(u^2bb'), v(u'b)) \geq \min(\gamma + \int \gamma, -\int s\gamma + \gamma) = \gamma - \int s\gamma > \gamma.$$

Finally, by [Lemma 7.4](#), it follows that $\gamma - \int s\gamma \in S$.

If $I(K) = \partial\mathcal{O}$, then we can arrange $u = 1$ above and thus

$$s - \frac{\delta'}{1+\delta} = \frac{bb'}{1+b} \asymp bb',$$

and so $v(bb') = \gamma + \int \gamma$. The claim about $v(y - K)^{>0}$ now follows from [Lemma 7.5](#) and [Proposition 3.19](#). \square

[Proposition 7.2](#) now follows immediately from [Lemma 7.6](#) and [Proposition 6.19](#).

8. Small integration

In this section K is a henselian pre-d-valued field of H -type and we fix an element $s \in K$ such that $v(s) \in (\Gamma^{>})'$ and $s \notin \partial\mathcal{O}$. In particular, K does not have small integration. Define the following nonempty set:

$$S := \{v(s - \varepsilon') : \varepsilon \in K^{<1}\} \subseteq (\Gamma^{>})' \subseteq \Gamma_\infty.$$

As K is pre-d-valued, we have the following, which elaborates on [\[ADH 2017, 10.2.5\(iii\)\]](#):

Lemma 8.1. *S has no largest element and is a downward closed subset of $(\Gamma^{>})'$; in particular, S is convex*

Proof. First note that $v(s) \in S$. Next take $\gamma \in S$ with $\gamma \geq v(s)$, and write $\gamma = v(s - \varepsilon')$ for some $\varepsilon < 1$ in K . As $\gamma \in (\Gamma^{>})'$, we take $b < 1$ in K such that $v(b') = \gamma$. Thus for some $u \asymp 1$ in K we have $v(s - \varepsilon' - ub') > \gamma$. By [\(PDV\)](#), $v(u'b) > v(b') = \gamma$ and so $v(s - \varepsilon' - (ub)') > \gamma$. This shows that S has no largest element. The claim that S is a downward closed subset of $(\Gamma^{>})'$ follows similarly from $S \subseteq (\Gamma^{>})'$. \square

Take a field extension $L = K(y)$ with y transcendental over K , equipped with the unique derivation extending the derivation of K such that $y' = s$.

ADH 8.2 [ADH 2017, 10.2.4 and 10.5.8]. *There is a unique valuation of L that makes it an H -asymptotic extension of K with $y \not\asymp 1$. With this valuation L is an immediate extension of K with $y < 1$ and L is pre- d -valued. Furthermore, if K is equipped with an ordering making it a pre- H -field, then there is a unique ordering on L making it a pre- H -field extension of K .*

For the rest of this section equip L with this valuation. The main result of this section is the following:

Proposition 8.3. *If K is λ -free, then so is $L = K(y)$.*

We will delay the proof of [Proposition 8.3](#) until the end of the section.

Lemma 8.4. $(v(y - K)^{>0})' = S$, and equivalently $v(y - K)^{>0} = \int S$.

Proof. (\subseteq) Let $\varepsilon \in K$ be such that $y - \varepsilon < 1$. Then necessarily $\varepsilon < 1$ because $y < 1$. Let $\alpha = v(y - \varepsilon)$. We want to show that $\alpha' \in S$. Note that because $y - \varepsilon \not\asymp 1$, we get

$$\alpha' = (v(y - \varepsilon))' = v(y' - \varepsilon') = v(s - \varepsilon') \in S.$$

For the (\supseteq) direction, let $\varepsilon < 1$ be such that $\alpha = v(s - \varepsilon')$ is an arbitrary element of S . Then by arguing as above, $v(y - \varepsilon) > 0$ and $(v(y - \varepsilon))' = \alpha$. \square

Lemma 8.5. *Suppose $\gamma \in S$. Then $\gamma < \gamma - \int s\gamma \in S$. If $\mathbf{I}(K) = (1 + \circ)^\dagger$, then $\gamma < \gamma + \int \gamma \in S$. Thus S has the derived yardstick property and so $v(y - K)^{>0}$ and $v(y - K)$ both have the yardstick property.*

Proof. Suppose $\gamma \in S$ and take $\varepsilon < 1$ in K such that $\gamma = v(s - \varepsilon')$. As $\gamma \in (\Gamma^{>})'$, we may take $b < 1$ in K such that $b' \asymp s - \varepsilon'$. Thus there is $u \asymp 1$ in K such that $ub' = s - \varepsilon'$. By (PDV), it follows that $v(u') > \Psi$. Thus

$$\begin{aligned} v(s - (\varepsilon - ub)') &= v(s - \varepsilon' - ub' - ub') = v(u'b) \\ &= v(u'b'(b^\dagger)^{-1}) = v(u') - \psi \int \gamma + \gamma \\ &> s^2\gamma - s\gamma + \gamma = -\int s\gamma + \gamma. \end{aligned}$$

Next, assume that $(1 + \circ)^\dagger = \mathbf{I}(K)$. Since $s - \varepsilon' \in \mathbf{I}(K)$, there is $\delta < 1$ such that $s - \varepsilon' = (1 + \delta)^\dagger$, i.e.,

$$s - \varepsilon' = \frac{\delta'}{1 + \delta}.$$

Now note that

$$s - (\varepsilon + \delta)' = s - \varepsilon' - \delta' = \frac{\delta'}{1 + \delta} - \delta' = \frac{-\delta'\delta}{1 + \delta} \asymp \delta'\delta,$$

and so

$$S \ni v(s - (\varepsilon + \delta)') = v(\delta'\delta) = \gamma + \int \gamma.$$

The claim about $v(y - K)^{>0}$ now follows from [Lemma 8.4](#) and [Proposition 3.19](#). \square

[Proposition 8.3](#) now follows immediately from [Lemma 8.5](#) and [Proposition 6.19](#).

9. Big integration

In this section K is a henselian pre-d-valued field of H -type and we fix an element $s \in K$ such that

$$S := \{v(s - a') : a \in K\} \subseteq (\Gamma^<)' \subseteq \Gamma_\infty.$$

It will necessarily be the case that $s \notin \partial K$ and $v(s) \in (\Gamma^<)'$.

Lemma 9.1. *S is downward closed and does not have a largest element.*

Proof. Let $\gamma = v(s - a') \in S$ for some $a \in K$. Suppose $\delta < \gamma$ in Γ . Then there is $f \in K$ such that $v(f') = \delta$ and so $\delta = v(s - (a + f)') \in S$. Next, by $S \subseteq (\Gamma^<)'$, take $b \in K$ such that $b' \asymp s - a'$, and then take $u \asymp 1$ in K with $ub' = s - a'$. By (PDV), $u'b < b'$ and thus $\gamma < v(s - a' - (ub)') \in S$. \square

Take a field extension $L = K(y)$ with y transcendental over K , equipped with the unique derivation extending the derivation of K such that $y' = s$.

ADH 9.2 [ADH 2017, 10.2.6 and 10.5.8]. *There is a unique valuation of L making it an H -asymptotic extension of K . With this valuation L is an immediate extension of K with $y \succ 1$ and L is pre-d-valued. Furthermore, if K is equipped with an ordering making it a pre- H -field, then there is a unique ordering on L making it a pre- H -field extension of K .*

For the rest of this section equip L with this valuation. The main result of this section is the following:

Proposition 9.3. *If K is λ -free, then so is $L = K(y)$.*

We will delay the proof of [Proposition 9.3](#) until the end of the section.

Lemma 9.4. *$v(y - K)' = S$, and equivalently $v(y - K) = \int S$.*

Proof. Let $\gamma = v(y - x)$ with $x \in K$. Then $v(y' - x') = v(s - x') \in S \subseteq (\Gamma^<)'$ and so $y - x \succ 1$. Thus $\gamma' = (v(y - x))' = v(y' - x') = v(s - x') \in S$. Conversely, if $\gamma = v(s - x') \in S$, then $\gamma = v(y' - x') = (v(y - x))'$. \square

By [Lemma 9.1](#), we fix $g \in K^{\succ 1}$ such that $g' \sim s$.

Lemma 9.5. *$S^{>v(s)}$ is cofinal in S , and*

$$S^{>v(s)} = \{v((g(1 + \varepsilon))' - s) : \varepsilon < 1\}.$$

Proof. $S^{>v(s)}$ is cofinal in S since $v(s) \in S$ and S does not have a largest element. Suppose $\varepsilon < 1$. Then by (PDV), $(g(1 + \varepsilon))' = g' + \varepsilon'g + \varepsilon g' \sim g' \sim s$ and so $(g(1 + \varepsilon))' - s < s$. Conversely, suppose $\gamma = v(x' - s) > vs$. Then $x' \sim s$ and so $x' \sim g'$, i.e., $x' - g' < g'$. As $g \succ 1$, we get $x - g < g$ and so $x = g(1 + \varepsilon)$ for some $\varepsilon < 1$. \square

Lemma 9.6. *If $\gamma \in S^{>v(s)}$, then $\gamma < \gamma - \int s\gamma \in S$. Thus S has the derived yardstick property and so $v(y - K)$ has the yardstick property.*

Proof. Let $\gamma = v((g(1 + \varepsilon))' - s)$ for some $\varepsilon < 1$. Note that

$$(g(1 + \varepsilon))' - s = g' + g\varepsilon' + g'\varepsilon - s.$$

Next take $\delta > 1$ such that

$$\delta' \succ g' + g\varepsilon' + g'\varepsilon - s,$$

so $v(\delta') = \gamma$, and take $u \succ 1$ such that

$$u\delta' = g' + g\varepsilon' + g'\varepsilon - s.$$

Then $\delta' < g' \prec s$ and so $\delta < g$, i.e., $\delta/g < 1$. Furthermore, $u^\dagger < \delta^\dagger$ implies that $u'\delta < u\delta'$. Now consider the following element of $S^{>v(s)}$:

$$\beta = v\left(\left(g\left(1 + \varepsilon - \frac{u\delta}{g}\right)\right)' - s\right).$$

Note that

$$\begin{aligned} \left(g\left(1 + \varepsilon - \frac{u\delta}{g}\right)\right)' - s &= (g + g\varepsilon - u\delta)' - s \\ &= g' + g\varepsilon' + g'\varepsilon - u'\delta - u\delta' - s \\ &= (g' + g\varepsilon' + g'\varepsilon - s - u\delta') - u'\delta \\ &= -u'\delta. \end{aligned}$$

Thus we can use that $v(u') > \Psi$ and $\gamma = v(\delta) + v(\delta^\dagger)$ to get the yardstick:

$$\begin{aligned} v(-u'\delta) &= v(u'(\delta^\dagger)^{-1}\delta') = v(u'(\delta^\dagger)^{-1}) + \gamma \\ &= v(u') - \psi \int \gamma + \gamma = v(u') - s\gamma + \gamma \\ &> s^2\gamma - s\gamma + \gamma = -\int s\gamma + \gamma. \end{aligned}$$

The claim about $v(y - K)$ now follows from [Lemma 9.4](#) and [Proposition 3.19](#). \square

[Proposition 9.3](#) now follows immediately from [Lemma 9.6](#) and [Proposition 6.19](#).

10. The differential-valued hull and H -field hull

In this section K is a pre-d-valued field of H -type.

ADH 10.1 [[ADH 2017](#), 10.3.1]. *K has a d-valued extension $\text{dv}(K)$ of H -type such that any embedding of K into any d-valued field L of H -type extends uniquely to an embedding of $\text{dv}(K)$ into L .*

The d-valued field $\text{dv}(K)$ as in [ADH 10.1](#) above is called the *differential-valued hull* of K .

Theorem 10.2. *If K is λ -free, then $\text{dv}(K)$ is λ -free.*

Proof. By iterating applications of [ADH 6.10](#), [Proposition 8.3](#), and [Lemma 6.9](#), we get an immediate henselian λ -free H -asymptotic extension L of K which has small integration. By [Lemma 5.7](#), L will also be d -valued. Thus by [ADH 10.1](#), $\text{dv}(K)$ can be identified with a subfield of L which contains K . Finally, by [ADH 6.7](#) it follows that $\text{dv}(K)$ is λ -free. \square

Definition 10.3. A gap β in K is said to be a *true gap* if no $b \asymp 1$ in K satisfies $v(b') = \beta$, and is said to be a *fake gap* otherwise (that is, if there is $b \asymp 1$ in K such that $v(b') = \beta$).

Remark 10.4. Suppose K has a gap β . Then the asymptotic couple (Γ, ψ) “believes” it can make a choice about β , in the sense of [Remark 3.5](#). However, if β is a fake gap, then this choice is completely predetermined by K itself. Indeed, if L is a d -valued extension of K of H -type and β is a fake gap, then there is $\varepsilon \in \mathcal{O}_L$ such that $v(\varepsilon') = \beta$. However, if β is a true gap, then both options of this choice are still available to K , see [[ADH 2017](#), 10.3.2(ii), 10.2.1, and 10.2.2].

Lemma 10.5. *If K is d -valued and has a gap β , then β is a true gap.*

Proof. Let K be a d -valued field and consider $\beta \in \Gamma$. Suppose that there is $b \asymp 1$ in K such that $v(b') = \beta$. Then there are $c \in C^\times$ and $\varepsilon < 1$ in K^\times such that $b = c + \varepsilon$ and thus $v(b') = v(\varepsilon') = \beta \in (\Gamma^{>})'$. In particular, β is not a gap. \square

Corollary 10.6. *The differential-valued hull of K has the following properties:*

- (1) *If K is grounded, then $\text{dv}(K)$ is grounded.*
- (2) *If K has a fake gap, then $\text{dv}(K)$ is grounded.*
- (3) *If K has a true gap, then $\text{dv}(K)$ has a true gap.*
- (4) *If K has asymptotic integration and is not λ -free, then $\text{dv}(K)$ has asymptotic integration and is not λ -free.*
- (5) *If K is λ -free, then $\text{dv}(K)$ is λ -free.*

Proof. (1)–(4) are a restatement of [[ADH 2017](#), 10.3.2]. (5) is [Theorem 10.2](#). \square

The H -field hull of a pre- H -field. *In this subsection we further assume that K is equipped with an ordering making it a pre- H -field.*

ADH 10.7 [[ADH 2017](#), 10.5.13]. *A unique field ordering on $\text{dv}(K)$ makes $\text{dv}(K)$ a pre- H -field extension of K . Let $H(K)$ be $\text{dv}(K)$ equipped with this ordering. Then $H(K)$ is an H -field and embeds uniquely over K into any H -field extension of K .*

The H -field $H(K)$ in [ADH 10.7](#) above is called the *H -field hull of K* . We have the following H -field analogues of [Theorem 10.2](#) and [Corollary 10.6](#):

Corollary 10.8. *If K is λ -free, then $H(K)$ is λ -free.*

Corollary 10.9. *The H -field hull of K has the following properties:*

- (1) *If K is grounded, then $H(K)$ is grounded.*
- (2) *If K has a fake gap, then $H(K)$ is grounded.*
- (3) *If K has a true gap, then $H(K)$ has a true gap.*
- (4) *If K has asymptotic integration and is not λ -free, then $H(K)$ has asymptotic integration and is not λ -free.*
- (5) *If K is λ -free, then $H(K)$ is λ -free.*

11. The integration closure

In this section K is a d -valued field of H -type with asymptotic integration.

ADH 11.1 [ADH 2017, 10.2.7]. *K has an immediate asymptotic extension $K(f)$ that is henselian, has integration, and embeds over K into any henselian d -valued H -asymptotic extension of K that has integration.*

Given any $K(f)$ with these properties, the only henselian asymptotic subfield of $K(f)$ containing K and having integration is $K(f)$.

Theorem 11.2. *If K is λ -free, then so is $K(f)$.*

Proof. By iterating Lemma 6.9, ADH 6.10, and Propositions 8.3 and 9.3, we obtain a λ -free d -valued immediate H -asymptotic extension L of K that is henselian and has integration. By ADH 11.1, $K(f)$ can be identified with a subfield of L which contains K . Finally, by ADH 6.7, $K(f)$ is also λ -free. \square

12. The number of Liouville closures

In this section K is a pre- H -field. K is said to be Liouville closed if it is a real closed H -field with integration and exponential integration. A Liouville closure of K is a Liouville closed H -field extension of K which is also a Liouville extension of K .

Theorem 12.1. *Suppose K is an H -field. Then K has at least one and at most two Liouville closures up to isomorphism over K . In particular,*

- (1) *K has exactly one Liouville closure up to isomorphism over K if and only if*
 - (a) *K is grounded, or*
 - (b) *K is λ -free.*
- (2) *K has exactly two Liouville closures up to isomorphism over K if and only if*
 - (c) *K has a gap, or*
 - (d) *K has asymptotic integration and is not λ -free.*

Theorem 12.1 will follow from the following Proposition, whose proof we delay until later in the section:

Proposition 12.2. *Suppose K is an H -field.*

- (1) *If K is λ -free, then K has exactly one Liouville closure up to isomorphism over K .*
- (2) *If K has asymptotic integration and is not λ -free, then K has at least two Liouville closures up to isomorphism over K .*

Proof of Theorem 12.1 assuming Proposition 12.2. It is clear that K will be in case (a), (b), (c) or (d), and all four cases are mutually exclusive. If K is in case (a), then K has exactly one Liouville closure up to isomorphism over K , by [ADH 2017, 10.6.23]. If K is in case (c), then K has exactly two Liouville closures up to isomorphism over K , by [ADH 2017, 10.6.25]. Cases (b) and (d) are taken care of by Proposition 12.2 and [ADH 2017, 10.6.12]. \square

In general, a pre- H -field which is not also an H -field might not have any Liouville closures at all. For instance, the pre- H -field L from Example 5.8 cannot have any Liouville closures: a Liouville closure of L would necessarily contain $H(L)$, but $H(L)$ cannot be contained inside any Liouville extension of L because $C_{H(L)}$ is not an algebraic extension of $C_L = \mathbb{R}$. In such a situation, the next best thing is to consider Liouville closures of the H -field hull:

Corollary 12.3. *$H(K)$ has at least one and at most two Liouville closures up to isomorphism over K . In particular,*

- (1) *$H(K)$ has exactly one Liouville closure up to isomorphism over K if and only if*
 - (a) *K is grounded, or*
 - (b) *K has a fake gap, or*
 - (c) *K is λ -free.*
- (2) *$H(K)$ has exactly two Liouville closures up to isomorphism over K if and only if*
 - (d) *K has a true gap, or*
 - (e) *K has asymptotic integration and is not λ -free.*

Proof. If we replace in the statement of Corollary 12.3 all instances of “up to isomorphism over K ” with “up to isomorphism over $H(K)$ ”, then this would follow from Corollary 10.9 and Theorem 12.1. Now, to strengthen the statements to “up to isomorphism over K ”, use that $H(K)$ is determined up to unique isomorphism in ADH 10.7. \square

Liouville towers. *In this subsection K is an H -field.* The primary method of constructing Liouville closures of an H -field is with a *Liouville tower*. A *Liouville tower on K* is a strictly increasing chain $(K_\lambda)_{\lambda \leq \mu}$ of H -fields, indexed by the ordinals less than or equal to some ordinal μ , such that

- (1) $K_0 = K$;
- (2) if λ is a limit ordinal, $0 < \lambda \leq \mu$, then $K_\lambda = \bigcup_{\iota < \lambda} K_\iota$;
- (3) for $\lambda < \lambda + 1 \leq \mu$, *either*
 - (a) K_λ is not real closed and $K_{\lambda+1}$ is a real closure of K_λ ,
or K_λ is real closed, $K_{\lambda+1} = K_\lambda(y_\lambda)$ with $y_\lambda \notin K_\lambda$ (so y_λ is transcendental over K_λ), and one of the following holds, with $(\Gamma_\lambda, \psi_\lambda)$ the asymptotic couple of K_λ and $\Psi_\lambda := \psi_\lambda(\Gamma_\lambda^\neq)$:
 - (b) $y'_\lambda = s_\lambda \in K_\lambda$ with $y_\lambda < 1$ and $v(s_\lambda)$ is a gap in K_λ ,
 - (c) $y'_\lambda = s_\lambda \in K_\lambda$ with $y_\lambda > 1$ and $v(s_\lambda)$ is a gap in K_λ ,
 - (d) $y'_\lambda = s_\lambda \in K_\lambda$ with $v(s_\lambda) = \max \Psi_\lambda$,
 - (e) $y'_\lambda = s_\lambda \in K_\lambda$ with $y_\lambda < 1$, $v(s_\lambda) \in (\Gamma_\lambda^{>})'$, and $s_\lambda \neq \varepsilon'$ for all $\varepsilon \in K_\lambda^{<1}$,
 - (f) $y'_\lambda = s_\lambda \in K_\lambda$ such that $S_\lambda := \{v(s_\lambda - a') : a \in K_\lambda\} < (\Gamma_\lambda^{>})'$, and S_λ has no largest element,
 - (g) $y_\lambda^\dagger = s_\lambda \in K_\lambda$ with $y_\lambda \sim 1$, $v(s_\lambda) \in (\Gamma_\lambda^{>})'$, and $s_\lambda \neq a^\dagger$ for all $a \in K_\lambda^\times$,
 - (h) $y_\lambda^\dagger = s_\lambda \in K_\lambda^{<}$ with $y_\lambda > 0$, and $v(s_\lambda - a^\dagger) \in \Psi_\lambda^\downarrow$ for all $a \in K_\lambda^\times$.

The H -field K_μ is called the *top* of the tower $(K_\lambda)_{\lambda \leq \mu}$. We say that a Liouville tower $(K_\lambda)_{\lambda \leq \mu}$ is *maximal* if it cannot be extended to a Liouville tower $(K_\lambda)_{\lambda \leq \mu+1}$ on K . Given a Liouville tower $(K_\lambda)_{\lambda \leq \mu}$ on K , $0 \leq \lambda < \lambda + 1 \leq \mu$, we say $K_{\lambda+1}$ is an *extension of type* $(*)$ for $(*) \in \{(a), (b), \dots, (h)\}$ if $K_{\lambda+1}$ and K_λ satisfy the properties of item $(*)$ as in the definition of Liouville tower.

ADH 12.4. (1) *Let $(K_\lambda)_{\lambda \leq \mu}$ be a Liouville tower on K . Then:*

- (a) K_μ is a Liouville extension of K .
 - (b) The constant field C_μ of K_μ is a real closure of C if $\mu > 0$.
 - (c) $|K_\mu| = |K|$, hence $\mu < |K|^+$.
- (2) *There is a maximal Liouville tower on K .*
- (3) *The top of a maximal Liouville tower on K is Liouville closed, and hence a Liouville closure of K .*
- (4) *If $(K_\lambda)_{\lambda \leq \mu}$ is a Liouville tower on K such that no K_λ with $\lambda < \mu$ has a gap, and if K_μ is Liouville closed, then K_μ is the unique Liouville closure of K up to isomorphism over K .*

Proof. (1) is [ADH 2017, 10.6.13], (2) follows from (1)(c), (3) is [ADH 2017, 10.6.14], and (4) is [ADH 2017, 10.6.17]. \square

For a set $\Lambda \subseteq \{(a), (b), \dots, (h)\}$ with $(a) \in \Lambda$, the definition of a Λ -tower on K is identical to that of a Liouville tower on K , except that in clause (3) of the above definition only the items from Λ occur. Thus every Λ -tower on K is also a Liouville tower on K . Maximal Λ -towers exist on K by Zorn's Lemma and ADH 12.4(1)(c).

Proof of Proposition 12.2. (1) Assume K is λ -free. By ADH 12.4(4), it suffices to find a Liouville tower $(K_\lambda)_{\lambda \leq \mu}$ on K such that K_μ is Liouville closed and no K_λ with $\lambda < \mu$ has a gap. Take a maximal $\{(a),(e),(f),(g),(h)\}$ -tower $(K_\lambda)_{\lambda \leq \mu}$ on K . By Lemmas 6.9, 6.12, Propositions 7.2, 8.3, 9.3 and ADH 6.14, K_λ is λ -free for every $\lambda \leq \mu$. In particular, no K_λ with $\lambda < \mu$ has a gap. Finally, by maximality, it follows that K_μ is Liouville closed.

(2) Assume that K has asymptotic integration and is not λ -free. First consider the case that K does not have rational asymptotic integration. Then $K_1 = K^{\text{rc}}$ has a gap. By [ADH 2017, 10.6.25] K_1 has two Liouville closures which are not isomorphic over K_1 . As K_1 is a real closure of K , they are not isomorphic over K either because the real closure is unique up to unique isomorphism. Thus K has at least two Liouville closures which are not isomorphic over K .

Next, consider the case that K is real closed. In this case, if L is a Liouville closure of K , then $C_L = C$ since C is necessarily real closed. As K is not λ -free, there is $\lambda \in K$ such that $\lambda_\rho \rightsquigarrow \lambda$. Next, let $K_1 = K(f)$ be the H -field extension from ADH 6.18 such that $f^\dagger = -\lambda$ and $v(f)$ is a gap in K_1 . Again by [ADH 2017, 10.6.25], K_1 has two Liouville closures L_1 and L_2 which are not isomorphic over K_1 . There is $\tilde{y} \in L_1^{\leftarrow 1}$ such that $\tilde{y}' = f$ whereas every $y \in L_2$ such that $y' = f$ has the property that $y > 1$. Furthermore, as both L_1 and L_2 are Liouville closed, they both contain nonconstant elements y such that $y'' = -\lambda y'$.

Claim. *If $y \in L_1 \setminus C$ is such that $y'' = -\lambda y'$, then $y \preccurlyeq 1$. If $y \in L_2 \setminus C$ is such that $y'' = -\lambda y'$, then $y > 1$.*

Proof of Claim. Suppose $y \in L_1 \setminus C$ is such that $y'' = -\lambda y'$. Let $\tilde{y} \in L_1^{\leftarrow 1}$ be such that $\tilde{y}' = f$. Then $\tilde{y} \in L_1 \setminus C$ since $f \neq 0$. Furthermore $\tilde{y}'' = -\lambda \tilde{y}'$ so there are $c_0 \in C^\times$ and $c_1 \in C$ such that $y = c_0 \tilde{y} + c_1$, by Lemma 5.4. It follows that $y \preccurlyeq 1$.

Next, let $y \in L_2 \setminus C$ and let $\tilde{y} \in L_2$ be such that $\tilde{y}' = f$. Then $\tilde{y} \notin C$ because $\tilde{y} > 1$ and $\tilde{y}'' = -\lambda \tilde{y}'$. As in the first case, it will follow from Lemma 5.4 that $y > 1$. \square

It follows from the claim that L_1 and L_2 are not isomorphic over K .

Finally, consider the case that K is not real closed, and has rational asymptotic integration. By the above case, the real closure K^{rc} has two Liouville closures L_1 and L_2 which are not isomorphic over K^{rc} . These two Liouville closures will also not be isomorphic over K , as real closures are unique up to unique isomorphism. \square

The next lemma concerns the appearances of gaps in arbitrary Liouville H -field extensions, not necessarily extensions occurring as the tops of Liouville towers.

Lemma 12.5. *Suppose K is grounded or is λ -free and L is a Liouville H -field extension of K . Then L does not have a gap.*

Proof. We first consider the case that K is λ -free. Let M be the Liouville closure of K which was constructed in the proof of Proposition 12.2. We claim that Ψ

is cofinal in Ψ_M . This follows from the fact that M is constructed as the top of an $\{(a),(e),(f),(g),(h)\}$ -tower on K : the Ψ -set remains unchanged when passing to extensions of type (a), (e), (f) or (g) and for extensions of type (h), the original Ψ -set is cofinal in the larger Ψ -set by [ADH 6.14](#). Finally, as M is the unique Liouville closure of K up to isomorphism over K , we may identify L with a subfield of M which contains K . Thus Ψ_L is cofinal in Ψ_M . As M is λ -free, so is L by [ADH 6.7](#). In particular, L has rational asymptotic integration and so it does not have a gap.

We next consider the case that K is grounded. Let M be the Liouville closure of K as constructed in the proof of [\[ADH 2017, 10.6.24\]](#) and the remarks following it. In particular, using the notation from the remarks following that proof, we have $M = \bigcup_{n < \omega} \ell^n(K)$ where $\ell^0(K) = K$ and $\ell^{n+1}(K)$, for each n , is a grounded Liouville H -field extension of K such that $\max \Psi_{\ell^{n+1}(K)} = s(\max \Psi_{\ell^n(K)})$. Thus the set $\{s^n(\max \Psi) : n < \omega\}$ is a cofinal subset of Ψ_M . We now identify L with a subfield of M that contains K and consider two cases:

Case 1: $\{s^n(\max \Psi) : n < \omega\} \not\subseteq \Psi_L$

In this case there is a least $N < \omega$ such that $s^N(\max \Psi) \in \Psi_L$ but $s(s^N(\max \Psi)) \in \Psi_M \setminus \Psi_L$. This implies that the element $s^N(\max \Psi) \in \Psi_L$ cannot be asymptotically integrated. The only way this can happen is if $s^N(\max \Psi) = \max \Psi_L$. Thus L is grounded and does not have a gap.

Case 2: $\{s^n(\max \Psi) : n < \omega\} \subseteq \Psi_L$

In this case Ψ_L is cofinal in Ψ_M and so L is λ -free by [ADH 6.7](#). This implies that L has rational asymptotic integration and therefore does not have a gap. \square

We also give a characterization of the dichotomy of [Theorem 12.1](#) entirely in terms of gaps appearing in Liouville towers and arbitrary Liouville extensions:

Corollary 12.6. *The following are equivalent:*

- (1) K has exactly two Liouville closures up to isomorphism over K .
- (2) There is a Liouville tower $(K_\lambda)_{\lambda \leq \mu}$ on K such that some K_λ has a gap.
- (3) Every maximal Liouville tower $(K_\lambda)_{\lambda \leq \mu}$ on K has some K_λ with a gap.
- (4) There is a Liouville tower $(K_\lambda)_{\lambda \leq \mu}$ on K with $\mu \geq \omega$ such that either K_0 , K_1 or K_2 has a gap.
- (5) There is an H -field L which has a gap and is a Liouville extension of K .

Proof. (4) \implies (2) and (3) \implies (2) are clear. (1) \implies (3) and (1) \implies (5) follow from [ADH 12.4\(4\)](#).

(1) \implies (4): If K has exactly two Liouville closures up to isomorphism over K , then in particular K itself is not Liouville closed. A routine argument shows that every maximal Liouville tower $(K_\lambda)_{\lambda \leq \mu}$ has $\mu \geq \omega$. By [Theorem 12.1](#) either K has a gap or K has asymptotic integration and is not λ -free. If K has a gap, then

for any maximal Liouville tower $(K_\lambda)_{\lambda \leq \mu}$, K_0 has a gap. Otherwise, the proof of [Proposition 12.2](#) shows how we can arrange either K_1 or K_2 to have a gap.

(2) \implies (1): We will prove the contrapositive. Suppose that K has exactly one Liouville closure up to isomorphism over K and let $(K_\lambda)_{\lambda \leq \mu}$ be a Liouville tower on K . We will prove by induction on λ that K_λ is either grounded or λ -free, and thus no K_λ has a gap. The case $\lambda = 0$ is clear and the limit ordinal case is taken care of by [ADH 6.8](#) and [Lemma 6.9](#). Suppose $\lambda = \nu + 1$ for some ordinal $0 \leq \nu < \mu$. If K_λ is a real closure of K_ν , then K_λ will be grounded if K_ν is by [Definition 3.6 \(1\)](#) and K_λ will be λ -free if K_ν is by [Lemma 6.12](#). By the inductive hypothesis, K_λ will never be an extension of type (b) or (c). If K_λ is an extension of type (d), then K_λ will also be grounded by [[ADH 2017](#), 10.2.3]. Extensions of type (e), (f) and (g) are necessarily immediate extensions, so if K_ν is grounded, then so is K_λ and if K_ν is λ -free, then so is K_λ by [Propositions 7.2](#), [8.3](#), and [9.3](#). Finally, if K_λ is an extension of type (h), and if K_ν is grounded, then so is K_λ by [[ADH 2017](#), 10.5.20], and if K_ν is λ -free then so is K_λ by [ADH 6.14](#).

(5) \implies (1): Suppose K has a Liouville H -field extension with a gap. Then by [Lemma 12.5](#), K has a gap or K has asymptotic integration and is not λ -free. By [Theorem 12.1](#), it follows that K has exactly two Liouville closures up to isomorphism over K . \square

Remark 12.7. The implication (2) \implies (1) of our [Corollary 12.6](#) above occurs without proof in [[Aschenbrenner and van den Dries 2002](#)] (see item (II) before their 6.11). Also, (1) \iff (5) of our [Corollary 12.6](#) is stated without proof in [[Aschenbrenner and van den Dries 2005](#)] (see the paragraph after their 4.3).

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