Pacific Journal of Mathematics

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Vol. 65, No. 2 October 1976

PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 2, 1976

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MAXWELL ROSENLICHT

Some recent results of Ax have made possible greatly simplified demonstrations of Liouville's basic results on the elementary integration of functions and the elementary solution of transcendental equations, together with their generalizations in various directions. An essentially self-contained exposition of this material is given here.

1. For the convenience of the reader, we provide in this section a succinct and somewhat simplified treatment of the necessary parts of Ax's paper [1].

Let $k \rightarrow K$ be a fixed homomorphism of commutative rings. (Thus K is a k-algebra. In all our applications K will be a field and k a subfield, but we may as well begin with the extra generality.) If M is a K-module, by a *k-derivation of K into M* is meant a k-linear map $D: K \rightarrow M$ such that $D(xy) = x(Dy) + y(Dx)$ for all $x, y \in K$. In such a situation we have $Dx^n = nx^{n-1}Dx$ for all $x \in K$ and all positive integers n; taking $x = 1$, $n = 2$, we get $D1 = 0$, and hence D vanishes on the image of k in K. The k-derivations of K into M form a K-submodule $\mathrm{Der}_k(K, M)$ of $\mathrm{Hom}_k(K, M)$. A *derivation on* (or *of*) *the ring* K is simply a $\boldsymbol{Z}$-derivation of K into K (that is, we take $k = \boldsymbol{Z}$, $K = M$). A derivation on an integral domain extends to a unique derivation on its field of quotients, by means of the equation $D(x/y) = (yDx - xDy)/y^2$.

PROPOSITION 1. *Let $k \rightarrow K$ be a homomorphism of commutative rings. Then there exists a K-module $\Omega_{K/k}$ and a k-derivation d of K into $\Omega_{K/k}$ such that for any k-derivation D of K into a K-module M there exists a unique K-homomorphism $\Omega_{K/k} \rightarrow M$ which composed with d gives D.*

This well-known result is most simply proved by trying for $\Omega_{K/k}$ the K-module Φ/Ψ, where Φ is the free K-module generated by the symbols $\{\delta x\}_{x \in K}$ and Ψ the K-submodule of Φ generated by all $\delta(x + y) - \delta x - \delta y$ and all $\delta(xy) - x\delta y - y\delta x$ for $x, y \in K$, and all δx for x in the image of k in K, with d the obvious composition with δ, and noting that this works.

The pair $(\Omega_{K/k}, d)$, clearly unique to within isomorphism, is called the *module of k-differentials of K*. Each element of $\Omega_{K/k}$ can be written as a finite sum $\sum_i x_i dy_i$, with $x_i, y_i \in K$. For any K-module

M, there is a natural K-module isomorphism between $\mathrm{Der}_k(K, M)$ and $\mathrm{Hom}_K(\Omega_{K/k}, M)$.

PROPOSITION 2. *Let $\iota: k \to K$ be a homomorphism of commutative rings, D a derivation on K such that there exists a map $D_k: k \to k$ such that $\iota D_k = D\iota$. Then there exists a unique map $D^1: \Omega_{K/k} \to \Omega_{K/k}$ such that for all $\omega, \eta \in \Omega_{K/k}$ and all $f \in K$ we have $D^1(\omega + \eta) = D^1\omega + D^1\eta$, $D^1(f\omega) = (Df)\omega + f(D^1\omega)$, and $D^1(df) = d(Df)$.*

Since each element of $\Omega_{K/k}$ is of the form $\Sigma_i x_i dy_i$, with $x_i, y_i \in K$, the uniqueness of D^1 is clear. To prove the existence of D^1 we first define an additive map D' on the free K-module Φ of the proof of Proposition 1 by setting $D'(\Sigma_i x_i \delta y_i) = \Sigma_i((Dx_i)\delta y_i + x, \delta(Dy_i))$ and then note that $D'\Psi \subset \Psi$, so that D' defines an induced map D^1 on Φ/Ψ, which is isomorphic to $\Omega_{K/k}$. The verification that D^1 has the desired properties is straightforward.

From now on K will be a field, usually of characteristic zero, k a subfield of K.

LEMMA. *Let k be a field, K a separable algebraic extension field of k. Then any derivation of k has a unique extension to a derivation of K.*

This is another standard result. A proof may be found in [2, §3], for example.

PROPOSITION 3. *Let $k \subset K$ be fields, $\{x_\alpha\}_{\alpha \in A}$ elements of K that are algebraically independent over k and such that K is separably algebraic over $k(\{x_\alpha\}_{\alpha \in A})$. Then $\{dx_\alpha\}_{\alpha \in A}$ is a K-basis for $\Omega_{K/k}$.*

Each element of K satisfies a separable polynomial equation with coefficients in the ring $k[\{x_\alpha\}_{\alpha \in A}]$, and by applying d to these equations we see that $dK \subset \Sigma_{\alpha \in A} K dx_\alpha$. In other words, $\{dx_\alpha\}_{\alpha \in A}$ spans $\Omega_{K/k}$. To show that $\{dx_\alpha\}_{\alpha \in A}$ are linearly independent over K, we use the existence, for each $\beta \in A$, of a derivation $\partial/\partial x_\beta$ of K which annuls k and each x_α, $\alpha \neq \beta$, and takes on the value 1 at x_β; the derivation $\partial/\partial x_\beta$ is first constructed for the ring $k[\{x_\alpha\}_{\alpha \in A}]$, then extended to its field of quotients $k(\{x_\alpha\}_{\alpha \in A})$, and then to K, using the lemma.

COROLLARY. *If $k \subset K \subset L$ are fields of characteristic zero, then the natural homomorphism $\Omega_{K/k} \to \Omega_{L/k}$ is injective.*

The "natural homomorphism" is of course that of Proposition 1. Injectivity results from Proposition 3, noting that any transcendence base for K/k is part of a transcendence base for L/k.

PROPOSITION 4. *Let $k \subset K$ be fields of characteristic zero, let $u_1, \cdots, u_n, v$ be elements of K, with $u_1, \cdots, u_n$ nonzero, and let $c_1, \cdots, c_n$ be elements of k that are linearly independent over the natural numbers $\boldsymbol{Q}$. Then the element*

$$c_1\frac{du_1}{u_1} + \cdots + c_n\frac{du_n}{u_n} + dv$$

of $\Omega_{K/k}$ is zero if and only if each $u_1, \cdots, u_n, v$ is algebraic over k.

The element dv of $\Omega_{k(v)/k}$ is zero if and only if v is algebraic over k, by Proposition 3. The Corollary implies that the element dv of $\Omega_{K/k}$ is zero if and only if v is algebraic over k. It remains only to prove that if u_1 is not algebraic over k then $c_1du_1/u_1 + \cdots + c_ndu_n/u_n + dv$ is nonzero. For this we may replace K, if necessary, by $k(u_1, \cdots, u_n, v)$, to reduce ourselves to the case where K is a finite extension of k. Let $x_1, \cdots, x_m$ be a transcendence base for K/k, with $x_1 = u_1$. Considering the natural homomorphism $\Omega_{K/k} \rightarrow \Omega_{K/k(x_2,\cdots,x_m)}$ and replacing k by $k(x_2, \cdots, x_m)$ if necessary, we see that we may suppose K algebraic over $k(u_1)$. Enlarge K, as we may if necessary, so that K is normal over $k(u_1)$. If $c_1du_1/u_1 + \cdots + c_ndu_n/u_n + dv = 0$ then for any $\sigma \in \mathrm{Aut}\,(K/k(u_1))$ we have $c_1d\sigma u_1/\sigma u_1 + \cdots + c_nd\sigma u_n/\sigma u_n + d\sigma v = 0$, and adding up over all $\sigma \in \mathrm{Aut}\,(K/k(u_1))$ produces an equation similar to our original one, but with c_1 replaced by $[K: k(u_1)]c_1$, with the same $c_2, \cdots, c_n, u_1$, and with $u_2, \cdots, u_n, v$ replaced by elements of $k(u_1)$. We therefore have to show that $c_1du_1/u_1 + \cdots + c_ndu_n/u_n + dv$ is nonzero in the special case where $u_2, \cdots, u_n, v \in K = k(u_1)$, with u_1 transcendental over k. This fact follows immediately upon expressing each u_i as a power product of monic irreducible elements of $k[u_1]$ and an element of k and v in its partial fraction form relative to $k[u_1]$, for we then get a non-cancelling partial fraction term du_1/u_1.

PROPOSITION 5. *Let $k \subset K$ be fields of characteristic zero, D a derivation of K such that $Dk \subset k$, C the field $\{x \in k: Dx = 0\}$, and u and t elements of K that are algebraically dependent over C. Then in $\Omega_{K/k}$ we have $D^1(udt) = d(uDt)$.*

For $D^1(udt) = (Du)dt + udDt$, while $d(uDt) = (Dt)du + udDt$, so that we have to show that $(Du)dt = (Dt)du$. Corresponding to parts of the inclusions $C \subset k \subset k(u, t) \subset K$ we have the homomorphisms $\Omega_{K/C} \rightarrow \Omega_{K/k}$ and $\Omega_{k(u,t)/k} \rightarrow \Omega_{K/k}$, so that we can reduce ourselves first to the case $k = C$ and next to the case $K = k(u, t) = C(u, t)$. In this case D is a multiple of the derivation $\partial/\partial t$ of $C(u, t)$ and our proof reduces to the known equality $du = (\partial u/\partial t)dt$.

PROPOSITION 6. *Let $k \subset K$ be fields, Δ a set of derivations of K such that $Dk \subset k$ for each $D \in \Delta$, and let C be the field $\bigcap_{D \in \Delta} \ker D$. Suppose $\omega_1, \cdots, \omega_n \in \Omega_{K/k}$ are annulled by each D^1, for $D \in \Delta$. Then if $\omega_1, \cdots, \omega_n$ are linearly dependent over K they are linearly dependent over C.*

For suppose that there are $a_1, \cdots, a_n \in K$, not all zero, such that $a_1\omega_1 + \cdots + a_n\omega_n = 0$. Choose $a_1, \cdots, a_n$ so that the number of nonzero a_i's is minimal, and that one of them, say a_1, is 1. For each $D \in \Delta$ we get $0 = D^1(a_1\omega_1 + \cdots + a_n\omega_n) = (Da_1)\omega_1 + \cdots + (Da_n)\omega_n = (Da_2)\omega_2 + \cdots + (Da_n)\omega_n$. Since the number of nonzero a_i's was minimal and $a_1 = 1$, we get $Da_2 = \cdots = Da_n = 0$. Hence each $a_i \in C$.

2. We come now to the applications of the previous material to differential algebra. The reader interested in the earlier literature should consult the bibliographies of the references listed at the end.

In what follows, by a *differential field* will be meant a field k, together with an indexed family $\{D_i\}_{i \in I}$ of derivations of k. For simplicity, one speaks of "the differential field k", instead of "the differential field $\{k, \{(i, D_i)\}_{i \in I}\}$". The *constants* of the differential field k are $\bigcap_{i \in I} \ker D_i$, a subfield of k. A *differential extension field of k* is an extension field K of k together with a family of derivations $\{D'_i\}_{i \in I}$ of K indexed by the same set such that the restriction of each D'_i to k is D_i.

THEOREM 1. *Let k be a differential field of characteristic zero, K a differential extension field of k with the same constants C. For each $i = 1, \cdots, n$ and $j = 1, \cdots, \nu$ let $c_{ij} \in C$ and let v_i be an element of K, u_j a nonzero element of K. Suppose that for each $i = 1, \cdots, n$ and each given derivation D of K we have*

$$\sum_{j=1}^{\nu} c_{ij} \frac{Du_j}{u_j} + Dv_i \in k .$$

Then either $\deg. \mathrm{tr}.\, k(u_1, \cdots, u_\nu, v_1, \cdots, v_n)/k \geqq n$ *or the n elements of $\Omega_{K/k}$ given by $\sum_{j=1}^{\nu} c_{ij} du_j/u_j + dv_i$, $i = 1, \cdots n$, are linearly dependent over C.*

Working in $\Omega_{K/k}$ and using Propositions 2 an 5, for each given derivation D of K and each $i = 1, \cdots, n$ we obtain

$$D^1\Big(\sum_{j=1}^{\nu} c_{ij}\frac{du_j}{u_j} + dv_i\Big) = d\Big(\sum_{j=1}^{\nu} c_{ij}\frac{Du_j}{u_j} + Dv_i\Big) = 0 .$$

If the differentials $\sum_{j=1}^{\nu} c_{ij}du_j/u_j + dv_i$, $i = 1, \cdots, n$, of $\Omega_{K/k}$ are linearly independent over C, then by Proposition 6 they are also linearly independent over K. Hence the n differentials $\sum_{j=1}^{\nu} c_{ij}du_j/u_j + dv_i$ of $\Omega_{k(u_1,\cdots,u_\nu,v_1,\cdots,v_n)/k}$ are linearly independent over $k(u_1, \cdots, u_\nu, v_1, \cdots, v_n)$. Since $\Omega_{k(u_1,\cdots,u_\nu,v_1,\cdots,v_n)k}$ is a vector space over $k(u_1, \cdots, u_\nu, v_1, \cdots, v_n)$ of dimension deg. tr. $k(u_1, \cdots, u_\nu, v_1, \cdots, v_n)/k$, this latter number must be at least n.

COROLLARY. *Let k be a differential field of characteristic zero, K a differential extension field of k with the same constants. Suppose that $u_1, \cdots, u_n, v_1, \cdots, v_n \in K$, with $u_1, \cdots, u_n$ nonzero, and that for each $i = 1, \cdots, n$ and each given derivation D of K we have $Du_i/u_i + Dv_i \in k$. Then either* deg. tr. $k(u_1, \cdots, u_n, v_1, \cdots, v_n)/k \geqq n$ *or some linear combination of $v_1, \cdots, v_n$ with constant coefficients that are not all zero and some power product of $u_1, \cdots, u_n$ with exponents not all zero are algebraic over k.*

This is a slight generalization of the main result Theorem 4 of [1]. To prove it, note that if the transcendence degree in question is not at least n then there exist $\gamma_1, \cdots, \gamma_n \in C$, not all zero, such that

$$\gamma_1\frac{du_1}{u_1} + \cdots + \gamma_n\frac{du_n}{u_n} + \gamma_1 dv_1 + \cdots + \gamma_n dv_n = 0 ,$$

choose a basis $c_1, \cdots, c_r$ for the vector space $\boldsymbol{Q}\gamma_1 + \cdots + \boldsymbol{Q}\gamma_n$ so that each γ_i can be written as $\gamma_i = \sum_{j=1}^{r} \nu_{ij}c_j$ with each $\nu_{ij} \in \boldsymbol{Z}$, hence (using "logarithmic derivatives") rewrite the displayed equation as

$$\sum_{j=1}^{r} c_j\frac{d(u_1^{\nu_{1j}}\cdots u_n^{\nu_{nj}})}{u_1^{\nu_{1j}}\cdots u_n^{\nu_{nj}}} + d(\gamma_1 v_1 + \cdots + \gamma_n v_n) = 0 ,$$

and quote Proposition 4.

The next theorem generalizes the main result of [3], to which paper we refer for its applications to the question of elementary solutions of transcendental equations. The lemma is an immediate consequence of Theorem 1.

LEMMA. *Let k be a differential field of characteristic zero, K a differential extension field of k having the same constants and such that* deg. tr. $K/k = 1$. *Then any two k-differentials of K which can be written in the form $c_1du_1/u_1 + \cdots + c_ndu_n/u_n + dv$, where $c_1, \cdots, c_n, u_1, \cdots, u_n, v \in k$, $c_1, \cdots, c_n$ being constants, in such a way that for each given derivation D of K we have $c_1Du_1/u_1 + \cdots +$*

$c_nDu_n/u_n + Dv \in k$, are linearly dependent over the subfield of constants.

THEOREM 2. *Let k be a differential field of characteristic zero, K a differential extension field of k with the same constants, with K algebraic over $k(t)$ for some given $t \in K$. Suppose that $c_1, \cdots, c_n$ are constants of k that are linearly independent over $\boldsymbol{Q}$, that $u_1, \cdots, u_n, v$ are elements of K, with $u_1, \cdots, u_n$ nonzero, and that for each given derivation D of K we have $\sum_{i=1}^{n} c_i Du_i/u_i + Dv \in k$. If for each given derivation D of K we have $Dt \in k$, then $u_1, \cdots, u_n$ are algebraic over k and there exists a constant c of k such that $v - ct$ is algebraic over k. If for each given derivation D of K we have $Dt/t \in k$, then v is algebraic over k and there are integers $\nu_0, \nu_1, \cdots, \nu_n$, with $\nu_0 \neq 0$, such that each $u_i^{\nu_0}/t^{\nu_i}$ is algebraic over k.*

We may suppose t transcendental over k, so that $dt \neq 0$. In either of the two cases, each $Dt \in k$ or each $Dt/t \in k$, the Lemma is applicable, and we have $c_1 du_1/u_1 + \cdots + c_n du_n/u_n + dv$ equal to either cdt or cdt/t, for some constant c of k. In the former case we have $c_1 du_1/u_1 + \cdots + c_n du_n/u_n + d(v - ct) = 0$, and Proposition 4 tells us that $u_1, \cdots, u_n, v - ct$ are all algebraic over k. In the latter case Proposition 4 first implies the linear dependence of $c_1, c_2, \cdots, c_n$ over $\boldsymbol{Q}$, so that we can write $c = (\sum_{i=1}^{n} \nu_i c_i)/\nu_0$, for suitable integers $\nu_0, \nu_1, \cdots, \nu_n$, with $\nu_0 \neq 0$, and so obtain

$$c_1 \frac{d(u_1^{\nu_0}/t^{\nu_1})}{u_1^{\nu_0}/t^{\nu_1}} + \cdots + c_n \frac{d(u_n^{\nu_0}/t^{\nu_n})}{u_n^{\nu_0}/t^{\nu_n}} + d(\nu_0 v) = 0 .$$

A final application of Proposition 4 to this last equation completes the proof.

If k is a differential field and $x, y \in k$, with $y \neq 0$, and the relation $Dx = Dy/y$ holds for each given derivation D of k, we call x a *logarithm of* y or y an *exponential of* x. A differential extension field of k is called an *elementary extension of* k if it is of the form $k(t_1, \cdots, t_N)$, where for each $i = 1, \cdots, N$, t_i is either a logarithm of an element of $k(t_1, \cdots, t_{i-1})$, or an exponential of an element of $k(t_1, \cdots, t_{i-1})$, or is algebraic over $k(t_1, \cdots, t_{i-1})$. In this case note that each field $k(t_1, \cdots, t_{i-1})$ is a differential extension field of k.

The following result generalizes Liouville's theorem on the elementary integrability of functions.

THEOREM 3. *Let k be a differential field of characteristic zero and for each given derivation D of k let $\alpha_D \in k$. Then there exists an elementary differential extension field of k having the same* *constants and containing an element y such that $Dy = \alpha_D$ for each given derivation D if and only if there are constants $c_1, \cdots, c_n \in k$ and elements $u_1, \cdots, u_n, v \in k$, such that for each given derivation D we have*

$$\alpha_D = \sum_{i=1}^{n} c_i \frac{Du_i}{u_i} + Dv .$$

First suppose that there is a differential extension field $k(t_1, \cdots, t_N)$ of k having the same constants, with each t_i a logarithm or an exponential of an element of $k(t_1, \cdots, t_{i-1})$, or algebraic over the latter field, that contains an element y such that $Dy = \alpha_D$ for each given derivation D. We shall prove by induction on N that elements $c_1, \cdots, c_n, u_1, \cdots, u_n, v$ of k exist as indicated. Since the case $N = 0$ is trivial, we assume that $N > 0$ and that the result holds for $N - 1$. If we apply the $N - 1$ case to the differential fields $k(t_1) \subset k(t_1, \cdots, t_N)$ we deduce immediately that there are constants $c_1, \cdots, c_n$ of k and elements $u_1, \cdots, u_n, v$ of $k(t_1)$ such that for each given derivation D we have $\alpha_D = \sum_{i=1}^{n} c_i Du_i/u_i + Dv$. Thus we are reduced to proving only the rather general statement that if there is an element t in a differential extension field of k having the same constants as k such that t is a logarithm or an exponential of an element of k or algebraic over k and if there exist an integer n, constants $c_1, \cdots, c_n$ of k, and elements $u_1, \cdots, u_n, v$ of $k(t)$ such that $\alpha_D = \sum_{i=1}^{n} c_i Du_i/u_i + Dv$ for each given derivation D, then such n and $c_1, \cdots, c_n, u_1, \cdots, u_n, v$ can be found with the latter all in k. If t is algebraic over k, we can assume $k(t)$ to be a normal extension of k. Then for each $\sigma \in \operatorname{Aut}(k(t)/k)$ we have $\alpha_D = \sum_{i=1}^{n} c_i D\sigma u_i/\sigma u_i + D\sigma v$ and summing over all σ we get $[k(t): k]\alpha_D = \sum_{i=1}^{n} c_i D\Pi_\sigma \sigma u_i/\Pi_\sigma \sigma u_i + D\Sigma_\sigma \sigma v$, with each element $\Pi_\sigma \sigma u_i$ and $\Sigma_\sigma \sigma v$ in k. Thus we may assume t transcendental over k. We claim that we may suppose $c_1, \cdots, c_n$ to be linearly independent over $\boldsymbol{Q}$. For if, say, c_n depends linearly on $c_1, \cdots, c_{n-1}$ we write $c_n = (m_1 c_1 + \cdots + m_{n-1} c_{n-1})/m$, with $m_1, \cdots, m_{n-1}, m \in \boldsymbol{Z}$, $m \neq 0$ and we obtain for each given derivation D the equation $\alpha_D = \sum_{i=1}^{n-1} (c_i/m) D(u_i^m u_n^{m_i})/u_i^m u_n^{m_i} + Dv$, similar to what we had before but with smaller n. Therefore we may assume that $c_1, \cdots, c_n$ are linearly independent over $\boldsymbol{Q}$. If t is a logarithm of an element of k, say $Dt = Da/a$ for some $a \in k$ and each given derivation D, then it is an immediate consequence of Theorem 2 that $u_1, \cdots, u_n \in k$, while $v = ct + w$, for some constant c and some $w \in k$, so that for each D we have $\alpha_D = c_1 Du_1/u_1 + \cdots + c_n Du_n/u_n + cDa/a + Dw$, a relation of the type desired, since all the terms here are in k. If t is an exponential of an element of k, say $Dt/t = Db$ for some $b \in k$ and each given derivation D, Theorem 2 tells us that $v \in k$ and there are integers $\nu_0, \nu_1, \cdots, \nu_n$, with $\nu_0 \neq 0$, such that each $u_i^{\nu_0}/t^{\nu_i} \in k$.

Thus for each D we have

$$Du_i/u_i = (1/\nu_0)Du_i^{\nu_0}/u_i^{\nu_0} = (1/\nu_0)D(u_i^{\nu_0}/t^{\nu_i})/(u_i^{\nu_0}/t^{\nu_i}) + (\nu_i/\nu_0)Dt/t\ .$$

Noting that v and each $u_i^{\nu_0}/t^{\nu_i}$ are in k and that $Dt/t = Db$, with $b \in k$, we get an expression for each α_D of the type desired, with all terms in k. It therefore remains only to prove the converse of what we have shown so far, namely that if there exist constants $c_1, \cdots, c_n$ in k and elements $u_1, \cdots, u_n, v$ of k such that for each D we have $\alpha_D = \sum_{i=1}^n c_i Du_i/u_i + Dv$, then there is an element y in some elementary extension field of k having the same constants such that for each D we have $Dy = \alpha_D$. It suffices to prove that for each i, $\{Du_i/u_i\}$ can be integrated in turn, without introducing new constants. In other words, it remains to show that if $a \in k$, $a \neq 0$, then there exists a differential extension field $k(t)$ of k having the same constants and such that $Dt = Da/a$ for each given derivation D. To do this, take t transcendental over k and make $k(t)$ a differential extension field of k by defining, for each given derivation D of k, $Dt = Da/a$. We are all done, unless it happens that $k(t)$ has a constant not in k. So suppose that f/g is a constant in $k(t)$, with f, g relatively prime elements of $k[t]$, not both in k, and g monic. For each given derivation D of k we have $D(f/g) = 0$, so that $gDf = fDg$. Now Df, $Dg \in k[t]$, with degrees respectively $\leqq$ (degree of f), $<$ (degree of g). Relative primeness implies $g \mid Dg$, so that $Dg = 0$, hence also $Df = 0$. Therefore there is a constant in $k[t]$ that is not in k. Say that $b_0, b_1, \cdots, b_n \in k$, $n > 0$, $b_0 \neq 0$, with $D(b_0 t^n + b_1 t^{n-1} + \cdots + b_n) = 0$ for all D. Then

$$(Db_0)t^n + (nb_0 Da/a + Db_1)t^{n-1} + \cdots = 0$$

for all D. Therefore b_0 is a constant in k and $Da/a = D(-b_1/nb_0)$. In this case a has a logarithm in k itself and we are done.

Added in proof. Another proof of the main part of this theorem is given in B. F. Caviness and M. Rothstein, "A Liouville theorem on integration in finite terms for line integrals," Communications in Algebra, 3 (1975), 781–795.

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Received September 20, 1974. Research supported by National Science Foundation grant number GP-37492X.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 65, No. 2 October, 1976