Orbifold Hecke algebras

Pavel Etingof (MIT)

Let W be a Coxeter group of rank r with generators s_i and defining relations

$$s_i^2 = 1,$$

$$(s_i s_j)^{m_{ij}} = 1 \text{ for } m_{ij} < \infty,$$

where $m_{ij} = m_{ji}$ are integers ≥ 2 or ∞, defined for $1 \leq i \neq j \leq r$.

The classical Hecke algebra $H_q(W)$ of W is generated over $\mathbb{C}[q, q^{-1}]$ by the same generators s_i with deformed defining relations:

$$(s_i - q)(s_i + q^{-1}) = 0,$$

$$(s_i s_j)^{m_{ij}} = 1 \text{ for } m_{ij} < \infty.$$

This algebra is a 1-parameter deformation of the group algebra $\mathbb{C}[W]$. Moreover, it is a **flat** deformation:

$H_q(W)$ is a **free** $\mathbb{C}[q, q^{-1}]$-module. This "PBW" property is not obvious from the relations; its proof is nontrivial and uses reduced presentations of elements of W.

The goal of my talk is to show that group algebras of many other discrete groups admit similar flat deformations, obtained by replacing relations of the form $T^m = 1$ by $(T - t_1)...(T - t_m) = 0$. This includes a number of known types of Hecke algebras (cyclotomic Hecke algebras of complex reflection groups, Cherednik's double affine Hecke algebras), as well as many new examples. In all cases, the proof of flatness is a nontrivial problem, usually more difficult than in the case of classical Hecke algebras, because of absence of a good notion of a reduced presentation. In this situation, geometric methods are often helpful (D-modules, constructible sheaves).

Remark. The results of this talk are taken from several of my papers, some of them joint with W.L.Gan, A.Oblomkov, and E.Rains.

1. Orbifold Hecke algebras

Let X be a connected, simply connected complex manifold, and G a discrete group of automorphisms of X. In this case the quotient X/G is a complex orbifold. Let $X' \subset X$ be the set of points having trivial stabilizer (it is a nonempty open subset of X). Define the braid group $\widetilde{G}$ of the orbifold X/G to be the fundamental group of the manifold X'/G with some base point x_0. We have a surjective homomorphism $\phi : \widetilde{G} \to G$, which corresponds to gluing back the points which have a nontrivial stabilizer. Let K be the kernel of this homomorphism.

For every $g \in G$, the fixed set X^g of g in Y is smooth, and consists of connected components X^g_j, possibly of different dimensions. Such a component is said to be a **reflection hypersurface** if it has codimension 1 in X.

The kernel K can be described by simple relations, corresponding to reflection hypersurfaces in X. Namely, for a reflection hypersurface $Y \subset X$, we have a conjugacy class C_Y in $\widetilde{G}$ which corresponds to the loop in X'/G which goes counterclockwise around the image of Y in X/G. Let T_Y be a representative of C_Y. Also, let $G_Y \subset G$ be the stabilizer of a generic point on Y; this is a cyclic group of some order n_Y. Then it follows from basic topology (van Kampen's theorem) that the elements $T_Y^{n_Y}$ belong to K, and K is the smallest normal subgroup of $\widetilde{G}$ containing all of them. In other words, the group G is the quotient of the braid group $\widetilde{G}$ by the relations

$$T_Y^{n_Y} = 1. \tag{1}$$

Now let $A_0 = \mathbb{C}[G]$, and let us define a (formal) deformation A of A_0

to be the quotient of the group algebra of the braid group $\widetilde{G}$ by a deformation of relations (1). Namely, for every reflection hypersurface $Y \subset X$ we introduce formal parameters $\tau_{Y,k}$, $k = 1, ..., n_Y$ (which are conjugation invariant), and replace relations (1) by the relations

$$\prod_{k=1}^{n_Y} (T_Y - e^{\frac{2\pi i k}{n_Y} + \tau_{Y,k}}) = 0. \tag{2}$$

The quotient A of $\mathbb{C}[\widetilde{G}][[\tau]]$ by these relations is called the **orbifold Hecke algebra** of X/G, and denoted by $\mathcal{H}_\tau(X, G)$.

Theorem 1.1. *If $H^2(X, \mathbb{C}) = 0$ then $A = \mathcal{H}_\tau(X, G)$ is a flat deformation of $\mathbb{C}[G]$.*

Here by saying that A is flat we mean that $A = A_0[[\tau]]$ as a $\mathbb{C}[[\tau]]$-module, with deformed multiplication.

2. Examples of orbifold Hecke algebras

Example 2.1. Let $\mathfrak{h}$ be a finite dimensional vector space, and W be a complex reflection group in $GL(\mathfrak{h})$. Then $\mathcal{H}_\tau(\mathfrak{h}, W)$ is the Hecke algebra of W studied by Broué, Malle, and Rouquier. It follows from Theorem 1.1 that this Hecke algebra is flat. This proof of flatness is in fact the same as the original proof of this result by Broué, Malle, and Rouquier (based on the Dunkl-Opdam-Cherednik operators). In the case when W is a real reflection group, we recover the classical Hecke algebra of W.

Example 2.2. Let $\mathfrak{h}$ be the Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$, W its Weyl group, $Q^\vee$ the dual root lattice, and $\widehat{W} := W \ltimes Q^\vee$ the corresponding affine Weyl group. Then $\mathcal{H}_\tau(\mathfrak{h}, \widehat{W})$ is the affine Hecke algebra. Its flatness, which is a consequence of Theorem 1.1, also follows from the fact that $\widehat{W}$ is a Coxeter group.

Example 2.3. Let $\mathfrak{h}, W, Q^\vee$ be as in the previous example, and $\tau \in \mathbb{C}^+$. Consider the double affine Weyl group $\widehat{\widehat{W}} := W \ltimes (Q^\vee \oplus \tau Q^\vee)$. Then $\mathcal{H}_\tau(\mathfrak{h}, \widehat{\widehat{W}})$ is the double affine Hecke algebra of Cherednik (in the generalized form due to Sahi and Stokman) and it is flat by Theorem 1.1. This algebra "controls" the theory of Macdonald and Koornwinder polynomials. The fact that this algebra is flat was proved by Cherednik (and Sahi) using a different approach (q-deformed Dunkl operators).

Example 2.4. Let H be a simply connected complex Riemann surface (i.e., Riemann sphere, Euclidean plane, or Lobachevsky plane), and Γ be a cocompact lattice in $\mathrm{Aut}(H)$. Let $\Sigma = H/\Gamma$. Then Σ is a compact complex Riemann surface. When Γ contains elliptic elements (i.e., nontrivial elements of finite order), Σ is an orbifold: it has special points P_i, $i = 1, ..., m$ with stabilizers $\mathbb{Z}_{n_i}$.

Let g be the genus of Σ, and $a_l, b_l, l = 1, ..., g$, be the a-cycles and b-cycles of Σ. Let c_j be the counterclockwise loops around P_j. Then Γ is generated by a_l, b_l, c_j with relations

$$c_j^{n_j} = 1,\ c_1 c_2 ... c_m = \prod_{l=1}^{g} a_l b_l a_l^{-1} b_l^{-1}.$$

For each j, introduce formal parameters τ_{kj}, $k = 1, ..., n_j$. Then the Hecke algebra $\mathcal{H}_\tau(H, \Gamma)$ is generated over $\mathbb{C}[[\tau]]$ by the same generators a_l, b_l, c_j with defining relations

$$\prod_{k=1}^{n_j} (c_j - e^{\frac{2\pi i k}{n_j} + \tau_{kj}}) = 0,$$

$$c_1c_2...c_m = \prod_{l=1}^{g} a_l b_l a_l^{-1} b_l^{-1}.$$

Therem 1.1 says that this deformation is flat if H is a Euclidean plane or a Lobachevsky plane, but it tells us nothing about the case when H is a sphere, since in that case the assumption $H^2(X, \mathbb{C}) = 0$ is violated. And indeed, it turns out that in these cases $\mathcal{H}_\tau(H, \Gamma)$ fails to be flat!

To see this, let us compute the determinant of the product $c_1...c_m$ in the regular representation of this algebra (which is finite dimensional if H is the sphere). On the one hand, it is 1, as $c_1...c_m$ is a product of commutators. On the other hand, the eigenvalues of c_j in this representation are $e^{\frac{2\pi ik}{n_j}+\tau_{kj}}$ with multiplicity $|\Gamma|/n_j$. Computing determinants as products of eigenvalues, we get a nontrivial equation on τ_{kj}, which means that the deformation $\mathcal{H}_\tau$ is not flat.

Thus, we see that $\mathcal{H}_\tau(H,\Gamma)$ fails to be flat in the following "forbidden" cases:

$$g=0,\ m=2,(n_1,n_2)=(n,n);$$

$$m=3,(n_1,n_2,n_3)=$$

$$(2,2,n),(2,3,3),(2,3,4),(2,3,5).$$

Remark. If $g=0$, then finite dimensional representations of $\mathcal{H}_\tau(H,\Gamma)$ are (essentially) the same things as solutions of the "multiplicative Deligne-Simpson problem".

The case when H is the Euclidean plane deserves special attention. If there are elliptic elements, the reduces to the following configurations: $g = 0$ and

$$m = 3, (n_1, n_2, n_3) =$$
$$(3, 3, 3), (2, 4, 4), (2, 3, 6),$$

(cases E_6, E_7, E_8) or

$m = 4, (n_1, n_2, n_3, n_4) = (2, 2, 2, 2)$. (case D_4).

In these cases, the algebra $\mathcal{H}_\tau(H, \Gamma)$ (for numerical τ) has Gelfand-Kirillov dimension 2, so it can be interpreted in terms of the theory of noncommutative surfaces. More specifically, let $\hbar = \sum_{j,k} n_j^{-1} \tau_{kj}$. Also let n be the largest of n_j, and c be the corresponding c_j. Let $\mathbf{e} \in \mathbb{C}[c] \subset \mathcal{H}_\tau(H, \Gamma)$ be the projector to an eigenspace of c. Consider the "spherical" subalgebra $B_\tau(H,\Gamma) := \mathbf{e}\mathcal{H}_\tau(H,\Gamma)\mathbf{e}$.

Theorem 2.5. *(E.-Oblomkov-Rains) (i) If $\hbar = 0$ then the algebra $B_\tau(H,\Gamma)$ is commutative, and its spectrum is an affine del Pezzo surface. More precisely, in the case (2,2,2,2) it is a del Pezzo surface of degree 3 (=a cubic surface) with a triangle of lines removed; in the cases (3,3,3),(2,4,4),(2,3,6) it is a del Pezzo surface of degrees 3,2,1 respectively with a nodal rational curve removed.*

(ii) The algebra $B_\tau(H,\Gamma)$ for $\hbar \neq 0$ is a quantization of the unique algebraic symplectic structure on the surface from (i) with Planck's constant $\hbar$.

Remark. In the case $(2,2,2,2)$, $\mathcal{H}_\tau(H,\Gamma)$ is the Cherednik-Sahi algebra of rank 1; it controls the theory of Askey-Wilson polynomials.

Example 2.6. This is a "multivariate" version of the previous example. Namely, letting H,Γ be as in the previous example, and $N \geq 1$, we consider the manifold $X = H^N$ with the action of $\Gamma_N = S_N \ltimes \Gamma^N$. If H is a Euclidean or Lobachevsky plane, then by Theorem 1.1 $\mathcal{H}_\tau(X^N,\Gamma_N)$ is a flat deformation of the group algebra $\mathbb{C}[\Gamma_N]$. If $N > 1$, this algebra has one more essential parameter than for $N = 1$ (corresponding to reflections in S_N). In the Euclidean case, one expects that an appropriate "spherical" subalgebra of this algebra is a quantization of the Hilbert scheme of a del Pezzo surface.

Example 2.7. (This is a generalization of double affine Hecke algebras). Let $\mathbf{L}$ be a symplectic lattice of rank $2n$, and G a finite subgroup of $Sp(\mathbf{L})$. Let $X = \mathbb{R} \otimes \mathbf{L}$ be the corresponding symplectic vector space. Let ω be the symplectic form on this space. To make things simple, assume that ω is a unique, up to scaling, G-invariant symplectic form on X. Pick a G-invariant Kähler structure on X, such that its imaginary part is ω (this can be done by averaging any Kähler structure with imaginary part ω over G). This makes X

into a complex n-dimensional vector space, on which G acts $\mathbb{C}$-linearly. Let $\Gamma = G \ltimes \mathbf{L}$. Then Γ acts holomorphically on X, so we can define the Hecke algebra $\mathcal{H}_\tau(X, G)$. By Theorem 1.1, this algebra is flat, and its essential parameters are a complex number q and a function on the set of conjugacy classes of affine reflections in $G \ltimes \mathbf{L}$, i.e., elements whose fixed set in X has real codimension 2 (or complex codimension 1). This is thus a trigonometric generalization of symplectic reflection algebras, attached to a finite subgroup of $Sp(V)$, where V is a complex vector space.

Example 2.8. This is a multidimensional version of Example 2.4. Let

X be the n-dimensional complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$, and Γ be a finitely generated discrete group of motions of X, i.e. a discrete subgroup of $PSU(n,1)$. Then the Hecke algebra $\mathcal{H}_\tau(X,G)$ is flat (since $H^2(X,\mathbb{C}) = 0$). A nice example of this is the algebras attached to Mostow groups in $PSU(2,1)$: they are generated by a, b, c with defining relations

$$P(a) = P(b) = P(c) = 0,$$

$$aba = bab, aca = cac, bcb = cbc,$$

where

$$P(x) = \prod_{j=1}^{p} (x - t_j),$$

with $t_j = e^{\frac{2\pi i j}{p} + \tau_j}$, and $p = 3, 4, 5$.

3. Hecke algebras of even subgroups of Coxeter groups

A very interesting question about the orbifold Hecke algebras $\mathcal{H}_\tau(X, G)$ is the following. Suppose instead of making the variables $\tau_{Y,k}$ formal, we introduce variables $t_{Y,k} := e^{\tau_{Y,k}}$, and define the algebra $\mathcal{H}(X, G, t)$ by the same generators and relations as $\mathcal{H}_\tau(X, G)$, but over the ring $R = \mathbb{C}[t, t^{-1}]$.

Question. Is the algebra $\mathcal{H}(X, G, t)$ algebraically flat, i.e. free as an R-module?

Algebraic flatness clearly implies the flatness of the formal deformation $\mathcal{H}_\tau(X, G)$, and is in fact a much stronger property. For this reason, it is unknown in most cases, including even the case of a finite complex reflection group (where algebraic flatness is a conjecture by Broué, Malle, and Rouquier). However, in Example 2.4 (H is a Euclidean or hyperbolic plane, and Γ acts on H), algebraic flatness can be established. This is due to the fact that in this case Γ is the group of even elements in a Coxeter group, and for such groups one can define certain deformations which are algebraically flat.

Let W be a Coxeter group of rank r, as defined above. Let W_+ be the subgroup of even elements of W. It is easy to see that W_+ is generated by the elements $a_{ij} := s_i s_j$, with defining relations

$$a_{ij}a_{ji} = 1,\ a_{ij}a_{jk}a_{ki} = 1,\ a_{ij}^{m_{ij}} = 1.$$

Define a deformation of $A_0 = \mathbb{C}[W_+]$ as follows. Introduce invertible parameters $t_{ij,k} = t_{ji,-k}^{-1}$, $k \in \mathbb{Z}/m_{ij}\mathbb{Z}$ for $m_{ij} < \infty$. Let $R = \mathbb{C}[t_{ij,k}]$, and A be the R-algebra generated by a_{ij} with defining relations

$$a_{ij}a_{ji} = 1, a_{ij}a_{jk}a_{ki} = 1,$$

$$\prod_{k=1}^{m_{ij}} (a_{ij} - t_{ij,k}) = 0.$$

For any $x \in W_+$, fix a reduced word $w(x)$ representing x. Let $T_{w(x)}$ be the element of A corresponding to this word.

Theorem 3.1. *(E.-Rains) (i) The elements $T_{w(x)}$ for $x \in W_+$ span A over R.*

(ii) These elements form a basis of A over R (so that A is algebraically flat) if and only if W has no finite parabolic subgroups of rank 3, i.e. iff for each i, j, l,

$$\frac{1}{m_{ij}} + \frac{1}{m_{jl}} + \frac{1}{m_{li}} \leq 1.$$

Note that the groups W_+ for Coxeter groups of rank 3 (with $m_{12} = p$, $m_{23} = q$, $m_{31} = r$) are triangle groups $F_{p,q,r}$ generated by rotations around vertices of a triangle with angles $\pi/p, \pi/q, \pi/r$ by twice the angle at the vertex. Such a group is finite iff this triangle lies on the sphere (rather than on the Euclidean or hyperbolic plane), i.e. iff

$$1/p + 1/q + 1/r > 1.$$

Thus the only cases when algebraic flatness of A fails are the "forbidden" triples (2,2,n),(2,3,3), (2,3,4),(2,3,5), for which, as we know, flatness fails already at the formal level.

4. About the proof of Theorem 1.1

Let us conclude by saying a few words about the proof of Theorem 1.1. We will do so in the special case when W is a finite group acting on a linear representation $\mathfrak{h}$ (which was our first example). Let $S \subset W$ be the set of reflections. For $s \in S$ let $\alpha_s \in \mathfrak{h}^*$ be an eigenvector of s with eigenvalue $\lambda_s \neq 1$, and $\alpha_s^\vee \in \mathfrak{h}$ be the eigenvector of s with eigenvalue λ_s^{-1} such that $(\alpha_s, \alpha_s^\vee) = 2$. Let $c : S \to \mathbb{C}$ be a W-equivariant function.

Definition 4.1. The rational Cherednik algebra $H_c(W, \mathfrak{h})$ is the quotient of $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = [y, y'] = 0,$$

$$[y, x] = (y, x) - \sum_{s \in S} c_s (x, \alpha_s^\vee)(y, \alpha_s) s,$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$.

Note that if $c = 0$ then $H_c(W, \mathfrak{h}) = \mathbb{C}W \ltimes D(\mathfrak{h})$. Let $\mathfrak{h}_{\mathrm{reg}}$ be the complement of the reflection hyperplanes, and $H_c(W, \mathfrak{h})_{\mathrm{loc}}$ be the localization of $H_c(W, \mathfrak{h})$ obtained by inverting the elements α_s, $s \in S$.

Theorem 4.2. *(i) (the PBW theorem) The multiplication map*

$$S\mathfrak{h}^* \otimes \mathbb{C}W \otimes S\mathfrak{h} \to H_c(W, \mathfrak{h})$$

is an isomorphism.

(ii) For any c there is an isomorphism $H_c(W,\mathfrak{h})_{\rm loc} \cong \mathbb{C}W \ltimes D(\mathfrak{h}_{\rm reg})$.

The proof of this theorem is based on representing $H_c(W,\mathfrak{h})$ by Dunkl operators.

Now let M_c be the $H_c(W,\mathfrak{h})$-module defined by $M_c = H_c(W,\mathfrak{h}) \otimes_{S\mathfrak{h}} \mathbb{C}$. Then M_c is a flat deformation of M_0, and its localization $(M_c)_{\rm loc}$ is a module over $\mathbb{C}W \ltimes D(\mathfrak{h}_{\rm reg})$ which is free of rank $|W|$ as a module over $\mathbb{C}[\mathfrak{h}_{\rm reg}]$. Hence it defines a local system on $\mathfrak{h}_{\rm reg}/W$ of rank $|W|$. The monodromy of this system is a representation of the braid group $\widetilde{W}$, and it is not hard to show that it factors through the Hecke algebra $\mathcal{H}_\tau(\mathfrak{h}, W)$ where τ is an appropriate linear function of c. This monodromy representation is a flat τ-deformation of the regular representation of W to a representation of $\mathcal{H}_\tau(\mathfrak{h}, W)$, which implies the result.

The general proof is based on similar ideas, although it is more complicated and requires defining the sheaf of Cherednik algebras attached to the action of G on X. In general, there is an obstruction to deforming M_0 to M_c, which lies in $H^2(X, \mathbb{C})$, which is why the condition on vanishing of $H^2(X, \mathbb{C})$ is needed.