THE LAURENT POLYNOMIALS $M^s_{y,w}$ IN THE HECKE ALGEBRA WITH UNEQUAL PARAMETERS

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Abstract. Let (W,S) be a Coxeter system and $\mathcal{H}$ the associated Hecke algebra with unequal parameters. The Laurent polynomials $M^s_{y,w}$ and $p_{y,w}$ for $y,w\in W$ and $s\in S$ play an important role in the representations of $\mathcal{H}$. We study the properties of $M^s_{y,w}$ and $p_{y,w}$, the relations among them, as well as with the left, right and two-sided cells of W.

In his book [5], Lusztig gave a systematic introduction to the Hecke algebras $\mathcal{H}$ associated to a Coxeter system (W,S) with unequal parameters, where the Laurent polynomials $M^s_{y,w}$ and $p_{y,w}$ for $y,w\in W$ and $s\in S$ play an important role in the structure theory and the representation theory of $\mathcal{H}$. However, owing to the lack of their explicit expressions, we know very little about the properties of $M^s_{y,w}$'s and $p_{y,w}$'s. In the present paper, we give some closed investigation for those Laurent polynomials.

We establish some criteria for the vanishing and the non-vanishing of $M^s_{y,w}$. In particular, we generalize some results of Kazhdan and Lusztig in [2].

In [5. Corollary 6.5], Lusztig showed that for any $y,w\in W$ with $sy<y<w<sw$ and $L(s)=1$, $M^s_{y,w}$ is equal to the coefficient of v^{-1} in $p_{y,w}$. In this paper, we generalize this result to unequal parameter case (see Proposition 3.1). We study the relation between

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the coefficients of v^{-1} in $p_{y,w}$ and $p_{y',w'}$, where y', y (resp. w', w) are in a left $\{s,t\}$-string for some $s, t \in S$ with $o(st) > 2$ (see Propositions 3.4, 3.9, Corollary 3.5 and Theorem 3.11).

We express $M^s_{y,w}$ in terms of $p_{\alpha,\beta}$'s modulo $\mathcal{A}_{<0}$ (see Theorem 4.1). Some properties of $M^s_{y,w}$ are deduced from such expressions.

Assume that (W, S) is an irreducible Coxeter system which is either finite or affine. Assume that $\emptyset \neq I_1 := \{s \in S \mid L(s) = 1\} \subsetneqq S$ and that $\min\{L(s) \mid s \in S \setminus I_1\} = k$. Guilhot showed in [1] that if k is greater than the length of the longest element in W_{I_1} then any two-sided cell of W_{I_1} is also a two-sided cell of W. We conjecture that any two-sided cell Ω of W_{I_1} with $a(\Omega) < k$ is also a two-sided cell of W, which strengthens Guilhot's result (see Conjecture 5.5). We verify our conjecture in the cases where $k \leqslant 2$ (see Propositions 5.3 and 5.6).

The contents of the paper are organized as follows. Section 1 is the preliminaries, we collect some concepts, notation and known results there for later use. We deduce some criteria for the vanishing and non-vanishing of $M^s_{y,w}$ in Section 2. In Section 3, we study the relation between the coefficients of v^{-1} in $p_{y,w}$ and $p_{y',w'}$, where y', y (resp. w', w) are in a left $\{s,t\}$-string for some $s, t \in S$ with $o(st) > 2$. We express $M^s_{y,w}$ in terms of $p_{\alpha,\beta}$'s modulo $\mathcal{A}_{<0}$ in Section 4. Finally, we propose a conjecture to strengthen a result of Guilhot and verify it in some special cases in Section 5.

§1. Preliminaries.

In this section, we collect some concepts and known results for later use, most of them follow Lusztig in [5].

1.1. Let (W, S) be a Coxeter system with ℓ its length function and $\leqslant$ the Bruhat-Chevalley ordering on W. An expression $w = s_1 s_2 \cdots s_r \in W$ with $s_i \in S$ is called *reduced* if $r = \ell(w)$. By a *weight function* on W, we mean a map $L : W \longrightarrow \mathbb{Z}$ satisfying that $L(s) = L(t)$ for any $s, t \in S$ conjugate in W and that $L(w) = L(s_1) + L(s_2) + \cdots +$

$L(s_r)$ for any reduced expression $w = s_1 s_2 \cdots s_r$ in W with $s_i \in S$.

Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v with integer coefficients. Denote $v_w = v^{L(w)}$ for any $w \in W$.

1.2. The Hecke algebra $\mathcal{H} := \mathcal{H}(W; L)$ of W with respect to a weight function L is, by definition, an associative algebra over $\mathcal{A}$ with $\{T_w \mid w \in W\}$ a free $\mathcal{A}$-basis, subject to the multiplication rule:

$$(T_s - v_s)(T_s + v_s^{-1}) = 0, \qquad \text{if } \ s \in S;$$

$$T_w T_y = T_{wy}, \qquad \text{if } \ \ell(wy) = \ell(w) + \ell(y).$$

1.3. Define a ring involution $a \longrightarrow \bar{a}$ of $\mathcal{A}$ by setting $\overline{\sum_i a_i v^i} = \sum_i a_i v^{-i}$ where $a_i \in \mathbb{Z}$ in the sum. Extend it to a ring involution $h \longrightarrow \bar{h}$ of $\mathcal{H}(W; L)$ by setting $\overline{\sum a_w T_w} = \sum \overline{a_w} T_{w^{-1}}^{-1}$ $(a_w \in \mathcal{A})$. Note that T_w is invertible for $w \in W$ since $T_s^{-1} = T_s + (v_s^{-1} - v_s)$ for $s \in S$.

From now on, we assume that $L(s) > 0$ for any $s \in S$.

Define $\mathcal{A}_{\leqslant m} = v^m \mathbb{Z}[v^{-1}]$ and $\mathcal{A}_{<m} = \{f \in \mathcal{A} \mid \deg f < m\}$ and $\mathcal{A}_{\geqslant m} = v^m \mathbb{Z}[v]$ and $\mathcal{A}_{>m} = \{\overline{f} \mid f \in \mathcal{A}_{<-m}\}$ for any $m \in \mathbb{Z}$ (here and later, when we use the notation " $\deg f$ " , we always regard f as a Laurent polynomial in v). By [5, Subsection 5.3], there is a unique $C_w \in \mathcal{H}(W; L)$ for each $w \in W$ such that

$$\overline{C_w} = C_w,$$

$$C_w = \sum_{y \leqslant w} p_{y,w} T_y,$$

where $p_{y,w} \in \mathcal{A}_{<0}$ for $y < w$, and $p_{w,w} = 1$ and $p_{y,w} = 0$ if $y \nleqslant w$. Moreover, $v_y^{-1} v_w p_{y,w} \in \mathbb{Z}[v^2]$.

Note that if the weight function L is constant on S, then the $p_{y,w}$'s are essentially the same as the Kazhdan-Lusztig polynomials $P_{y,w}$ defined in [2, Theorem 1.1]. For example, if $L(s)=1$ for any $s\in S$, then $P_{y,w}=v^{\ell(w)-\ell(y)}p_{y,w}\in\mathbb{Z}[v^2]$ for any $y,w\in W$. However, if L is not constant on S, then the relation between $p_{y,w}$ and $P_{y,w}$ becomes quite complicated, where the coefficients of $P_{y,w}$ are conjecturally non-negative for any $y,w\in W$ (see [2, Subsection 1.1]), while those of $p_{y,w}$ might be negative for some $y,w\in W$ (see Example 4.9).

1.4. For $y,w\in W$ and $s\in S$ with $sy<y<w<sw$, define $M^s_{y,w}\in\mathcal{A}$ recurrently by

$$\sum_{\substack{y\leqslant z<w\\ sz<z}} M^s_{z,w}p_{y,z}\equiv v_s p_{y,w} \pmod{\mathcal{A}_{<0}}, \tag{1.4.1}$$

$$\overline{M^s_{y,w}}=M^s_{y,w}. \tag{1.4.2}$$

The condition (1.4.1) determines uniquely the coefficients of v^k in $M^s_{y,w}$ for all $k\geqslant 0$; then (1.4.2) determines all the other coefficients. We have $v_s^{-1}v_y^{-1}v_wM^s_{y,w}\in\mathbb{Z}[v^2,v^{-2}]$ (see [5, Chapter 6]).

1.5. By [5, Theorem 6.6], we have, for $s\in S$ and $w\in W$, the equalities:

$$C_sC_w=\begin{cases} C_{sw}+\sum\limits_{\substack{z<w\\ sz<z}} M^s_{z,w}C_z, & \text{if } w<sw,\\ (v_s^{-1}+v_s)C_w, & \text{if } w>sw.\end{cases} \tag{1.5.1}$$

Let j be the anti-automorphism of the $\mathcal{A}$-algebra $\mathcal{H}(W;L)$ defined by $j(\sum_w a_wT_w)=\sum_w a_wT_{w^{-1}}$, where $a_w\in\mathcal{A}$. It is easily seen that $j(C_w)=C_{w^{-1}}$.

For $y,w\in W$ and $s\in S$ with $ys<y<w<ws$, define $N^s_{y,w}\in\mathcal{A}$ recurrently by

$$\sum_{\substack{y\leqslant z<w\\ zs<z}} N^s_{z,w}p_{y,z}\equiv v_s p_{y,w} \pmod{\mathcal{A}_{<0}}, \tag{1.5.2}$$

$$\overline{N^s_{y,w}}=N^s_{y,w}. \tag{1.5.3}$$

Then we can deduce by applying j that

$$N^s_{y,w}=M^s_{y^{-1},w^{-1}} \qquad \text{for any } y,w\in W. \tag{1.5.4}$$

and that

$$(1.5.5)\qquad C_wC_s=\begin{cases} C_{ws}+\sum\limits_{\substack{z<w\\ zs<z}} N^s_{z,w}C_z, & \text{if } w<ws,\\ (v_s^{-1}+v_s)C_w, & \text{if } w>ws.\end{cases}$$

(see [5, Corollary 6.7])

1.6. Define a preorder $\underset{\mathrm{L}}{\leqslant}$ (respectively, $\underset{\mathrm{R}}{\leqslant}$) on W which is transitively generated by the relation $y \underset{\mathrm{L}}{\leqslant} w$ (respectively, $\underset{\mathrm{R}}{\leqslant}$), where $w<sw$, and either $y=sw$ or $M^s_{y,w}\neq 0$ (respectively, $w<ws$, and either $y=ws$ or $N^s_{y,w}\neq 0$) holds for some $s\in S$. The equivalence relation associated to this preorder is denoted by $\underset{\mathrm{L}}{\sim}$ (respectively, $\underset{\mathrm{R}}{\sim}$). The corresponding equivalence classes in W are called *generalized left cells* (respectively, *generalized right cells*) of W. Write $y \underset{\mathrm{LR}}{\leqslant} w$ in W, if there exists a sequence of elements $y_0=y, y_1,\cdots,y_r=w$ in W with some $r\geqslant 0$ such that for every $1\leqslant i\leqslant r$, either $y_{i-1}\underset{\mathrm{L}}{\leqslant} y_i$ or $y_{i-1}\underset{\mathrm{R}}{\leqslant} y_i$ holds. The equivalence relation associated to the preorder $\underset{\mathrm{LR}}{\leqslant}$ is denoted by $\underset{\mathrm{LR}}{\sim}$ and the corresponding equivalence classes in W are called *generalized two-sided cells* of W. It is well known that for $y,w\in W$, the relation $y\underset{\mathrm{L}}{\leqslant} w$ (resp. $y\underset{\mathrm{LR}}{\leqslant} w$) holds if and only if there exists some $h\in\mathcal{H}(W;L)$ (resp. $h,h'\in\mathcal{H}(W;L)$) such that $a_y\neq 0$ in the expansion $hC_w=\sum_z a_zC_z$ (resp. $hC_wh'=\sum_z a_zC_z$), where $a_z\in\mathcal{A}$ (see [5, Subsection 8.1]).

In the subsequent discussion, we usually call the generalized left (respectively, right, two-sided) cells of W simply by left (respectively, right, two-sided) cells when no danger of confusion will cause in the context.

1.7. Following Lusztig, we state the following results:

(1) If $y,w\in W$ satisfy $y\underset{\mathrm{L}}{\leqslant} w$ (respectively, $y\underset{\mathrm{R}}{\leqslant} w$), then $\mathcal{R}(y)\supseteq\mathcal{R}(w)$ (respectively, $\mathcal{L}(y)\supseteq\mathcal{L}(w)$). In particular, if $y\underset{\mathrm{L}}{\sim} w$ (respectively, $y\underset{\mathrm{R}}{\sim} w$), then $\mathcal{R}(y)=\mathcal{R}(w)$ (respectively, $\mathcal{L}(y)=\mathcal{L}(w)$) (see [5, Lemma 8.6]).

Now assume that $y,w\in W$ and $s\in S$ satisfy $sy<y<w<sw$.

(2) $M^s_{y,w}$ is a $\mathbb{Z}$-linear combination of powers v^n with $-L(s)+1 \leqslant n \leqslant L(s)-1$ and $n \equiv L(w)-L(y)-L(s) \pmod 2$ (see [5, Proposition 6.4]).

(3) Assume that $L(s)=1$. Then $M^s_{y,w} \in \mathbb{Z}$ is equal to the coefficient of v^{-1} in $p_{y,w}$. In particular, it is 0 unless $L(w)-L(y)$ is odd (see [5, Corollary 6.5]). Note that when $L(s)=1$ for any $s \in S$, $M^s_{y,w}$ is the same as the integer $\mu(y,w)$ defined in [2, Definition 1.2]. Hence $M^s_{y,w}$ can be regarded as a generalization of the function $\mu : W \times W \to \mathbb{Z}$ to the unequal parameter case.

(4) For $y, w \in W$, $p_{y,w} \in v^{L(w)-L(y)}\mathbb{Z}[v^2, v^{-2}]$ and $p_{y,w} \equiv v^{L(y)-L(w)} \pmod{\mathcal{A}_{>L(y)-L(w)}}$ (see [5, Proposition 5.4]).

1.8. From (1.5.1), we get the following recurrence formula:

(1.8.1)
$$p_{y,w} = v_s^{\epsilon} p_{y,sw} + p_{sy,sw} - \sum_{\substack{y \leqslant z < sw \\ sz < z}} M^s_{z,sw} p_{y,z} \qquad \text{for } y < w \text{ and } sw < w;$$

(1.8.2)
$$p_{y,w} = v_s^{\epsilon} p_{y,ws} + p_{ys,ws} - \sum_{\substack{y \leqslant z < ws \\ zs < z}} N^s_{z,ws} p_{y,z} \qquad \text{for } y < w \text{ and } ws < w;$$

where $\epsilon = 1$, if $sy < y$ (respectively, $ys < y$), and -1, if $sy > y$ (respectively, $ys > y$) (see [5, The proof of Theorem 6.6]). From 1.5 and (1.8.1)-(1.8.2), we get the following results immediately:

(1) $p_{y,w} = v_s^{-1} p_{sy,w}$ if $y < sy \leqslant w$ and $sw < w$. Also, $p_{y,w} = v_s^{-1} p_{ys,w}$ if $y < ys \leqslant w$ and $ws < w$.

When W is finite, let w_0 be the longest element of W. Then $p_{y,w_0} = v^{-1}_{yw_0}$ for any $y \in W$.

(2) $p_{y,w} = v_s^{-1}$ if $\ell(w) = \ell(y)+1$ and if y can be obtained from a reduced expression of w by deleting a factor $s \in S$.

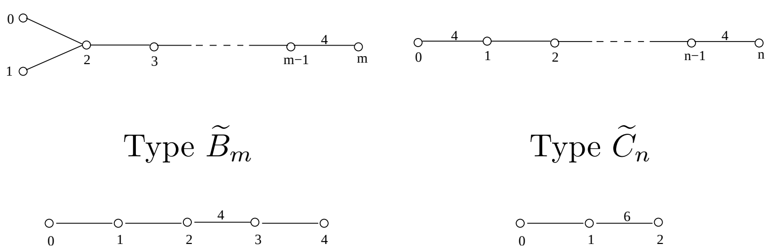
(3) In the case of (2), if $ry < y < w < rw$ (respectively, $yr < y < w < wr$) for some $r \in S$, then

$$M^r_{y,w} \text{ (respectively, } N^r_{y,w}) = \begin{cases} 0, & \text{if } v_r < v_s, \\ 1, & \text{if } v_r = v_s, \\ v_s v_r^{-1} + v_s^{-1} v_r, & \text{if } v_r > v_s. \end{cases} \tag{1.8.3}$$

(4) If $y < w$, $sw < w$ and $y \not< sw$, then $p_{y,w} = p_{sy,sw}$ (note that in this case, we have $sy < y$).

Note that $\deg p_{y,w} \leqslant -1$ for any $y < w$ in W.

1.9. In Figure 1, we display the Coxeter graphs of types $\widetilde{B}_m$, $\widetilde{C}_n$, $\widetilde{F}_4$, $\widetilde{G}_2$ for $m > 2$ and $n > 1$.



Type $\widetilde{B}_m$ Type $\widetilde{C}_n$

Type $\widetilde{F}_4$ Type $\widetilde{G}_2$

Figure 1. Coxeter graphs

§2. Some criteria for the vanishing and the non-vanishing of $M^s_{y,w}$.

In this section, we establish some criteria for the vanishing and the non-vanishing of $M^s_{y,w}$. In particular, we generalize some results of Kazhdan and Lusztig in [2, Subsection 2.3 (e)-(f)].

Lemma 2.1. *Assume that $y, w \in W$ and $s, t \in S$ satisfy $sy < y < w < sw$ and $L(s) < L(t)$ and $t \in (\mathcal{L}(w) \setminus \mathcal{L}(y)) \cup (\mathcal{R}(w) \setminus \mathcal{R}(y))$. Then $M^s_{y,w} = 0$.*

Proof. If $M^s_{y,w} \neq 0$ then $y \leqslant_L w$ and hence $\mathcal{R}(y) \supseteq \mathcal{R}(w)$ by 1.7 (1). So $M^s_{y,w} = 0$ in the case of $t \in \mathcal{R}(w) \setminus \mathcal{R}(y)$. Now assume $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$. Apply induction on

$k = \ell(w) - \ell(y) \geqslant 1$. When $k = 1$, we have $M^s_{y,w} = 0$ by (1.8.3). Now assume $k > 1$. To show $M^s_{y,w} = 0$, it is enough to show

$$\sum_{\substack{y<z<w\\ sz<z}} M^s_{z,w} p_{y,z} \equiv v_s p_{y,w} \ (\mathrm{mod}\ \mathcal{A}_{<0}) \tag{2.1.1}$$

by (1.4.1)-(1.4.2). We have $v_s p_{y,w} = v_s v_t^{-1} p_{ty,w} \in \mathcal{A}_{<0}$. Consider the term $f_z = M^s_{z,w} p_{y,z}$ occurring in (2.1.1). If $tz > z$ then $M^s_{z,w} = 0$ by inductive hypothesis, hence $f_z = 0$. If $tz < z$, then $f_z = v_t^{-1} M^s_{z,w} p_{ty,z} \in \mathcal{A}_{<0}$ by 1.7 (2) and 1.8 (1) and the assumption $L(s) < L(t)$. This proves the equation in (2.1.1). So our result follows by induction. $\square$

Lemma 2.2. *Let $w, y \in W$ and $s, t \in S$ satisfy $st = ts$ and $sy < y < w < sw$ and $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$. Then $M^s_{y,w} = 0$.*

Proof. By (1.5.1), we have $C_t C_s C_w = C_s C_t C_w = (v_t^{-1} + v_t) C_s C_w$ and hence

$$(v_t^{-1} + v_t) C_s C_w = C_t C_s C_w = C_t C_{sw} + \sum_{\substack{z<w\\ sz<z}} M^s_{z,w} C_t C_z. \tag{2.2.1}$$

Since the right hand-side of (2.2.1) is an $\mathcal{A}$-linear combination of C_u with $tu < u$, the coefficient $(v_t^{-1} + v_t) M^s_{y,w}$ of C_y on the left hand-side of (2.2.1) must be zero, hence $M^s_{y,w} = 0$. $\square$

Proposition 2.3. *Suppose that $y, w \in W$ and $s \in S$ satisfy that*

(i) $sy < y < w < sw$;

(ii) there is some $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$ satisfying one of the following three conditions: (a) $L(t) > L(s)$; (b) $st = ts$; (c) $st \neq ts$ and $L(t) = L(s)$ and $y \neq tw$.

Then $M^s_{y,w} = 0$.

Proof. We have $M^s_{y,w} = 0$ in the case where $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$ satisfies either $L(t) > L(s)$ or $st = ts$ by Lemmas 2.1 and 2.2, respectively. Now assume that $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$ satisfies $L(t) = L(s)$ and $st \neq ts$ and $y \neq tw$.

We have $\ell(w) - \ell(y) > 1$ by the assumptions of $y < w$ and $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$ and $w \neq ty$. Apply induction on $\ell(w) - \ell(y) \geqslant 2$. If $\ell(w) - \ell(y) = 2$, then (1.4.1) becomes $M^s_{y,w} \equiv v_s p_{y,w} = p_{ty,w} \equiv 0 \,(\bmod\ \mathcal{A}_{<0})$, hence $M^s_{y,w} = 0$ by (1.4.2). Now assume $\ell(w) - \ell(y) > 2$. By inductive hypothesis, we have $M^s_{z,w} = 0$ for any $z \in W$ with $y < z < w$ and $sz < z$ and $tz > z$ and $z \neq tw$. By (1.4.1), we have

$$(2.3.1)\quad M^s_{y,w} + \epsilon(w,s,t) M^s_{tw,w} p_{y,tw} + \sum_{\substack{ty \leqslant z < w \\ sz < z \\ tz < z}} M^s_{z,w} v_t^{-1} p_{ty,z} \equiv v_s v_t^{-1} p_{ty,w} \ (\bmod\ \mathcal{A}_{<0}),$$

where $\epsilon(w,s,t)$ is 1 if $s \in \mathcal{L}(tw)$ and 0 otherwise. Since $L(t) = L(s)$, the terms $M^s_{z,w} v_t^{-1} p_{ty,z}$ in the above sum and the terms $\epsilon(w,s,t) M^s_{tw,w} p_{y,tw}$, $v_s v_t^{-1} p_{ty,w}$ are all contained in $\mathcal{A}_{<0}$ by 1.7 (2) and by the assumption of $y \neq tw$. This implies that $M^s_{y,w} = 0$ by (2.3.1). Our result follows by induction. □

Corollary 2.4. *Suppose that $y, w \in W$ and $s \in S$ satisfy $M^s_{y,w} \neq 0$. Then any $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$ satisfies that $st \neq ts$ and $L(s) \geqslant L(t)$ and that $w = ty$ when $L(s) = L(t)$.*

Proof. The condition $M^s_{y,w} \neq 0$ implies $sy < y < w < sw$. So our result is a direct consequence of Proposition 2.3. □

Corollary 2.5. *Let (W, S) be an irreducible finite or affine Coxeter group with $W \neq \widetilde{C}_2$. Suppose that $y, w \in W$ and $s \in S$ satisfy that $sy < y < w < sw$ and $|\mathcal{L}(w)| > |\mathcal{L}(y)| = 1$. Then $M^s_{y,w} = 0$.*

Proof. We argue by contradiction. Suppose $M^s_{y,w} \neq 0$. By the classification of irreducible finite and affine Coxeter groups, there must exist some $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$ such that either $st = ts$ or $L(t) \geqslant L(s)$ by the assumptions of $|\mathcal{L}(w)| > |\mathcal{L}(y)| = 1$ and

$W \neq \widetilde{C}_2$. By Corollary 2.4 and the assumption $|\mathcal{L}(y)| = 1$, any $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$ satisfies $st \neq ts$ and that either $L(s) > L(t)$, or $w = ty$ with $L(s) = L(t)$. However, there exists at most one element $t \in S$ satisfying those conditions for a given $s \in S$, i.e., $|\mathcal{L}(w) \setminus \mathcal{L}(y)| \leqslant 1$. Thus $s \in \mathcal{L}(w)$ by the assumption $|\mathcal{L}(w)| > 1$. But this contradicts the assumption of $w < sw$. So $M^s_{y,w} = 0$. $\square$

Note that the assumption of $W \neq \widetilde{C}_2$ is necessary for the assertion $M^s_{y,w} = 0$ in Corollary 2.5. For otherwise, assume $W = \widetilde{C}_2$ and $L(s_1) > L(s_0) + L(s_2)$ (see Figure 1). Let $y = s_1$ and $w = s_0s_2s_1$. Then $\mathcal{L}(y) = \{s_1\}$ and $\mathcal{L}(w) = \{s_0, s_2\}$ and $M^{s_1}_{y,w} = v^{L(s_1)-L(s_0)-L(s_2)} + v^{-L(s_1)+L(s_0)+L(s_2)} \neq 0$ by (1.4.1)-(1.4.2).

In the case where the weight function L is constant on S, we see by [2, Subsection 2.3 (e)] that for any $y, w \in W$ with $s \in \mathcal{L}(y) \setminus \mathcal{L}(w)$ and $\mathcal{L}(w) \setminus \mathcal{L}(y) \neq \emptyset$, we have $M^s_{y,w} \neq 0$ if and only if $w = ty$ and $\mathcal{L}(w) \setminus \mathcal{L}(y) = \{t\}$. When the equivalent conditions hold, we have $st \neq ts$. We shall extend this result to the case where L is not constant on S.

Proposition 2.6. *Suppose that $y < w$ in W and $s, t \in S$ satisfy that $t \in \mathcal{L}(w) \setminus \mathcal{L}(y)$ and $s \in \mathcal{L}(y) \setminus \mathcal{L}(w)$ and $L(t) \geqslant L(s)$. Then $M^s_{y,w} \neq 0$ if and only if $w = ty$ and $L(s) = L(t)$. When the equivalent conditions hold, we have $st \neq ts$.*

Proof. The implication "$\Longleftarrow$" follows directly by (1.4.1), while the implication "$\Longrightarrow$" is a direct consequence of Corollary 2.4. $\square$

The right-handed version of Proposition 2.6 also holds.

Proposition 2.7. *Suppose that $y < w$ in W and $s, t \in S$ satisfy that $t \in \mathcal{R}(w) \setminus \mathcal{R}(y)$ and $s \in \mathcal{R}(y) \setminus \mathcal{R}(w)$ and $L(t) \geqslant L(s)$. Then $N^s_{y,w} \neq 0$ if and only if $w = yt$ and $L(s) = L(t)$. When the equivalent conditions hold, we have $st \neq ts$.*

Propositions 2.6 and 2.7 can be regarded as a generalization of the results in [2, Subsection 2.3 (e)-(f)].

§3. Some relations between the coefficients in $M^s_{y,w}$ and in $p_{y,w}$.

In [5. Corollary 6.5], Lusztig showed that for any $y, w \in W$ with $sy < y < w < sw$ (respectively, $ys < y < w < ws$) and $L(s) = 1$, $M^s_{y,w}$ (respectively, $N^s_{y,w}$) is equal to the coefficient of v^{-1} in $p_{y,w}$. We shall generalize this result in the present section.

For any $w, x, y \in W$, the notation $w = x{\cdot}y$ means that $w = xy$ and $\ell(w) = \ell(x)+\ell(y)$.

Proposition 3.1. *Let $y, w \in W$ and $s \in S$ satisfy $sy < y < w < sw$.*

(1) The coefficient of v^{-1} in $p_{y,w}$ is equal to the coefficient of $v^{L(s)-1}$ in $M^s_{y,w}$.

(2) If the coefficient of v^{-1} in $p_{y,w}$ is non-zero, then $M^s_{y,w} \neq 0$.

Proof. By 1.7 (2), we have $\deg M^s_{y,w} \leqslant L(s) - 1$. Consider the terms in (1.4.1). We see that $\deg M^s_{z,w} p_{y,z} \leqslant L(s) - 2$ for any $z \in W$ with $y < z < w$ and $sz < z$. Hence the coefficient of $v^{L(s)-1}$ in $M^s_{y,w}$ is equal to the coefficient of v^{-1} in $p_{y,w}$ by (1.4.1). This proves (1). Then (2) is an immediate consequence of (1). □

3.2. Given $s, t \in S$ with $o(st) = m > 2$ and $L(s) = L(t)$. A sequence of elements in W of the form

(3.2.1) $$\xi : \underbrace{sy,\ tsy,\ stsy,\ \ldots}_{m-1 \text{ terms}} \qquad (\text{respectively, } \underbrace{ys,\ yst,\ ysts,\ \ldots}_{m-1 \text{ terms}})$$

is called a *left* $\{s, t\}$*-string* or just a *left string* (respectively, a *right* $\{s, t\}$*-string* or just a *right string*) if $y \in W$ satisfies $\mathcal{L}(y) \cap \{s, t\} = \emptyset$ (respectively, $\mathcal{R}(y) \cap \{s, t\} = \emptyset$). Clearly, when (3.2.1) is a left (respectively, right) $\{s, t\}$-string, the sequence

(3.2.2) $$\xi' : \underbrace{ty,\ sty,\ tsty,\ \ldots}_{m-1 \text{ terms}} \qquad (\text{respectively, } \underbrace{yt,\ yts,\ ytst,\ \ldots}_{m-1 \text{ terms}})$$

is also a left (respectively, right) $\{s, t\}$-string.

Clearly, any left (respectively, right) $\{s, t\}$-string is wholly contained in some left (respectively, right) cell of W.

3.3. For any $s, t \in S$ with $o(st) > 2$, denote by $D_L(s,t)$ (respectively $D_R(s,t)$) the set of all elements w in W such that $|\mathcal{L}(w) \cap \{s,t\}| = 1$ (respectively, $|\mathcal{R}(w) \cap \{s,t\}| = 1$). If $w \in D_L(s,t)$, then the left $\{s,t\}$-string ξ_w containing w is wholly contained in $D_L(s,t)$; we denote the set $\{sw, tw\} \cap D_L(s,t)$ by ${}^*w$, which contains either one or two elements according to whether or not w is a terminal term in the string ξ_w. In particular, when $o(st) = 3$, ${}^*w$ consists of a single element (in this case, we identify ${}^*w$ with the element it contains) and the map $w \mapsto {}^*w$ is an involution of $D_L(s,t)$, called *a left* $\{s,t\}$*-star operation* (or a *left star operation* in short). Similarly, we have a map $w \mapsto w^*$ of $D_R(s,t)$: $w^* = D_R(s,t) \cap \{ws, wt\}$, called *a right* $\{s,t\}$*-star operation* (or a right star operation in short) if $o(st) = 3$. Let $\langle s,t \rangle$ be the subgroup of W generated by s, t.

Star operations on a Coxeter group were first introduced by Kazhdan and Lusztig in [2, Section 4] in equal parameter case (i.e., when L is constant on S). Here we shall generalize them to the unequal parameter case (i.e., when L is not constant on S).

In the subsequent discussion of this section, the notation " $\equiv$ " always stands for the congruence relation modulo $\mathcal{A}_{<-1}$ unless otherwise specified (note the difference from the same symbol in Section 4, where it will be modulo $\mathcal{A}_{<0}$). We usually omit the symbol " (mod $\mathcal{A}_{<-1}$) " after the notation " $\equiv$ " when no danger of confusion in the context.

The following result generalizes the result in [2, Theorem 4.2] to the unequal parameter case.

Proposition 3.4. *Let $s, t \in S$ satisfy $o(st) = 3$ (so $L(s) = L(t)$). Let $y < w$ in W.*

Assume $y, w \in D_L(s,t)$.

(1) If $yw^{-1} \notin \langle s,t \rangle$, then $p_{y,w} \equiv p_{{}^*y,{}^*w}$; in particular, $p_{y,w} \not\equiv 0$ if and only if $p_{{}^*y,{}^*w} \not\equiv 0$.

(2) If $yw^{-1} \in \langle s,t \rangle$, then $p_{y,w} = p_{{}^*w,{}^*y} = v_s^{-1}$.

Now assume $y, w \in D_R(s,t)$.

(3) If $y^{-1}w\notin\langle s,t\rangle$*, then* $p_{y,w}\equiv p_{y^*,w^*}$*; in particular,* $p_{y,w}\not\equiv 0$ *if and only if* $p_{y^*,w^*}\not\equiv 0$.

(4) If $y^{-1}w\in\langle s,t\rangle$*, then* $p_{y,w}=p_{w^*,y^*}=v_s^{-1}$.

Proof. By symmetry, it is enough to prove (1)-(2). When $yw^{-1}\in\langle s,t\rangle$ and $y<w$ in $D_L(s,t)$, we have $\ell(w)=\ell(y)+1$ and $\ell({}^*y)=\ell({}^*w)+1$, hence $p_{y,w}=p_{{}^*w,{}^*y}=v_s^{-1}$ by 1.8 (2). This proves (2). In the remainder of the proof, we shall assume that $y,w\in D_L(s,t)$ satisfy $yw^{-1}\notin\langle s,t\rangle$. When $\{s,t\}\cap(\mathcal{L}(y)\cap\mathcal{L}(w))=\emptyset$, we have $\{s,t\}\cap(\mathcal{L}({}^*y)\cap\mathcal{L}({}^*w))=\emptyset$ and hence $p_{y,w}\equiv 0\equiv p_{{}^*y,{}^*w}$ by 1.8 (1). Now assume $\{s,t\}\cap(\mathcal{L}(y)\cap\mathcal{L}(w))\neq\emptyset$.

There are two cases to consider.

Case 1: $y=sty_0$ and $w=stw_0$ for some $y_0\neq w_0$ in W with $s,t\notin\mathcal{L}(y_0)\cup\mathcal{L}(w_0)$.

By (1.8.1), we have

$$p_{sty_0,stw_0}=p_{ty_0,tw_0}+v_sp_{sty_0,tw_0}-\sum_{\substack{sty_0\leqslant z<tw_0\\ sz<z}}M^s_{z,tw_0}p_{sty_0,z}. \tag{3.4.1}$$

By Proposition 2.6, we have $M^s_{z,tw_0}\neq 0$ for z in the sum of (3.4.1) only if $z=stsz_0$ for some $z_0\in W$ with $s,t\notin\mathcal{L}(z_0)$; in the latter case, we have $M^s_{stsz_0,tw_0}p_{sty_0,stsz_0}=v_s^{-1}M^s_{stsz_0,tw_0}p_{tsty_0,stsz_0}$ by 1.8 (1) and the assumption $L(s)=L(t)$. By 1.7 (2), we see that $v_s^{-1}M^s_{stsz_0,tw_0}p_{tsty_0,stsz_0}\not\equiv 0$ only if $z_0=y_0$. Since

$$v_sp_{sty_0,tw_0}-v_s^{-1}M^s_{tsty_0,tw_0}=p_{tsty_0,tw_0}-v_s^{-1}M^s_{tsty_0,tw_0}\equiv 0$$

by 1.8 (1) and Proposition 3.1 (1), we get $p_{sty_0,stw_0}\equiv p_{ty_0,tw_0}=p_{{}^*y,{}^*w}$ by (3.4.1).

Case 2: $y=sy_0$ and $w=stw_0$ for some $y_0\neq w_0$ in W with $s,t\notin\mathcal{L}(y_0)\cup\mathcal{L}(w_0)$.

By (1.8.1), we have

$$p_{sy_0,stw_0}=p_{y_0,tw_0}+v_sp_{sy_0,tw_0}-\sum_{\substack{sy_0\leqslant z<tw_0\\ sz<z}}M^s_{z,tw_0}p_{sy_0,z}. \tag{3.4.2}$$

By Proposition 2.6, we have $M^s_{z,tw_0}\neq 0$ for z in the sum of (3.4.2) only if $z=stsz_0$ for some $z_0\in W$ with $s,t\notin\mathcal{L}(z_0)$; in the latter case, we have $M^s_{stsz_0,tw_0}p_{sy_0,stsz_0}=v_s^{-2}M^s_{stsz_0,tw_0}p_{stsy_0,stsz_0}\equiv 0$ by 1.8 (1) and 1.7 (2) and the assumption $L(s)=L(t)$. On the other hand, we have $p_{y_0,tw_0}=v_t^{-1}p_{ty_0,tw_0}\equiv 0$ by the assumption of $y_0\neq w_0$ (i.e., $yw^{-1}\notin\langle s,t\rangle$). So $p_{sy_0,stw_0}\equiv v_sp_{sy_0,tw_0}=p_{tsy_0,tw_0}=p_{^*y,^*w}$ by (3.4.2) and 1.8 (1) and the assumption $L(s)=L(t)$.

This proves (1) and so our proof is complete. $\square$

Corollary 3.5. *Suppose that $s,t\in S$ satisfy $o(st)=3$ (hence $L(s)=L(t)$).*

(1) Assume that $y,w\in D_L(s,t)$ and $r\in S$ satisfy $yr<y<w<wr$ and $yw^{-1}\notin\langle s,t\rangle$. Then the coefficient of $v^{L(r)-1}$ in $N^r_{y,w}$ is equal to that in $N^r_{^*y,^*w}$. If the coefficient of v^{-1} in $p_{y,w}$ is non-zero, then $N^r_{y,w}\neq 0\neq N^r_{^*y,^*w}$.

(2) If $y,w\in D_L(s,t)$ and the coefficient of v^{-1} in $p_{y,w}$ or in $p_{w,y}$ is non-zero, then $y\underset{\mathrm{R}}{\sim}w$ if and only if $^*y\underset{\mathrm{R}}{\sim}{}^*w$.

(3) Assume that $y,w\in D_R(s,t)$ and $r\in S$ satisfy $ry<y<w<rw$ and $y^{-1}w\notin\langle s,t\rangle$. Then the coefficient of $v^{L(r)-1}$ in $M^r_{y,w}$ is equal to that in $M^r_{y^*,w^*}$; if the coefficient of v^{-1} in $p_{y,w}$ is non-zero, then $M^r_{y,w}\neq 0\neq M^r_{y^*,w^*}$.

(4) If $y,w\in D_R(s,t)$ and the coefficient of v^{-1} in $p_{y,w}$ or in $p_{w,y}$ is non-zero, then $y\underset{\mathrm{L}}{\sim}w$ if and only if $y^*\underset{\mathrm{L}}{\sim}w^*$.

Proof. By symmetry, we need only to prove (1)-(2). By the right-handed version of Proposition 3.1, we see that for any $y,w\in W$ with $yr<y<w<wr$, the coefficient of $v^{L(r)-1}$ in $N^r_{y,w}$, resp., $N^r_{^*y,^*w}$, is equal to the coefficient of v^{-1} in $p_{y,w}$, resp., $p_{^*y,^*w}$. So (1) follows by Proposition 3.4.

Now let us show (2). By symmetry and Proposition 3.4, we need only to show that if $y\underset{\mathrm{R}}{\leqslant}w$ then $^*y\underset{\mathrm{R}}{\leqslant}{}^*w$. To do so, we need only to consider the following two special cases of $y\underset{\mathrm{R}}{\leqslant}w$:

(a) There exists some $r \in \mathcal{R}(y)\setminus\mathcal{R}(w)$ with the coefficient of $v^{L(r)-1}$ in $N^r_{y,w}$ non-zero;

(b) $y = w \cdot r$ for some $r \in S$ with $L(r) = 1$.

We see that the coefficient of v^{-1} in $p_{y,w}$ or $p_{w,y}$ is non-zero in either of the cases (a) and (b) by Proposition 3.1. We must show that we are in the case either (a) or (b) with ${}^*y$, ${}^*w$ in the places of y, w respectively. By 1.7 (1), we may assume $s \in \mathcal{L}(y) \cap \mathcal{L}(w)$ and $t \notin \mathcal{L}(y) \cup \mathcal{L}(w)$ since $y, w \in D_L(s,t)$ and $\mathcal{L}(y) \supseteq \mathcal{L}(w)$ for the sake of definiteness. By Proposition 3.4, we see that if $yw^{-1} \notin \langle s,t\rangle$ then the coefficient of v^{-1} in $p_{{}^*y,{}^*w}$ is non-zero and that if $yw^{-1} \in \langle s,t\rangle$ then ${}^*w < {}^*y$ and $p_{{}^*w,{}^*y} = p_{y,w} = v_s^{-1} = v^{-1}$ by our assumption. That is, the coefficient of v^{-1} in $p_{{}^*y,{}^*w}$ or in $p_{{}^*w,{}^*y}$ is non-zero in either case. In case (a), we see by Propositions 3.1 and 3.4 that the coefficient of $v^{L(r)-1}$ in $N^r_{{}^*y,{}^*w}$ is non-zero if $yw^{-1} \notin \langle s,t\rangle$, and that ${}^*y = {}^*w \cdot r$ if $yw^{-1} \in \langle s,t\rangle$, where $y = sy_0$ and $w = sty_0$ with $y_0 \in W$ satisfying $\mathcal{L}(y_0) \cap \{s,t\} = \emptyset$ and $sy_0 = y_0 r$. In case (b), we have either $w = sy_0$, $y = sy_0 r$, or $w = sty_0$, $y = sty_0 r$, where $y_0 \in W$ satisfies $\mathcal{L}(y_0) \cap \{s,t\} = \emptyset$; in either case, we have $L(r) = 1$ by our hypothesis. First assume $w = sy_0$, $y = sy_0 r$. Then ${}^*w = tsy_0$, ${}^*y = tsy_0 r$ if $y_0 r \neq ty_0$, and ${}^*w = tsy_0$, ${}^*y = ty_0$ if $y_0 r = ty_0$; in the latter case, we have $L(t) = L(r) = 1$ and $N^r_{{}^*y,{}^*w} = 1$. Next assume $w = sty_0$, $y = sty_0 r$. Then ${}^*w = ty_0$, ${}^*y = ty_0 r$. Thus either ${}^*y = {}^*w \cdot r$ or the coefficient of $v^{L(r)-1}$ in $N^r_{{}^*y,{}^*w}$ is non-zero. So we are in case either (a) or (b) with ${}^*y$, ${}^*w$ in the places of y, w respectively. $\square$

3.6. Define a preorder $\leqslant'_R$ on W as follows. Write $x \leqslant'_R y$ in W, if there exists a sequence of elements $x_0 = x, x_1, ..., x_t = y$ in W with some $t \geqslant 0$ such that for every $1 \leqslant i \leqslant t$, either $x_{i-1} = x_i \cdot r$ for some $r \in S$ with $L(r) = 1$, or $\deg N^r_{x_{i-1},x_i} = L(r) - 1$ for some $r \in \mathcal{R}(x_{i-1})\setminus\mathcal{R}(x_i)$. Write $x \sim'_R y$ if $x \leqslant'_R y \leqslant'_R x$. This defines an equivalence relation on W, the corresponding equivalence classes of W are called *strictly right cells.* It is easily seen that any right cell of W is a union of some strictly right cells. Also, for any $s, t \in S$ with $o(st) = 3$, the set $D_L(s,t)$ is a union of some strictly right cells by 1.7

(1). A left $\{s,t\}$-star operation on $D_L(s,t)$ gives rise to a permutation on those strictly right cells by Corollary 3.5.

Remark 3.7. For $s,t,r\in S$ with $o(st)=3$, let $y,w\in D_L(s,t)$ satisfy $yr<y<w<wr$, then the coefficient of v^{-1} in $p_{y,w}$ is equal to that in $p_{{}^*y,{}^*w}$ or in $p_{{}^*w,{}^*y}$ by Proposition 3.4. Thus, once we know that the coefficient of v^{-1} in $p_{y,w}$ is non-zero, let y',w' be obtained from y,w respectively by applying the same sequence of left $\{s,t\}$-star operations with the pairs $\{s,t\}$, $o(st)=3$, varying over S, in other words, there exist two sequences of elements $y_0=y,y_1,...,y_u=y'$ and $w_0=w,w_1,...,w_u=w'$ in W with some $u\geqslant 0$ such that for every $1\leqslant i\leqslant u$, the elements y_i,w_i are obtained from y_{i-1},w_{i-1}, respectively, by a left $\{s_i,t_i\}$-star operation for some $s_i,t_i\in S$ with $o(s_it_i)=3$. We can conclude that the coefficient of v^{-1} in $p_{y',w'}$ or $p_{w',y'}$ is non-zero by Corollary 3.5 (1). Since $\mathcal{R}(y')=\mathcal{R}(y)$ and $\mathcal{R}(w')=\mathcal{R}(w)$ by Corollary 3.5 (4) and 1.7 (1), we have $r\in\mathcal{R}(y')\setminus\mathcal{R}(w')$ and hence either $N^r_{y',w'}\neq 0$ or $y'=w'\cdot r$ by Propositions 3.1 (1) and 2.7.

3.8. Let $s,t\in S$ satisfy $o(st)=4$ and $L(s)=L(t)$. Let $y_0\neq w_0$ in W satisfy $s,t\notin\mathcal{L}(y_0)\cup\mathcal{L}(w_0)$. For $1\leqslant i,j\leqslant 3$ and $r\in\{s,t\}$, denote by a^r_{ij} the coefficient of v^{-1} in the polynomial p_{xy_0,zw_0} for some $x,z\in\langle s,t\rangle$ with $(\ell(x),\ell(z))=(i,j)$ and $r\in\mathcal{L}(x)\cap\mathcal{L}(z)$, and let $\bar{r}$ satisfy $\{r,\bar{r}\}=\{s,t\}$.

We shall generalize a result in [4, Subsection 10.4] to the unequal parameter case.

Proposition 3.9. *Let $y_0\neq w_0$ in W and $s,t\in S$ satisfy $o(st)=4$ and $L(s)=L(t)$ and $s,t\notin\mathcal{L}(y_0)\cup\mathcal{L}(w_0)$. Let a^r_{ij}* ($r\in\{s,t\}$ and $1\leqslant i,j\leqslant 3$) *be defined as in 3.8.*

(a) $a^r_{11}=a^r_{33}$ *and* $a^r_{13}=a^r_{31}$.

(b) $a^r_{22}=a^{\bar{r}}_{11}+a^{\bar{r}}_{31}$.

(c) $a^r_{12}=a^{\bar{r}}_{21}=a^{\bar{r}}_{23}=a^r_{32}$.

Proof. (1) $a^s_{33}+a^s_{31}=a^t_{22}$.

(3.9.1) $$p_{stsy_0,stsw_0} = p_{tsy_0,tsw_0} + v_s p_{stsy_0,tsw_0} - \sum_{\substack{stsy_0 \leqslant z < tsw_0 \\ sz < z}} M^s_{z,tsw_0} p_{stsy_0,z}.$$

By Proposition 2.6, we have $M^s_{z,tsw_0} \neq 0$ for z in the sum of (3.9.1) only if either $z = sw_0$ or $z = ststz_0$ for some $z_0 \in W$ with $s,t \notin \mathcal{L}(z_0)$. When $z = sw_0$, we have $M^s_{z,tsw_0} p_{stsy_0,z} = p_{stsy_0,sw_0}$; when $z = ststz_0$, we have, by 1.8 (1) and Proposition 3.1 (1), that $M^s_{z,tsw_0} p_{stsy_0,z} = v_s^{-1} M^s_{ststz_0,tsw_0} p_{tstsy_0,ststz_0}$ and the assumption $L(s) = L(t)$, which is not congruent to 0 only if $z_0 = y_0$. Since

$$v_s p_{stsy_0,tsw_0} - v_s^{-1} M^s_{ststy_0,tsw_0} = p_{tstsy_0,tsw_0} - v_s^{-1} M^s_{ststy_0,tsw_0} \equiv 0$$

by 1.8 (1) and Proposition 3.1 (1), we get $p_{stsy_0,stsw_0} \equiv p_{tsy_0,tsw_0} - p_{stsy_0,sw_0}$ by (3.9.1).

(2) $a^s_{22} = a^t_{11} + a^t_{31}$.

(3.9.2) $$p_{sty_0,stw_0} = p_{ty_0,tw_0} + v_s p_{sty_0,tw_0} - \sum_{\substack{sty_0 \leqslant z < tw_0 \\ sz < z}} M^s_{z,tw_0} p_{sty_0,z}.$$

By Proposition 2.6, we have $M^s_{z,tw_0} \neq 0$ for z in the sum of (3.9.2) only if $z = ststz_0$ for some $z_0 \in W$ with $s,t \notin \mathcal{L}(z_0)$; in the latter case, we have $M^s_{ststz_0,tw_0} p_{sty_0,ststz_0} = v_s^{-2} M^s_{ststz_0,tw_0} p_{ststy_0,ststz_0} \equiv 0$ by 1.8 (1) and the assumption $L(s) = L(t)$ and 1.7 (2). Since $v_s p_{sty_0,tw_0} = p_{tsty_0,tw_0}$ by 1.8 (1), we get $p_{sty_0,stw_0} \equiv p_{ty_0,tw_0} + p_{tsty_0,tw_0}$ by (3.9.2).

(3) $a^s_{13} + a^s_{11} = a^t_{22}$.

(3.9.3) $$p_{sy_0,stsw_0} = p_{y_0,tsw_0} + v_s p_{sy_0,tsw_0} - \sum_{\substack{sy_0 \leqslant z < tsw_0 \\ sz < z}} M^s_{z,tsw_0} p_{sy_0,z}.$$

By Proposition 2.6, we have $M^s_{z,tsw_0} \neq 0$ for z in the sum of (3.9.3) only if either $z = sw_0$ or $z = ststz_0$ for some $z_0 \in W$ with $s,t \notin \mathcal{L}(z_0)$. We have $M^s_{z,tsw_0} p_{sy_0,z} =$

p_{sy_0,sw_0} if $z = sw_0$ and $M^s_{z,tsw_0} p_{sy_0,z} = v_s^{-3} M^s_{ststz_0,tsw_0} p_{tstsy_0,ststz_0} \equiv 0$ if $z = ststz_0$ by 1.8 (1) and the assumption $L(s) = L(t)$ and 1.7 (2). Since $v_s p_{sy_0,tsw_0} = p_{tsy_0,tsw_0}$ and $p_{y_0,tsw_0} = v_s^{-1} p_{ty_0,tsw_0} \equiv 0$ by 1.8 (1), we get $p_{sy_0,stsw_0} \equiv p_{tsy_0,tsw_0} - p_{sy_0,sw_0}$ by (3.9.3).

(4) $a^s_{32} = a^t_{21}$.

$$p_{stsy_0,stw_0} = p_{tsy_0,tw_0} + v_s p_{stsy_0,tw_0} - \sum_{\substack{stsy_0 \leqslant z < tw_0 \\ sz<z}} M^s_{z,tw_0} p_{stsy_0,z}. \tag{3.9.4}$$

By Proposition 2.6, we have $M^s_{z,tw_0} \neq 0$ for z in the sum of (3.9.4) only if $z = ststz_0$ for some $z_0 \in W$ with $s,t \notin \mathcal{L}(z_0)$; in the latter case, we have $M^s_{z,tw_0} p_{stsy_0,z} = v_s^{-1} M^s_{ststz_0,tw_0} p_{tstsy_0,ststz_0} \not\equiv 0$ only if $z_0 = y_0$ and $\deg p_{ststy_0,tw_0} = -1$ by 1.8 (1) and the assumption $L(s) = L(t)$ and 1.7 (2). Since $v_s p_{stsy_0,tw_0} - v_s^{-1} M^s_{tstsy_0,tw_0} = p_{tstsy_0,tw_0} - v_s^{-1} M^s_{tstsy_0,tw_0} \equiv 0$ by 1.8 (1) and Proposition 3.1 (1), we get $p_{stsy_0,stw_0} \equiv p_{tsy_0,tw_0}$ by (3.9.4).

(5) $a^s_{23} = a^t_{12}$.

$$p_{sty_0,stsw_0} = p_{ty_0,tsw_0} + v_s p_{sty_0,tsw_0} - \sum_{\substack{sty_0 \leqslant z < tsw_0 \\ sz<z}} M^s_{z,tsw_0} p_{sty_0,z}. \tag{3.9.5}$$

By Proposition 2.6, we have $M^s_{z,tsw_0} \neq 0$ for z in the sum of (3.9.5) only if either $z = sw_0$ or $z = tstsz_0$ for some $z_0 \in W$ with $s,t \notin \mathcal{L}(z_0)$. We have $M^s_{z,tsw_0} p_{sty_0,z} = p_{sty_0,sw_0}$ if $z = sw_0$ and $M^s_{z,tsw_0} p_{sty_0,z} = v_s^{-2} M^s_{tstsz_0,tsw_0} p_{ststy_0,tstsz_0} \equiv 0$ if $z = tstsz_0$ by 1.8 (1) and the assumption $L(s) = L(t)$ and 1.7 (2). Since $v_s p_{sty_0,tsw_0} = p_{tsty_0,tsw_0}$, we get $p_{sty_0,stsw_0} \equiv p_{ty_0,tsw_0} + p_{tsty_0,tsw_0} - p_{sty_0,sw_0} \equiv p_{ty_0,tsw_0}$ by (3.9.5) and by the equation $a^t_{32} = a^s_{21}$, the latter is obtained from (4) by the symmetry on s and t.

(6) $a^s_{12} = a^t_{21}$.

(3.9.6) $$p_{sy_0,stw_0} = p_{y_0,tw_0} + v_s p_{sy_0,tw_0} - \sum_{\substack{sy_0 \leqslant z < tw_0 \\ sz<z}} M^s_{z,tw_0} p_{sy_0,z}.$$

By Proposition 2.6, we have $M^s_{z,tw_0} \neq 0$ for z in the sum of (3.9.6) only if $z = ststz_0$ for some $z_0 \in W$ with $s,t \notin \mathcal{L}(z_0)$; in the latter case, we have $M^s_{z,tw_0} p_{sy_0,z} = v_s^{-3} M^s_{ststz_0,tw_0} p_{tstsy_0,ststz_0} \equiv 0$ by 1.8 (1) and the assumption $L(s) = L(t)$ and 1.7 (2). By 1.8 (1), we have $v_s p_{sy_0,tw_0} = p_{tsy_0,tw_0}$ and $p_{y_0,tw_0} = v_s^{-1} p_{ty_0,tw_0} \equiv 0$ by the assumption $y_0 \neq w_0$. So by (3.9.6) and Proposition 3.1, we have $p_{sy_0,stw_0} \equiv p_{tsy_0,tw_0}$.

By the symmetry on s and t, we get (a)-(b) from (1)-(3) and (c) from (4)-(6). □

Remark 3.10. (1) The right-handed version of Proposition 3.9 also holds.

(2) Under the hypothesis in Proposition 3.9 (i.e., $s,t \in S$ satisfy $o(st) = 4$ and $L(s) = L(t)$), the weight function L of an irreducible finite or an affine Coxeter group W is not constant on S only if W is of type $\widetilde{C}_n$, $n \geqslant 2$. However, L could be not constant on S in many other cases where W is neither finite nor affine.

(3) Keep the notation in 3.8 but with " $o(st) = 4$ " and " $1 \leqslant i,j \leqslant 3$ " replaced by " $o(st) = m \in \{3,4\}$ " and " $1 \leqslant i,j \leqslant m-1$ ", respectively. Then the results in Propositions 3.4 (1) and 3.9 can be summarized as below.

Theorem 3.11. (Comparing with [4, Subsection 10.4]) *Under the setup of Remark 3.10 (3), let $1 \leqslant i,j \leqslant m-1$ and $r \in \{s,t\}$.*

(1) $a^r_{ij} = a^r_{m-i,m-j}$ *if* $m = 4$;

(2) $a^r_{ij} = a^{\bar{r}}_{m-i,m-j}$ *if* $m = 3$;

(3) $a^r_{i,i+1} = a^{\bar{r}}_{i+1,i}$ *if* $1 \leqslant i < m-1$.

Corollary 3.12. *Suppose that $s,t \in S$ satisfy $o(st) = 4$ with $L(s) = L(t)$.*

(1) Assume that $y,w \in D_L(s,t)$ and that the coefficient of v^{-1} in $p_{y,w}$ or $p_{w,y}$ is non-zero. Then there exist some y',w' in the left $\{s,t\}$-strings ξ_y, ξ_w containing y,w

respectively with $\{y', w'\} \neq \{y, w\}$ *such that the following two conditions are satisfied:*

(1a) either any or none of the sets $\{y, y'\}$ *and* $\{w, w'\}$ *consists of neighboring terms in the left* $\{s, t\}$*-string containing it;*

(1b) the coefficient of v^{-1} *in* $p_{y',w'}$ *or* $p_{w',y'}$ *is non-zero.*

(2) Let $y, w, y', w' \in D_L(s, t)$ *be as in (1). If* $y \underset{\mathrm{R}}{\sim}' w$ *then* $y' \underset{\mathrm{R}}{\sim}' w'$.

(3) Assume that $y, w \in D_R(s, t)$ *and that the coefficient of* v^{-1} *in* $p_{y,w}$ *ot* $p_{w,y}$ *is non-zero. Then there exist some* y'', w'' *in the right* $\{s, t\}$*-strings* ζ_y, ζ_w *containing* y, w *respectively with* $\{y'', w''\} \neq \{y, w\}$ *such that the following two conditions are satisfied:*

(3a) either any or none of the sets $\{y, y''\}$ *and* $\{w, w''\}$ *consists of neighboring terms in the right* $\{s, t\}$*-string containing it;*

(3b) the coefficient of v^{-1} *in* $p_{y'',w''}$ *or* $p_{w'',y''}$ *is non-zero.*

(4) Let $y, w, y'', w'' \in D_R(s, t)$ *be as in (3). If* $y \underset{\mathrm{L}}{\sim}' w$ *then* $y'' \underset{\mathrm{L}}{\sim}' w''$.

Proof. By symmetry, we need only to prove (1)-(2). The assertion (1) in the case of $yw^{-1} \in \langle s, t\rangle$ is obvious, while the assertion (1) in the case of $yw^{-1} \notin \langle s, t\rangle$ follows by Proposition 3.9.

Now let us show the assertion (2). By symmetry and Proposition 3.9, we need only to show that if $y \underset{\mathrm{R}}{\leqslant}' w$ then $y' \underset{\mathrm{R}}{\leqslant}' w'$. To do so, we need only to consider the following two special cases of $y \underset{\mathrm{R}}{\leqslant}' w$:

(a) There exists some $r \in \mathcal{R}(y)\backslash\mathcal{R}(w)$ with the coefficient of $v^{L(r)-1}$ in $N^r_{y,w}$ non-zero;

(b) $y = w \cdot r$ for some $r \in S$ with $L(r) = 1$.

We see that the coefficient of v^{-1} in $p_{y,w}$ or $p_{w,y}$ is non-zero in either of the cases (a) and (b) by Proposition 3.1. We must show that it holds for either (a) or (b) with y', w' in the places of y, w respectively. Since y', w' are the terms in the left $\{s, t\}$-strings ξ_y, ξ_w respectively, we have $\mathcal{R}(y') = \mathcal{R}(y)$ and $\mathcal{R}(w') = \mathcal{R}(w)$. So $r \in \mathcal{R}(y') \setminus \mathcal{R}(w')$. By the assumption that the coefficient of v^{-1} in $p_{y',w'}$ or $p_{w',y'}$ is non-zero, we see by Proposition 3.1 and 1.8 (1) that either that $y' < w'$ and the coefficient of $v^{L(r)-1}$ in

$N^r_{y',w'}$ is non-zero, or that $y' = w' \cdot r$. This completes our proof. $\square$

When the weight function L is constant on S, the requirement (1a) (respectively, (3a)) of Corollary 3.12 on y', w' can be replaced by the condition (1a$'$) as follows.

(1a$'$) (respectively, (3a$'$)) any of the sets $\{y, y'\}$ and $\{w, w'\}$ consists of neighboring terms in the left (respectively, right) $\{s, t\}$-string containing it.

This is because the inequality $a^r_{ij} \geqslant 0$ holds in this case for any $r \in \{s, t\}$ and $1 \leqslant i, j \leqslant 3$. For example, we have the equation $a^r_{22} = a^{\bar{r}}_{11} + a^{\bar{r}}_{13}$ by Proposition 3.9. If the coefficient of v^{-1} in $p_{y,w}$ is either a^r_{11} or a^r_{13}, which is non-zero, take $y', w' \in W$ to satisfy the condition (1a$'$), then the coefficient of v^{-1} in $p_{y',w'}$ should be $a^{\bar{r}}_{22}$, which is non-zero by the above equality. However, when L is not constant on S, the inequality $a^r_{ij} \geqslant 0$ does not hold in general for any $r \in \{s, t\}$ and $1 \leqslant i, j \leqslant 3$. Thus the condition $a^r_{11} \neq 0$ or $a^r_{13} \neq 0$ does not always imply $a^{\bar{r}}_{22} \neq 0$. It might happen that $a^r_{11} = -a^r_{13} \neq 0$ and $a^{\bar{r}}_{22} = 0$.

§4. Expressing $M^s_{y,w}$ in terms of $p_{x,z}$'s.

In the present section, we shall express the Laurent polynomials $M^s_{y,w}$ in terms of polynomials $p_{\alpha,\beta}$'s modulo $\mathcal{A}_{<0}$. Some properties of $M^s_{y,w}$ are deduced from such expressions.

In the subsequent discussion of the section, the symbol " $\equiv$ " always denotes the congruence relation modulo $\mathcal{A}_{<0}$ unless otherwise specified (Note the difference from the same symbol in Section 3, where it was modulo $\mathcal{A}_{<-1}$.)

For any sequence $\xi : z_1, z_2, ..., z_r$ in W, set $\ell(\xi) = r$ and $P_\xi = p_{z_1,z_2} p_{z_2,z_3} \cdots p_{z_{r-1},z_r}$. Clearly, we have $P_\xi \neq 0$ if and only if $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_r$.

For any $y, w \in W$ and $s \in S$ with $sy < y < w$, define $I(y, w; s)$ to be the set of all sequences $\xi : z_1, z_2, ..., z_r$ in W with some $r > 1$ such that $z_1 = y < z_2 < \cdots < z_r = w$ and $s \in \mathcal{L}(z_i)$ for any $1 \leqslant i < r$.

Theorem 4.1. *For any $y, w \in W$ and $s \in S$ with $sy < y < w < sw$, we have*

$$M_{y,w}^s \equiv v_s \sum_{\xi \in I(y,w;s)} (-1)^{\ell(\xi)} P_\xi \pmod{\mathcal{A}_{<0}}. \tag{4.1.1}$$

Proof. By (1.4.1), we have

$$M_{y,w}^s = -\sum_{\substack{y<z<w \\ sz<z}} M_{z,w}^s p_{y,z} + v_s p_{y,w} + h_{y,w} \tag{4.1.2}$$

for some $h_{y,w} \in \mathcal{A}_{<0}$. Applying induction on $\ell(w) - \ell(y) \geqslant 1$. We have, for any z, $y < z < w$, in the sum of (4.1.2), that

$$M_{z,w}^s = v_s \sum_{\xi \in I(z,w;s)} (-1)^{\ell(\xi)} P_\xi + h_{z,w} \tag{4.1.3}$$

for some $h_{z,w} \in \mathcal{A}_{<0}$ by inductive hypothesis. Substituting (4.1.3) into (4.1.2), we get (4.1.1) immediately by the fact that $p_{\alpha,\beta} \in \mathcal{A}_{<0}$ for any $\alpha < \beta$ in W. $\square$

Remark 4.2. (1) Only the sequences $\xi \in I(y, w; s)$ with $\ell(\xi) \leqslant L(s) + 1$ are effective in the formula (4.1.1). Hence the formula (4.1.1) becomes simpler when $L(s)$ is getting smaller. For example, when $L(s) = 1$, (4.1.1) becomes $M_{y,w}^s \equiv v p_{y,w}$, i.e., $M_{y,w}^s$ is just the coefficient of v^{-1} in $p_{y,w}$ (see 1.7 (3)). Now assume $L(s) = 2$. (4.1.1) becomes

$$M_{y,w}^s \equiv v_s \left(p_{y,w} - \sum_z p_{y,z} p_{z,w} \right), \tag{4.2.1}$$

where the sum takes over all $z \in W$ with $y < z < w$ and $sz < z$ and $\mathcal{R}(z) \supseteq \mathcal{R}(w)$; we can further require z in the sum to satisfy $\deg p_{z,w} = \deg p_{y,z} = -1$; in particular, when $L(w) - L(y)$ is odd, we have $M_{y,w}^s \equiv v_s p_{y,w}$ modulo $\mathcal{A}_{<0}$ by 1.7 (4).

(2) In the setup of Theorem 4.1, let $z \in W$ satisfy $y \leqslant z < w$ and $sz < z$. Let $I_z(y,w;s)$ be the set of all sequences $\xi : z_1, z_2, ..., z_r$ in $I(y,w;s)$ which contains z as its term. For any $\xi : z_1, z_2, ..., z_r$ and $\xi' : z_1', z_2', ..., z_t'$ in $I_z(y,w;s)$, we write $\xi \approx \xi'$ if there exists some $i \geqslant 1$ such that $z_i = z_i' = z$ and $z_j = z_j'$ for any $1 \leqslant j < i$. This defines an equivalence relation on the set $I_z(y,w;s)$. Let E be an equivalence class in $I_z(y,w;s)$ with respect to $\approx$. Take any $\xi : z_1, z_2, ..., z_r$ in E with $z_i = z$. Then the sequence $z_1, z_2, ..., z_i$ is independent of the choice of ξ in E, denote it by ξ_E. We have

$$v_s \sum_{\zeta \in E} (-1)^{\ell(\zeta)} P_\zeta \equiv (-1)^{\ell(\xi_E)-1} P_{\xi_E} M_{z,w}^s \tag{4.2.2}$$

by Theorem 4.1. This further implies that

$$v_s \sum_{\xi \in I_z(y,w;s)} (-1)^{\ell(\xi)} P_\xi \equiv M_{z,w}^s \sum_{\zeta \in I(y,z;s)} (-1)^{\ell(\zeta)-1} P_\zeta. \tag{4.2.3}$$

The congruence formula (4.1.1) remains valid if we remove some summands as follows.

Theorem 4.3. *Let $y, w \in W$ and $s \in S$ satisfy the relation $sy < y < w < sw$. Let I be a set of some elements z of W such that $y < z < w$ and $sz < z$ and $M_{z,w}^s = 0$ (note that we don't require I to be the full set of such elements z in general). Then the congruence formula (4.1.1) remains valid if the sequence $\xi : z_1, z_2, ..., z_r$ in the sum ranges over all those in $I(y,w;s)$ with $z_i \notin I$ for any $1 \leqslant i < r$.*

Proof. The proof for the new version of the congruence formula (4.1.1) is almost the same as before, except that in (4.1.3), we require the sequence $z_1, z_2, ..., z_r$ to satisfy one additional condition $z_i \notin I$ for any $1 \leqslant i < r$. By (4.2.3), we see that we loss nothing in (4.1.1) by removing all the summands corresponding to the sequences containing some terms in I since $M_{z,w}^s = 0$ for any $z \in I$. $\square$

Note that in Theorem 4.3, we may take I to be the set of all the elements z of W such that $y < z < w$ and $sz < z$ and $\mathcal{R}(z) \not\supseteq \mathcal{R}(w)$ since we always have $M^s_{z,w} = 0$ for any such element z by 1.7 (1). We have taken this fact into account in the expression (4.2.1).

4.4. Let $y, w \in W$ and $s \in S$ be as in Theorem 4.3. For any $\xi : z_1 = y, z_2, ..., z_r = w$ in $I(y, w; s)$ and any $1 < j < r$ with $\mathcal{R}(z_j) \setminus \mathcal{R}(z_{j-1}) \neq \emptyset$, we see by the fact $z_j > z_{j-1}$ that exactly one of the following three cases occurs: (a) $z_j > z'_{j-1} \cdot w_{\mathcal{R}(z_j)}$; (b) $z_j = z'_{j-1} \cdot w_{\mathcal{R}(z_j)}$ and $\mathcal{R}(z_j) \subseteq \mathcal{R}(z_{j+1})$; (c) $z_j = z'_{j-1} \cdot w_{\mathcal{R}(z_j)}$ and $\mathcal{R}(z_j) \not\subseteq \mathcal{R}(z_{j+1})$, where z'_{j-1} is the shortest element in the left coset $z_{j-1}W_{\mathcal{R}(z_j)}$.

Lemma 4.5. *In the above setup, let J be the set of all sequences $\xi : z_1, z_2, ..., z_r$ in $I(y, w; s)$ satisfying the following conditions: there exists some $1 < i < r$ with $\mathcal{R}(z_i) \setminus \mathcal{R}(z_{i-1}) \neq \emptyset$ such that either $z_i > z'_{i-1} \cdot w_{\mathcal{R}(z_i)}$, or $z_i = z'_{i-1} \cdot w_{\mathcal{R}(z_i)}$ and $\mathcal{R}(z_i) \subseteq \mathcal{R}(z_{i+1})$, where z'_{i-1} is the shortest element in the left coset $z_{i-1}W_{\mathcal{R}(z_i)}$. Then the resulting congruence remains valid after removing all the summands of (4.1.1) corresponding to the sequences in J.*

Proof. Let J_0 be the set of all sequences $\xi : z_1, z_2, ..., z_r$ in J satisfying the following conditions: for any $1 < j < r$,

(∗) if $\mathcal{R}(z_j) \setminus \mathcal{R}(z_{j-1}) \neq \emptyset$ and $z_j = z'_{j-1} \cdot w_{\mathcal{R}(z_j)}$ then $\mathcal{R}(z_j) \not\subseteq \mathcal{R}(z_{j+1})$.

For each $\xi : z_1, z_2, ..., z_r$ in J_0, let $J(\xi)$ be the set of all j, $1 < j < r$, such that $\mathcal{R}(z_j) \setminus \mathcal{R}(z_{j-1}) \neq \emptyset$ and $z_j > z'_{j-1} \cdot w_{\mathcal{R}(z_j)}$, where z'_{j-1} is the shortest element in the left coset $z_{j-1}W_{\mathcal{R}(z_j)}$. Then $J(\xi) \neq \emptyset$. For any $E \subseteq J(\xi)$, let ξ_E be the sequence obtained from ξ by inserting the term $z'_{j-1} \cdot w_{\mathcal{R}(z_j)}$ between z_{j-1} and z_j for any $j \in E$. Then $\xi_E \in J$. Moreover,

$$J = \dot{\bigcup}_{\xi \in J_0} \{\xi_E \mid E \subseteq J(\xi)\} \tag{4.5.1}$$

is a partition of J. For any $\xi \in J_0$, let $m = |J(\xi)|$, then

$$\sum_{E \subseteq J(\xi)} (-1)^{\ell(\xi_E)} P_{\xi_E} = \sum_{E \subseteq J(\xi)} (-1)^{\ell(\xi)+|E|} P_\xi$$

$$= (-1)^{\ell(\xi)} P_\xi \cdot \sum_{k=0}^{m} \binom{m}{k} (-1)^k = (-1)^{\ell(\xi)} P_\xi \cdot (1-1)^m = 0 \tag{4.5.2}$$

by the fact $J(\xi) \neq \emptyset$. This implies that $\sum_{\xi \in J} (-1)^{\ell(\xi)} P_\xi = 0$ by (4.5.1)-(4.5.2). So our result follows. □

4.6. The congruence (4.1.1) still holds if we remove all the terms corresponding to the sequences $\xi : z_1, z_2, ..., z_r$ in $I(y, w; s)$ satisfying one of the following conditions:

(a) Let $R(\xi)$ be the set of all integers j, $1 < j \leqslant r$, such that $\mathcal{R}(z_j) \setminus \mathcal{R}(z_{j-1}) \neq \emptyset$ and $z_j = z'_{j-1} \cdot w_{\mathcal{R}(z_j)}$, where z'_{j-1} is the shortest element in the left coset $z_{j-1} W_{\mathcal{R}(z_j)}$. Then $\sum_{j \in R(\xi)} (L(z_j) - L(z_{j-1})) + (r - 1 - |R(\xi)|) > L(s)$.

(b) Let $L(\xi)$ be the set of all integers i, $1 < i \leqslant r$, such that $\mathcal{L}(z_i) \setminus \mathcal{L}(z_{i-1}) \neq \emptyset$ and $z_i = w_{\mathcal{L}(z_i)} \cdot z''_{i-1}$, where z''_{i-1} is the shortest element in the right coset $W_{\mathcal{L}(z_i)} z_{i-1}$. Then $\sum_{i \in L(\xi)} (L(z_i) - L(z_{i-1})) + (r - 1 - |L(\xi)|) > L(s)$.

since those terms all belong to $\mathcal{A}_{<0}$ by 1.8 (1).

4.7. By 4.6 and Lemma 4.5 and Theorem 4.1, we see that after a certain term-removing, all the sequences $\xi : z_1 = y, z_2, ..., z_r = w$ of $I(y, w; s)$ remained in the sum of (4.1.1) satisfy that,

(i) $\mathcal{R}(z_j) \supseteq \mathcal{R}(w)$ and $s \in \mathcal{L}(z_j)$ for any $1 \leqslant j < r$;

(ii) For any $1 < j \leqslant r$, either $\mathcal{R}(z_{j-1}) \supseteq \mathcal{R}(z_j)$, or $\mathcal{R}(z_j) \setminus \mathcal{R}(z_{j-1}) \neq \emptyset$ and $\mathcal{R}(z_j) \nsubseteq \mathcal{R}(z_{j+1})$ and $z_j = z'_{j-1} \cdot w_{\mathcal{R}(z_j)}$, where z'_{j-1} is the shortest element in the left coset $z_{j-1} W_{\mathcal{R}(z_j)}$;

(iii) For any $1 < j \leqslant r$, either $\mathcal{L}(z_{j-1}) \supseteq \mathcal{L}(z_j)$, or $\mathcal{L}(z_j) \setminus \mathcal{L}(z_{j-1}) \neq \emptyset$ and $\mathcal{L}(z_j) \nsubseteq \mathcal{L}(z_{j+1})$ and $z_j = w_{\mathcal{L}(z_j)} \cdot z''_{j-1}$, where z''_{j-1} is the shortest element in the right coset

$W_{\mathcal{L}(z_j)}z_{j-1}$;

(iv) Let $R(\xi)$ and $L(\xi)$ be defined as in 4.6. Then $\sum_{j\in R(\xi)}(L(z_j)-L(z_{j-1}))+(r-1-|R(\xi)|)\leqslant L(s)$. Also, $\sum_{i\in L(\xi)}(L(z_i)-L(z_{i-1}))+(r-1-|L(\xi)|)\leqslant L(s)$.

4.8. For $y,w\in W$ and $s\in S$ with $sy<y<w<sw$ and $\mathcal{R}(y)\supseteq\mathcal{R}(w)$, let $[y,w)$ be the set of all elements z satisfying $y\leqslant z<w$ and $sz<z$ and $\mathcal{R}(z)\supseteq\mathcal{R}(w)$. For any $z\in[y,w)$, denote by $n(z)$ the largest number k such that there exists some sequence $z_1=z,z_2,...,z_k$ in $[y,w)$ with $z_1<z_2<\cdots<z_k<w$. Let $[y,w)'_k=\{z\in[y,w)\mid n(z)=k\}$.

Clearly, we have $n(z)>n(z')$ for any $z<z'$ in $[y,w)$. In particular, if $n(y)=m$ then $[y,w)'_m=\{y\}$ and $[y,w)=\dot\cup_{k=1}^m[y,w)'_k$.

By (1.4.1) and Theorem 4.1, we have the following algorithm for computing $M^s_{y,w}$:

(1) Compute the sets $[y,w)'_k$ for any $1\leqslant k\leqslant n(y)$.

(2) For any $z\in[y,w)'_1$, we set $M^s_{z,w}\in\mathcal{A}$ by the requirements:

$$M^s_{z,w}\equiv v_s p_{z,w}\quad\text{and}\quad\overline{M^s_{z,w}}=M^s_{z,w}.$$

(3) If $n(y)=1$ then the algorithm terminates. If $n(y)>1$, then let $[y,w)_1=\{z\in[y,w)'_1\mid M^s_{z,w}\neq 0\}$.

(4) Take i with $1\leqslant i\leqslant n(y)$. Suppose that we have got all the sets $[y,w)_h=\{z\in[y,w)'_h\mid M^s_{z,w}\neq 0\}$ $(1\leqslant h<i)$ and the $M^s_{z,w}$'s in $\mathcal{A}$ for any $z\in(\bigcup_{k=1}^{i-1}[y,w)_k)\cup[y,w)'_i$. If $n(y)=i$ then the algorithm terminates. If $n(y)>i$, then let $[y,w)_i=\{z\in[y,w]'_i\mid M^s_{z,w}\neq 0\}$ and for any $z\in[y,w)'_{i+1}$, find $M^s_{z,w}\in\mathcal{A}$ by the requirements

$$M^s_{z,w}\equiv v_s\sum_{z_1=z<z_2<\cdots<z_r=w}(-1)^r p_{z_1,z_2}\cdots p_{z_{r-1},z_r}$$

and $\overline{M^s_{z,w}}=M^s_{z,w}$, where the sum is taken over all the sequences $z_2<z_3<...<z_{r-1}$ in the set $\bigcup_{k=1}^i[y,w)_k$. Let $[y,w)_{i+1}=\{z\in[y,w)'_{i+1}\mid M^s_{z,w}\neq 0\}$.

Example 4.9. Let $W = \widetilde{F}_4$ and $m = L(s_4) = L(s_3) > L(s_2) = L(s_1) = L(s_0) = 1$. Take $y = s_3$ and $w = s_2s_3s_2s_4s_3$. We have $[y,w)' = \{s_3, s_3s_2s_3, s_3s_4s_3, s_3s_2s_4s_3, s_2s_3s_2s_3\}$. By a direct computation, we get $p_{s_3,s_2s_3s_2s_4s_3} = v^{-2m-2} + v^{-2m} + v^{-2}$ and $p_{s_3,s_3s_2s_4s_3} = v^{-2m-1} + v^{-1}$ and $p_{s_3,s_3s_2s_3} = (v^{-1} - v)v^{-m}$ and $M^{s_3}_{s_3s_2s_4s_3,s_2s_3s_2s_4s_3} = v^{-1+m} + v^{-m+1}$ and $M^{3}_{s_2s_3s_2s_3,s_2s_3s_2s_4s_3} = 1$ and $M^{s_3}_{s_3s_2s_3,s_2s_3s_2s_4s_3} = M^{s_3}_{s_3s_4s_3,s_2s_3s_2s_4s_3} = 0$.

By Theorem 4.1, we have

$$\begin{aligned}
M^{s_3}_{s_3,s_2s_3s_2s_4s_3} \equiv{}& v^m[p_{s_3,s_2s_3s_2s_4s_3} - p_{s_3,s_3s_2s_4s_3}p_{s_3s_2s_4s_3,s_2s_3s_2s_4s_3} \\
&- p_{s_3,s_2s_3s_2s_3}p_{s_2s_3s_2s_3,s_2s_3s_2s_4s_3} - p_{s_3,s_3s_2s_3}p_{s_3s_2s_3,s_2s_3s_2s_4s_3} \\
&- p_{s_3,s_3s_4s_3}p_{s_3s_4s_3,s_2s_3s_2s_4s_3} + p_{s_3,s_3s_2s_3}p_{s_3s_2s_3,s_2s_3s_2s_3}p_{s_2s_3s_2s_3,s_2s_3s_2s_4s_3} \\
&+ p_{s_3,s_3s_2s_3}p_{s_3s_2s_3,s_3s_2s_4s_3}p_{s_3s_2s_4s_3,s_2s_3s_2s_4s_3} \\
&+ p_{s_3,s_3s_4s_3}p_{s_3s_4s_3,s_3s_2s_4s_3}p_{s_3s_2s_4s_3,s_2s_3s_2s_4s_3}] \\
\equiv{}& 0.
\end{aligned}$$

On the other hand, by Theorem 4.3 with $J = \{s_3s_2s_3, s_3s_4s_3\}$, we have

$$\begin{aligned}
M^{s_3}_{s_3,s_2s_3s_2s_4s_3} &\equiv v^m[p_{s_3,s_2s_3s_2s_4s_3} - p_{s_3,s_3s_2s_4s_3}p_{s_3s_2s_4s_3,s_2s_3s_2s_4s_3} \\
&\quad - p_{s_3,s_2s_3s_2s_3}p_{s_2s_3s_2s_3,s_2s_3s_2s_4s_3}] \\
&\equiv v^m[v^{-2m-2} + v^{-2} + v^{-2m} - (v^{-2m-1} + v^{-1})v^{-1} - v^{-2m-2}] \\
&\equiv 0.
\end{aligned}$$

Clearly, the latter is simpler.

§5. Cells in W_{I_1} with $L(I_1) = 1$.

In the present section, we assume (W,S) to be an irreducible Coxeter system which is either finite or affine. Let ∇ be the set of all $y \in W \setminus \{e\}$ (e the identity element of W) which have a unique reduced expression as a product of elements in S. When the weight function L of W is constant on S, Lusztig showed in [3, Proposition 3.8] that ∇ forms a single two-sided cell of W. This result no longer holds in general when L is not constant on S. For example, when W be a dihedral group D_{2n} of order $4n$ with $n \in \{2, 3, 4, ...\} \cup \{\infty\}$, Lusztig showed in [5, Subsection 8.8] that ∇ is a union of two two-sided cells of W if $n = \infty$, and is a union of three two-sided cells of W if $n < \infty$.

It is natural to ask if ∇ is always a union of some two-sided cells of W. The answer is negative.

5.1 Example. Consider the affine Weyl group $\widetilde{F}_4$ with the distinguished generator set $S = \{s_0, s_1, s_2, s_3, s_4\}$, where $o(s_0s_1) = o(s_1s_2) = o(s_3s_4) = 3$ and $o(s_2s_3) = 4$ (see 1.9). Let $L : W \longrightarrow \mathbb{Z}$ be a weight function satisfying $L(s_4) = L(s_3) > 2L(s_2) = 2L(s_1) = 2L(s_0) = 2$. Take $y = s_3s_2s_3$ and $w = s_2s_1s_3s_2s_3$ and $s = s_3$. Then $y \in \nabla$ and $w \in W \setminus \nabla$. By (1.4.1)-(1.4.2), we get $M^s_{y,w} = -v_sv^{-2} - v_s^{-1}v^2 \neq 0$. So $y \underset{\mathrm{L}}{\leqslant} w \underset{\mathrm{L}}{\leqslant} s_1s_3s_2s_3 \underset{\mathrm{L}}{\leqslant} y$. i.e., $y \underset{\mathrm{L}}{\sim} w$. So ∇ is not a union of some two-sided cells of W.

5.2. Assume that $\min\{L(r) \mid r \in S\} = 1$ and that $I_1 = \{s \in S \mid L(s) = 1\} \subsetneqq S$. Let $I_2 = S \setminus I_1$. Then the Coxeter system (W_{I_1}, I_1) is irreducible unless $W = \widetilde{C}_l$ and $I_1 = \{s_0, s_l\}$, where $s_0, s_l \in S$ correspond to two terminal nodes in the Coxeter graph of W (see 1.9). We can talk about the left, right and two-sided cells of W_{I_1} with respect to the weight function $L_1 : W_{I_1} \longrightarrow \mathbb{N}$, where L_1 is the restriction of L to W_{I_1}, which is constant on I_1. Let $\nabla_1 = \nabla \cap W_{I_1}$. Assume that there exists some two-sided cell Ω in W_{I_1} with $a(\Omega) = 2$ (note that such a two-sided cell, when it exists, need not be unique in W_{I_1}). With respect to the partial order $\underset{\mathrm{LR}}{\leqslant}$ on the set of two-sided cells of W_{I_1}, the set $\{e\}$ forms the highest two-sided cell of W_{I_1} (and also of W). By [3, Proposition 3.8], we know that the set ∇_1 forms the second highest two-sided cell of W_{I_1} in the case where W_{I_1} is irreducible. The set Ω forms a third highest two-sided cell of W_{I_1}.

Proposition 5.3. *In the setup of 5.2 with W_{I_1} irreducible, the set ∇_1 forms a single two-sided cell of W.*

Proof. By [3, Proposition 3.8], we see that the set ∇_1 is a two-sided cell of the Coxeter group W_{I_1}, hence it is contained in some two-sided cell of W. By symmetry, to show our assertion, it is enough to show that if $y \in \nabla_1$ and $w \in W \setminus \nabla_1$ and $t \in S$ satisfy $ty < y < w < tw$, then $M_{y,w}^t = 0$, or equivalently, the coefficient $c(y,w)$ of v^{-1} in $p_{y,w}$ is zero by 1.7 (3) and by the fact $t \in I_1$. The assertion follows by [3, Proposition 3.8] if $w \in W_{I_1}$. Now assume $w \notin W_{I_1}$. Take any $s \in \mathcal{L}(w)$. Then $s \notin \mathcal{L}(y)$ by the facts that $|\mathcal{L}(y)| = 1$ and $t \in \mathcal{L}(y) \setminus \mathcal{L}(w)$. Hence $p_{y,w} = v_s^{-1} p_{sy,w}$ by 1.8 (1). If $s \in I_2$, then $c(y,w) = 0$ by the facts $p_{sy,w} \in \mathcal{A}_{\leqslant 0}$ and $L(s) > 1$. If $s \in I_1$, then $sy \in W_{I_1}$, hence $sy \neq w$ since $w \notin W_{I_1}$, so $p_{sy,w} \in \mathcal{A}_{<0}$, we again have $c(y,w) = 0$. □

Remark 5.4. (1) Proposition 5.3 generalizes the result in [3, Proposition 3.8] to the unequal parameter case, and also generalizes the result in [5, Subsection 8.8] to the case where W is an arbitrary Coxeter group (i.e., not necessarily a dihedral group).

(2) Let m be the length of the longest element in W_{I_1}. In [1, Theorem 1.1], Guilhot showed that if $L(s) > m$ for any $s \in I_2$, then any left (respectively, right, two-sided) cell of W_{I_1} is also a left (respectively, right, two-sided) cell of W. One may propose the following conjecture to strengthen the result of Guilhot.

Conjecture 5.5. *In the setup of 5.2, suppose that Ω is a left (respectively, right, two-sided) cell of W_{I_1} with $a(\Omega) = k$ and that $L(s) > k$ for any $s \in I_2$. Then Ω is also a left (respectively, right, two-sided) cell of W.*

Proposition 5.3 supports Conjecture 5.5 in the case of $k = 1$. The following result provides one more evidence, i.e., the case of $k = 2$, to support the conjecture.

We say that I_1 is *exceptional*, if $W = \widetilde{C}_l$, $l \geqslant 2$, and I_1 is one of the sets $\{s_0, s_l\}$ and $\{s_0, s_1, ..., s_{l-1}\}$ and $\{s_1, s_2, ..., s_l\}$.

Proposition 5.6. *In the setup of 5.2, assume that Ω is a two-sided cell of W_{I_1} with $a(\Omega) = 2$ and that $L(s) > 2$ for any $s \in I_2$. Then Ω is also a two-sided cell of W.*

Proof. By [6, Theorem 3.1], we see that any $y \in \Omega$ has an expression of the form $y = x' \cdot w_I \cdot y'$ for some $x', y' \in W_{I_1}$ and some $I = \{s,t\} \subset I_1$ with $st = ts$ and that if $y \in \Omega$ has an expression of the form $y = x'' \cdot w_{I'} \cdot y''$ with $x'', y'' \in W_{I_1}$ and $I' \subseteq S$, $|I'| > 1$, then $I' = \{s', t'\}$ for some $s', t' \in I_1$ with $s't' = t's'$. If $y \in \Omega$ is in a left $\{s,t\}$-string ξ for some $s, t \in S$ with $o(st) > 2$, then ξ is contained in Ω (see 3.2, note that $s, t \in I_1$ in this case).

Let $E_1 = \Omega \cup \nabla_1 \cup \{e\}$. Since Ω is a third highest two-sided cell of W_{I_1}, to show our result, we need only to show that if $y \in \Omega$ and $w \in W \setminus E_1$ and $u \in S$ satisfy $uy < y < w < uw$ (hence $u \in I_1$), then $M^u_{y,w} = 0$, or equivalently, the coefficient of v^{-1} in $p_{y,w}$ is zero by 1.7 (3).

If $w \in W_{I_1} \setminus E_1$, then $M^u_{y,w} = 0$ since Ω is a third highest two-sided cell in W_{I_1} (see 5.2). Now assume $w \in W \setminus W_{I_1}$. If $\mathcal{L}(w) \not\subseteq \mathcal{L}(y)$, then we can prove the equation $M^u_{y,w} = 0$ by the same argument as that in the proof of Proposition 5.3. Now assume $\mathcal{L}(w) \subseteq \mathcal{L}(y)$. By the facts of $u \in \mathcal{L}(y) \setminus \mathcal{L}(w)$ and $|\mathcal{L}(y)| \leqslant 2$ and $\mathcal{L}(w) \neq \emptyset$, we have $\mathcal{L}(y) = \{u, t\}$ and $\mathcal{L}(w) = \{t\}$ for some $t \in I_1$ with $tu = ut$.

(1) First assume that I_1 is not exceptional. Then the full subgraph Γ' of the Coxeter graph Γ of W with the node set I_1 is connected and simply-laced (where by Γ' being simply-laced, we mean that any $s, t \in I_1$ satisfy $o(st) \leqslant 3$).

By our assumptions on W and on I_1, we can write $w = t_1 t_2 \cdots t_r \cdot w'$ with some $r \geqslant 1$ such that $t_1 = t, t_2, ..., t_r$ are all in I_1 and satisfy $o(t_i t_{i+1}) = 3$ and $\mathcal{L}(t_j t_{j+1} \cdots t_r \cdot w') = \{t_j\}$ for any $1 \leqslant i < r$ and any $1 \leqslant j \leqslant r$ and that either $\mathcal{L}(w') \cap I_2 \neq \emptyset$ or $|\mathcal{L}(w') \cap I_1| > 1$.

(1a) First assume $r = 1$. Then by (1.8.1), we have

$$p_{ty',tw'} = p_{y',w'} + v p_{ty',w'} - \sum_{\substack{ty' \leqslant z < w' \\ tz < z}} M^t_{z,w'} p_{ty',z}. \tag{5.6.1}$$

where $y = ty'$ for some $y' \in E_1$, and either $\mathcal{L}(w') \cap I_2 \neq \emptyset$ or that there exist some $s \neq s'$ in $\mathcal{L}(w')$ satisfying $o(st) = o(s't) = 3$ (hence $u \notin \{s, s'\} \subseteq I_1$). We claim that in either case, any of $p_{y',w'}$, $vp_{ty',w'}$ and $M^t_{z,w'}p_{ty',z}$ in (5.6.1) is in $\mathcal{A}_{<-1}$. For, assume $\mathcal{L}(w') \cap I_2 \neq \emptyset$. Take $s \in \mathcal{L}(w') \cap I_2$. Then by 1.8 (1) and the assumption $L(s) > 2$, we see that both $p_{y',w'} = v_s^{-1}p_{sy',w'}$ and $vp_{ty',w'} = v^{1-L(s)}p_{sty',w'}$ are in $\mathcal{A}_{<-1}$. On the other hand, if $s \notin \mathcal{L}(z)$ then $M^t_{z,w'} = 0$ by Proposition 2.3; if $s \in \mathcal{L}(z)$ then $M^t_{z,w'}p_{ty',z} = v_s^{-1}M^t_{z,w'}p_{sty',z}$ is in $\mathcal{A}_{<-1}$. Assume that $s \neq s'$ in $\mathcal{L}(w')$ satisfy $o(st) = o(s't) = 3$ (hence $s, s' \in I_1$). Since $u \in \mathcal{L}(y') \setminus \{s, s'\}$ and $y' \in \Omega \cup \nabla_1$, at least one of s and s' is not in $\mathcal{L}(y')$ (say $s \notin \mathcal{L}(y')$ for the sake of definiteness), hence $p_{y',w'} = v^{-1}p_{sy',w'}$ and $vp_{ty',w'} = v^{-1}p_{ss'ty',w'}$, both of which are in $\mathcal{A}_{<-1}$ by the facts $sy' \neq w' \neq ss'ty'$ (note that $sy', ss'ty' \in W_{I_1}$ and $w' \notin W_{I_1}$). On the other hand, if $\mathcal{L}(z) \cap \{s, s'\} = \emptyset$, then $M^t_{z,w'} = 0$ by Proposition 2.3. If $\{s, s'\} \subset \mathcal{L}(z)$, then $M^t_{z,w'}p_{ty',z} = v^{-2}M^t_{z,w'}p_{ss'ty',z}$. If $|\mathcal{L}(z) \cap \{s, s'\}| = 1$ (say $s \notin \mathcal{L}(z)$ for the sake of definiteness), then $M^t_{z,w'} \neq 0$ if and only if $w' = sz$ by Corollary 2.4, when the equivalent conditions hold, we have $M^t_{z,w'}p_{ty',z} = v^{-1}p_{s'ty',z}$ with $s'ty' \neq z$ (since $s'ty' \in W_{I_1}$ and $z \notin W_{I_1}$). So $M^t_{z,w'}p_{ty',z} \in \mathcal{A}_{<-1}$ in either case. This proves our claim. So the coefficient of v^{-1} in $p_{y,w}$ is zero by (5.6.1).

(1b) Next assume $r > 1$. Apply left $\{t_1, t_2\}$-, $\{t_2, t_3\}$-, ..., $\{t_{r-1}, t_r\}$-star operations successively on both elements w and y, we get two sequences of elements: $w_1 = w, w_2, ..., w_r$ in $W \setminus W_{I_1}$ and $y_1 = y, y_1, ..., y_r$ in Ω, respectively, where $w_i = t_i t_{i+1} \cdots t_r \cdot w'$ for $1 \leqslant i \leqslant r$ (see Remark 3.7). Since the set $\mathcal{L}(y_i)$ consists of either a single element or two commutative elements in I_1 with $\mathcal{L}(y_i) \cap \{t_i, t_{i+1}\} = \mathcal{L}(w_i) \cap \{t_i, t_{i+1}\} = \{t_i\}$ for any $1 \leqslant i < r$, such left star operations on y can always be carried through. Eventually, we have $y_r = t_r y' \in \Omega$ for some $y' \in E_1$. By Proposition 3.4, we see that the coefficient of v^{-1} in $p_{y,w}$ is equal to that in $p_{t_r y', t_r w'}$. By (1.8.1), we have

$$p_{t_ry',t_rw'} = p_{y',w'} + vp_{t_ry',w'} - \sum_{\substack{t_ry' \leqslant z < w' \\ t_rz < z}} M^{t_r}_{z,w'} p_{t_ry',z}. \tag{5.6.2}$$

Again, we see that the coefficient of v^{-1} in any of $p_{y'w'}$, $vp_{t_ry',w'}$ and $M^{t_r}_{z,w'}p_{t_ry',z}$ in (5.6.2) is zero. Hence the coefficient of v^{-1} in $p_{t_ry',t_rw'}$ is zero by (5.6.2). This implies that the coefficient of v^{-1} in $p_{y,w}$ is zero.

(2) Next assume I_1 exceptional. Thus $W = \widetilde{C}_l$, $l > 1$. If $I_1 = \{s_0, s_l\}$, then $\Omega = \{s_0s_l\}$. Our result follows by [1, Theorem 1.1]. Now assume $I_1 = \{s_0, s_1, ..., s_{l-1}\}$ (hence $I_2 = \{s_l\}$). Then w with $\mathcal{L}(w) = \{t\} \subseteq I_1$ is one of the elements x_k, z_h, x'_i as follows:

(i) $x_k = s_ks_{k+1}\cdots s_{l-1}\cdot w'$, $z_h = s_hs_{h-1}\cdots s_1s_0s_1\cdots s_{l-1}\cdot w'$ for some $1 \leqslant k \leqslant l-1$ and $0 \leqslant h \leqslant l-1$, where $w' \in W$ satisfies $\mathcal{L}(w') = \{s_l\}$.

(ii) $x'_i = s_i \cdot w'$ for some $1 \leqslant i < l$ and $w' \in W$ with $\mathcal{L}(w') = \{s_{i-1}, s_{i+1}\}$.

The cases of w being x'_i and x_k can be dealt with in the same way as that in (1a) and (1b) respectively (see (1)). Now assume $w = z_h$ for some $0 \leqslant h \leqslant l-1$.

(2a) If $h = 0$, then $w = z_0$ and y are in some left $\{s_0, s_1\}$-strings ξ and ζ, respectively, where $\xi : x_1, z_0, z_1$ (notation as in (i)). By Proposition 3.9, we can find some term y_1 in ζ such that the coefficient of v^{-1} in p_{y_1,x_1} is non-zero whenever that in $p_{y,w}$ is non-zero. In fact, if y is a terminal term of ζ, then take y_1 to be the middle term of ζ; if y is the middle term of ζ, then take y_1 to be one of two terminal terms y_{11}, y_{13} of ζ in such a way that the absolute value of the coefficient of v^{-1} in p_{y_1,x_1} is the largest among those in p_{y_{11},x_1} and p_{y_{13},x_1}. Then by the same argument as that in (1b), we can prove that the coefficient of v^{-1} in p_{y_1,x_1} is zero. This implies that the coefficients of v^{-1} in $p_{y,w}$ is zero.

(2b) Now assume $h \geqslant 1$. Apply left $\{s_h, s_{h-1}\}$-, $\{s_{h-1}, s_{h-2}\}$-, ..., $\{s_2, s_1\}$-star operations successively on both elements w and y, we get two sequences of elements:

$z_h = w, z_{h-1}, ..., z_1$ (notation as in (i)) in $W \setminus W_{I_1}$ and $y_h = y, y_{h-1}, ..., y_1$ in Ω, respectively (see Remark 3.7). Then z_1 and y_1 are in some left $\{s_0, s_1\}$-strings ξ, ζ, respectively, where $\xi : x_1, x_0, z_1$ (notation as in (i)). The coefficient of v^{-1} in p_{y_1,z_1} is equal to that in $p_{y,w}$ by Proposition 3.4. By Proposition 3.9, we can choose y'_1 in ζ such that the coefficient of v^{-1} in $p_{y'_1,x_1}$ is equal to that of p_{y_1,z_1}. In fact, if y_1 is a terminal term of ζ, then take y'_1 to be another terminal term of ζ; if y_1 is the middle term of ζ, then take y'_1 to be y_1. Now we apply left $\{s_1, s_2\}$-, $\{s_2, s_3\}$-, ..., $\{s_{l-2}, s_{l-1}\}$-star operations successively on both elements x_1 and y'_1, we get two sequences of elements: $x_1, x_2, ..., x_{l-1}$ (notation as in (i)) in $W \setminus W_{I_1}$ and $y'_1, y'_2, ..., y'_{l-1}$ in Ω, respectively. Then $x_{l-1} = s_{l-1} \cdot w'$ and y'_{l-1} satisfy $\mathcal{L}(x_{l-1}) = \{s_{l-1}\} \subseteq \mathcal{L}(y'_{l-1})$ and $\mathcal{L}(w') = \{s_l\} = I_2$ and that the coefficient of v^{-1} in $p_{y'_{l-1},x_{l-1}}$ is equal to that of $p_{y'_1,x_1}$ by Proposition 3.4. By the result in (1a), we see that the coefficients of v^{-1} in $p_{y'_{l-1},x_{l-1}}$ is zero. This implies that the coefficient of v^{-1} in $p_{y,w}$ is zero.

The case of $I_1 = \{s_1, s_2, ..., s_l\}$ can be dealt with similarly.

So our proof is completed. $\square$

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