AA2. Lecture 20

03.02.2022.

Last time:

- $(A \circlearrowright)$ A quotient module U/V is simple $\Longleftrightarrow$ V is a maximal submodule of U.

- The simple modules of $k[x]/\langle f\rangle$, $f \in k[x]$, are precisely the modules $k[x]/\langle h\rangle$ where h is an irr. poly. dividing f.

  the action $k[x]/\langle f\rangle \circlearrowright k[x]/\langle h\rangle$ $\quad (g+\langle f\rangle).(g'+\langle h\rangle) = gg'+\langle h\rangle$.

  "reason": S is a simple module of $k[x]/\langle f\rangle =: A$ $\Longleftrightarrow$ $S = A/M$, M maximal ideal in A

  $$S = \frac{k[x]/\langle f\rangle}{\langle h\rangle/\langle f\rangle} = k[x]/\langle h\rangle. \Longleftarrow \quad M \text{ has to be } \langle h\rangle/\langle f\rangle \text{ for irr. } h \mid f$$

Today:

- Prop 3.23.(2). why different "h"s give nonisomorphic simples

- Simple modules of path algebras

1. A preservation-of-scalar-action argument

eg. $\mathbb{C}[x]/\langle x^2+1\rangle$ $\quad$ $\mathbb{C}[x]/\langle x-i\rangle$, $\mathbb{C}[x]/\langle x+i\rangle$

Prop: (= Prop. 3.23.(2)) Let $A = k[x]/\langle f\rangle$ for some poly $f \in k[x]$ of positive degree. Write $f = f_1^{a_1}\cdots f_r^{a_r}$ as a product of irreducible polynomials $f_1, \dots, f_r$ that are pairwise coprime. Then A has precisely r nonisomorphic simple modules, namely the modules $S_i := k[x]/\langle f_i\rangle$.

Pf: It remains to show that $S_i \not\cong S_j$ for distinct i, j. If $i \neq j$, then $f_i + \langle f\rangle$ acts as 0 on $S_i = k[x]/\langle f_i\rangle$ because $(f_i+\langle f\rangle)\cdot(g+\langle f_i\rangle) = f_i g + \langle f_i\rangle = 0$

but $f_i + \langle f\rangle$ does not act as 0 on S_j, because, for example, for $g = 1$,

$$(f_i+\langle f\rangle)(g+\langle f_j\rangle) = f_i g + \langle f_j\rangle = f_i + \langle f_j\rangle \neq 0 \quad \text{because } f_j \nmid f_i. \ \square$$

Summary: We now have complete, irredundant classifications of simple modules of

$k[x]/\langle f\rangle$. $\rightsquigarrow$ $k[x]/\langle f_i\rangle$,

Note: · Recall that $\dim_k\left(k[x]/\langle g\rangle\right) = \deg(g)$ $\forall g \in k[x]$.

· If $k = \mathbb{C}$ or other algebraically closed field. every $f \in k[x]$ factors into linear (= deg. 1) irreducibles of the form $(x-\alpha)$, so all simples of $k[x]/\langle f\rangle$ will be of the form $k[x]/\langle x-\alpha\rangle$ and thus have dimension 1.

· If $k = \mathbb{R}$, then (fact:) every irreducible in $k[x]$ have degree 1 or 2. So any simple module of an algebra of the form $k[x]/\langle f\rangle$ must have dimension 1 or 2.

2. Simple modules of path algebras

no oriented cycle (and loops)

We study the simple modules of the path algebras of acyclic quivers $Q = (Q_0, Q_1)$.

Note: Let $A = kQ$. For all $i \in Q_0$,

- The space Ae_i is a left ideal / submodule of (the regular module) A, and equals the span of all paths on Q starting at i.
- The space $Ae_i^{\geq 1} := \mathrm{Span}(\text{all paths of positive length starting at } i)$. We'll write $J_i = Ae_i^{\geq 1}$.

  (submodule of Ae_i)
- Let $S_i := Ae_i / Ae_i^{\geq 1} = \mathrm{Span}(e_i + Ae_i^{\geq 1}) = \mathrm{Span}(e_i + J_i)$.

  Then since $\dim(S_i) = 1$, S_i is a simple module of A.
- $e_j \cdot S_i$: The only possibly nonzero elt (up to scalar) in $e_j S_i$ is

$$\underbrace{e_j}_{\in A} \cdot \underbrace{(e_i + J_i)}_{\text{only nonzero elt in } S_i \text{ up to scalar}} = \underbrace{(e_j e_i) + J_i}_{\in S_i} = \begin{cases} 0 + J_i = 0 & \text{if } i \neq j \\ e_i + J_i & \text{if } i = j \end{cases}$$

So, e_j acts as the scalar 0 on S_i if $i \neq j$ and as the scalar 1 on S_i if $i = j$. So if $i \neq j$ then $S_i \not\cong S_j$ by the "preservation-of-scaling" principle.

We can now conclude:

Prop: The modules S_i ($i \in Q_0$) are pairwise nonisomorphic simple modules of $A = kQ$.

Thm. (Thm 3.26.) Let Q be an acyclic cycle. Let $A = kQ$.

Then every simple A-module is isomorphic to S_i for some $i \in Q_0$.

(So $\{S_i : i \in Q_0\}$ is a complete and irredundant list of simples of A up to iso.)

Pf of the theorem:

We first prove a lemma:

Lemma (Lemma 3.27): (a) $\forall i \in Q_0$, we have $e_i A e_i = \mathrm{Span}(e_i)$, and hence $\dim(e_i A e_i) = 1$.

(b) The only maximal submodule of Ae_i is $J_i = Ae_i^{\geq 1}$, and $e_i J_i = 0$.

Pf: (a) This follows from the fact that $e_i A e_i = \mathrm{Span}(\text{all path from } i \text{ to } i)$ and the fact the only path from i to i is e_i since Q is acyclic.

(b) · J_i is a submodule of Ae_i and has dim. $\dim(J_i) = \dim(Ae_i) - 1$, so J_i is a maximal submodule of Ae_i.

· To prove J_i is the only maximal submodule of Ae_i, we prove that any proper submodule U of Ae_i is contained in J_i. If not, then U contains some elt v not in J. Thus, we can write $v = ce_i + w$ for some $w \in J_i$ and $c \in k \setminus \{0\}$.

But then since $u = ce_i + w \in U$ and $e_i \in A$,

$$e_i u = e_i(ce_i + w) = ce_i^2 + \underbrace{e_i w}_{\in e_i J_i} \in U$$

Note that $e_i J_i = e_i(Ae_i^{\geq 1}) = \mathrm{Span}(\text{path of pos. length from } i \text{ to } i) = \mathrm{Span}(\phi) = 0$.

So $e_i w = 0$. It follows that $e_i u = ce_i^2 = ce_i \in U$.

But then $e_i = \frac{1}{c}(ce_i) \in U$, so $Ae_i \leq U$. a contradiction (to $U \subsetneq Ae_i$).

It follows that $U \subseteq J_i$. $\square$

Pf of the theorem: next time.