

# DIFFERENTIAL CHOW VARIETIES EXIST

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**ABSTRACT.** Chow varieties are a parameter space for cycles of a given variety of a given codimension and degree. We construct their analog for differential algebraic varieties with differential algebraic subvarieties, answering a question of [11]. The proof uses the construction of classical algebro-geometric Chow varieties, the theory of characteristic sets of differential varieties, the theory of prolongation spaces, and the theory of differential Chow forms. In the course of the proof several definability results from the theory of algebraically closed fields are required. Elementary proofs of these results are given in the appendix.

**Keywords:** Differential Chow variety, Differential Chow form, Prolongation admissibility, Model theory, Chow variety

## 1. INTRODUCTION

For simplicity in the following discussion, let  $k$  be an algebraically closed field. The  $r$ -cycles over  $k$  of a  $k$ -variety  $V$  are elements of the free  $\mathbb{Z}$ -module generated by the integral subvarieties of  $V$  of dimension  $r$  defined over  $k$ . If the coefficients are taken over  $\mathbb{N}$  then the cycle is said to be positive or effective. For a given cycle  $\sum n_i V_i$ , the degree of  $\sum n_i V_i$  is given by  $\sum n_i \cdot \deg(V_i)$ . The positive  $r$ -cycles of degree  $d$  of a  $k$ -variety  $V$  are parameterized by a  $k$ -variety, which we denote  $\text{Chow}_{r,d}(V)$ . For background information on Chow varieties and Chow forms, see [3] (or [14] or [4] for a modern exposition). The purpose of this article is to carry out the construction of the differential algebraic analog of the Chow variety, whose construction was begun in [11], but was completed only in certain very special cases. For our purposes, one can view Chow varieties and their differential counterparts as parameter spaces for cycles with particular characteristics (degree and codimension in the algebraic case). The algebraic theory of Chow varieties also has numerous applications and deeper uses (e.g. Lawson (co)homology [9] and various counting problems in geometry [6]).

Given a differential algebraic variety  $V$  over a differentially closed field  $K$ , the group of differential cycles of dimension  $d$  and order  $h$  is the free  $\mathbb{Z}$ -module generated by irreducible differential subvarieties  $W \subseteq V$  so that the dimension,  $\dim(W)$ , is  $d$  and the order,  $\text{ord}(W)$ , is  $h$ . The differential cycles of index  $(d, h, g, m)$  are those cycles with leading differential degree  $g$  and differential degree  $m$ . These invariants have a very natural definition and are a suitable notion of degree for differential cycles; see section 3 for the definitions. Our main result establishes the existence

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of a differential variety which parameterizes this particular set of differential cycles of  $V$ .

There are various foundational approaches to differential algebraic geometry (e.g. the scheme-theoretic approaches [26] or the Weil-style definition of an abstract differential algebraic variety [23]). Such abstract settings will not be pertinent here, since we work exclusively with affine differential algebraic varieties over a differential field. In this setting, beyond the basic development of the theory, there are two approaches relevant to our work. The first is the classical theory using characteristic sets [22]. We also use the more recent geometric approach using the theory of jet and prolongation spaces [31]. This approach allows one to replace a differential algebraic variety by an associated sequence of algebraic varieties, but owing to the Noetherianity of the Kolchin topology, some finite portion of the sequence contains all of the data of the sequence. This allows the importation of various results and techniques from the algebraic category.

We use the two approaches in the following manner. We use classical algebraic Chow varieties to parameterize prolongation sequences. We then use the generic components of these prolongation sequences to parameterize the *characteristic sets* of differential algebraic cycles with given index. There are essentially two steps to the construction. First, degree bounds are used to restrict the space of Chow varieties in which we must look for the points which generate the prolongation sequences parameterizing differential cycles of a given index; this development uses basic intersection theory (e.g. [16]) and the theory of differential Chow forms [11]. Within the appropriate Chow varieties which parameterize these prolongation spaces, only a subset of the points will correspond to differential cycles with the specified index. We show that this collection of points (such that the characteristic set corresponding to the generic components of the prolongation sequence has the specified numerical invariants) is in fact a Kolchin-constructible subset.

The construction of differential Chow varieties is related to *canonical parameters* in the sense of model theory. In the theory of differentially closed fields, canonical parameters manifest themselves as the generators of fields of definition of differential varieties. In recent years, detailed analyses of canonical parameters have been undertaken in analogy with results of Campana [1] and Fujiki [10] from compact complex manifolds (for instance, see [33, 2, 29]). The following is essentially pointed out by Pillay and Ziegler [33]. Let  $K$  be a differential field and  $x$  an  $n$ -tuple of elements from some differential field extension. Let  $X$  be the differential locus of  $x$  over  $K$ . Let  $L$  be a differential field extension of  $K$  and let  $Z$  be the differential locus of  $x$  over  $L$ , and assume that the inclusion  $Z \subseteq X$  is proper. Let  $b$  be a generator for the differential field of definition of  $Z$  over  $K$ . That is, there is some differential algebraic subvariety  $Y \subseteq \mathbb{A}^n \times \mathbb{A}^m$  so that  $b \in \mathbb{A}^m(L)$ ,  $(x, b) \in Y_b$  and the second projection map  $\pi : Y \rightarrow \mathbb{A}^m$  is differentially birational over its image. Consider  $\mathcal{Y} := {}_x Y$ , the fibre of  $Y$  over  $x$  via the first projection  $Y \rightarrow \mathbb{A}^n$ , which is subvariety of a certain differential Chow variety of  $X$  in the sense of this paper. The main result of [33] is that  $\mathcal{Y}$  is internal to the constants. One could certainly expand upon these observations to give statements about the structure of differential Chow varieties, but we will not pursue these matters in further detail in this paper.

In [33, page 581], Pillay and Ziegler write of the above situation,

We are unaware of any systematic development of machinery and language (such as “differential Hilbert spaces”) in differential algebraic geometry which is adequate for the geometric translation above. This is among the reasons why we will stick with the language of model theory in our proofs below. The issue of algebraizing the content and proofs is a serious one which will be considered in future papers.

Subsequent work by Moosa and Scanlon [31] did algebraize and generalize much of the work by Pillay and Ziegler, but no systematic development of differential Hilbert schemes or differential Chow varieties appears to have occurred in the decade following Pillay and Ziegler’s work. One should view [11] as the beginning of such a systematic development, where the theory of the differential Chow form was developed, and the existence of differential Chow varieties was established in certain very special cases. In [11, section 5], the authors write that they are unable to prove the existence of the differential Chow variety in general. The work here is an extension of [11], in which we will establish the existence of the differential Chow variety in general, answering the most natural question left open by [11]. As we have pointed out above, our general technique is also the descendant of a line thinking that originated (at least in the model theoretic context) with Pillay and Ziegler’s work on jet spaces and the linearization of differential equations.

The rest of the paper is organized as follows. In section 2, we give background definitions and some preliminary results which we use later in the paper. In addition, we describe the relationship of the problems we consider to the Ritt problem. Following this interlude, we prove the results which eventually allow us to work around the issues involved in the Ritt problem (whose solution would allow for a simplification of the proofs of the results in this paper). In section 3, we describe the necessary background from the classical theory of Chow varieties. Our approach is slightly nonstandard in this section, owing to the fact that we work with affine varieties. In section 4, we establish various bounds on the order and degree of the varieties we consider using the theory of differential Chow forms. In section 5, we establish the existence of differential Chow varieties, proving the main result of the paper.

The appendix gives elementary proofs of several facts from algebraic geometry which we require. The facts proved in the appendix are well known and frequently used in model theory (for instance, see the citation in appendix 3.1 of [20]), however, the proof given here seems to be new. Constructive proofs of the result (which give additional information about certain bounds, rather than simply proving that a certain bound exists) are much more involved (see [35], which corrected the proof given in [17]). Various other non-constructive proofs of the theorem are given in the literature [13, 15.5.3] [5, where a nonstandard approach is taken] [19, where a model theoretic approach is taken to give an elementary proof].

## 2. PRELIMINARIES AND PROLONGATIONS

We fix  $\mathcal{U}$  a saturated differentially closed field. Implicitly, all differential fields we consider are substructures of  $\mathcal{U}$ . If  $W$  is an algebraic variety or a differential variety, then an expression of the form “ $a \in W$ ” is shorthand for “ $a \in V(\mathcal{U})$ ”. Throughout,  $K$  will be a small differential subfield of  $\mathcal{U}$  and  $\delta$  denotes the distinguished derivation on  $K$ ,  $\mathcal{U}$ , or, indeed, any differential ring that we consider. Unless explicitly stated

to the contrary, all varieties and differential varieties are defined over  $K$ . Often, we will use phrases like “ $\delta$ -variety”, “ $\delta$ -constructible”, “ $\delta$ -field”, *et cetera* as synonyms for “differential variety”, “differentially constructible”, “differential field”, *et cetera*.

If  $f : X \rightarrow Y$  is a morphism of varieties, then by  $f(X)$  we mean the scheme theoretic image of  $X$  under  $f$ . That is,  $f(X)$  is the smallest subvariety  $Z$  of  $Y$  for which  $f$  factors through the inclusion  $Z \hookrightarrow Y$ . On points,  $f(X)$  is the Zariski closure of  $\{f(a) : a \in X(\mathcal{U})\}$ .

We write  $K\{x_1, \dots, x_n\}$  for the differential polynomial ring in the variables  $x_1, \dots, x_n$  over  $K$ . For  $m \in \mathbb{N}$ , we write

$$K\{x_1, \dots, x_n\}_{\leq m} = K[\{x_i^{(j)} : 1 \leq i \leq n, 0 \leq j \leq m\}]$$

for the subring of differential polynomials of order at most  $m$  where we have written  $x_i^{(j)}$  for  $\delta^j x_i$ .

If  $\mathcal{J} \subseteq K\{x_1, \dots, x_n\}$  is a differential ideal, then we write  $\mathbb{V}(\mathcal{J})$  for the differential subvariety of  $\mathbb{A}_K^n$  defined by the vanishing of all  $f \in \mathcal{J}$ , and for  $S \subseteq \mathbb{A}^n(K)$ , we let  $\mathbb{I}(S) \subseteq K\{x_1, \dots, x_n\}$  be the differential ideal of all differential polynomials vanishing on  $V$ . On the other hand, if  $I \subseteq K[x_1, \dots, x_n]$  is an ideal, then we write  $V(I)$  for the variety defined by the vanishing of all  $f \in I$ , and for  $S \subseteq \mathbb{A}^n(K)$ , we write  $I(S)$  for the ideal of polynomial functions in  $K[x_1, \dots, x_n]$  which vanish on  $S$ . By convention, when we speak of an irreducible variety (irreducible differential variety, respectively), we mean an absolutely irreducible variety (absolutely irreducible differential variety, respectively).

In general, if  $R$  is a reduced ring, we write  $\mathcal{Q}(R)$  for its total ring of fractions. When  $R$  is a differential ring, so is  $\mathcal{Q}(R)$ . We write  $K\langle V \rangle$  for  $\mathcal{Q}(K\{x_1, \dots, x_n\}/\mathbb{I}(V))$ . When  $\mathbb{I}(V)$  is prime, that is, when  $V$  is irreducible, this is called the differential function field of  $V$ . For  $S \subseteq K\{x_1, \dots, x_n\}$  we write  $(S)$  for the ideal generated by  $S$  and  $[S]$  for the *differential* ideal generated by  $S$ . When  $S = \{f\}$  is a singleton, we write  $(f) := (S)$  and  $[f] := [S]$ . Likewise, we write  $\mathbb{V}(f)$  for  $\mathbb{V}([f])$ .

We sometimes speak about “generic points”. These should be understood in the sense of Weil-style algebraic (or differential algebraic) geometry. That is, if  $V$  is a variety (respectively, differential algebraic variety) over  $K$ , then  $\eta \in V(\mathcal{U})$  is generic if there is no proper subvariety (respectively, differential subvariety)  $W \subsetneq V$  defined over  $K$  with  $\eta \in W(\mathcal{U})$ . Provided that  $V$  is irreducible, this is equivalent to asking that the field  $K(\eta)$  (respectively, differential field  $K\langle\eta\rangle$ ) be isomorphic over  $K$  to  $K(V)$  (respectively,  $K\langle V \rangle$ ).

Next, we follow the notation of section 2 of [30]. There, the authors define a sequence of functors  $\tau_m$  indexed by the natural numbers from varieties over  $K$  to varieties over  $K$  (to be honest, the functor may return a nonreduced scheme, but the distinction between a scheme and its reduced subscheme is immaterial here). For affine space itself, one has  $\tau_m(\mathbb{A}^n) \cong \mathbb{A}^{n(m+1)}$  where if we present  $\mathbb{A}^n$  as  $\text{Spec}(K[x_1, \dots, x_n])$ , then  $\tau_m(\mathbb{A}^n) = \text{Spec}(K\{x_1, \dots, x_n\}_{\leq m})$ . If  $V \subseteq \mathbb{A}^m$  is a subvariety of affine space, then  $\tau_m V = \text{Spec}(K\{x_1, \dots, x_n\}_{\leq m}/(\{\delta^j f : f \in \mathbb{I}(V), j \leq m\}))$ . Note that the ideal  $(\{\delta^j f : f \in \mathbb{I}(V), j \leq m\})$  is contained in  $[\mathbb{I}(V)] \cap K\{x_1, \dots, x_n\}_{\leq m}$ , but the inclusion may be proper.

There is a natural differential algebraic map  $\nabla_m : V \rightarrow \tau_m V$  given on points valued in a differential ring by

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n; \delta(a_1), \dots, \delta(a_n); \dots; \delta^m(a_1), \dots, \delta^m(a_n)) .$$

We call points in the image of  $\nabla_m$  *differential points*.

The image of the map  $\nabla_m$  need not be Zariski dense, even on  $\mathcal{U}$ -valued points. For any differential subvariety  $W \subseteq V$  of the algebraic variety  $V$ , we define  $B_m(W)$  to be the Zariski closure in  $\tau_m V$  of  $\nabla_m(W(\mathcal{U}))$ .

The functors  $\tau_m$  form a projective system with the natural transformation  $\pi_{m,\ell} : \tau_m \rightarrow \tau_\ell$  for  $\ell \leq m$  given by projecting onto the coordinates corresponding to the first  $\ell$  derivatives. We write  $\pi_{m,\ell} : \tau_m V \rightarrow \tau_\ell V$  rather than  $\pi_{m,\ell}^V$ . Moreover,  $\tau_0$  is simply the identity functor so that we write  $V$  rather than  $\tau_0(V)$ . From the definition, for  $W \subseteq V$  a differential subvariety, it is clear that  $\pi_{m,\ell}$  restricts to make the sequence of varieties  $(B_m(W))$  into a projective system of algebraic varieties in which each map in the system is dominant.

We write  $\tau^m$  for the result of composing the functor  $\tau_1$  with itself  $m$  times. There is a natural transformation  $\rho_m : \tau_m \rightarrow \tau^m$  which for any algebraic variety  $V$  gives a closed embedding  $\rho_m : \tau_m V \hookrightarrow \tau^m V$ . To ease notation, let us write the map  $\rho$  in coordinates only for the case of  $V = \mathbb{A}^1$  and  $m = 3$ . The general case requires one to decorate the variables with further subscripts and to nest the coordinates more deeply. Here

$$\rho(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) = (((x^{(0)}, x^{(1)}), (x^{(1)}, x^{(2)})), ((x^{(1)}, x^{(2)}), (x^{(2)}, x^{(3)})))$$

For the formal definitions of the items discussed above, please refer to [31, 30].

**Definition 2.1.** A sequence of varieties  $X_\ell \subseteq \tau_\ell \mathbb{A}^n$  ( $\ell \in \mathbb{N}$ ) is called a *prolongation sequence* if

- (1) The map induced by projection  $X_{\ell+1} \rightarrow X_\ell$  is dominant.
- (2) For all  $\ell$ ,  $\rho_{\ell+1}(X_{\ell+1})$  is a closed subvariety of  $\tau_1(\rho_\ell(X_\ell))$ .

Given a prolongation sequence  $(X_\ell)_{\ell \geq 0}$ , the  $\mathcal{U}$ -points in the differential variety  $V$  given by  $(X_\ell)_{\ell \geq 0}$  is the set

$$\{b \in \mathbb{A}^n(\mathcal{U}) : (\forall \ell) \nabla_\ell(b) \in X_\ell(\mathcal{U})\}.$$

From the point of view of differential ideals, if  $\mathcal{J} = \bigcup_{\ell=0}^{\infty} I(X_\ell) \subseteq K\{x_1, \dots, x_n\}$ , then  $V = \mathbb{V}(\mathcal{J})$ .

There is a bijective correspondence between irreducible prolongation sequences (by which we mean each variety in the sequence is irreducible) and irreducible differential varieties. Given a differential variety  $V$ , the prolongation sequence corresponding to  $V$  is given by  $X_\ell = B_\ell(V)$  for all  $\ell \geq 0$  in [30, discussion proceeding Definition 2.8]. Thus, prolongation sequences are in one-to-one correspondence with differential algebraic varieties, and the Noetherianity of the Kolchin topology guarantees that a finite portion of a prolongation sequence determines the entire sequence.

**Definition 2.2.** Given an algebraic variety  $V \subseteq \tau_\ell \mathbb{A}^n$ , we say that  $V$  is prolongation admissible if  $\rho_\ell(V) \subseteq \tau^{\ell-d}(\rho_d(\pi_{\ell,d}(V)))$  for all  $0 \leq d < \ell$ .

The following fact is the basis of the well known geometric axioms for differentially closed fields, written in our language:

**Fact 2.3.** *Given irreducible varieties  $V$  and  $W$  over  $\mathcal{U}$  with  $W \subseteq \tau_1(V)$  so that the restriction of  $\pi_{1,0}$  to  $W$  is a dominant map to  $V$ , then for any  $U \subseteq W(\mathcal{U})$ , a Zariski open set, there is a  $a \in V(\mathcal{U})$  such that  $\nabla_1(a) \in U$ .*

Indeed, Fact 2.3 characterizes differentially closed fields; for details, see [32].

**Lemma 2.4.** *Suppose that  $V \subseteq \tau_\ell(\mathbb{A}^n)$  is an irreducible prolongation admissible variety. Then for any open subset  $U \subseteq V$ , there is some  $a \in \mathbb{A}^n(\mathcal{U})$  such that  $\nabla_\ell(a) \in U$ .*

*That is, a variety is prolongation admissible if and only if the differential points form a dense subset.*

*Proof.* Since  $V$  is prolongation admissible,  $\rho_\ell(V) \subseteq \tau_1(\rho_{\ell-1}(\pi_{\ell,\ell-1}(V)))$ . So, we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho_\ell} & \rho_\ell(V) \subseteq \tau_1(\rho_{\ell-1}(\pi_{\ell,\ell-1}(V))) \\ \pi_{\ell,\ell-1} \downarrow & & \downarrow \pi_{1,0} \\ \pi_{\ell,\ell-1}(V) & \xrightarrow{\rho_{\ell-1}} & \rho_{\ell-1}(\pi_{\ell,\ell-1}(V)). \end{array}$$

Since each  $\rho$  is an embedding, we must have that  $\pi_{1,0}$  is dominant to  $\rho_{\ell-1}(\pi_{\ell,\ell-1}(V))$ . Thus, by Fact 2.3, the set of points

$$\{\nabla_1(a) \in \rho_{\ell-1}(\pi_{\ell,\ell-1}(V))(\mathcal{U}) \mid \nabla_1(a) \in \rho_\ell(V)(\mathcal{U})\}$$

is Zariski dense in  $\rho_\ell(V)$ , so the set  $\rho_\ell^{-1}(\nabla_1(\{a \in \pi_{\ell,\ell-1}(V)(\mathcal{U}) \mid \nabla_1(a) \in \rho_\ell(V)(\mathcal{U})\}))$  is Zariski dense in  $V$ . Every such point has the form  $\nabla_\ell(b)$  for  $b \in \mathbb{A}^n(\mathcal{U})$ , proving the claim.  $\square$

From the definition, it is easy to see that if  $\mathcal{V}$  is a set of prolongation sequences  $V := (V_\ell \subseteq \tau_\ell \mathbb{A}^n)_{\ell=0}^\infty$ , then the sequence  $(\bigcup_{V \in \mathcal{V}} \overline{V_\ell})_{\ell=0}^\infty$  is also a prolongation sequence. This justifies the following definition.

**Definition 2.5.** Given a variety  $V \subseteq \tau_h(\mathbb{A}^n)$ , the prolongation sequence generated by  $V$  is the maximal sequence of subvarieties  $V_\ell \subseteq \tau_\ell \mathbb{A}^n$  which is a prolongation sequence and for which  $V_h \subseteq V$ .

The following lemma follows from the observation above that the closure of an arbitrary union of prolongation admissible varieties is also prolongation admissible.

**Lemma 2.6.** *Given  $V \subseteq \tau_h \mathbb{A}^n$ , there is a finite set of irreducible maximal prolongation admissible subvarieties of  $V$ .*

**Lemma 2.7.** *If  $V \subseteq \tau_h(\mathbb{A}^n)$  is an irreducible prolongation admissible variety and  $(V_\ell)_{\ell=0}^\infty$  is the prolongation sequence generated by  $V$ , then for each  $\ell \geq h$  there is a unique component,  $U_\ell$ , of  $V_\ell$  which projects dominantly to  $V = V_h$  and moreover,  $U_\ell = \dim(V) + (\ell - h)d$  where  $d = \dim(V) - \dim(V_{h-1})$ .*

*Proof.* At the cost of permuting the coordinates of  $\mathbb{A}^n$ , we may assume that the sequence  $x_1^{(h)}, \dots, x_d^{(h)}$  gives a transcendence basis of  $K(V)$  over  $K(V_{h-1})$ . Let  $Q \in \mathcal{O}(V_{h-1})[x_1^{(h)}, \dots, x_d^{(h)}]$  so that each  $x_i^{(h)}$  is integral over the localized ring  $\mathcal{O}(V_{h-1})[x_1^{(h)}, \dots, x_d^{(h)}, \frac{1}{Q}]$ . For each  $i$  with  $d < i \leq n$ , let  $g_i(X)$  be the minimal monic polynomial of  $x_i^{(h)}$  in  $\mathcal{O}(V_{h-1})[x_1^{(h)}, \dots, x_d^{(h)}, \frac{1}{Q}][X]$  and let  $S_i := g_i'(x_i^{(h)})$  be its separant. Let  $R := Q \prod_{i=d+1}^n S_i$  and let  $Z$  be the differential variety  $\mathbb{V}([R])$ . Let  $W$  be the differential subvariety of  $\mathbb{A}^n \setminus Z$  given by  $V$ . We claim that for  $\ell \geq h$  if  $Y$  is a component of  $V_\ell$  then either  $Y$  is the Zariski closure of  $B_\ell(W)$  or  $\pi_{\ell,h}(Y) \subseteq Z$ . Indeed, after inverting  $R$ , it is clear that for each  $\ell \geq h$ , the ideal generated by  $\{\delta^j f : j \leq (\ell - h), f \in \mathbb{I}(V)\}$  is prime and is generated by

$\mathbb{I}(V)$  and expressions of the form  $x_i^{(j)} - A_{i,j}$  for  $d < i \leq n$ ,  $h < j \leq \ell$ , and  $A_{i,j} \in \mathcal{O}(V)[\{x_s^{(t)} : 1 \leq s \leq d, h \leq t \leq \ell\}][\frac{1}{R}]$ .

Thus, the differential ring  $K\{x_1, \dots, x_n\}[\frac{1}{R}]/[\mathbb{I}(V)]$  is an integral domain and therefore embeds into  $\mathcal{U}$  over  $K$ . The image of  $(x_1, \dots, x_n)$  is thus a generic point in  $W$  and its image under  $\nabla_\ell$  is the generic point of  $B_\ell(W)$ .  $\square$

*Remark 2.8.* From the proof of Lemma 2.7, the differential variety corresponding to the prolongation sequence generated by a prolongation admissible variety  $V \subseteq \tau_h(\mathbb{A}^n)$  has a unique component with maximal differential dimension and order. The previous result has a partial differential analog, but the situation is more complicated. See [28] for details.

We shall speak of definable families of definable sets and of certain properties being definable in families. These are general notions but we shall use them only for the theories of algebraically closed fields of characteristic zero and of differentially closed fields of characteristic zero. In these cases, “definable” is synonymous with “constructible” or “differentially constructible”, respectively.

**Definition 2.9.** We say that a family of sets  $\{X_a\}_{a \in B}$  is a *definable family* if there are formulae  $\psi(x; y)$  and  $\theta(y)$  so that  $B$  is the set of realizations of  $\theta$  and for each  $a \in B$ ,  $X_a$  is the set of realizations of  $\psi(x; a)$ .

Given a property  $\mathcal{P}$  of definable sets, we say that  $\mathcal{P}$  is definable in families if for any family of definable sets  $\{X_a\}_{a \in B}$  given by the formulae  $\psi(x; y)$  and  $\theta(y)$ , there is a formula  $\phi(y)$  so that the set  $\{a \in B : X_a \text{ has property } \mathcal{P}\}$  is defined by  $\phi$ .

Given an operation  $\mathcal{F}$  which takes a set and returns another set, we say that  $\mathcal{F}$  is definable in families if for any family of definable sets  $\{X_a\}_{a \in B}$  given by the formulae  $\psi(x; y)$  and  $\theta(y)$ , there is formula  $\phi(z; y)$  so that for each  $a \in B$ , the set  $\mathcal{F}(X_a)$  is defined by  $\phi(z; a)$ .

We will require the following facts about definability in algebraically closed fields.

**Fact 2.10.** *We work relative to the theory of algebraically closed fields.*

- (1) *The Zariski closure is definable in families.*
- (2) *The dimension and degree of the Zariski closure of a set are definable in families.*
- (3) *Irreducibility of the Zariski closure is a definable property. More generally, the number of components of the Zariski closure is definable in families.*
- (4) *If the Zariski closure is an irreducible hypersurface given by the vanishing of some nonzero polynomial, then the degree of that polynomial in any particular variable is definable in families.*
- (5) *The family of irreducible components of the Zariski closure is definable in families.*

Fact 2.10 is established in the Appendix 7. As we noted in the Introduction, other proofs appear in the literature.

**2.1. Methods of algebraic and differential characteristic sets.** In this paper, the Wu-Ritt characteristic set method is a basic tool for establishing a correspondence between differential algebraic cycles and algebraic cycles satisfying certain conditions. In this section we recall the definition and basic properties of differential and algebraic characteristic sets.

Fix a sequence of variable  $x_1, x_2, x_3, \dots$ . A differential ranking is a total order  $\prec$  on the set  $\Theta := \{x_i^{(j)}\}_{i,j \in \mathbb{N}}$  satisfying

- For all  $\alpha \in \Theta$   $\delta\alpha \succ \alpha$  for all and
- If  $\alpha_1 \succ \alpha_2$ , then  $\delta\alpha_1 \succ \delta\alpha_2$ .

An orderly ranking is a differential ranking which satisfies in addition

- If  $k > \ell$ , then  $\delta^k x_i \succ \delta^\ell x_j$  for all  $i$  and  $j$ .

Throughout the paper, we fix some orderly ranking.

Let  $f$  be a differential polynomial in  $K\{x_1, \dots, x_n\}$ . The *leader* of  $f$ , denoted by  $\text{ld}(f)$ , is the greatest variable with respect to  $\prec$  which appears effectively in  $f$ . Regarding  $f$  as a univariate polynomial in  $\text{ld}(f)$ , its leading coefficient is called the *initial* of  $f$ , denoted by  $\text{init}_f$ ,

and the partial derivative of  $f$  with respect to  $\text{ld}(f)$  is called the *separant* of  $f$ , denoted by  $S_f$ . For any two differential polynomials  $f, g$  in  $\mathcal{U}\{x_1, \dots, x_n\}$ ,  $f$  is said to be of lower rank than  $g$ , denoted by  $f < g$ , if

- $\text{ld}(f) < \text{ld}(g)$  or
- $\text{ld}(f) = \text{ld}(g)$  and  $\deg(f, \text{ld}(f)) < \deg(g, \text{ld}(f))$  (here  $\deg(h, y)$  means the degree of  $h$  as a polynomial in the variable  $y$ ).

The differential polynomial  $f$  is said to be reduced with respect to  $g$  if no proper derivative of  $\text{ld}(g)$  appears in  $f$  and  $\deg(f, \text{ld}(g)) < \deg(g, \text{ld}(g))$ .

Let  $\mathcal{A}$  be a set of differential polynomials. Then  $\mathcal{A}$  is said to be an auto-reduced set if each differential polynomial in  $\mathcal{A}$  is reduced with respect to any other element of  $\mathcal{A}$ . Every auto-reduced set is finite [34].

Let  $\mathcal{A}$  be an auto-reduced set. We denote  $H_{\mathcal{A}}$  to be the set of all initials and separants of  $\mathcal{A}$  and  $H_{\mathcal{A}}^\infty$  the minimal multiplicative set containing  $H_{\mathcal{A}}$ . The saturation differential ideal of  $\mathcal{A}$  is defined to be

$$\text{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^\infty = \{f \in K\{x_1, \dots, x_n\} : \exists h \in H_{\mathcal{A}}^\infty \text{ for which } hf \in [\mathcal{A}]\}.$$

An auto-reduced set  $\mathcal{C}$  contained in a differential polynomial set  $\mathcal{S}$  is said to be a *characteristic set* of  $\mathcal{S}$  if  $\mathcal{S}$  does not contain any nonzero element reduced with respect to  $\mathcal{C}$ . A characteristic set  $\mathcal{C}$  of a differential ideal  $\mathcal{I}$  reduces all elements of  $\mathcal{I}$  to zero. Furthermore, if  $\mathcal{I}$  is prime, then  $\mathcal{I} = \text{sat}(\mathcal{C})$ .

**Definition 2.11.** For an auto-reduced set  $\mathcal{A} = \{A_1, \dots, A_t\}$  with  $\text{ld}(A_i) = x_{c_i}^{(o_i)}$ , the order of  $\mathcal{A}$  is defined as  $\text{ord}(\mathcal{A}) = \sum_{i=1}^t o_i$ .

Let  $\mathcal{I}$  be a prime differential ideal in  $K\{x_1, \dots, x_n\}$ . The *differential dimension* of  $\mathcal{I}$  is defined as the differential transcendence degree of the differential extension field  $K\langle \mathbb{V}(\mathcal{I}) \rangle$  over  $K$ .

**Definition 2.12.** [24] Let  $\mathcal{I}$  be a prime differential ideal of  $K\{x_1, \dots, x_n\}$ . Then there exists a unique numerical polynomial  $\omega_{\mathcal{I}}(t)$  such that

$$\omega_{\mathcal{I}}(t) = \text{tr. deg}(\mathcal{Q}(K\{x_1, \dots, x_n\}_{\leq t} / (\mathcal{I} \cap (K\{x_1, \dots, x_n\}_{\leq t}))) / K)$$

for all sufficiently large  $t \in \mathbb{N}$ . The polynomial  $\omega_{\mathcal{I}}(t)$  is called the *Kolchin polynomial* of  $\mathcal{I}$ .

In the present paper, we also need the Wu-Ritt algebraic characteristic method to deal with the prolongation ideals of a differential ideal, regarded as pure algebraic ideals. Below, we conclude this subsection by recalling basic concepts about algebraic characteristic sets [36].



Consider the polynomial ring  $K[x_1, \dots, x_n]$  and fix an ordering on  $x_1, \dots, x_n$ , say,  $x_1 \prec \dots \prec x_n$ . Given  $f \in K[x_1, \dots, x_n] \setminus K$ , the leading variable of  $f$  is the greatest variable  $x_k$  effectively appearing in  $f$ , denoted by  $\text{lv}(f)$ . A sequence of polynomials  $A_1, \dots, A_r$  is said to be an ascending chain, if

- $r = 1$  and  $A_1 \neq 0$ , or
- all  $A_i$  are nonconstant,
- $\text{lv}(A_i) < \text{lv}(A_j)$  for  $1 \leq i < j$  and
- $\deg(A_k, \text{lv}(A_k)) > \deg(A_m, \text{lv}(A_k))$  for  $m > k$ .

Suppose  $\mathcal{A} = \langle A_1, \dots, A_r \rangle$  and  $\mathcal{B} = \langle B_1, \dots, B_s \rangle$  are two ascending chains in  $K[x_1, \dots, x_n]$ . We say  $\mathcal{A}$  is of lower rank than  $\mathcal{B}$ , denoted by  $\mathcal{A} \prec \mathcal{B}$ , if either

- there exists  $k \leq \min\{r, s\}$  such that  $\text{lv}(A_i) = \text{lv}(B_i)$  for  $i < k$  and  $\text{lv}(A_k) < \text{lv}(B_k)$ , or
- $r > s$  and  $\text{lv}(A_i) = \text{lv}(B_i)$  for  $i \leq s$ .

Given an ideal  $I$  in  $K[x_1, \dots, x_n]$ , an ascending chain contained in  $I$  which is of lowest rank is called an algebraic characteristic set of  $I$ . If  $V \subseteq \mathbb{A}^n$  is an irreducible variety, then an algebraic characteristic set of  $V$  is defined as the characteristic set of  $I(V)$ .

**Lemma 2.13.** *Algebraic characteristic sets with respect to an arbitrary ranking are definable in families.*

*Proof.* Let  $(V_b)_{b \in B}$  be a constructible family of algebraic subvarieties of  $\mathbb{A}^n$ . By the algebraic version of [15, Corollary 6.2], there is a number  $r$  only depending on the degrees of the defining equations of  $(V_b)_{b \in B}$  and  $n$  such that  $V_b$  has a characteristic set of degree bounded by  $r$ .

One can naturally view the space of sets of  $n - d$  polynomials of degree at most  $r$  as a definable quotient of  $\mathbb{A}^{\binom{n+r}{r} \cdot (n-d)}$ . We claim that the set of points in this space which correspond to algebraic characteristic sets of  $V_b$  is a definable set.

By definition, algebraic characteristic sets are those ascending chains which are minimal with respect to our fixed ranking. Ordering the coordinates by the rank of the corresponding monomials in each of the  $n - d$  copies of  $\mathbb{A}^{\binom{n+r}{r}}$ , the property for a finite set of polynomials to be an ascending chain is definable by quantifying over the support of the polynomials in the set. Minimality among ascending chains can be seen easily by quantification over the (definable) set of ascending chains and quantifier free statements regarding the support of the sets of polynomials.  $\square$

**2.2. The Ritt problem.** Irreducibility of *differential* varieties is *not* known to be a definable in families. This essentially comes down to the fact that it might not be possible to bound the order the differential polynomials which witness the non-primality of the differential ideal only from geometric data. Developing such a bound is equivalent to several problems considered by Ritt [15, for instance, see the statement of Theorem 5.7 along with the references in the following remark], and we will refer to the development of such a bound as the *Ritt problem*.

Characteristic sets are *an* answer to this problem; various properties become definable in families of characteristic sets. The drawback is that for points  $p$  such that the product of the separants of a given characteristic set vanish at  $p$ , determining if  $p$  is in the differential variety with a given characteristic set is an open problem [20, see the discussion in the appendices]. In this paper, we will parameterize characteristic sets of  $\delta$ -cycles rather than parameterizing generators of  $\delta$ -ideals. One

might seek a more direct parameterization by generators of differential ideals, but doing so while following our general strategy would, at least on the surface, seem to require a solution to the Ritt problem.

Here is a specific indication of the problems that can arise when working directly with the generating sets of differential ideals; the following example shows that the order,  $h$ , will not suffice for the bound described in the previous paragraphs.

**Example 2.14.** [20] Let  $V = \mathbb{V}(2x^{(1)}x^{(3)} - (x^{(2)})^2 - 2x)$ . Differentiating the defining equation results in the equation  $2x^{(1)}(x^{(4)} - 1) = 0$ . From this, it is easy to see that  $V$  consists of two components,  $x = 0$  and the generic component.

Of course, more differentiations might be necessary:

**Example 2.15.** [34] Consider  $V = \mathbb{V}(f)$  where  $f = (y^{(2)})^2 - y \in K\{y\}$ . Differentiating  $f$  successively 3 times, one obtains

$$\begin{aligned}\delta f &= 2y^{(2)}y^{(3)} - y^{(1)} \\ \delta^2 f &= 2y^{(2)}y^{(4)} + 2(y^{(3)})^2 - y^{(2)} \\ \delta^3 f &= 2y^{(2)}y^{(5)} + 6y^{(3)}y^{(4)} - y^{(3)}\end{aligned}$$

Then  $2y^{(3)} \cdot \delta^3 f - (6y^{(4)} - 1)f^{(2)} = y^{(2)}(4y^{(3)}y^{(5)} - 12(y^{(4)})^2 + 8y^{(4)} - 1) \in [f]$ . Thus,  $V = \mathbb{V}(f, y^{(2)}) \cup \mathbb{V}(f, 4y^{(3)}y^{(5)} - 12(y^{(4)})^2 + 8y^{(4)} - 1)$  is reducible.

Informally, the Ritt problem asks if there is an upper bound to the number of required differentiations in terms of the “shape” of the equations. An equivalent form of the Ritt problem [20, Appendix 1] is testing when a given point (say 0) at which the separants of the characteristic set vanish is in the generic component of the differential ideal generated by the characteristic set.

**2.3. Skirting around the Ritt problem.** As we described in the previous section, the facts corresponding to 2.10 are not known in the differential setting. Our replacement for these facts will be the results of this section and the use of prolongation sequences. The next lemma follows easily from Fact 2.10 and the definition of prolongation admissible.

**Lemma 2.16.** *Prolongation admissibility is definable in families.*

Given a system of differential equations of order  $h$ , one might use our characterization of prolongation admissibility to express the system of differential equations via a prolongation admissible variety. For a prolongation admissible variety, the order and dimension of the corresponding differential variety are clearly definable, applying Lemma 2.7. A more direct geometric argument not using prolongation sequences is also possible, which we give here.

In [11], intersections of differential varieties with  $\delta$ -generic hyperplanes were analyzed. The coefficients of the defining equations of the hyperplanes were taken to be sets of differential indeterminates  $u$  over the differential field  $K$  in which the variety was defined and various aspects of the geometry of the resulting intersection were established over the field  $K\langle u \rangle$ . The geometry of generic intersections of differential varieties was analyzed in [7], where the following result was proved.

**Theorem 2.17.** *Let  $V \subseteq \mathbb{A}^n$  be a geometrically irreducible affine differential variety which is not an algebraic curve. Let  $H$  be a hyperplane defined by an inhomogeneous linear form over  $\mathcal{U}$  whose coefficients are  $\delta$ -generic over  $K$ . Then  $V \cap H$*

is a geometrically irreducible differential variety, which is nonempty just in case  $\dim(V) > 0$ . In that case,  $V \cap H$  has Kolchin polynomial:

$$\omega_{V/K}(t) = (t + 1) .$$

One can use Theorem 2.17 to prove the definability of dimension and order; as we have remarked above, there seem to be various other ways to prove these results; also see section 5.

**Lemma 2.18.** *Given a  $\delta$ -constructible family of  $\delta$ -varieties  $(X_s)_{s \in S}$ , with  $\dim(S) = 0$ , the set  $\{s \in S(\mathcal{U}) : \dim(X_s) = d\}$  is a  $\delta$ -constructible subset of  $S$*

*Proof.* Fix  $d$  and  $n + 1$ -tuples  $(c_{i,j})_{1 \leq i \leq d, 1 \leq j \leq n+1}$  such that the elements in the tuple are differentially independent over  $K$ . Then by Theorem 2.17, for  $s \in S$  one has

$$\dim(X_s) \geq d \iff X_s \cap V(\{\sum_{j=1}^n c_{i,j}y_j - c_{1,n+1}\}_{i=1}^d) \neq \emptyset .$$

□

One should note that Theorem 2.17 applies in this case only because over any base of  $S$ , we know that any point on  $S$  is of differential transcendence degree 0. So, choosing some collection of independent differential transcendentals over the base of all of the definable sets,  $S$ , the collection is independent and differentially transcendental over any given point in  $S$ . We will refer to an inhomogeneous linear form,  $\sum_{j=1}^n c_j y_j - c_{n+1}$ , whose coefficients are independent differential transcendentals as a generic inhomogeneous linear form. The zero set of such a form will be called a generic hyperplane. A collection of such forms whose coefficients are independent will be called a set of independent generic forms and the corresponding set of hyperplanes will be called a generic independent set of hyperplanes.

**Lemma 2.19.** *Differential dimension is definable in families. That is, given a  $\delta$ -constructible family of  $\delta$ -varieties  $(X_s)_{s \in S}$  and a number  $d \in \mathbb{N}$  the set  $\{s \in S(\mathcal{U}) : \dim(X_s) = d\}$  is a  $\delta$ -constructible subset of  $S$ .*

*Proof.* Adopt the notation of Lemma 2.18. Suppose that  $\dim(S) = n_1$ . Then pick  $2n_1 + 1$  systems of  $d(n + 1)$ -tuples of mutually independent  $\delta$ -transcendentals (equivalently, fix an indiscernible set in the generic type, over  $K$ ; then pick any  $(2n_1 + 1)d(n + 1)$  elements from this set). Denote the chosen elements

$$\{c_{k,i,j} : 1 \leq k \leq 2n_1 + 1, 1 \leq i \leq d, 1 \leq j \leq n + 1\}$$

Of course, over any given fiber of  $S$  some of the  $2n_1 + 1$  systems do not give generic independent sets of hyperplanes. But, because  $\dim(S) = n_1$  and the systems are mutually independent, at least  $n_1 + 1$  of the systems are generic over any given fiber  $\phi^{-1}(s)$ .

Now, the requirement that  $\dim(\phi^{-1}(s)) \geq d$  is equivalent to the condition that for at least  $n_1 + 1$  values of the  $k$ ,

$$\phi^{-1}(s) \cap V(\sum_{j=1}^n c_{k,1,j}y_j - c_{k,1,n+1}, \dots, \sum_{j=1}^n c_{k,d,j}y_j - c_{k,d,n+1}) \neq \emptyset .$$

□

Recall that the set of numerical polynomials can be totally ordered with respect to the ordering:  $\omega_1 \leq \omega_2$  if and only if  $\omega_1(s) \leq \omega_2(s)$  for all sufficiently large  $s \in \mathbb{N}$ . Given a differential variety  $V$ , we define a *generic component* of  $V$  to be a component which has the maximal Kolchin polynomial among all the components of  $V$ . By the order of a differential variety  $V$ , we mean the order of the generic components of  $V$ .

The order of a family of finite rank differential varieties is definable in families by [20, Appendix A.1]. The general result follows by reducing to this case via an argument similar to the proof of Lemma 2.19. See [8] for complete details.

**Lemma 2.20.** *The order of a definable set is definable in families.*

**Lemma 2.21.** *The number of generic components of a differential variety is definable in families. Characteristic sets of the generic components and the product of their initials and separants are definable in families.*

*Proof.* Let  $(V_b)_{b \in B}$  be a  $\delta$ -constructible family of  $\delta$ -subvarieties of  $\mathbb{A}^n$ .

Throughout this proof we will be repeatedly using Fact 2.10. By Theorem 6.1 of [15], given a finite set  $S$  of differential polynomials, there is a bound on the order and degree of the elements in the characteristic sets of the minimal prime differential ideals  $\mathfrak{p}_i$  ( $i = 1, \dots, \ell$ ) containing the set  $S$ . Indeed, combining Theorems in [34, p.135] and [11, Theorem 2.11], the order of each  $\mathfrak{p}_i$  does not exceed  $\max_{i=1}^n \text{ord}(S, x_i)$ . Thus, the order of each differential polynomial in a characteristic set of  $\mathfrak{p}_i$  with respect to our orderly ranking is also bounded by  $\max_{i=1}^n \text{ord}(S, x_i)$ . Let  $h$  be the order bound for the differential polynomials which define our family of differential varieties. The order of the differential varieties corresponding to the minimal primes is thus bounded by  $n \cdot h$ . These bounds are independent of the choice of  $b \in B$ .

Let  $h_b$  be the order of  $V_b$  and let  $d_b$  be the dimension of  $V_b$ . Both of these quantities are definable in families by Lemmas 2.19 and 2.20. Then

$$\dim(B_{nh}(V_b)) = (nh + 1)(d_b) + h_b.$$

We claim that the irreducible prolongation admissible components of  $B_{nh}(V_b)$  of dimension  $(nh + 1)(d_b) + h_b$  are in one to one correspondence with the generic components of  $V_b$ . Every generic component of  $V_b$  has the same Kolchin polynomial as  $V_b$ . Since  $nh$  is at least as large as the order of any component of  $V_b$ , if  $X$  is any component of  $V_b$ , then  $X$  is a generic component if and only if  $\dim(B_{nh}(X)) = (nh + 1)(d_b) + h_b$ .

On the other hand, prolongation admissible varieties are characterized by the fact that for  $Y \subseteq \tau_h \mathbb{A}^n$ , points of the form  $\{u \in \mathbb{A}^n(\mathcal{U}) \mid \nabla_h(u) \in Y(\mathcal{U})\}$  are Zariski dense in  $Y$ . So, by Lemma 2.7, each irreducible prolongation admissible component  $Y$  of  $B_{nh}(V_b)$  of dimension  $(nh + 1)(d_b) + h_b$  will determine a generic component of  $V_b$ , which is just the unique generic component of the differential variety associated to the prolongation sequence generated by  $Y$ . Applying Fact 2.10 and Lemma 2.16, the number of generic components of  $V_b$  is definable as a function of  $b \in B$ .

By [15, Theorem 6.1], there is a uniform bound on the degree of the elements of a characteristic set of the minimal prime differential ideals containing  $\mathbb{I}(V_b)$ . Using this bound we may parametrize the characteristic sets of the generic components of  $V_b$  uniformly.  $\square$

*Remark 2.22.* Let us note a subtlety in the proof of Lemma 2.21: while there is a correspondence between the irreducible prolongation admissible components of  $B_{nh}(V_b)$  and the generic components of  $V_a$ , it need not be the case that the differential variety given by such a prolongation admissible component be irreducible itself; it may be necessary to consider higher derivatives in order to single out a generic component. In particular, we do not know whether there is a finite bound on the number of additional differentiations required over the family. In general, given a characteristic set of a differential ideal, we do not know a bound for the order of the generators of the differential ideal determined by the characteristic set.

Many of the arguments and bounds in this paper are theoretical, but could be made effective (for instance, many of the definability arguments could be made effective using differential elimination algorithms.). Our proof of Lemma 2.21, by its reliance on nonstandard methods [15], is an exception; producing explicit equations for differential Chow varieties in specific cases would require effectivizing Theorem 6.1 of [15]. The process used in the proof of Lemma 2.21 will not produce all of the components of  $V_a$  in all circumstances. For instance, consider the equation  $(y')^2 - 4y = 0$ . Because  $\delta((y')^2 - 4y) = y'(2y'' - 4)$ , the differential ideal  $[(y')^2 - 4y] \subseteq K\{y\}$  is not prime. The minimal primes in  $K\{y\}/[(y')^2 - 4y]$  correspond to the generic component of the solution set and the special component  $y = 0$ . Of course,  $h = 1$  is a bound for the order of the components of the equation (in the notation of the previous proof). The only maximal irreducible prolongation admissible subvariety of  $(y')^2 - 4y = 0$  considered in  $\tau_1\mathbb{A}^1$  is the entire variety. In this case, the differential variety corresponding to the prolongation sequence generated by  $(y')^2 - 4y = 0$  is not irreducible, but the differential variety corresponding to  $[(y')^2 - 4y] : (y')^\infty$  is irreducible.

There is a way of *partially* remedying this defect given in [12]; let  $S$  be a collection of differential polynomials. Then there is an algorithm which produces a finite set of characteristic sets such that the radical differential ideal  $I$  generated by  $S$  is the intersection of the prime differential ideals given by the characteristic sets. Then the minimal primes of the differential radical ideal generated by  $S$  are among the prime differential ideals given by the finite list of characteristic sets. However, there is no known algorithm for testing which of these ideals are actually minimal. For complete details, see [12].

### 3. CHOW FORMS AND VARIETIES

In this section we recall the definitions of Chow forms, Chow varieties, and their differential algebraic analogs. Suppose  $K$  is an algebraically closed field. The algebraic Chow form was first defined for projective varieties by Chow [3]. When  $\sum_{i=0}^n c_i y_i$  is a linear form with  $\{c_i\}_{i=0}^n$ , a tuple of independent transcendentals, we call the form algebraically generic, and we call the zero set of such a form a generic hyperplane.

**Definition 3.1.** [3, 18] Let  $V \subseteq \mathbb{P}^n$  be an irreducible projective variety of dimension  $d$ . Take  $d$  independent generic linear forms  $L_i = v_{i0}y_0 + \cdots + v_{in}y_n$  for  $1 \leq i \leq d$ , then  $V$  intersects  $V(L_1, \dots, L_d)$  in a finite set of points, say  $(\xi_{\tau_0}, \dots, \xi_{\tau_n})$  ( $\tau = 1, \dots, m$ ). Then there exists a polynomial  $A \in K[\mathbf{v}_1, \dots, \mathbf{v}_d]$  such that  $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = A \prod_{\tau=1}^m (\sum_{j=0}^n v_{0j} \xi_{\tau j})$  is an irreducible polynomial in  $K[\mathbf{v}_0, \dots, \mathbf{v}_d]$  where  $\mathbf{v}_i = (v_{i0}, v_{i1}, \dots, v_{in})$ . This  $F$  is called the *algebraic Chow form* of  $V$ .

The Chow form is that  $F$  is homogeneous in each  $\mathbf{v}_i$  of degree  $m$ . We call  $m$  the *degree* of  $V$ , denoted by  $\deg(V)$ . Throughout the remainder of this paper, unless otherwise indicated, varieties and differential varieties are affine. First, we introduce the concept of algebraic Chow form for irreducible varieties in  $\mathbb{A}^n$ .

**Definition 3.2.** Let  $V$  be an irreducible affine variety of dimension  $d$  in  $\mathbb{A}^n$ . Let  $V'$  be the projective closure of  $V$  with respect to the usual inclusion of  $\mathbb{A}^n$  in  $\mathbb{P}^n$ . We define the algebraic Chow form of  $V$  to be the algebraic Chow form of  $V'$ .

An (effective) *algebraic cycle* in  $\mathbb{A}^n$  of dimension  $d$  over  $K$  is of the form  $V = \sum_{i=1}^l t_i V_i$  ( $t_i \in \mathbb{Z}_{\geq 0}$ ) where each  $V_i$  is an irreducible variety of dimension  $d$  in  $\mathbb{A}^n$ . We define the algebraic Chow form of  $V$  to be  $F(\mathbf{v}_0, \dots, \mathbf{v}_d) = \prod_{i=1}^l (F_i(\mathbf{v}_0, \dots, \mathbf{v}_d))^{t_i}$  where  $F_i$  is the algebraic Chow form of  $V_i$ , and define the degree of  $V$  to be  $\sum_{i=1}^l t_i \deg(V_i)$ , which is the homogenous degree of  $F$  in each  $\mathbf{v}_i$ . The coefficient vector of  $F$ , regarded as a point in a product of projective spaces, is correspondingly called the *Chow coordinate* of  $V$ . Each algebraic cycle is uniquely determined by its algebraic Chow form, in other words, determined by its Chow coordinate.

In [3], Chow proved that the set of all algebraic cycles in  $\mathbb{P}^n$  of dimension  $d$  and degree  $m$  in the Chow coordinate space is a projective variety, called Chow variety of index  $(d, m)$ . In general, the set of all algebraic cycles in  $\mathbb{A}^n$  of dimension  $d$  and degree  $m$  is not closed in the Chow coordinate space. Below, we give a simple example.

**Example 3.3.** Consider the set  $X$  of all algebraic cycles in  $\mathbb{A}^2$  of dimension 0 and degree 1. Each  $V \in X$  can be represented by two linear equations  $a_{i0} + a_{i1}y_1 + a_{i2}y_2 = 0$  ( $i = 0, 1$ ) with  $a_{01}a_{12} - a_{02}a_{11} \neq 0$ . Then the Chow form of  $V$  is  $F(v_{00}, v_{01}, v_{02}) = (a_{01}a_{12} - a_{02}a_{11})v_{00} - (a_{00}a_{12} - a_{02}a_{10})v_{01} + (a_{00}a_{11} - a_{01}a_{10})v_{02}$ . So the Chow coordinate of  $V$  is  $(a_{01}a_{12} - a_{02}a_{11}, -a_{00}a_{12} + a_{02}a_{10}, a_{00}a_{11} - a_{01}a_{10})$ . Thus, the Chow coordinates of cycles in  $X$  is the set  $\{(c_0, c_1, c_2) : c_0 \neq 0\} = \mathbb{P}^n \setminus V(c_0)$ , which is not a closed variety, but is a constructible set.

The following result shows that the set of all cycles with given degree and dimension is always a constructible set in the Chow coordinate space.

**Proposition 3.4.** *The set of all algebraic cycles in  $\mathbb{A}^n$  of dimension  $d$  and degree  $m$  is a constructible set in a higher dimensional projective space. We call this set the affine Chow variety of index  $(d, m)$ , denoted by  $\text{Chow}_n(d, m)$ , or  $\text{Chow}(d, m)$  if the space  $\mathbb{A}^n$  is clear from the context.*

*Proof.* Let  $M$  be the set of all monomials in  $\mathbf{v}_0, \dots, \mathbf{v}_d$  which are of degree  $m$  in each  $\mathbf{v}_i$ . That is,  $M = \{\prod_{i=0}^d \prod_{j=0}^n v_{ij}^{\sigma_{ij}} \mid \sigma_{ij} \in \mathbb{Z}_{\geq 0}, \sum_{j=0}^n \sigma_{ij} = m\}$ . Let  $F_0 = \sum_{\phi \in M} c_\phi \phi$  where  $c_\phi$  are algebraic indeterminates over  $K$ . By [3, 18], there exists a projective variety  $W \subseteq \mathbb{P}^{|M|-1}$  such that  $(\bar{c}_\phi : \phi \in M) \in W$  if and if  $\bar{F}_0 = \sum_{\phi \in M} \bar{c}_\phi \phi$  is the algebraic Chow form of an algebraic cycle in  $\mathbb{P}^n$  of dimension  $d$  and degree  $m$ .

Let  $N = \{v_{00}^m \prod_{i=1}^d \prod_{j=0}^n v_{ij}^{\sigma_{ij}} \mid \sigma_{ij} \in \mathbb{Z}_{\geq 0}, \sum_{j=0}^n \sigma_{ij} = m\} \subseteq M$  and let  $\{c_1, \dots, c_{|N|}\}$  be the set of all coefficients of  $F_0$  with respect to monomials contained in  $N$ . Let  $W_1 = W \setminus V(c_1, \dots, c_{|N|})$ . We claim that there is a one-to-one correspondence between  $\text{Chow}_n(d, m)$  and  $W_1$  via algebraic Chow forms. On the one hand, for each point in  $\text{Chow}_n(d, m)$  corresponding to an algebraic cycle  $V$ , the algebraic Chow form  $F = \sum_{\phi \in M} \bar{c}_\phi \phi$  of  $V$  has the following Poisson-type product formula:

$F = A \prod_{\tau=1}^m (v_{00} + \sum_{j=1}^n v_{0j} \xi_{\tau j})$  where  $A \in k[\mathbf{v}_1, \dots, \mathbf{v}_d]$  and  $(\xi_{\tau 1}, \dots, \xi_{\tau n})$  is a generic point of a component of  $V$ . Thus, there exists at least one monomial  $\phi \in N$  such that  $\phi$  appears effectively in  $F$ . As a consequence,  $(\bar{c}_\phi) \in W_1$ . On the one hand, for each  $(\bar{c}_\phi : \phi \in M) \in W_1$ ,  $\bar{F}_0 = \sum_{\phi \in M} \bar{c}_\phi \phi$  is the algebraic Chow form of an algebraic cycle  $\bar{V}' = \sum_i t_i V'_i$  in  $\mathbb{P}^n$  of dimension  $d$  and degree  $m$ . Also from the Poisson-product formula, we can see that each  $V'_i \not\subseteq \mathbb{V}(x_0)$ . Thus,  $\bar{F}_0$  is the algebraic Chow form of the algebraic cycle  $\bar{V} = \sum_i t_i V_i \in \text{Chow}_n(d, m)$ . Hence, we have proved that  $\text{Chow}_n(d, m)$  is a constructible set.  $\square$

Let  $V \subseteq \mathbb{A}^n$  be an irreducible differential variety defined over  $K$  of dimension  $d$  and

$$L_i = u_{i0} + u_{i1}y_1 + \dots + u_{in}y_n \quad (i = 0, \dots, d)$$

be  $d+1$  differentially generic inhomogeneous linear forms. For each  $i$ , denote  $\mathbf{u}_i = (u_{i0}, u_{i1}, \dots, u_{in})$ . Let

$$(1) \quad \mathcal{J}_{\mathbf{u}} = [\mathbb{I}(V), L_0, \dots, L_d]_{K\{y_1, \dots, y_n, \mathbf{u}_0, \dots, \mathbf{u}_d\}} \cap K\{\mathbf{u}_0, \dots, \mathbf{u}_d\}.$$

Then by [11, Lemma 4.1],  $\mathcal{J}_{\mathbf{u}}$  is a prime differential ideal in  $K\{\mathbf{u}_0, \dots, \mathbf{u}_d\}$  of codimension one.

**Definition 3.5.** The *differential Chow form* of  $V$  or  $\mathbb{I}(V)$  is defined as the unique (up to appropriate scaling) irreducible differential polynomial  $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$  such that  $\mathcal{J}_{\mathbf{u}} = \text{sat}(F)$  under any rankings.

Differential Chow forms uniquely characterize their corresponding differential ideals. The following theorem gives some basic properties of differential Chow forms.

**Theorem 3.6.** [11] *Let  $V$  be an irreducible differential variety defined over  $K$  with differential dimension  $d$  and order  $h$ . Suppose  $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$  is the differential Chow form of  $V$ . Then  $F$  has the following properties.*

- 1)  $\text{ord}(F) = h$ . In particular,  $\text{ord}(F, u_{i0}) = h$  for each  $i = 0, \dots, d$ .
- 2)  $F$  is differentially homogenous of the same degree  $m$  in each  $\mathbf{u}_i$ . This  $m$  is called the differential degree of  $V$ .
- 3) Let  $g = \deg(F, u_{00}^{(h)})$ . There exist differential extension fields  $K_\tau$  ( $\tau = 1, \dots, g$ ) of  $K$  and  $\xi_{\tau j} \in K_\tau$  ( $j = 1, \dots, n$ ) such that  $F = A \prod_{\tau=1}^g (u_{00} + u_{01}\xi_{\tau 1} + \dots + u_{0n}\xi_{\tau n})^{(h)}$  where  $A$  is a differential polynomial free from  $u_{00}^{(h)}$ . Moreover, each  $\xi_\tau = (\xi_{\tau 1}, \dots, \xi_{\tau n})$  are generic points of  $V$  and  $L_1, \dots, L_d$  all vanish at  $\xi_\tau$ .
- 4) The algebraic variety  $B_h(V) \cap V(L_1^{(h)}, \dots, L_d^{(h)}, L_0^{(h-1)}) \subseteq \tau_h \mathbb{A}^n$  is of dimension zero. Its size,  $g$ , is called the leading differential degree of  $V$ .

A differential variety is called *order-unmixed* if all its components have the same differential dimension and order. Let  $V$  be an order-unmixed differential variety of dimension  $d$  and order  $h$  and  $V = \bigcup_{i=1}^l V_i$  its minimal irreducible decomposition with  $F_i(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)$  the Chow form of  $V_i$ . Let

$$(2) \quad F(\mathbf{u}_0, \dots, \mathbf{u}_d) = \prod_{i=1}^l F_i(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_d)^{s_i}$$

with  $s_i$  arbitrary nonnegative integers. In [11], a *differential algebraic cycle* is defined associated to (2) similar to its algebraic analog, that is,  $\mathbf{V} = \sum_{i=1}^l s_i V_i$  is

a differential algebraic cycle with  $s_i$  as the multiplicity of  $V_i$  and  $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$  is called the differential Chow form of  $\mathbf{V}$ .

Suppose each  $V_i$  is of differential degree  $m_i$  and leading differential degree  $g_i$ , then the leading differential degree and differential degree of  $\mathbf{V}$  is defined to be  $\sum_{i=1}^l s_i g_i$  and  $\sum_{i=1}^l s_i m_i$  respectively.

**Definition 3.7.** A differential cycle  $\mathbf{V}$  in the  $n$  dimensional affine space with dimension  $d$ , order  $h$ , leading differential degree  $g$ , and differential degree  $m$  is said to be of index  $(d, h, g, m)$  in  $\mathbb{A}^n$ .

**Definition 3.8.** Let  $\mathbf{V}$  be a differential cycle of index  $(d, h, g, m)$  in  $\mathbb{A}^n$ . The *differential Chow coordinate* of  $\mathbf{V}$  is the coefficient vector of the differential Chow form of  $\mathbf{V}$  considered as a point in a higher dimensional projective space determined by  $(d, h, g, m)$  and  $n$ .

**Proposition 3.9.** *The differential Chow form of an order-unmixed differential variety  $V$  is definable in families. The index  $(d, h, g, m)$  of a  $\delta$ -cycle in  $\mathbb{A}^n$  is definable in families.*

*Proof.* The definability of  $d$  and  $h$  in families is given by Propositions 2.19 and 2.20, respectively. It is clear that  $g$  and  $m$  are definable in families, given that the differential Chow form is definable in families.

Suppose now that  $V \subseteq \mathbb{A}^n$  has  $\delta$ -dimension  $d$  and order  $h$ . The characteristic sets of the generic components of a family of differential varieties are definable in families by Lemma 2.21. Let  $\mathbf{u} = (\mathbf{u}_0, \dots, \mathbf{u}_d)$  be  $\delta$ -indeterminates with  $\mathbf{u}_i = (u_{i0}, \dots, u_{in})$  for  $0 \leq i \leq d$ . Let  $L_i := u_{i,0} + \sum_{j=1}^n u_{ij}x_j \in K\{\mathbf{u}, x_1, \dots, x_n\}$ . Let  $\mathcal{J}$  be the differential ideal generated by  $\mathbb{I}(V)$  and  $\{L_0, \dots, L_d\}$ .

Let  $W := \mathbb{V}(\mathcal{J})$ . Visibly,  $W$  is uniformly defined from  $V$ . Moreover,  $B_h(W)$  is uniformly defined from  $W$ . To see this, note that by [21, Remark 3], the degree of  $B_h(W)$  is uniformly bounded in terms of a number of variables, and the degree and order of the equations defining  $W$ . The variety  $B_h(W)$  is defined by at most  $(n + (n+1)(d+1))(h+1) + 1$  polynomials over  $K$  of degree at most  $\deg(B_h(W))$  (see [16], for instance). Among all zero sets of such collections, the ones which contain the  $\delta$ -points of  $W$  and are minimal with respect to containment give  $B_h(W)$  since the  $\delta$ -points are dense in  $B_h(W)$ . Now take the projection of  $B_h(W)$  to the coordinates corresponding to  $\mathbf{u}$  and their derivatives. By the definition and properties of differential Chow form,  $I(B_h(W)) \cap k\{\mathbf{u}_0, \dots, \mathbf{u}_d\}_{\leq h} = (F)$ , where  $F$  is the differential Chow form of  $V$ . Thus, the image of the projection is a hypersurface of degree at most  $\deg(B_h(W))$ , given by the vanishing of the differential Chow form.  $\square$

**Definition 3.10.** Fix an index  $(d, h, g, m)$  and  $n$  and consider the set

$$\mathbb{V}_{(n,d,h,g,m)} = \{\mathbf{V} : \mathbf{V} \text{ is a differential cycle of index } (d, h, g, m) \text{ in } \mathbb{A}^n\}.$$

If  $\mathbb{V}_{(n,d,h,g,m)}$  has the structure of a  $\delta$ -constructible in some affine differential variety of finite type, then  $\mathbb{V}_{(n,d,h,g,m)}$  is called the *differential Chow Variety* of index  $(d, h, g, m)$  of  $\mathbb{A}^n$ , denoted by  $\delta\text{-Chow}(n, d, h, g, m)$ .

**Theorem 3.11.** [11, Theorem 5.7] *In the case  $g = 1$ , the differential Chow variety  $\delta\text{-Chow}(n, d, h, 1, m)$  exists.*

**Remark 3.12.** In the algebraic setting, for an arbitrary tuple  $(d, m)$ ,  $\text{Chow}_n(d, m)$  is always a nonempty constructible set. However, it is more subtle in the differential



case and  $\mathbb{V}_{(n,d,h,g,m)}$  may be empty for certain values  $(n, d, h, g, m)$ . For example, when a differential algebraic cycle is of order 1, its differential degree is at least 2, so  $\mathbb{V}_{(n,d,1,g,1)} = \emptyset$ .

#### 4. DEGREE BOUND FOR PROLONGATION SEQUENCES

We are interested in the space of all differential cycles in  $n$  dimensional affine space of some fixed index  $(d, h, g, m)$ . Ultimately, the point in our parameter space corresponding to a differential cycle  $\sum_i a_i V_i$  will be given by the point representing  $\sum_i a_i B_h(V_i)$  in an appropriate algebraic Chow variety. In order to ensure that the space of such algebraic varieties has the structure of a definable set, we must establish degree bounds for the corresponding algebraic cycles. This is the topic of the present section.

**Proposition 4.1.** *Suppose  $V$  is an irreducible differential variety of index  $(d, h, g, m)$  in  $\mathbb{A}^n$ . Then there is a natural number  $D$  depending only on  $(d, h, g, m)$  such that  $B_h(V) \subseteq \tau_h \mathbb{A}^n$  is an irreducible algebraic variety with degree satisfying  $\deg(B_h(V)) \leq D$ .*

*Proof.* The irreducibility of  $B_h(V)$  follows from the fact that  $B_h(V) = V(\mathbb{I}(V) \cap K\{x_1, \dots, x_n\}_{\leq h})$ . It remains for us to show that there is  $D$  with the claimed properties.

Suppose  $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$  is the differential Chow form of  $V$  where  $\mathbf{u}_i = (u_{i0}, \dots, u_{in})$  ( $i = 0, \dots, d$ ). Let  $\mathbf{u}$  be the tuple of variables  $(u_{i,j})_{i=0,j=1}^{d,n}$ . That is, we are omitting the variables of the form  $u_{i,0}$ . Set  $K_1 = K\langle \mathbf{u} \rangle$ . Let  $W$  be the differential variety in  $\mathbb{A}^{d+1}$  defined by  $\text{sat}(F)$  considered as a differential ideal in  $K_1\{u_{00}, \dots, u_{d0}\}$ . Then by Theorem 3.6,  $B_h(W) = V(F) \subseteq \tau_h \mathbb{A}^{d+1}$  is an irreducible variety.

By Theorem [11, Theorem 4.13], the map given by

$$f(\mathbf{u}_0) = \left( \frac{\partial F}{\partial u_{0,i}^{(h)}} / S_F \right)_{i=1}^n$$

gives a differential birational map from  $W$  to  $V_{K_1}$ . By quantifier elimination in  $\text{DCF}_0$ , the image is given by the vanishing and non vanishing of some collection of differential polynomials. By the compactness theorem, the number, degree and order of these equations and inequations must be bounded uniformly depending only on the degrees, orders, and number of variables of  $F$  and  $f$ . The results of [21] (see Remark 3.2) give a uniform upper bound,  $D$  for the degree of  $B_h(V)$ .  $\square$

**Corollary 4.2.** *Suppose  $\mathbf{V} = \sum_i a_i V_i \subseteq \mathbb{A}^n$  ( $a_i \in \mathbb{Z}_{\geq 0}$ ) is an order-unmixed differential variety of index  $(d, h, g, m)$ . Then there is a natural number  $D$  such that  $\sum_i a_i B_h(V_i)$  is an algebraic cycle in  $\tau_h \mathbb{A}^n$  of dimension  $d(h+1) + h$  and degree satisfying  $\deg(B_h(V)) \leq D$ .*

It is possible to give effective versions of Proposition 4.1 and Corollary 4.2 with a more complicated proof; the following proposition gives such detailed effective bounds. In the following section of the paper, we will use Corollary 4.2 to restrict the space of algebraic Chow varieties which we consider. A more detailed analysis of the particular defining equations of the differential Chow variety might be undertaken by applying the more detailed effective bounds of the following Proposition (or improving upon them), but the main thrust of our results in the next section concerns the *existence* of differential Chow varieties, so the following result is given

primarily given to indicate that the construction of differential Chow varieties can be made effective *in principle*.

**Proposition 4.3.** *Suppose  $V$  is an irreducible differential variety of index  $(d, h, g, m)$  in  $\mathbb{A}^n$ . Then  $B_h(V) \subseteq \tau_h \mathbb{A}^n$  is an irreducible algebraic variety with degree satisfying*

$$(3) \quad \max\{g, m/(h+1)\} \leq \deg(B_h(V)) \leq [(d+1)m]^{nh+n+1}.$$

*Proof.* Suppose  $F(\mathbf{u}_0, \dots, \mathbf{u}_d)$  is the differential Chow form of  $V$  where  $\mathbf{u}_i = (u_{i0}, \dots, u_{in})$  ( $i = 0, \dots, d$ ). Let  $\mathbf{u} = (u_{ij})_{i=0, j=1}^{d, n}$  and  $K_1 = K(\mathbf{u})$ .

Let  $\mathcal{J} = [\mathbb{I}(V), L_0, \dots, L_d] \subset K_1\{x_1, \dots, x_n, u_{00}, \dots, u_{d0}\}$ . Then by the proof of [11, Theorem 4.36], the polynomials  $g_{jk} = \frac{\partial F}{\partial u_{00}^{(h)}} x_j^{(k)} + \sum_{\ell=1}^k \binom{h-\ell}{k-\ell} \binom{h}{k} \frac{\partial F}{\partial u_{00}^{(h-\ell)}} x_j^{(k-\ell)} - \frac{\partial F}{\partial u_{0j}^{(h-k)}} (j = 1, \dots, n; k = 0, \dots, h)$  are contained in  $\mathcal{J}$ . Fix an ordering of algebraic indeterminates so that  $x_1 < \dots < x_n < x_1^{(1)} < \dots < x_n^{(1)} < \dots < x_1^{(h)} < \dots < x_n^{(h)}$  and  $u_{ij}^{(k)} < x_\ell^{(m)}$  for all  $i, j, k, \ell$ , and  $m$ .

Let  $\mathcal{J}^{[h]} := \mathcal{J} \cap K_1\{x_1, \dots, x_n, u_{00}, \dots, u_{d0}\}_{\leq h}$ . Since for each  $f \in \mathcal{J}^{[h]}$ , the algebraic remainder of  $f$  with respect to  $g_{jk}$  is a polynomial in  $\mathcal{J} \cap K_1\{u_{00}, \dots, u_{d0}\}_{\leq h} = (F), \{F\} \cup \{g_{jk} : 1 \leq j \leq n, 0 \leq k \leq h\}$  constitutes an algebraic characteristic set of  $\mathcal{J}^{[h]}$ . Thus,

$$\mathcal{J}^{[h]} = (F, (g_{jk})_{1 \leq j \leq n; 0 \leq k \leq h}) : \left( \frac{\partial F}{\partial u_{00}^{(h)}} \right)^\infty.$$

Since the variety defined by the ideal  $(F, (g_{jk})_{1 \leq j \leq n; 0 \leq k \leq h}) : \left( \frac{\partial F}{\partial u_{00}^{(h)}} \right)^\infty$  is a component of the closed set given by the vanishing of  $(F, (g_{jk})_{1 \leq j \leq n; 0 \leq k \leq h})$ , by [16, Theorem 1],

$$\begin{aligned} \deg(\mathcal{J}^{[h]}) &\leq \deg((F, (g_{jk})_{1 \leq j \leq n; 0 \leq k \leq h})) \\ &\leq \deg(F)^{n(h+1)+1} \leq [(d+1)m]^{nh+n+1}. \end{aligned}$$

Since  $\mathcal{J}^{[h]} \cap K_1\{x_1, \dots, x_n\}_{\leq h} = I(B_h(V)_{K_1})$ , by [16, 27],  $\deg(B_h(V)) \leq \deg(\mathcal{J}^{[h]})$ . Hence,  $\deg(B_h(V)) \leq [(d+1)m]^{nh+n+1}$ .

Since  $\dim(B_h(V)) = d(h+1) + h$  and by [11],  $B_h(V)$  and some  $d(h+1) + h$  hyperplanes defined by  $L_0^{(i)}$  for  $0 \leq i < h$  and  $L_1^{[i]}, \dots, L_d^{[i]}$  for  $0 \leq i \leq h$  intersect in  $g$  points,  $\deg(B_h(V)) \geq g$ . On the other hand,  $F$  can be obtained from the algebraic Chow form of  $B_h(V)$  using the strategy of specializations in [27, Theorem 4.2]. So  $\deg(F) \leq (h+1)(d+1)\deg(B_h(V))$  and  $\deg(F) = m(d+1)$ . Thus, (3) follows.  $\square$

## 5. ON DIFFERENTIAL CHOW VARIETIES

In this section, we will show that for a fixed  $n \in \mathbb{N}$  and a fixed index  $(d, h, g, m)$ ,  $\mathbb{V}_{(n, d, h, g, m)}$  is a  $\delta$ -constructible set. That is, the differential Chow variety  $\delta\text{-Chow}(n, d, h, g, m)$  exists.

Consider the disjoint union of algebraic constructible sets

$$\mathcal{C} = \bigcup_{e \leq D} \text{Chow}(d(h+1) + h, e)(\tau_h \mathbb{A}^n)$$

where  $D$  is the bound of Corollary 4.2. Let  $\mathcal{C}_1$  be the subset consisting of all points  $a \in \mathcal{C}$  such that

- $a$  is the Chow coordinate of an algebraic cycle  $\sum_i t_i W_i$  where each  $W_i$  is irreducible and prolongation admissible and
- the generic component of the differential variety corresponding to the prolongation sequence generated by  $W_i$  is of index  $(d, h, g_i, m_i)$  and  $\sum_i t_i g_i = g, \sum_i t_i m_i = m$ .

**Theorem 5.1.** *The set  $\mathcal{C}_1$  is constructible and the map which associates a differential algebraic cycle  $\mathbf{V} = \sum s_i V_i$  of index  $(d, h, g, m)$  in  $\mathbb{A}^n$  with the Chow coordinate of the algebraic cycle  $\sum s_i B_h(V_i)$  identifies  $\delta\text{-Chow}(n, d, h, g, m)$  with  $\mathcal{C}_1$ . In particular, the differential Chow variety  $\delta\text{-Chow}(n, d, h, g, m)$  exists.*

*Proof.* From the very definition of prolongation admissibility, it is  $\delta$ -constructible condition. Thus, for each  $e$  the set of Chow coordinates of positive cycles of degree  $e$  built from prolongation admissible varieties of dimension  $d(h+1) + h$  is a  $\delta$ -constructible subset of  $\text{Chow}(d(h+1) + h, e)(\tau_h \mathbb{A}^n)$ . By Lemma 2.19 and Lemma 2.20, the set of Chow coordinates in which each irreducible variety in a given cycle corresponds generates a prolongation sequence whose corresponding differential variety has dimension  $d$  and order  $h$  is a  $\delta$ -constructible set. Thus, by Proposition 3.9,  $\mathcal{C}_1$  is  $\delta$ -constructible.

By Lemma 2.7 the differential algebraic cycle  $\mathbf{V} = \sum s_i V_i$  is determined by the algebraic cycle  $\sum s_i B_h(V_i)$ . By Corollary 4.2, this algebraic cycle belongs to  $\mathcal{C}_1$ .  $\square$

*Remark 5.2.* For the special case  $d = n - 1$ , the existence of the differential Chow variety of index  $(n - 1, h, g, m)$  can be easily shown from the point of view of differential characteristic sets. Indeed, note that each order-unmixed radical differential ideal  $\mathcal{I}$  of dimension  $n - 1$  and order  $h$  has the prime decomposition  $\mathcal{I} = \bigcap_{i=1}^t \text{sat}(f_i) = \text{sat}(\prod_{i=1}^t f_i)$ , where  $f_i \in K\{x_1, \dots, x_n\}$  is irreducible and of order  $h$ . Thus, there is a one-to-one correspondence between  $\mathbb{V}_{(n-1, h, g, m)}$  and the set of all differential polynomials  $f \in K\{x_1, \dots, x_n\}$  such that each irreducible component of  $f$  is of order  $h$ ,  $\deg(f, \{x_1^{(h)}, \dots, x_n^{(h)}\}) = g$  and the denomination of  $f$  is equal to  $m$ . Here, the *denomination* of  $f$  is the smallest number  $r$  such that  $x_0^r p(x_1/x_0, \dots, x_n/x_0) \in \mathcal{F}\{x_0, x_1, \dots, x_n\}$  [25]. Since all these characteristic numbers are definable for differential polynomials,  $\mathbb{V}_{(n-1, h, g, m, n)}$  is a definable subset of  $\mathbb{A}^{\binom{m+n(h+1)}{n(h+1)}}$ . Hence, the differential Chow variety of index  $(n - 1, h, g, m)$  exists.

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APPENDIX: GEOMETRIC IRREDUCIBILITY AND ZARISKI CLOSURE ARE  
DEFINABLE IN FAMILIES  
BY WILLIAM JOHNSON

In this appendix we establish the results on definability in algebraically closed fields stated as Fact 2.10 in the main text. We follow standard model theoretic notations and conventions. For example, we write  $RM(a/B)$  for the Morley rank of the type of  $a$  over  $B$  and use the nonforking symbol freely.

**A.1. Irreducibility in Projective Space.** Let  $\mathbb{C}$  be a monster model of  $ACF$ . For  $\vec{x} \in \mathbb{P}^n(\mathbb{C})$ , let  $\mathbb{P}_{\vec{x}}$  be the  $n - 1$ -dimensional projective space of lines through  $\vec{x}$ , and let  $\pi_{\vec{x}} : \mathbb{P}^n \setminus \{\vec{x}\} \rightarrow \mathbb{P}_{\vec{x}}$  be the projection.

**Lemma 1.** *Let  $A$  be a small set of parameters, and suppose  $\vec{x} \in \mathbb{P}^n(\mathbb{C})$  is generic over  $A$ . Suppose  $V$  is an  $A$ -definable Zariski closed subset of  $\mathbb{P}^n$ , of codimension greater than 1. Then  $\pi_{\vec{x}}(V) \subseteq \mathbb{P}_{\vec{x}}$  is well-defined, Zariski closed, of codimension one less than the codimension of  $V$ . Moreover,  $\pi_{\vec{x}}(V)$  is irreducible if and only if  $V$  is irreducible.*

*Proof.* Replacing  $A$  with  $\text{acl}(A)$ , we may assume  $A$  is algebraically closed, implying that the irreducible components of  $V$  are also  $A$ -definable.

Since  $\vec{x}$  is generic, and  $V$  has codimension at least 1,  $\vec{x} \notin V$  so  $\pi_{\vec{x}}(V)$  is well-defined. It is Zariski closed because  $\mathbb{P}^n$  is a complete variety, so  $V$  is complete and the image of  $V$  under any morphism of varieties is closed.

**Claim 2.** *Let  $C$  be any irreducible component of  $V$ , and let  $\vec{c} \in V$  realize the generic type of  $C$ , over  $A\vec{x}$ . Then  $\vec{c}$  is the sole preimage in  $V$  of  $\pi_{\vec{x}}(\vec{c})$ .*

*Proof.* The generic type of  $C$  is  $A$ -definable, so  $\vec{c} \perp_A \vec{x}$ , and therefore  $RM(\vec{x}/A\vec{c}) = RM(\vec{x}/A) = n$ . Suppose for the sake of contradiction that there was a second point  $\vec{d} \in V$ ,  $\vec{d} \neq \vec{c}$ , satisfying

$$\pi_{\vec{x}}(\vec{d}) = \pi_{\vec{x}}(\vec{c}).$$

This means exactly that the three points  $\vec{c}$ ,  $\vec{d}$ , and  $\vec{x}$  are colinear. Then  $\vec{x}$  is on the 1-dimensional line determined by  $\vec{c}$  and  $\vec{d}$ , so

$$RM(\vec{x}/A\vec{c}\vec{d}) \leq 1.$$

But then

$$n = RM(\vec{x}/A\vec{c}) \leq RM(\vec{x}\vec{d}/A\vec{c}) = RM(\vec{x}/A\vec{c}\vec{d}) + RM(\vec{d}/A\vec{c}) \leq 1 + RM(V) < n,$$

by the codimension assumption.  $\square$

Using the claim, we see that  $\pi_{\vec{x}}(V)$  and  $V$  have the same dimension (= Morley rank). Indeed, let  $\vec{v} \in V$  have Morley rank  $RM(V)$  over  $A\vec{x}$ . Then  $\vec{v}$  realizes the generic type of *some* irreducible component  $C$ , so by the claim,  $\vec{v}$  is interdefinable over  $A\vec{x}$  with  $\pi_{\vec{x}}(\vec{v})$ . But then

$$RM(\pi_{\vec{x}}(V)) \geq RM(\pi_{\vec{x}}(\vec{v})/A\vec{x}) = RM(\vec{v}/A\vec{x}) = RM(V),$$

and the reverse inequality is obvious. So the codimension of  $\pi_{\vec{x}}(V)$  is indeed one less.

Let  $C_1, \dots, C_m$  enumerate the irreducible components of  $V$  (possibly  $m = 1$ ). Each of the components  $C_i$  is a closed subset of  $\mathbb{P}^n$ , and so by completeness each of the images  $\pi_{\vec{x}}(C_i)$  is a Zariski closed subset of  $\mathbb{P}_{\vec{x}}$ . The image of each of the

components is irreducible, on general grounds. If  $\pi_{\vec{x}}(C_i) \subseteq \pi_{\vec{x}}(C_j)$  for some  $i \neq j$ , then the generic type of  $C_i$  would have the same image under  $\pi_{\vec{x}}$  as some point in  $C_j$ , contradicting the Claim. So  $\pi_{\vec{x}}(C_i) \not\subseteq \pi_{\vec{x}}(C_j)$  for  $i \neq j$ . It follows that the images  $\pi_{\vec{x}}(C_i)$  are the irreducible components of

$$\pi_{\vec{x}}(V) = \bigcup_{i=1}^m \pi_{\vec{x}}(C_i).$$

Therefore,  $\pi_{\vec{x}}(V)$  and  $V$  have the same number of irreducible components, proving the last point of the lemma.  $\square$

**Theorem 3.** *Let  $X_{\vec{a}} \subseteq \mathbb{P}^n$  be a definable family of Zariski closed subsets of  $\mathbb{P}^n$ . Then the set of  $\vec{a}$  for which  $X_{\vec{a}}$  is irreducible, is definable.*

*Proof.* Dimension is definable in families, because *ACF* is strongly minimal. So we may assume that all (non-empty)  $X_{\vec{a}}$  have the same (co)dimension. We proceed by induction on codimension, allowing  $n$  to vary.

For the base case of codimension one, we note the following:

- (1) The family of Zariski closed subsets of  $\mathbb{P}^n$  is ind-definable, that is a small (i.e. less than the size of the monster model) union of definable families, because the Zariski closed subsets are exactly the zero sets of finitely-generated ideals.
- (2) Using 1, the family of *reducible* Zariski closed subsets of  $\mathbb{P}^n$  is also ind-definable, because a definable set is a *reducible* Zariski closed set if and only if it is the union of two incomparable (with respect to containment) Zariski closed sets.
- (3) Whether or not a polynomial in  $\mathbb{C}[x_1, \dots, x_{n+1}]$  is irreducible, is definable in terms of the coefficients, because we only need to quantify over lower-degree polynomials.
- (4) A hypersurface in  $\mathbb{P}^n$  is irreducible if and only if it is equal to the zero-set of an irreducible homogeneous polynomial. It follows by 3 that the family of irreducible codimension 1 closed subsets of  $\mathbb{P}^n$  is ind-definable.
- (5) By 2 (resp. 4), the set of  $\vec{a}$  such that  $X_{\vec{a}}$  is reducible (resp. irreducible) is ind-definable. Since these two sets are complementary, both are definable, proving the base case.

For the inductive step, suppose that irreducibility is definable in families of codimension one less than  $X_{\vec{a}}$ . By choosing an isomorphism between  $\mathbb{P}_{\vec{x}}$  and  $\mathbb{P}^{n-1}$ , one easily verifies the definability of the set of  $(\vec{x}, \vec{a})$  such that  $\pi_{\vec{x}}(X_{\vec{a}})$  is irreducible and has codimension one less.

By Lemma 1,  $X_{\vec{a}}$  is irreducible if and only if  $(\vec{x}, \vec{a})$  lies in this set, for generic  $\vec{x}$ . Definability of types in stable theories then implies definability of the set of  $\vec{a}$  such that  $X_{\vec{a}}$  is irreducible.  $\square$

**Corollary 4.** *The family of irreducible closed subsets of  $\mathbb{P}^n$  is ind-definable.*

*Proof.* The family of closed subsets is ind-definable, and by Theorem 3 we can select the irreducible ones within any definable family.  $\square$

**Corollary 5.** *The family of pairs  $(X, \overline{X})$  with  $X$  definable and  $\overline{X}$  its Zariski-closure, is ind-definable.*

*Proof.* By quantifier elimination in  $ACF$ , any definable set  $X$  can be written as a union of sets of the form  $C \cap U$  with  $C$  closed and  $U$  open. Replacing  $V$  with a union of irreducible components, and distributing, we can write  $X$  as a union  $\bigcup_{i=1}^m C_i \cap U_i$ , with  $C_i$  Zariski closed and  $U_i$  Zariski open. We may assume that  $C_i \cap U_i \neq \emptyset$  for each  $i$ , or equivalently, that  $C_i \setminus U_i \neq C_i$ .

In any topological space, closure commutes with unions, so

$$\overline{X} = \bigcup_{i=1}^n \overline{C_i \cap U_i}.$$

Now  $\overline{C_i \cap U_i} \subseteq \overline{C_i} = C_i$ , and

$$C_i = \overline{C_i \cap U_i} \cup (C_i \setminus U_i),$$

so by irreducibility of  $C_i$ ,  $\overline{C_i \cap U_i} = C_i$ . Therefore,

$$\overline{X} = \bigcup_{i=1}^n C_i.$$

Corollary 4 implies the ind-definability of the family of pairs

$$\left( \bigcup_{i=1}^n \overline{C_i \cap U_i}, \bigcup_{i=1}^n C_i \right)$$

with  $C_i$  irreducible closed,  $U_i$  open, and  $C_i \cap U_i \neq \emptyset$ . We have seen that this is the desired family of pairs.  $\square$

The following corollary is an easy consequence:

**Corollary 6.** *Let  $X_{\vec{a}}$  be a definable family of subsets of  $\mathbb{P}^n$ . Then the Zariski closures  $\overline{X_{\vec{a}}}$  are also a definable family.*

## A.2. Irreducibility in Affine Space.

**Theorem 7.** *Let  $X_{\vec{a}}$  be a definable family of subsets of affine  $n$ -space.*

- (1) *The family of Zariski closures  $\overline{X_{\vec{a}}}$  is also definable.*
- (2) *The set of  $\vec{a}$  such that  $\overline{X_{\vec{a}}}$  is irreducible is definable. More generally, the number of irreducible components of  $\overline{X_{\vec{a}}}$  is definable in families (and bounded in families).*
- (3) *Dimension and Morley degree of  $X_{\vec{a}}$  are definable in  $\vec{a}$ .*
- (4) *If each  $\overline{X_{\vec{a}}}$  is a hypersurface given by the irreducible polynomial  $F_{\vec{a}}(x_1, \dots, x_n)$ , then the degree of  $F_{\vec{a}}$  in each  $x_i$  is definable in  $\vec{a}$ . In fact, the polynomials  $F_{\vec{a}}$  have bounded total degree and the family of  $F_{\vec{a}}$  (up to scalar multiples) is definable.*
- (5) *The family of irreducible components of the Zariski closure is definable in families.*

*Proof.* (1) Embed  $\mathbb{A}^n$  into  $\mathbb{P}^n$ . Then the Zariski closure of  $X_{\vec{a}}$  within  $\mathbb{A}^n$  is the intersection of  $\mathbb{A}^n$  with the closure within  $\mathbb{P}^n$ . Use Corollary 6.  
 (2) The number of irreducible components of the Zariski closure is the same whether we take the closure in  $\mathbb{A}^n$  or  $\mathbb{P}^n$ . This proves the first sentence. The first sentence yields the ind-definability of the family of irreducible Zariski closed subsets of  $\mathbb{A}^n$ , from which the second statement is an exercise in compactness.

- (3) We may assume  $X_{\vec{a}}$  is closed, since taking the closure changes neither Morley rank nor Morley degree. The family of  $d$ -dimensional Zariski irreducible closed subsets of  $\mathbb{A}^n$  is ind-definable, making this an exercise in compactness.
- (4) Whether or not an  $n$ -variable polynomial is irreducible is definable in the coefficients, because to check reducibility one only needs to quantify over the (definable) set of lower-degree polynomials. This makes the family of irreducible polynomials ind-definable. Therefore, the set of pairs  $(\vec{a}, F_{\vec{a}})$  where  $F_{\vec{a}}$  cuts out  $\overline{X_{\vec{a}}}$ , is ind-definable. For any given  $\vec{a}$ , all the possibilities for  $F_{\vec{a}}$  are essentially the same, differing only by scalar multiples. So the total degree of  $F_{\vec{a}}$  only depends on  $\vec{a}$ , and compactness yields a bound on the total degree. This in turn makes the set of pairs  $(\vec{a}, F_{\vec{a}})$  definable.
- (5) Every irreducible subvariety of  $\mathbb{A}^n$  which is of codimension  $d$  is given (set-theoretically) by the intersection the zero sets of  $d + 1$  polynomials, whose degrees are bounded by the degree of the variety. Since the degree of a family of varieties is uniformly bounded by the product,  $D$ , of the degrees of the defining polynomials and the number of components is bounded by the degree, there are at most  $D$  many maximal irreducible subvarieties, each of which has degree less than or equal to  $D$ . Among such zero sets, the components of the variety are those which are maximal and irreducible.

□

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