Assignment 2

PMAT 611

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Problem 1 (Aluffi III.7 Problem 11)**.** Let

$$0 \longrightarrow M_1 \longrightarrow N \longrightarrow M_2 \longrightarrow 0 \tag{$*$}$$

be an exact sequence of R-modules. Suppose there is *any* R-module homomorphism $\varphi : N \longrightarrow M_1 \oplus M_2$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d, dashed, "\varphi"] & M_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] & M_1 \oplus M_2 \arrow[r] & M_2 \arrow[r] & 0
\end{tikzcd}$$

commute, where the bottom sequence is the standard sequence of a direct sum. Prove that $(*)$ splits.

Solution. By the five lemma, φ must be an isomorphism. This is the definition of 'split' in the Aluffi-sense, which is equivalent to the definition of split in the other sense.

Problem 2 (Aluffi VI.I Problem 13)**.** Let A be and abelian group such that $\mathrm{End}_{\mathsf{Ab}}(A)$ is a field of characteristic zero. Prove that $A \cong \mathbb{Q}$.

Solution. Note that, for a field k, every k-module (i.e. a k-vector space) is free (see Proposition VI.1.7 in Aluffi). That is, an abelian group A is a k-vector space if and only if $A \cong k^S$ for some set S. Furthermore, for two k-vector spaces $A \cong k^S$ and $B \cong k^T$, we have $A \cong B$ if and only if $|S| = |T|$. This comes from the fact that k is an integral domain, and Exercise VI.1.8 (Corollary VI.1.11) in Aluffi, which we prove here.

Lemma 1 (Corollary VI.1.11)**.** *Let R be an integral domain and let S and T be sets. Then*

$$\mathrm{Free}_{R\text{-}\mathsf{Mod}}(S) \cong \mathrm{Free}_{R\text{-}\mathsf{Mod}}(T) \quad \text{if and only if} \quad S \cong T.$$

Proof. If $S \cong T$, then clearly $\mathrm{Free}_{\text{R-}\mathsf{Mod}}(S) \cong \mathrm{Free}_{\text{R-}\mathsf{Mod}}(T)$. Indeed, given a bijection $f : S \longrightarrow T$, we have the induced isomorphism $\mathrm{Free}_{\text{R-}\mathsf{Mod}}(S) \longrightarrow \mathrm{Free}_{\text{R-}\mathsf{Mod}}(T)$ given by

$$\sum_{a \in A} r_a a \longmapsto \sum_{a \in A} r_a f(a) = \sum_{b \in B} r_b b.$$

So suppose there is an isomorphism $\varphi : \mathrm{Free}_{\text{R-}\mathsf{Mod}}(S) \longrightarrow \mathrm{Free}_{\text{R-}\mathsf{Mod}}(T)$. Define the set $T' = \{\varphi(1_R s) \mid s \in S\}$. Since $\{1_R s \mid s \in S\}$ is a linearly independent set in $\mathrm{Free}_{\text{R-}\mathsf{Mod}}(S)$, T' mst be a linearly independent set in $\mathrm{Free}_{\text{R-}\mathsf{Mod}}(T)$ since $\mathrm{Free}_{\text{R-}\mathsf{Mod}}(S) \cong \mathrm{Free}_{\text{R-}\mathsf{Mod}}(T)$. But T is a maximal linearly independent set in $\mathrm{Free}_{\text{R-}\mathsf{Mod}}(T)$, so $|T'| \leq |T|$ by Proposition VI.1.7 in Aluffi. But $|S| = |T'|$ and thus $|S| \leq |T|$. Analogously, define the set $S' = \left\{\varphi^{-1}(1_R t) \,\middle|\, t \in T\right\}$, and we have $|T| = |S'| \leq |S|$. Hence $|S| = |T|$, and thus $S \cong T$. $\square$

Lemma 2. *Let A be a $\mathbb{Q}$-vector space.*

1. *If $A \ncong \mathbb{Q}$, then $\mathrm{End}_{\mathsf{Ab}}(A)$ is not a field.*
2. *If $A \cong \mathbb{Q}$, then $\mathrm{End}_{\mathsf{Ab}}(A) \cong \mathbb{Q}$.*

Proof. All vector spaces over $\mathbb{Q}$ are of the form $A \cong \mathbb{Q}^S$ for some set S. Then A is isomorphic to $\mathbb{Q}$ if and only if S has exactly one element.

1. If S is empty, then $A = \{0\}$ and $\mathrm{End}_{\mathsf{Ab}}(A) = \{0\}$ which is not a field. So suppose that $A \cong \mathbb{Q}^S$ and S is a set with more than one element. Let $s_0 \in S$ and define a map $\pi_{s_0} : \mathbb{Q}^S \longrightarrow \mathbb{Q}^S$

$$\pi_{s_0} : \sum_{s \in S} r_s s \longmapsto r_{s_0} s_0.$$

   This is a nonzero endomorphism on $\mathbb{Q}^S$ as an abelian group that is clearly not surjective and thus has no inverse in the ring $\mathrm{End}_{\mathsf{Ab}}(\mathbb{Q}^S)$. Hence $\mathrm{End}_{\mathsf{Ab}}(\mathbb{Q}^S) \cong \mathrm{End}_{\mathsf{Ab}}(A)$ is not a field.

2. Consider $\mathbb{Q}$ as an additive group. For each $f \in \mathrm{End}_{\mathsf{Ab}}(\mathbb{Q})$, we have

$$f(na) = f(a + \cdots + a) = f(a) + \cdots + f(a) = nf(a)$$

   for all $a \in \mathbb{Q}$ and $n \in \mathbb{Z}$. Similarly, for each nonzero $m \in \mathbb{Z}$ we have

$$f\left(\frac{1}{m}a\right) = \frac{1}{m}mf\left(\frac{1}{m}a\right) = \frac{1}{m}f\left(m\frac{1}{m}a\right) = \frac{1}{m}f(a).$$

   Hence $f(\frac{p}{q}) = \frac{p}{q}f(1)$ for each $f \in \mathrm{End}_{\mathsf{Ab}}(\mathbb{Q})$ and $\frac{p}{q} \in \mathbb{Q}$. Consider the isomorphism $\mathrm{End}_{\mathsf{Ab}}(\mathbb{Q}) \to \mathbb{Q}$ given by

$$f \longmapsto f(1).$$

   Indeed, this is injective, since $f(1) = 0$ implies $f(a) = 0$ for all $a \in \mathbb{Q}$ and thus $f = 0$. This is also surjective, since for each $\frac{p}{q} \in \mathbb{Q}$ there is a group endomorphism defined by $f(a) = \frac{p}{q}a$.

$\square$

Proposition 3. *If A is an abelian group such that $\mathrm{End}_{\mathsf{Ab}}(A)$ is a field of characteristic zero, then $A \cong \mathbb{Q}$.*

Proof. Note that A as an abelian group has a $\mathbb{Z}$-action given by the multiplication maps $\mu_n \in \mathrm{End}_{\mathsf{Ab}}(A)$ with $\mu_n(a) = na$. Since $\mathrm{End}_{\mathsf{Ab}}(A)$ is a field with characte1ristic zero, for each $n \neq 0$ there is an $a \in A$ such that $\mu_n(a) \neq 0$. Otherwise there is an n such that $n\mu_1 = \mu_n = 0$ and thus $\mathrm{End}_{\mathsf{Ab}}(A)$ would have characteristic at most n. Hence, for each $n \neq 0$, the multiplication map μ_n is not the zero map in the field $\mathrm{End}_{\mathsf{Ab}}(A)$, and thus each μ_n with $n \neq 0$ has an inverse $(\mu_n)^{-1} \in \mathrm{End}_{\mathsf{Ab}}(A)$. Define a $\mathbb{Q}$-action by extending the multiplication maps to $\mu_{\frac{p}{q}}$ for $\frac{p}{q} \in \mathbb{Q}$ by

$$\mu_{\frac{p}{q}} := \mu_p \circ (\mu_q)^{-1}.$$

Since A is an abelian group with a $\mathbb{Q}$-action, it is a $\mathbb{Q}$-module and thus a $\mathbb{Q}$-vector space.

By Lemma 2 above, we see that, for a $\mathbb{Q}$-vector space A, the endomorphism ring $\mathrm{End}_{\mathsf{Ab}}(A)$ is a field if and only if $A \cong \mathbb{Q}$. Since $\mathrm{End}_{\mathsf{Ab}}(A) \cong \mathbb{Q}$ and $\mathbb{Q}$ is a field of characteristic zero, we are done. $\square$

Problem 3 (Aluffi VI.I Problem 16)**.** Let M be a module over a ring R. A *finite composition series* for M (if it exists) is a decreasing sequence of submodules

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_m = \langle 0 \rangle$$

in which all quotients M_i/M_{i+1} are simple R-modules. The *length* of a series is the number of strict inclusions. The *composition factors* are the quotients M_i/M_{i+1}.

Prove a Jordan-Hölder theorem for modules: any two finite composition series of a module have the same length and the same composition factors.

Solution. We first show some important facts about submodules. (These were never actually shown in class, so I figured they should be proved here since I use them.)

Lemma 4 (Second Isomorphism Theorem' for Modules)**.** *Let R be a ring, M an R-module and $N_1, N_2 \subset M$ submodules. Then*

1. *$N_1 \cap N_2$ is a submodule of $M1$;*
2. *$N_1 + N_2$ is a submodule of M;*
3. $\dfrac{N_1}{N_1 \cap N_2} \cong \dfrac{N_1 + N_2}{N_2}$.

Proof. .

1. Let $a, b \in N_1 \cap N_2$. Then $a + b \in N_1$ and $a + b \in N_2$ since N_1 and N_2 are modules, and thus $a + b \in N_1 \cap N_2$. Similarly, for each $r \in R$ we have $rm \in N_1$ and $rm \in N_2$, and thus $rm \in N_1 \cap N_2$. So $N_1 \cap N_2 \subset M$ is a submodule.

2. Each element of $N_1 + N_2$ is of the form $a + b$ for some $a \in N_1 \subset M$ and $b \in N_2 \subset M$, so $a + b \in M$ and $r(a + b) \in M$ for each $r \in R$. For $(a + b), (a' + b') \in N_1 + N_2$ and $r \in R$,

$$(a + b) + (a' + b') \ = (a + a') + (b + b') \in N_1 + N_2 \quad \text{and} \quad r(a + b) = ra + rb \in N_1 + N_2$$

since N_1 and N_2 are submodules.

3. Let $\varphi : N_1 \longrightarrow M/N_2$ be the R-module homomorphism defined by $m \longmapsto x + T$. Then $\ker \varphi = N_1 \cap N_2$ and $\operatorname{im} \varphi = \frac{N_1 + N_2}{N_2}$. By the first isomorphism theorem, $\operatorname{im} \varphi \cong N_1 / \ker \varphi$, and thus

$$\frac{N_1 + N_2}{N_2} \cong \frac{N_1}{N_1 \cap N_2}$$

as deisred.

$\square$

Theorem 5 (Jordan-Hölder)**.** *Let R be a ring and M an R-module. Then any two finite composition series of M have the same length and the same composition factors.*

Proof (following the proof of Theorem IV.3.2 in Aluffi). We argue by induction. Let

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_m = \langle 0 \rangle \tag{†}$$

be a composition series for M. If $m = 0$, then M is the trivial module and we are done. Assume $m > 0$ and suppose that the theorem holds for all submodules M_k of the series with $k \neq 0$. Let

$$M = M_0' \supsetneq M_1' \supsetneq \cdots \supsetneq M_n' = \langle 0 \rangle \tag{††}$$

be another composition series for M. If $M_1' \cong M_1$, then the result follows from the induction hypothesis.

So we may suppose that $M_1' \not\cong M_1$, in which case we have $M_1 + M_1' = M$. Indeed, $M_1 + M_1'$ is a submodule of M, so we have $M_1 \subsetneq M_1 + M_1' \subset M$, but there are no proper submodules of M between M_1 and M since M/M_1 is simple. Note that $K = M_1 \cap M_1' \subset M$ is another submodule of M, and let

$$K = K_0 \supsetneq K_1 \supsetneq \cdots \supsetneq K_r = \langle 0 \rangle$$

be a composition series for K. By the second isomorphism theorem, we have

$$\frac{M_1}{K} = \frac{M_1}{M_1 \cap M_1'} \cong \frac{M_1 + M_1'}{M_1'} = \frac{M}{M_1'} \quad \text{and} \quad \frac{M_1'}{K} = \frac{M_1'}{M_1 \cap M_1'} \cong \frac{M_1 + M_1'}{M_1} = \frac{M}{M_1},$$

and thus M_1/K and M_2/K are both simple modules. Therefore, we have two new composition series for M

$$\begin{array}{ccccccccccccc} M & \supsetneq & M_1 & \supsetneq & K & \supsetneq & K_1 & \supsetneq & \cdots & \supsetneq & K_r & = & \langle 0 \rangle \\ \| & & & & \| & & \| & & & & \| & & \\ M & \supsetneq & M_1' & \supsetneq & K & \supsetneq & K_1 & \supsetneq & \cdots & \supsetneq & K_r & = & \langle 0 \rangle \end{array}$$

which only differ at the first step. These two series clearly have the same length and the same quotients. (In particular, the first two quotients get switched from one series to the other.)

By the induction hypothesis, the composition series

$$M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_m = \langle 0 \rangle$$

is equivalent to the composition series

$$M_1 \supsetneq K \supsetneq K_1 \supsetneq \cdots \supsetneq K_r = \langle 0 \rangle .$$

That is, they have the same length and quotients. In particular, the length is equal to $m - 1 = r + 1$. Similarly, the composition series

$$M_1' \supsetneq M_2' \supsetneq \cdots \supsetneq M_n' = \langle 0 \rangle$$

is equivalent to the composition series

$$M_1' \supsetneq K \supsetneq K_1 \supsetneq \cdots \supsetneq K_r = \langle 0 \rangle .$$

So we have $n - 1 = r + 1$, and thus $n = m$, and so the composition series (†) and (††) have the same length and the same quotients. $\square$