Annali di Matematica pura ed applicata
(IV), Vol. CLVI (1990), pp. 265-321

Cauchy Problem for Overdetermined Systems (*).

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Sunto. – *Si caratterizzano le nozioni di direzioni caratteristiche, formalmente caratteristiche, di iperbolicità e di evoluzione in una direzione per sistemi a coefficienti costanti, connesse allo studio delle soluzioni del problema di Cauchy in un semispazio.*

Introduction.

The Cauchy problem is of central importance in the theory of partial differential equations. In the framework of over-determined systems with constant coefficients, a very natural question is the following: given an ideal $\mathscr{I}$ in $\mathbb{C}[\zeta_1, \ldots, \zeta_n]$ and a partial differential operator

$$P(D) = D_0^m + \sum_0^{m-1} p_j(D_1, \ldots, D_n) D_0^j \tag{0.1}$$

in $\mathbb{R}^{n+1}$ we can ask whether, given functions $\varphi_0, \varphi_1, \ldots, \varphi_{m-1}$, that are C^∞ on $\mathbb{R}^n$ and satisfy

$$p(D_1, \ldots, D_n)\varphi_h = 0 \quad \text{on } \mathbb{R}^n \ \forall p \in \mathscr{I}, \qquad \text{for } h = 0, \ldots, m-1 \tag{0.2}$$

is it possible to find a C^∞ function u on

$$H = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \colon x_0 \geqq 0\} \tag{0.3}$$

such that

$$P(D)u = 0 \quad \text{on } H \tag{0.4}$$

and

$$D_0^j u|_{x_0=0} = \varphi_j \quad \text{for } j = 0, \ldots, n-1 \tag{0.5}$$

$$p(D_1, \ldots, D_n)u = 0 \quad \text{on } H \quad \forall p \in \mathscr{I}. \tag{0.6}$$

(*) Entrata in Redazione il 7 luglio 1988.

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When $\mathbb{R}^n = \mathbb{C}^k$ and the differential operators associated to a set of generators of $\mathscr{I}$ are the Cauchy-Riemann ones, we are lead to the Cauchy-Kowalewska problem and, in case P if of order m, the question is solved in the affirmative by the classical theorem of Cauchy-Kowalewska.

Actually, in § 8 we generalize this theorem to the characteristic case, showing that, in this instance, the problem (0.4), (0.5), (0.6) has always a solution (not unique if $P(D)$ has order larger than m) provided that the compatibility conditions (0.2) are satisfied. This will be used then to prove a general theorem that implies the solvability of (0.4), (0.5), (0.6) when $\mathscr{I}$ is elliptic, i.e. all solutions of (0.2) are real analytic on $\mathbb{R}^n$ and generalizes the classical conditions of Petrowski when $\mathscr{I}$ is the zero ideal.

In this paper we shall consider general Hilbert complexes of partial differential operators with constant coefficients. As they come from the resolutions of unitary left modules of finite type over the ring of polynomials, the results are stated in terms of their invariants. This leads to a fair ammount of commutative algebra. Hence in § 1 we list the algebraic preliminaries, together with rules to translate algebraic jargon into statements about systems of partial differential equations. Then two propositions are proved that allow to reduce to the simpler case of prime ideals. This means to state conditions for the solvability of systems of equations in terms of properties of reduced affine algebraic varieties associated to it.

Here we can make a further remark: these properties are apparently of two kinds. Some are « hereditary », in the sense that when they hold for a system, they also hold for a system containing additional equations. They can be often expressed by pointwise inequalities. Ellipticity, hypoellipticity, as the fact of being non-characteristic (§ 3), formally non-characteristic (§ 4), hyperbolic (§ 5) in a given direction, are examples of « hereditary » properties.

Examples of « non-hereditary » properties are provided by analytic convexity (cf. [15], [3]) and, as we show in § 7 and § 8, by being of evolution in some direction. The concept of evolution that we adopt in this paper is, for the system (0.4), (0.5), (0.6), the requirement of existence of a (non necessarily unique) solution for all data satisfying (0.2). The general definition will be given in § 7. This analogy with analytic convexity was a reason for searching a condition for evolution modules in terms of a Phragmén-Lindelöf principle on the associated zero varieties (§ 8).

After § 2, where the spaces of functions and distributions used in the paper are discussed in short, section 3 and 4 are dedicated to the notion of noncharacteristic and formally noncharacteristic systems, related to uniqueness and formal well posedness questions. While these notions can be traced back to the scalar case, hyperbolicity for systems exhibits some new special features. These are illustrated in § 6 by some examples. We note in particular that for systems it is no longer true that hyperbolicity with respect to a given direction implies hyperbolicity in the opposite direction, as most of the nice properties of hyperbolic polynomials do not hold for systems.

In the last sections we discuss evolution modules.

We restrict in this research to the categories of C^∞ functions and distributions and to constant coefficients. There are several interesting problems, that are not considered here, as that of correctedness and well-posedness classes, partly investigated in [9]; the theory developed in this paper could also be extended to the case of data on an affine subspace of higher codimension and should be a first step toward a comprehensive treatment of the case of variable coefficients. This project requires to complement the results obtained here by a stability theory for systems, that, at the moment, seems to present still some difficulties. I hope to treat these subjects in some future papers.

1. – Algebraic preliminaries and notations.

A *Primary decomposition.*

Let us denote by $\mathcal{T}$ the ring $\mathbb{C}[\zeta_1, \dots, \zeta_n]$ of polynomials with complex coefficients in the n indeterminates $\zeta_1, \dots, \zeta_n$. Let Spec $(\mathcal{T})$ be the set of all prime ideals of $\mathcal{T}$.

Given a unitary left $\mathcal{T}$-module M, the annihilator of an element m of M is the ideal

$$\operatorname{Ann}(m) = \{p \in \mathcal{T} : p \cdot m = 0\} . \tag{1.1}$$

The *annihilator* of M is then the ideal

$$\operatorname{Ann}(M) = \cap \{\operatorname{Ann}(m) : m \in M\} . \tag{1.2}$$

The *support* of M is the subset supp (M) of Spec $(\mathcal{T})$ of prime ideals containing Ann(M). We have, $M_{\mathcal{T}}$ denoting the localization of M at $\mathcal{T}$ (cf. [7]),

$$\operatorname{supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(\mathcal{T}) : M_{\mathcal{T}} \neq 0\} . \tag{1.3}$$

A unitary left $\mathcal{T}$-module M of finite type is *coprimary* if, for every $p \in \mathcal{T}$, the $\mathcal{T}$-homomorphism $M \ni m \to p \cdot m \in M$ is either injective or nilpotent. In this case

$$\mathfrak{p} = \sqrt{\operatorname{Ann}(M)}$$

is a prime ideal and M is said to be $\mathfrak{p}$-coprimary.

In general, there is a uniquely determined finite set Ass $(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\} \subset$ $\subset$ Spec $(\mathcal{T})$ to which we can associate submodules $N_1, \dots, N_k$ of M such that

(i) $\bigcap_{j=1}^{k} N_j = 0$,

(ii) N/N_j is $\mathfrak{p}_j$-coprimary for $j = 1, \dots, k$,

(iii) $\bigcap_{j \neq h} N_j \neq 0$ for $h = 1, \dots, k$.

We say then that $N_1, \dots, N_k$ is a *primary decomposition* of 0 in M.

We set $V(\mathfrak{p}_j)$ for the closed algebraic affine variety

$$V(\mathfrak{p}_j) = \{\zeta \in \mathbb{C}^n \colon p(\zeta) = 0 \;\; \forall p \in \mathfrak{p}_j\} \tag{1.4}$$

of common zeros in $\mathbb{C}^n$ of polynomials in $\mathfrak{p}_j$. The disjoint union

$$\dot{\bigcup} V(\mathfrak{p}_j) \text{ will be denoted by } V(M) \tag{1.5}$$

and will be called the *zero variety* of M.

With these notations we have $V(\mathfrak{p}_j) = V(\mathcal{P}/\mathfrak{p}_j) = V(M/N_j)$ for $j = 1, \ldots, k$.

B *Asymptotic and characteristic variety.*

Let us consider the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ described in homogeneous coordinates by

$$(\zeta_1, \ldots, \zeta_n) \to (1, \zeta_1, \ldots, \zeta_n)\,.$$

Given a closed algebraic affine variety V in $\mathbb{C}^n$, its closure $\tilde{V}$ in $\mathbb{CP}^n$ is a projective variety. We write $\tilde{V}(\mathfrak{p})$ and $\tilde{V}(M)$ for the projective varieties associated to $V(\mathfrak{p})$ and $V(M)$ respectively: note that $\tilde{V}(M) = \dot{\bigcup} \{\tilde{V}(\mathfrak{p}) \colon \mathfrak{p} \in \operatorname{Ass}(M)\}$ is a disjoint union. Then we can define the asymptotic cone of V by

$$W = \{\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n \colon (0, \zeta_1, \ldots, \zeta_n) \in \tilde{V}\} \cup \{0\}\,. \tag{1.6}$$

For $\mathfrak{p} \in \operatorname{Spec}(\mathcal{P})$, $W(\mathfrak{p})$ denotes the asymptotic cone of $V(\mathfrak{p})$ and

$$W(M) = \dot{\bigcup} \{W(\mathfrak{p}) \colon \mathfrak{p} \in \operatorname{Ass}(M)\} \tag{1.7}$$

is called the *asymptotic variety*, or *full characteristic variety*, of M.

A *characterstic direction for M* is a vector $\nu \in \mathbb{R}^n - \{0\}$ in $W(M)$.

C *Homological algebra.*

Let M be a unitary left $\mathcal{P}$-module. A free resolution of M is an exact sequence

$$0 \leftarrow M \leftarrow F_0 \xleftarrow{A_0} F_1 \xleftarrow{A_1} F_2 \leftarrow \ldots \tag{1.8}$$

of unitary left $\mathcal{P}$-modules and $\mathcal{P}$-homomorphisms where all the F_i's are free.

The fact that the sequence is exact means that the image of every homomorphism equals the kernel of the following one.

Given another unitary left $\mathcal{P}$-module N, the groups $\operatorname{Ext}^j_{\mathcal{P}}(M, N)$ are defined as the cohomology groups of the complex

$$0 \to \mathrm{Hom}_{\mathfrak{P}}(F_0, N) \xrightarrow{{}^tA_0} \mathrm{Hom}_{\mathfrak{P}}(F_1, N) \xrightarrow{{}^tA_1} \mathrm{Hom}_{\mathfrak{P}}(F_2, N) \to \dots . \tag{1.9}$$

If R is a unitary right $\mathfrak{P}$-module, the groups $\mathrm{Tor}^j_{\mathfrak{P}}(M, R)$ are defined as the cohomology groups of the complex

$$0 \leftarrow F_0 \otimes_{\mathfrak{P}} R \leftarrow F_1 \otimes_{\mathfrak{P}} R \leftarrow F_2 \otimes_{\mathfrak{P}} R \leftarrow \dots \tag{1.10}$$

We will also use the following fact from homological algebra: if

$$0 \to E \to F \to G \to 0 \tag{1.11}$$

is a short exact sequence of unitary left (resp. right) $\mathfrak{P}$-modules, then for every unitary left $\mathfrak{P}$-module M we obtain the long exact sequence for the Ext functor:

$$\begin{aligned} 0 &\to \mathrm{Ext}^0_{\mathfrak{P}}(M, E) \to \mathrm{Ext}^0_{\mathfrak{P}}(M, F) \to \mathrm{Ext}^0_{\mathfrak{P}}(M, G) \to \\ &\to \mathrm{Ext}^1_{\mathfrak{P}}(M, E) \to \mathrm{Ext}^1_{\mathfrak{P}}(M, F) \to \mathrm{Ext}^1_{\mathfrak{P}}(M, G) \to \\ &\cdots\cdots\cdots\cdots\cdots\cdots \to \mathrm{Ext}^j_{\mathfrak{P}}(M, G) \to \\ &\to \mathrm{Ext}^{j+1}_{\mathfrak{P}}(M, E) \to \mathrm{Ext}^{j+1}_{\mathfrak{P}}(M, F) \to \mathrm{Ext}^{j+1}_{\mathfrak{P}}(M, G) \to \cdots \end{aligned} \tag{1.12}$$

and, respectively, the long exact sequence for the Tor functor:

$$\begin{aligned} 0 &\leftarrow \mathrm{Tor}_0^{\mathfrak{P}}(M, G) \leftarrow \mathrm{Tor}_0^{\mathfrak{P}}(M, F) \leftarrow \mathrm{Tor}_0^{\mathfrak{P}}(M, E) \leftarrow \\ &\leftarrow \mathrm{Tor}_1^{\mathfrak{P}}(M, G) \leftarrow \mathrm{Tor}_1^{\mathfrak{P}}(M, F) \leftarrow \mathrm{Tor}_1^{\mathfrak{P}}(M, E) \leftarrow \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \leftarrow \mathrm{Tor}_j^{\mathfrak{P}}(M, E) \leftarrow \\ &\leftarrow \mathrm{Tor}_{j+1}^{\mathfrak{P}}(M, G) \leftarrow \mathrm{Tor}_{j+1}^{\mathfrak{P}}(M, F) \leftarrow \mathrm{Tor}_{j+1}^{\mathfrak{P}}(M, E) \leftarrow \cdots \end{aligned} \tag{1.13}$$

D *Hilbert resolutions.*

Given a $\mathfrak{P}$-module M of finite type, we can find a resolution of M by free modules of finite type, of the form

$$0 \to \mathfrak{P}^{a_d} \xrightarrow{A_{d-1}} \mathfrak{P}^{a_{d-1}} \to \dots \to \mathfrak{P}^{a_1} \xrightarrow{A_0} \mathfrak{P}^{a_0} \to M \to 0 \tag{1.14}$$

with $d \leq n$. Any such resolution is called a Hilbert resolution of M.

E *Differential modules.*

If $\mathcal{F}$ is a space of functions or distributions in an open set of $\mathbb{R}^n$, such that

$$\partial_j f = \frac{\partial f}{\partial x_j} \in \mathcal{F} \qquad \forall j = 1, \ldots, n \text{ and } f \in \mathcal{F} ,$$

then we shall consider $\mathcal{F}$ as a left and right $\mathfrak{I}$-module letting elements of $\mathfrak{I}$ operate on $\mathcal{F}$ as partial differential operators with constant coefficients.

If $p(\zeta) = \sum_{|\alpha| \leq m} a_\alpha \zeta^\alpha$ is a polynomial in $\mathfrak{I}$, we set

$$p(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

with $D = (D_1, \ldots, D_n)$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, being

$$D_j = \frac{1}{i} \frac{\partial}{\partial x_j} .$$

Then the action on $p(\zeta)$ of $f \in \mathcal{F}$ is described by

$$p(\zeta) \cdot f = f \cdot p(\zeta) = p(D) f . \tag{1.15}$$

Given a Hilbert resolution (1.14) of a $\mathfrak{I}$-module of finite type M, the maps A_j are represented by $a_j \times a_{j+1}$ matrices $\left(A_j^{r,s}(\zeta)\right)_{1 \leq r \leq a_j, 1 \leq s \leq a_{j+1}}$. We denote by $A_j(D)$ the corresponding matrix of linear partial differential operators with constant coefficients:

$$A_j(D) = \left(A_j^{r,s}(D)\right) .$$

We have natural isomorphisms

$$\operatorname{Hom}_{\mathfrak{I}}(\mathfrak{I}^j, \mathcal{F}) \simeq \mathcal{F}^j , \qquad \mathfrak{I}^j \otimes_{\mathfrak{I}} \mathcal{F} \simeq \mathcal{F}^j \tag{1.16}$$

and therefore we obtain the isomorphisms:

$$\begin{aligned}
&\operatorname{Ext}^0_{\mathfrak{I}}(M, \mathcal{F}) \simeq \{f \in \mathcal{F}^{a_0} \colon {}^t A_0(D) f = 0\} , \\
&\operatorname{Tor}^{\mathfrak{I}}_0(M, \mathcal{F}) \simeq \mathcal{F}^{a_0} / A_0(D) \mathcal{F}^{a_1} , \\
&\operatorname{Ext}^j_{\mathfrak{I}}(M, \mathcal{F}) \simeq \frac{\ker \left({}^t A_j(D) \colon \mathcal{F}^{a_j} \to \mathcal{F}^{a_{j+1}}\right)}{\operatorname{image} \left({}^t A_{j-1}(D) \colon \mathcal{F}^{a_{j-1}} \to \mathcal{F}^{a_j}\right)} \quad \text{for } j \geq 1 , \\
&\operatorname{Tor}^{\mathfrak{I}}_j(M, \mathcal{F}) \simeq \frac{\ker \left(A_{j-1}(D) \colon \mathcal{F}^{a_j} \to \mathcal{F}^{a_{j-1}}\right)}{\operatorname{image} \left(A_j(D) \colon \mathcal{F}^{a_{j+1}} \to \mathcal{F}^{a_j}\right)} \quad \text{for } j \geq 1 .
\end{aligned} \tag{1.17}$$

These isomorphisms translate statements about the solvability of systems of partial differential operators with constant coefficients into statements about invariant groups associated to the $\mathfrak{I}$-module M.

For the relationship between a given matrix of partial differential operators with constant coefficients and the $\mathfrak{I}$-module M in a resolution of Hilbert containing the associated matrix of polynomials as a $\mathfrak{I}$-homomorphism, we refer to [5].

F *Injective and flat $\mathfrak{I}$-modules.*

A unitary left $\mathfrak{I}$-module N is said to be injective if for every pair of $\mathfrak{I}$-modules N_1 and N_2, given a $\mathfrak{I}$-homomrphism $\alpha\colon N \to N_2$ and ad injective $\mathfrak{I}$-homomorphism $\iota\colon N \to N_1$, we can find a $\mathfrak{I}$-homomorphism $\beta\colon N_1 \to N_2$ making the diagram

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[dr, "\alpha"'] & N_1 \arrow[d, "\beta"] \\ & N_2 \end{tikzcd}$$

commute. The following equations are each one equivalent to the fact that N is injective:

(i) $\operatorname{Ext}^1_{\mathfrak{I}}(M, N) = 0$ for every unitary left $\mathfrak{I}$-module M

(ii) $\operatorname{Ext}^j_{\mathfrak{I}}(M, N) = 0$ for every unitary left $\mathfrak{I}$-module M and every $j > 0$.

A unitary right $\mathfrak{I}$-module R is said to be flat if for every short exact sequence of left-$\mathfrak{I}$-modules:

$$0 \to E \to F \to G \to 0$$

the sequence

$$0 \to E \otimes_{\mathfrak{I}} R \to F \otimes_{\mathfrak{I}} R \to G \otimes_{\mathfrak{I}} R \to 0 \tag{1.18}$$

is also exact. The following equations are each one equivalent to the fact that R is flat:

(i) $\operatorname{Tor}_1^{\mathfrak{I}}(M, R) = 0$ for every unitary left $\mathfrak{I}$-module M;

(ii) $\operatorname{Tor}_1^{\mathfrak{I}}(\mathfrak{I}/\mathfrak{p}, R) = 0$ for every ideal $\mathfrak{p}$ in $\mathfrak{I}$;

(iii) $\operatorname{Tor}_j^{\mathfrak{I}}(M, R) = 0$ for every unitary left $\mathfrak{I}$-module M and every $j > 0$.

We can restate the conditions for injectivity and flatness for a differential module $\mathcal{F}$ in the following way:

$\mathcal{F}$ is injective (resp. flat) iff, for every exact sequence

$$\mathfrak{I}^a \xrightarrow{A(\zeta)} \mathfrak{I}^b \xrightarrow{B(\zeta)} \mathfrak{I}^c \tag{1.19}$$

of free $\mathcal{T}$-modules of finite type and $\mathcal{T}$-homomorphisms we have, for $f \in \mathcal{F}^b$:
the condition

$$(1.20) \qquad {}^tA(D)f = 0 \qquad (\text{resp. } B(D)f = 0)$$

is necessary and sufficient in order that the equation

$$(1.21) \qquad {}^tB(D)u = f \text{ has a solution } u \in \mathcal{F}^c$$

$$(\text{resp. } A(D)u = f \text{ has a solution } u \in \mathcal{F}^a)\,.$$

G *Reduction to prime ideals.*

We give here two algebraic results that permit to reduce several vanishing theorems for $\mathcal{T}$-modules M of finite type to the simpler case in which $M = \mathcal{T}/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \in \operatorname{Spec}(\mathcal{T})$.

PROPOSITION 1.1. – *Let E, M be unitary left $\mathcal{T}$-modules, with M of finite type. Then a necessary and sufficient condition in order that, for a fixed integer $j_0 \geqq 0$,*

$$(1.22) \qquad \operatorname{Ext}^j_{\mathcal{T}}(M, E) = 0 \qquad \forall j \leqq j_0$$

is that

$$(.23) \qquad \operatorname{Ext}^j_{\mathcal{T}}(\mathcal{T}/\mathfrak{p}, E) = 0 \qquad \forall j \leqq j_0 \textit{ and } \forall \mathfrak{p} \in \operatorname{Supp}(M)\,.$$

PROOF. – Sufficiency. We can find a composition series:

$$(1.24) \qquad 0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_{k+1} = M$$

such that

$$(1.25) \qquad M_{h+1}/M_h \simeq \mathcal{T}/\mathfrak{p}_h \qquad \text{with } \mathfrak{p}_h \in \operatorname{Supp}(M) \text{ for } 0 \leqq h \leqq k\,.$$

For every h we have the long exact sequence for the Ext functor:

$$\begin{aligned} 0 &\to \operatorname{Ext}^0_{\mathcal{T}}(M_{h+1}/M_h, E) \to \operatorname{Ext}^0_{\mathcal{T}}(M_{h+1}, E) \to \operatorname{Ext}^0_{\mathcal{T}}(M_h, E) \to \\ &\to \operatorname{Ext}^1_{\mathcal{T}}(M_{h+1}/M_h, E) \to \cdots \\ \cdots &\to \operatorname{Ext}^j_{\mathcal{T}}(M_{h+1}/M_h, E) \to \operatorname{Ext}^j_{\mathcal{T}}(M_{h+1}, E) \to \operatorname{Ext}^j_{\mathcal{T}}(M_h, E) \to \\ &\to \operatorname{Ext}^{j+1}_{\mathcal{T}}(M_{h+1}/M_h, E) \to \cdots\,. \end{aligned}$$

By the assumption we have then

$$(1.26) \qquad \operatorname{Ext}^j_{\mathcal{T}}(M_h, E) \simeq \operatorname{Ext}^j_{\mathcal{T}}(M_{h+1}, E) \qquad \forall j < j_0 \text{ and } 1 \leqq h \leqq k\,,$$

while all maps

$$\operatorname{Ext}^{j_0}_{\mathcal{T}}(M_{h+1}, E) \to \operatorname{Ext}^{j_0}_{\mathcal{T}}(M_h, E) \quad \text{for } 1 \leqq h \leqq k \tag{1.27}$$

are injective.

Because

$$\operatorname{Ext}^{j}_{\mathcal{T}}(M_1, E) \simeq \operatorname{Ext}^{j}_{\mathcal{T}}(\mathcal{T}/\mathfrak{p}_1, E) = 0 \quad \text{for } j \leqq j_0 , \tag{1.28}$$

it follows that

$$\operatorname{Ext}^{j}_{\mathcal{T}}(M, E) = \operatorname{Ext}^{j}_{\mathcal{T}}(M_{k+1}, E) = 0 \quad \text{for } j \leqq j_0 . \tag{1.29}$$

Necessity. We argue by contradiction, assuming that

$$\operatorname{Ext}^{j}_{\mathcal{T}}(M, E) = 0 \quad \text{for } j \leqq j_0 \tag{1.30}$$

but

$$\operatorname{Ext}^{j}_{\mathcal{T}}(\mathcal{T}/\mathfrak{p}, E) \neq 0 \quad \text{for some } j \leqq j_0 \text{ and } \mathfrak{p} \in \operatorname{Supp}(M) . \tag{1.31}$$

Let j_1, with $0 \leqq j_1 \leqq j_0$, be the smallest integer for which we can find such a $\mathfrak{p}$ in $\operatorname{Supp}(M)$, and let us fix then $\mathfrak{p} \in \operatorname{Supp}(M)$ with

$$\operatorname{Ext}^{j_1}_{\mathcal{T}}(\mathcal{T}/\mathfrak{p}, E) \neq 0 \tag{1.32}$$

maximal with this property, i.e. such that

$$\operatorname{Ext}^{j_1}_{\mathcal{T}}(\mathcal{T}/\mathfrak{p}', E) = 0 \quad \text{if } \mathfrak{p}' \in \operatorname{Spec}(M) \text{ and } \mathfrak{p}' \subsetneq \mathfrak{p} . \tag{1.33}$$

This choice is possible because $\mathcal{T}$ is noetherian.

By Proposition 20, Ch. II, § 4 in [7], we can find a submodule $Q \neq 0$ of $\mathcal{T}/\mathfrak{p}$ and an exact sequence

$$0 \to N \to M \to Q \to 0 . \tag{1.34}$$

By the choice of j_1 we have $\operatorname{Ext}^{j}_{\mathcal{T}}(N, E) = 0$ for $j < j_1$ by the sufficiency part of the proof. Thus the long exact sequence of Ext yields:

$$0 = \operatorname{Ext}^{j_1 - 1}_{\mathcal{T}}(N, E) \to \operatorname{Ext}^{j_1}_{\mathcal{T}}(Q, E) \to \operatorname{Ext}^{j_1}_{\mathcal{T}}(M, E) = 0 \tag{1.35}$$

and hence

$$\operatorname{Ext}^{j_1}_{\mathcal{T}}(Q, E) = 0 . \tag{1.36}$$

We have

$$Q \simeq \mathscr{I}/\mathfrak{p}$$

for some ideal $\mathscr{I}$ of $\mathfrak{I}$ with

$$\mathscr{I} \supsetneq \mathfrak{p}\,.$$

Then from the exact sequence

$$0 \to Q \to \mathfrak{I}/\mathfrak{p} \to \mathfrak{I}/\mathscr{I} \to 0 \tag{1.37}$$

We deduce an exact sequence

$$\cdots \to \operatorname{Ext}^{j_1}_{\mathfrak{I}}(\mathfrak{I}/\mathscr{I}, E) \to \operatorname{Ext}^{j_1}_{\mathfrak{I}}(\mathfrak{I}/\mathfrak{p}, E) \to \operatorname{Ext}^{j_1}_{\mathfrak{I}}(Q, E) = 0\,. \tag{1.38}$$

Because all $\mathfrak{p}' \in \operatorname{Supp}(\mathfrak{I}/\mathscr{I})$ properly contain $\mathfrak{p}$, by the choice of $\mathfrak{p}$ and the sufficiency part of the proof we have

$$\operatorname{Ext}^{j_1}_{\mathfrak{I}}(\mathfrak{I}/\mathscr{I}, E) = 0\,,$$

which implies that

$$\operatorname{Ext}^{j_1}_{\mathfrak{I}}(\mathfrak{I}/\mathfrak{p}, E) = 0\,,$$

giving a contradiction.

PROPOSITION 1.2. – *Let E, M be unitary left $\mathfrak{I}$-modules, with M of finite type. Assume that, for a fixed $j_0 \geqq 0$, we have*

$$\operatorname{Ext}^{j}_{\mathfrak{I}}(\mathfrak{I}/\mathfrak{p}, E) = 0 \qquad \forall j > j_0 \text{ and } \forall \mathfrak{p} \in \operatorname{Supp}(M)\,. \tag{1.39}$$

Then a necessary and sufficient condition in order that

$$\operatorname{Ext}^{j_0}_{\mathfrak{I}}(M, E) = 0 \tag{1.40}$$

is that

$$\operatorname{Ext}^{j_0}_{\mathfrak{I}}(\mathfrak{I}/\mathfrak{p}, E) = 0 \qquad \text{for every } \mathfrak{p} \in \operatorname{Ass}(M)\,. \tag{1.41}$$

Clearly in this case

$$\operatorname{Ext}^{j}_{\mathfrak{I}}(M, E) = 0 \qquad \forall j \geqq j_0. \tag{1.42}$$

PROOF. – *Sufficiency.*

We argue by descending induction on j_0. Indeed the statement is trivial if $j_0 > n$. We consider then a fixed $j_0 \leqq n$ and assume the statement is true for larger j_0.

Let us consider first the case of a $\mathfrak{p}$-coprimary M. Then we use again induction on the smallest integer k such that

$$\mathfrak{p}^k M = 0\,.$$

Assume that $k = 1$, i.e. that

$$\mathfrak{p}M = 0 \, .$$

Then we can consider M as a torsion free $\mathfrak{I}/\mathfrak{p}$-module and we can construct an exact sequence

$$0 \to M \to (\mathfrak{I}/\mathfrak{p})^h \to Q \to 0 \, . \tag{1.43}$$

Then we have an exact sequence

$$0 = (\operatorname{Ext}^{j_0}_{\mathfrak{I}}(\mathfrak{I}/\mathfrak{p}, E))^h \to \operatorname{Ext}^{j_0}_{\mathfrak{I}}(M, E) \to \operatorname{Ext}^{j_0+1}_{\mathfrak{I}}(Q, E) \, . \tag{1.44}$$

The last group is zero because of the inductive assumption on j_0 since

$$\operatorname{Supp}(Q) \subset \operatorname{supp}(\mathfrak{I}/\mathfrak{p}) \subset \operatorname{supp}(M) \, . \tag{1.45}$$

Then also $\operatorname{Ext}^{j_0}_{\mathfrak{I}}(M, E) = 0$.

Assume now that, for some $k > 1$, the statement is true for all $\mathfrak{p}$-coprimary modules N for which

$$\mathfrak{p}^h N = 0 \qquad \text{for some } h < k \, .$$

Let

$$M_0 = \{m \in M : \mathfrak{p}m = 0\} \, .$$

Then M_0 and M/M_0 are both $\mathfrak{p}$-coprimary (1) and

$$\mathfrak{p}M_0 = 0 \, , \qquad \mathfrak{p}^{k-1}(M/M_0) = 0 \, .$$

From the exact sequence

$$\operatorname{Ext}^{j_0}_{\mathfrak{I}}(M/M_0, E) \to \operatorname{Ext}^{j_0}_{\mathfrak{I}}(M, E) \to \operatorname{Ext}^{j_0}_{\mathfrak{I}}(M_0, E) \tag{1.46}$$

and the inductive assumption we deduce then that

$$\operatorname{Ext}^{j_0}_{\mathfrak{I}}(M, E) = 0 \, .$$

To drop the assumption that M is $\mathfrak{p}$-coprimary, we note that, if φ is any part of $\operatorname{Ass}(M)$, we can find a sub-module N of M such that

$$\operatorname{Ass}(N) = \varphi \, , \qquad \operatorname{Ass}(M/N) = \operatorname{Ass}(M) - \varphi \, .$$

---

(1) If $p \in \mathfrak{I} - \mathfrak{p}$, $m \in M$ and $p \cdot m \in M_0$, then we have $p \cdot (qm) = q(pm) = 0$ for all $q \in \mathfrak{p}$, and therefore $qm = 0$ because M is $\mathfrak{p}$-coprimary. But this means that $m \in M_0$ and hence M/M_0 is $\mathfrak{p}$-coprimary.

As we have an exact sequence

$$\text{(1.47)} \qquad \operatorname{Ext}^{j_0}_{\mathcal{P}}(M/N, E) \to \operatorname{Ext}^{j_0}_{\mathcal{P}}(M, E) \to \operatorname{Ext}^{j_0}_{\mathcal{P}}(N, E) \to \operatorname{Ext}^{j_0+1}_{\mathcal{P}}(M/N, E)$$

the statement for a $\mathcal{P}$-module M for which $\operatorname{Ass}(M)$ contains more than one element follows from the statement about $\mathcal{P}$-modules M to which is associated a strictly lesser number of ideals.

Necessity. – By the sufficiency part of the proof, we have

$$\operatorname{Ext}^{j}_{\mathcal{P}}(Q, E) = 0 \qquad \forall j > j_0$$

for all left $\mathcal{P}$-modules Q of finite type such that

$$\operatorname{Supp}(Q) \subset \operatorname{Supp}(M)\,.$$

If $\mathfrak{p} \in \operatorname{Ass}(M)$, then we can find a $\mathcal{P}$-submodule N of M isomorphic to $\mathcal{P}/\mathfrak{p}$. From the exact sequence

$$\operatorname{Ext}^{j_0}_{\mathcal{P}}(M, E) \to \operatorname{Ext}^{j_0}_{\mathcal{P}}(N, E) \to \operatorname{Ext}^{j_0+1}_{\mathcal{P}}(M/N, E)\,,$$

because $\operatorname{Supp}(M/N) \subset \operatorname{Supp}(M)$, we deduce that

$$\operatorname{Ext}^{j_0}_{\mathcal{P}}(\mathcal{P}/\mathfrak{p}, E) \simeq \operatorname{Ext}^{j_0}_{\mathcal{P}}(N, E) = 0\,.$$

2. – Spaces of functions and distributions.

Given an open set Ω in $\mathbb{R}^n$, we denote by $\mathcal{E}(\Omega)$ the space of complex valued C^∞ functions on Ω, with the Fréchet-Schwartz topology of uniform convergence with all derivatives on compact sets.

If K is a closed subset of Ω, we denote by $\mathcal{E}_K(\Omega)$ the closed subspace of $\mathcal{E}(\Omega)$ of functions with support contained in K. As this space depends only on the relatively closed set K and not on its neighborhood Ω, we write simply $\mathcal{E}_K$ for $\mathcal{E}_K(\Omega)$. When K is compact, we write also $\mathcal{D}_K$ instead of $\mathcal{E}_K$.

If A is any subset of $\mathbb{R}^n$, we define

$$\mathcal{D}_A = \varinjlim_{K \subset\subset A} \mathcal{D}_K$$

with the Schwartz direct limit topology. Note that $\mathcal{D}_A$ can be identified to the space of functions in $\mathcal{E}(\mathbb{R}^n)$ having a compact support contained in A, but the topology of $\mathcal{D}_A$ is stronger than that induced by $\mathcal{E}(\mathbb{R}^n)$, unless A is compact.

When A is open, we write, as customary, $\mathcal{D}(A)$ for $\mathcal{D}_A$.

If Ω is an open set, we denote by $\mathcal{D}'(\Omega)$ the space of distributions in Ω, with the topology of strong dual of $\mathcal{D}(\Omega)$, and by $\mathcal{E}'(\Omega)$ the strong dual of $\mathcal{E}(\Omega)$, that can be identified to the space of distributions having a compact support contained in Ω.

For a subset A of Ω, we denote by $\mathcal{D}'_A(\Omega)$ (resp. $\mathcal{E}'_A$)the space of distributions in Ω (resp. of distributions with compact support) with support contained in A.

Given a closed subset K in Ω, let $\mathcal{F}_K(\Omega)$ (resp. $\mathcal{F}_K^{\text{comp}}(\Omega)$) denote the subspace of $\mathcal{E}(\Omega)$ (resp. $\mathcal{D}(\Omega)$) of functions vanishing with all derivatives on K.

Then we define the space W_K od Whitney function on K by the exact sequence

$$0 \to \mathcal{F}_K(\Omega) \to \mathcal{E}(\Omega) \to W_K \to 0 \tag{2.1}$$

and the space W_K^{comp} of Whitney functions on K with compact support by the exact sequence

$$0 \to \mathcal{F}_K^{\text{comp}}(\Omega) \to \mathcal{D}(\Omega) \to W_K^{\text{comp}} \to 0\,. \tag{2.2}$$

We endow these spaces with the natural quotient topologies. Note that W_K is a space of Fréchet-Schwartz.

With $F = \overline{\Omega - K} \cap \Omega$ we have

$$\mathcal{F}_K(\Omega) = \mathcal{E}_F \quad \text{ and } \mathcal{F}_K^{\text{comp}}(\Omega) = \mathcal{D}_F\,.$$

We also define the space $\check{\mathcal{D}}'_K$ of extendible distributions on K by the exact sequence:

$$0 \to \mathcal{D}'_F \to \mathcal{D}'(\Omega) \to \check{\mathcal{D}}'_K \to 0 \tag{2.3}$$

and the space $\check{\mathcal{E}}'_K$ of extendible distributions with compact support in K by the exact sequence

$$0 \to \mathcal{E}'_F \to \mathcal{E}'(\Omega) \to \check{\mathcal{E}}'_K \to 0\,. \tag{2.4}$$

Note that we have natural inclusions

$$0 \to \check{\mathcal{D}}'_K \to \mathcal{D}'(\operatorname{Int} K)$$
$$0 \to \check{\mathcal{E}}'_K \to \mathcal{D}'_K\,,$$

so that distributions in $\check{\mathcal{D}}'_K$ can be considered as those distributions in $\operatorname{int} K$ that can be continued beyond the points of $\partial K \cap K$.

In this paper, we will mostly consider the case where K is convex. Then the regularity properties are satisfied that guarantee that

$$\mathcal{E}'_K \text{ is the strong dual of } W_K$$

$$\mathcal{D}'_K \text{ is the strong dual of } W_K^{\text{comp}}$$

$$\check{\mathcal{D}}'_K \text{ is the strong dual of } \mathcal{D}_K$$

$$\check{\mathcal{E}}'_K \text{ is the strong dual of } \mathcal{E}_K\,.$$

If u is a distribution, or a C^∞ function, with compact support, we denote by $\hat{u}$ its Fourier-Laplace transform:

$$\hat{u}(\zeta) = \langle u, \exp[-i\langle\cdot,\zeta\rangle]\rangle \quad \text{for } \zeta\in\mathbb{C}^n\,.$$

It will be convenient, while studying the Cauchy problem in a half space, to consider euclidean spaces $\mathbb{R}^{n+1}$, where the dimension is written as the sum of 1 and a positive integer n. We shall denote by $x_0, x_1, \ldots, x_n$ the coordinates in $\mathbb{R}^{n+1}$ and by $\mathcal{P} = \mathbb{C}[\zeta_0, \zeta_1, \ldots, \zeta_n]$ the ring of polynomials in $n+1$ indeterminates $\zeta_0, \zeta_1, \ldots, \zeta_n$.

3. – Cauchy problem with initial data on a hypersurface. Uniqueness.

For a fixed vector $\nu\in\mathbb{R}^{n+1} - \{0\}$, we denote by $H = H(\nu)$ the half space:

$$H = \{x\in\mathbb{R}^{n+1}\colon \langle x, \nu\rangle \geqq 0\}$$

and by $\check{H}$ its symmetrical:

$$\check{H} = \{x\in\mathbb{R}^{n+1}\colon -x\in H\} = \{x\in\mathbb{R}^{n+1}\colon \langle x,\nu\rangle \leqq 0\}\,.$$

Let $S = S(\nu)$ denote the hypersurface

$$S = H\cap\check{H} = \{x\in\mathbb{R}^{n+1}\colon \langle x,\nu\rangle = 0\}\,.$$

Let us consider the exact sequences for functions

$$0\to\mathcal{E}_H\to\mathcal{E}(\mathbb{R}^{n+1})\to W_{\check{H}}\to 0 \tag{3.1}$$

and for distributions:

$$0\to\mathcal{D}'_H\to\mathcal{D}'(\mathbb{R}^{n+1})\to\check{\mathcal{D}}'_{\check{H}}\to 0 \tag{3.2}$$

If M is a unitary left $\mathcal{T}_{n+1}$-module of finite type, we have (cf. [18])

$$\begin{aligned}
&\operatorname{Ext}^j_{\mathcal{T}}\big(M, \mathcal{E}(\mathbb{R}^{n+1})\big) \simeq \operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_{\check{H}}) = 0 && \text{for } j \geqq 1\\
&\operatorname{Ext}^j_{\mathcal{T}}\big(M, \mathcal{D}'(\mathbb{R}^{n+1})\big) \simeq \operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_{\check{H}}) = 0 && \text{for } j \geqq 1\\
&\operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_H) = 0 && \text{for } j \geqq 2\\
&\operatorname{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_H) = 0 && \text{for } j \geqq 2 .
\end{aligned}$$

Therefore, of the long exact sequence of Ext, deduced from (3.1) and (3.2), the only parts that are not zero for all modules M are:

$$(3.3) \qquad 0 \to \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H) \to \operatorname{Ext}^0_{\mathcal{T}}\big(M, \mathcal{E}(\mathbb{R}^{n+1})\big) \to \operatorname{Ext}^0_{\mathcal{T}}(M, W_{\check{H}}) \to \operatorname{Ext}^1_{\mathcal{T}}(M, \mathcal{E}_H) \to 0$$

and

$$(3.4) \qquad 0 \to \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'_H) \to \operatorname{Ext}^0_{\mathcal{T}}\big(M, \mathcal{D}'(\mathbb{R}^{n+1})\big) \to \\ \to \operatorname{Ext}^0_{\mathcal{T}}\big(M, \check{\mathcal{D}}'(\check{H})\big) \to \operatorname{Ext}^1_{\mathcal{T}}(M, \mathcal{D}'_H) \to 0 .$$

DEFINITION. – *We say that ν is non-characteristic for M if ν does not belong to $W(M)$.*

We have:

THEOREM 3.1. – *The following statements for a left $\mathcal{T}$-module M of finite type and a direction $\nu \in \mathbb{R}^{n+1} - \{0\}$ are equivalent:*

(i) *ν is non-characteristic for M*

(ii) $\operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'_H) = 0$

(iii) $\operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H) = 0$.

PROOF. – (i) $\Rightarrow$ (ii). If $\nu \notin W(M)$, then $\operatorname{Ann}(M)$ contains a polynomial q with principal part q_0 non vanishing at ν. By Holmgren's uniqueness theorem, we have then

$$\operatorname{Ext}^0_{\mathcal{T}}\big(\mathcal{T}/(q), \mathcal{D}'_H\big) = 0 .$$

On the other hand there is a surjective $\mathcal{T}$-homomorphism

$$\big(\mathcal{T}/(q)\big)^k \to M \to 0 .$$

Indeed, if $m_1, \ldots, m_k$ are generators of M,

$$\mathcal{T}^k \ni (p_1, \ldots, p_k) \to p_1 m_1 + \ldots + p_k m_k \in M ,$$

by passing to the quotient, defines such a homomorphism.

This yields an injection

$$0 \to \operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{D}'_H) \to \left(\operatorname{Ext}^0_{\mathcal{P}}(P/(q), \mathcal{D}'_H)\right)^k$$

and therefore

$$\operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{D}'_H) = 0$$

because the last group is zero.

(ii) $\Rightarrow$ (iii). This follows from the inclusion

$$0 \to \mathcal{E}_H \to \mathcal{D}'_H$$

which implies also the inclusion

$$0 \to \operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{E}_H) \to \operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{D}'_H)\,.$$

(iii) $\Rightarrow$ (i). We show by contradiction that, if ν is characteristic, i.e. $\nu \in W(M)$, then we can find a non-zero element in $\operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{E}_H)$. In this construction we follow [13].

If $\nu \in W(M)$, then we can find a prime ideal $\mathfrak{p}$ in $\operatorname{supp}(M)$ with

$$\dim_{\mathbb{C}} V(\mathfrak{p}) = 1 \qquad \text{and } \nu \in W(\mathfrak{p})\,.$$

Note that $W(\mathfrak{p})$ contains only distinct lines. Choosing then real coordinates in $\mathbb{R}^{n+1}$ in such a way that $\nu = (1, 0, \ldots, 0)$, we have a Puiseux series representation of $V(\mathfrak{p})$ close to ν_∞ of the form

$$\varphi(s^{1/p}) : \begin{cases} \zeta_0 = s \\ \zeta_j = s \cdot \sum\limits_{h=1}^{\infty} a_{jh}\, s^{-h/p} \qquad \text{for } j = 1, \ldots, n \end{cases}$$

convergent for $|s^{1/p}| > M$. Having fixed ϱ with $1 - 1/p < \varrho < 1$ and defining $(is)^\varrho$ on the half plane $\operatorname{Im} s < 0$ so that it is real and positive when s is purely imaginary, for a fixed branch of $s^{1/p}$ on $\operatorname{Im} s < 0$, we set

$$u(x) = \int\limits_{-\infty - i\tau}^{+\infty - i\tau} \exp\left[i\langle x, \varphi(s^{1/p})\rangle - (is)^\varrho\right] ds \qquad \text{for } \tau > (2M)^p\,.$$

This function defines an element of $\operatorname{Ext}^0_{\mathcal{P}}(\mathcal{P}/\mathfrak{p}, \mathcal{E}_H)$ having support equal to H (cf. [11]).

By proposition 1.1 the implication (iii) $\Rightarrow$(i) follows.

We also have (cf. [13])

PROPOSITION 3.2. – *If $\nu \in W(\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Ass}(M)$, then*

$$\operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H) \to \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}(\overset{\circ}{H}))$$

has a dense image.

The proof, similar to that in [13], is omitted.

By the local character of Holmgren's uniqueness theorem, we deduce

COROLLARY 3.3. – *If M is a left $\mathcal{T}$-module of finite type and $\nu \in \mathbb{R}^{n+1} - \{0\}$, then the following are equivalent:*

(i) *ν is non-characteristic for M;*

(iv) $\operatorname{Ext}^0_{\mathcal{T}}(M, (\mathcal{D}'_H)_0) = 0$;

(v) $\operatorname{Ext}^0_{\mathcal{T}}(M, (\mathcal{E}_H)_0) = 0$.

Indeed, the argument that shows that (i) $\Rightarrow$ (ii) in the proof of Theorem 3.1 also shows the implication (i) $\Rightarrow$ (iv). Then (iv) $\Rightarrow$ (v) is obvious and (v) $\Rightarrow$ (i) is proved as the implication (iii) $\Rightarrow$ (i).

Theorem 3.1 has an application to the study of the Hartogs phenomenon.

The same proof as in [8] yields:

THEOREM 3.3. – *Assume that $\operatorname{Ext}^1_{\mathcal{T}}(M, \mathcal{T}) = 0$ (i.e. that M is overdetermined). Then a necessary and sufficient condition in order that the natural restriction map*

$$\operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}(\mathbb{R}^{n+1})) \to \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}(\mathbb{R}^{n+1} - K))$$

be an isomorphism for every compact K in $\mathbb{R}^{n+1}$ for which $\mathbb{R}^{n+1} - K$ is connected, is that M be elliptic, i.e.

$$W(\mathfrak{p}) \cap \mathbb{R}^{n+1} \subset \{0\} \qquad \textit{for every } \mathfrak{p} \in \operatorname{Ass}(M).$$

4. – Nonhomogeneous Cauchy problem with data on a hyperplane.

A) We first discuss the one-sided Cauchy problem on functions. The notations are the same of the previous section. Our starting point is the exact sequence:

$$0 \to \mathcal{E}_H \to W_H \to W_S \to 0. \tag{4.1}$$

For every left $\mathcal{T}$-module M of finite type we have:

$$\operatorname{Ext}^j_{\mathcal{T}}(M, W_H) = \operatorname{Ext}^j_{\mathcal{T}}(M, W_S) = 0 \qquad \text{for } j \geqq 1,$$

so that the exact sequence of Ext deduced from (4.1) reduces to:

$$(4.2) \qquad 0 \to \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H) \to \operatorname{Ext}^0_{\mathcal{T}}(M, W_H) \to \operatorname{Ext}^0_{\mathcal{T}}(M, W_S) \to \operatorname{Ext}^1_{\mathcal{T}}(M, \mathcal{E}_H) \to 0 .$$

Note that elements of W_H are functions on H that are smooth up to the boundary, while elements of W_S can be thought as formal power series, in a direction t transversal to S, with coefficients in the space $\mathcal{E}(S)$ of smooth functions on S. Therefore an element of W_S is the collection of all normal derivatives on S of a C^∞ function defined on a neighborhood of S.

Identifying by a Hilbert resolution the space $\operatorname{Ext}^0_{\mathcal{T}}(M, W_H)$ to a space of vector valued Whitney functions on H satisfying a system of linear partial differential equations with constant coefficients, by Theorem 3.1 and the exact sequence (4.2) we have:

THEOREM 4.1. – *A necessary and sufficient condition in order that the elements of* $\operatorname{Ext}^0_{\mathcal{T}}(M, W_H)$ *be uniquely determined by their normal derivatives on* S, *is that* ν *be non-characteristic for* M.

Indeed the map

$$\operatorname{Ext}^0_{\mathcal{T}}(M, W_H) \to \operatorname{Ext}^0_{\mathcal{T}}(M, W_S)$$

can be considered as defined by the map that associates to every element of $\operatorname{Ext}^0_{\mathcal{T}}(M, W_H)$ its formal power series on S in the direction normal to S.

B) We want to introduce now the notion of formally noncharacteristic direction for a unitary left $\mathcal{T}$-module M of finite type.

To simplify the notations, we assume that $\nu = (1, 0, \dots, 0)$. We consider

$$\mathcal{T}_n = \mathbb{C}[\zeta_1, \dots, \zeta_n]$$

as a subring of

$$\mathcal{T} = \mathcal{T}_{n+1} = \mathbb{C}[\zeta_0, \dots, \zeta_n] .$$

Then every $\mathcal{T}$-module M can also be considered as a $\mathcal{T}_n$-module by change of ring.

We write $(M)_n$ for the $\mathcal{T}_n$-module obtained from a $\mathcal{T}$-module M by change of ring.

We have in particular a natural identification

$$W_S \cong \operatorname{Ext}^0_{\mathcal{T}_n}((\mathcal{T}_{n+1})_n, \mathcal{E}(S)) = \operatorname{Hom}_{\mathcal{T}_n}((\mathcal{T})_n, \mathcal{E}(S)) \cong \mathcal{E}(S)\{\{x_0\}\} .$$

Therefore, applying the functor $\operatorname{Hom}_{\mathcal{T}_n}(\cdot, \mathcal{E}(S))$ to a Hilbert resolution of M, considered as a $\mathcal{T}$-module, that can be also considered as a resolution of $(M)_n$ by free

$\mathcal{P}_n$-modules, we obtain an isomorphism:

$$\mathrm{Ext}^0_{\mathcal{P}}(M, W_S) \cong \mathrm{Ext}^0_{\mathcal{P}_n}\big((M)_n, \mathcal{E}(S)\big) .$$

Let

$$\mathcal{P}^{a_1} \xrightarrow{{}^tA} \mathcal{P}^{a_0} \to M \to 0 \tag{4.3}$$

be a finite presentation of M.

We say that (4.3) is *formally non characteristic in the direction* ν if the (formal) Cauchy problem:

$$\begin{cases} A(D)u = 0 \\ D_j^0 u|S = 0 \qquad \text{for } j \leq h \\ u \in W_S^{a_0} = \mathcal{E}(S)\{\{x_0\}\} \end{cases} \tag{4.4}$$

for some integer $h \geq 0$ has only the solution $u = 0$.

LEMMA 4.2. – *Let M be a unitary left P-module of finite type. Then if M has a formally non-characteristic presentation, all finite presentations of M are formally non-characteristic.*

PROOF. – Assume that (4.3) is a formally non-characteristic presentation of M and let

$$\mathcal{P}^{b_1} \xrightarrow{{}^tB} \mathcal{P}^{b_0} \to M \to 0 \tag{4.3'}$$

be another finite presentation of M. Then we can find $\mathcal{P}$-homomorphisms

$${}^tL_j\colon \mathcal{P}^{b_j} \to \mathcal{P}^{a_j} \text{ and } {}^tR_j\colon \mathcal{P}^{a_j} \to \mathcal{P}^{b_j} \text{ for } j = 1, 2 \text{ and } {}^tG\colon \mathcal{P}^{a_0} \to \mathcal{P}^{a_1},\ {}^tK\colon \mathcal{P}^{a_1} \to \mathcal{P}^{b_1}$$

with the properties:

$${}^tL_0\,{}^tB = {}^tA\,{}^tL_1 \qquad {}^tB\,{}^tR_1 = {}^tR_0\,{}^tA$$
$${}^tL_0\,{}^tR_0 - \mathrm{Id}_{\mathcal{P}^{a_0}} = {}^tA\,{}^tG$$
$${}^tR_0\,{}^tL_0 - \mathrm{Id}_{\mathcal{P}^{b_0}} = {}^tB\,{}^tK .$$

Assume that (4.4) holds and let k be the maximum degree in ζ_0 of polynomials in the matrix ${}^tR_0(\zeta)$.

Let $w \in W_S^{b_0}$ satisfy

$$\begin{cases} B(D)w = 0 \\ D_0^j w|S = 0 \qquad \text{for } j \leq h + k . \end{cases}$$

Then, for $u = R_0(D)w$, we have

$$A(D)u = A(D)R_0(D)w = R_1(D)B(D)w = 0$$

and

$$D_0^j u|S = D_0^j R_0(D)w|S = 0 \qquad \text{for } j \leqq h \, .$$

Thus, by the assumption,

$$R_0(D)w = 0 \, .$$

But the homotopy formula yields then:

$$w = L_0(D)R_0(D)w - K(D)B(D)w = 0 \, ,$$

i.e. $w = 0$ and therefore also (4.3′) is formally non-characteristic.

We can give then the

DEFINITION. – *A unitary left $\mathfrak{T}$-module of finite type M is said to be formally non-characteristic in the direction ν if it admits a finite presentation* (4.3) *that is formally noncharacteristic in the direction ν.*

We assume that $\nu = (1, 0, \dots, 0)$. Then we define the (*ascending*) *ζ_0-filtration* of $\mathfrak{T}$:

$${}_j\mathfrak{T} = \{p \in \mathfrak{T} \colon \text{degree of } p \text{ in } \zeta_0 \leqq j\} \, , \qquad \text{for } j \in \mathbb{Z} \, .$$

Then ${}_j\mathfrak{T} = \{0\}$ for $j < 0$ and

$${}_0\mathfrak{T} = \mathfrak{T}_n = \mathbb{C}[\zeta_1, \dots, \zeta_n] \, .$$

A *ζ_0-filtration* of a left $\mathfrak{T}$-module M is a sequence $(M_j)_{j\in\mathbb{Z}}$ of abelian subgroups of M with

$$M_j \subset M_{j+1} \qquad \forall j \in \mathbb{Z}$$

and

$${}_h\mathfrak{T} \cdot M_j \subset M_{j+h} \qquad \forall j, h \in \mathbb{Z} \, .$$

The ζ_0-filtration $(M_j)_{j\in\mathbb{Z}}$ of M is *good* if we can find $j_0, j_1 \in \mathbb{Z}$ such that

(i) for each $j \in \mathbb{Z}$, M_j is a $\mathfrak{T}_n$-module of finite type

(i) $M_j = 0$ for $j < j_0$

(iii) ${}_1\mathfrak{T} M_j = M_{j+1}$ for $j \geqq j_1$.

Note that the last condition implies

(iii′) ${}_h\mathfrak{T} M_j = M_{j+h}$ $\forall j \geqq j_1$, $h \geqq 0$.

We also remark that (i), (ii) and (iii) are equivalent to the fact that

$$\operatorname{gr}(M) = \bigoplus_{j\in\mathbb{Z}} M_{j+1}/M_j$$

is finitely generated as a module over the graded ring

$$\operatorname{gr}(\mathcal{T}) = \bigoplus_{j\in\mathbb{Z}} {}_{j+1}\mathcal{T}/{}_j\mathcal{T} .$$

We have the following

LEMMA 4.3. – *Every unitary left $\mathcal{T}$-module M of finite type admits a good ζ_0-filtration. Two different good ζ_0-filtrations $(M_j)_{j\in\mathbb{Z}}$ and $(M'_j)_{j\in\mathbb{Z}}$ of the same $\mathcal{T}$-module M are compatible, in the sense that we can find $k\in\mathbb{Z}$ such that*

$$M_j \subset M'_{j+k} \quad \text{and} \quad M'_j \subset M_{j+k} \qquad \forall j \in \mathbb{Z} .$$

(For the proof of this lemma, cf. [6]).

Let $(M_j)_{j\in\mathbb{Z}}$ be a good filtration of M and let j_1 satisfy (iii) above. Then we have

$$M_{j+1}/M_j \cong M_{h+1}/M_h \quad \text{as } \mathcal{T}_n\text{-modules for } j, h \geqq j_1 .$$

Let us denote then by $\tilde{M}$ this $\mathcal{T}_n$-module associated to the given good filtration of M.

We have the following

THEOREM 4.4. – *Let M be a left $\mathcal{T}$-module of finite type and let $\nu = (1, 0, \ldots, 0)$. Then the following statements are equivalent:*

(i) *M is formally non-characteristic in the direction ν*

(ii) $\tilde{M} = 0$

(iii) *$(M)_n$ is a $\mathcal{T}_n$-module of finite type*

(iv) *For each $\mathfrak{p} \in \operatorname{Ass}(M)$ the map*

$$V(\mathfrak{p}) \in (\zeta_0, \zeta_1, \ldots, \zeta_n) \to (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$$

is finite and dominant.

PROOF. – (i) $\Leftrightarrow$ (ii). Corresponding to the good filtration $(M_j)_{j\in\mathbb{Z}}$ of M we can find a « correct » Hilbert resolution of M (cf. [4], [6])

$$\ldots \mathcal{T}^{a_2} \xrightarrow{{}^tA_1} \mathcal{T}^{a_1} \xrightarrow{{}^tA_0} \mathcal{T}^{a_0} \to M \to 0 .$$

This means the following. For every non negative integer j there are integers

$$\alpha_{j1}, \ldots, \alpha_{ja_j}$$

such that

$$A_j(\zeta) = \left(A_j^{r,s}(\zeta)\right)_{1\leq r\leq a_{j+1}, 1\leq s\leq a_j}$$

with $A_j^{r,s}$ of degree in ζ_0 less or equal to $\alpha_{js} - \alpha_{j+1,r}$; if we set

$$\mathfrak{T}_h^{a_j} = \{(p_1, \ldots, p_{a_j}) \in \mathfrak{T}^{a_j}: \text{degree of } p_k \text{ in } \zeta_0 \leq h + \alpha_{jk}\}\,,$$

then

$${}^tA_j(\zeta)\,\mathfrak{T}_h^{a_{j+1}} \subset \mathfrak{T}_h^{a_j}$$

and M_h is the image of $\mathfrak{T}_h^{a_0}$:

$$M_h \cong \mathfrak{T}_h^{a_0}/{}^tA_0\,\mathfrak{T}_h^{a_1}\,.$$

Let us write the homogeneous part of degree $\alpha_{js} - \alpha_{j+1,r}$ in ζ_0 of $A_j^{r,s}(\zeta)$ as

$$\zeta_0^{\alpha_{js}-\alpha_{j+1,r}}\tilde{A}_j^{r,s}(\zeta_1, \ldots, \zeta_n)$$

and set

$${}^t\tilde{A}_j = {}^t\tilde{A}_j(\zeta_1, \ldots, \zeta_n) = \left(\tilde{A}_j^{r,s}(\zeta_1, \ldots, \zeta_n)\right)_{1\leq r\leq a_{j+1}, 1\leq s\leq a_j}$$

Then

$$\ldots \to \mathfrak{T}_n^{a_2} \xrightarrow{{}^t\tilde{A}_1} \mathfrak{T}_n^{a_1} \xrightarrow{{}^t\tilde{A}_0} \mathfrak{T}_n^{a_0} \to \tilde{M} \to 0$$

is a Hilbert resolution of $\tilde{M}$.

We consider the (descending) filtration of W_S

$$W_S = W_S(0) \supset W_S(1) \supset W_S(2) \supset \ldots$$

defined by

$$W_S(h) = \{u \in W_S: |D_0^j u|S = 0 \text{ for } j < h\}\,, \quad \text{for } h \in \mathbb{Z}\,.$$

Then we set, corresponding to the correct resolution defined above:

$$F_j(h) = \{(u_1, \ldots, u_{a_j}) \in W_S^{a_j}: u_k \in W_S(h + \alpha_{jk}) \text{ for } k = 1, \ldots, a_j\}\,.$$

For every h we obtain a complex:

$$(F_*(h), A_*(D)) = \{0 \to F_0(h) \xrightarrow{A_0(D)} F_1(h) \xrightarrow{A_1(D)} F_2(h) \to \ldots\}\,.$$

From the exact sequences

$$0 \to F_j(h+1) \to F_j(h) \to F_j(h)/F_{j+1}(h) \to 0 \qquad (j,\, h \in \mathbb{Z})$$

we obtain long exact cohomology sequences:

$$0 \to H^0(F_*(h+1), A_*(D)) \to H^0(F_*(h), A_*(D)) \to H^0(F_*(h)/F_*(h+1), \dot{A}_*(D)) \to$$
$$\to H^1(F_*(h+1), A_*(D)) \to H^1(F_*(h), A_*(D)) \to H^1(F_*(h)/F_*(h+1), \dot{A}_*(D)) \to \dots$$

where the $\dot{A}_j$'(D)s denote the quotient maps

$$\dot{A}_j\colon F_j(h)/F_j(h+1) \to F_{j+1}(h)/F_{j+1}(h+1)\,,$$

defining the complex

$$(F_*(h)/F_*(h+1), \dot{A}_*(D)) = \left\{0 \to \frac{F_0(h)}{F_0(h+1)} \xrightarrow{\dot{A}_0(D)} \frac{F_1(h)}{F_1(h+1)} \xrightarrow{\dot{A}_1(D)} \frac{F_2(h)}{F_2(h+1)} \to \dots\right\}.$$

When condition (iii) for the filtration $(M_j)_{j\in\mathbb{Z}}$ holds, we have

$$H^s(F_*(h)/F_*(h+1), \dot{A}_*(D)) \cong \operatorname{Ext}^s_{\mathcal{P}_n}(M, \mathcal{E}(S)) = 0 \qquad \text{for } h \geqq j_1, \text{ for every } s \geqq 1$$

and therefore

$$H^1(F_*(h+1), A_*(D)) \to H^1(F_*(h), A_*(D)) \to 0 \tag{4.5}$$

is surjective for every $h \geqq j_1$.

This condition implies that

$$H^1(F_*(h), A_*(D)) = 0 \quad \text{for } h \geqq j_0\,. \tag{4.6}$$

Indeed let us fix $h \geqq j_0$ and let

$$f \in F_1(h) \text{ satisfy } A_1(D)f = 0\,.$$

By (4.5) we can find $u_h \in F_0(h)$ such that

$$f - A_0(D)u_h \in F_1(h+1)$$

Recursively we can define

$$u_h, u_{h+1}, \dots, u_{h+k}, \dots$$

such that

$$f - A_0(u_h + \dots + u_{h+k}) \in F_1(h+k+1) \quad \text{for } k = 0, 1, 2, \dots\,.$$

The series

$$u = \sum_{k=0}^{\infty} u_{h+k}$$

defines then an element u in $F_0(h)$ such that

$$A_0(D)u = f .$$

This proves that (4.6) holds.

We have therefore an exact sequence

$$(4.7) \quad 0 \to H^0(F_*(h+1), A_*(D)) \to H^0(F_*(h), A_*(D)) \to \mathrm{Ext}^0_{\mathcal{P}_n}(\tilde{M}, \mathcal{E}(S)) \to 0$$

for every $h \geqq j_0$.

Because

$$\mathrm{Ext}^0_{\mathcal{P}_n}(\tilde{M}, \mathcal{E}(S)) = \mathrm{Ext}^0_{\mathcal{P}_n}(\tilde{M}, \mathcal{E}(\mathbb{R}^n)) = 0$$

if and only if $\tilde{M} = 0$ (²) the equivalence (i) $\Leftrightarrow$ (ii) follows from (4.7).

(ii) $\Rightarrow$ (iii). Indeed the condition $\tilde{M} = 0$ implies that $M = M_{j_1}$ and therefore $(M)_n$ is of finite type because M_{j_1} is a $\mathcal{P}_n$-module of finite type.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). This follows because (iii) and (iv) are both equivalent to the fact that Ann (M) contains a polynomial that is monic in ζ_0.

To explain the meaning of the notion of formally noncharacteristic, we briefly rehearse a construction in [2]. Let

$$\ldots \mathcal{P}_n^{b_2} \xrightarrow{{}^tB_1} \mathcal{P}_n^{b_1} \xrightarrow{{}^tB_0} \mathcal{P}_n^{b_0} \to (M)_n \to 0$$

be a Hilbert resolution of $(M)_n$ as a P_n-module. Then one can define $\mathcal{P}_n$-homomorphisms (trace homomorphisms)

$${}^t\tau_j\colon \mathcal{P}_n^{b_j} \to \mathcal{P}^{a_j}$$

such that

$${}^tA_{j-1} \circ {}^t\tau_j = {}^t\tau_{j-1} \circ {}^tB_j \quad \text{for } j \geqq 1 ,$$

(²) Indeed the sequence

$$0 \to \mathcal{E}^{a_0}(S) \xrightarrow{\tilde{A}_0(D)} \mathcal{E}^{a_1}S)$$

is exact if and only if

$$0 \leftarrow \mathcal{P}_n^{a_0} \leftarrow \mathcal{P}_n^{a_1}$$

is exact (cf. [5]).

so that the diagram

$$\begin{array}{ccccccccc} \ldots \to \mathcal{P}^{a_2} & \xrightarrow{{}^tA_1} & \mathcal{P}^{a_1} & \xrightarrow{{}^tA_0} & \mathcal{P}^{a_0} & \to & M & \to 0 \\ \uparrow {}^t\tau_2 & & \uparrow {}^t\tau_1 & & \uparrow {}^t\tau_0 & & \| & \\ \ldots \to \mathcal{P}_n^{b_2} & \xrightarrow{{}^tB_1} & \mathcal{P}^{b_1} & \xrightarrow{{}^tB_0} & \mathcal{P}^{b_0} & \to & (M)_n & \to 0 \end{array}$$

commutes. Then the Cauchy problem

$$(4.4') \qquad \begin{cases} A_0(D)u = 0 \\ \tau_0(D)u\big|_{x_0=0} = u_0 \in \mathcal{E}^{b_0}(S) \end{cases}$$

has a unique solution in $(\mathcal{E}(S)\{\{x_0\}\})^{a_0}$ for every $u_0 \in \mathcal{E}^{b_0}(S)$ that satisfies the integrability condition

$$(4.4'') \qquad B_0(D_1, \ldots, D_n)u_0 = 0 .$$

We can also consider the non-homogeneous Cauchy problem:

$$(4.4''') \qquad \begin{cases} A_0(D)u = f \in W_H^{a_0} \\ \tau_0(D)u\big|_{x_0=0} = u_0 \in \mathcal{E}^{b_0}(S) \end{cases}$$

The compatibility conditions are now

$$(4.4^{\text{iv}}) \qquad \begin{cases} A_1(D)f = 0 & \text{on } H \\ B_0(D)u_0 = \tau_1(D)f & \text{on } S \end{cases}$$

and again the assumption that M is formally non-characteristic in the direction $\nu = (1, 0, \ldots, 0)$ guarantees existence and uniqueness of the solution in $(\mathcal{E}(S)\{\{x_0\}\})^{a_0}$.

C) We end this section by a brief discussion of the nonhomogeneous Cauchy problem for distributions.

From the exact sequence

$$(4.8) \qquad 0 \to \mathcal{D}'_S \to \mathcal{D}'_H \to \check{\mathcal{D}}'_H \to 0$$

we obtain, for every left $\mathcal{P}$-module M of finite type, a long exact sequence:

$$(4.9) \qquad \begin{aligned} 0 &\to \operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{D}'_S) \to \operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{D}'_H) \to \operatorname{Ext}^0_{\mathcal{P}}(M, \check{\mathcal{D}}'_H) \\ &\to \operatorname{Ext}^1_{\mathcal{P}}(M, \mathcal{D}'_S) \to \operatorname{Ext}^1_{\mathcal{P}}(M, \mathcal{D}'_H) \to 0 . \end{aligned}$$

The map

$$\tau\colon \operatorname{Ext}^0_{\mathcal{P}}(M, \check{\mathfrak{D}}'_H) \to \operatorname{Ext}^1_{\mathcal{P}}(M, \mathfrak{D}'_S)$$

is the one that defines « Cauchy data in the sense of Distributions ».

To show that it generalizes the usual definition, we consider an element $u \in \operatorname{Ext}^0_{\mathcal{P}}(M, W_H)$. By the natural inclusion

$$W_H \hookrightarrow \check{\mathfrak{D}}'_H$$

this u defines an element of $\operatorname{Ext}^0_{\mathcal{P}}(M, \check{\mathfrak{D}}'_H)$.

If u has zero Cauchy data in the classical sense, it belongs to the image of the map

$$\operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{E}_H) \to \operatorname{Ext}^0_{\mathcal{P}}(M, W_H)\,.$$

But the inclusion

$$\mathcal{E}_H \hookrightarrow \mathfrak{D}'_H$$

defines a map

$$\operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{E}_H) \to \operatorname{Ext}^0_{\mathcal{P}}(M, \mathfrak{D}'_H)$$

and clearly the diagram

$$\begin{array}{ccc} \operatorname{Ext}^0_{\mathcal{P}}(M, \mathcal{E}_H) & \to & \operatorname{Ext}^0_{\mathcal{P}}(M, W_H) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^0_{\mathcal{P}}(M, \mathfrak{D}'_H) & \to & \operatorname{Ext}^0_{\mathcal{P}}(M, \check{\mathfrak{D}}'_H) \end{array}$$

commutes. Therefore, still denoting by u the image of u in $\operatorname{Ext}^0_{\mathcal{P}}(M, \check{\mathfrak{D}}'_H)$, we have:

$$\tau(u) = 0 \quad \text{iff } u \text{ has zero Cauchy data in the classical sense}\,.$$

The vice versa is true if we assume that M is formally noncharacteristic in the direction ν. Assume indeed that this is the case and let

$$0 \leftarrow M \leftarrow \mathcal{P}^a \xleftarrow{{}^tA} \mathcal{P}^b$$

be a finite presentation of M. Assuming $\nu = (1, 0, \ldots, 0)$, then $\operatorname{Ann}(M)$ contains a polynomial p monic in ζ_0. If $u \in W^a_H$ is such that

$$A(D)u = 0\,,$$

the condition that

$$\tau(u) = 0$$

means that, for the distribution u_0 defined by

$$u^0 = u \quad \text{for } x_0 \geqq 0 ,$$
$$u^0 = 0 \quad \text{for } x_0 < 0 ,$$

we have

$$A(D) u^0 = 0 .$$

But then, for $u = (u_1, \ldots, u_a)$, we obtain

$$p(D) u_j^0 = 0 \quad \text{on } \mathbb{R}^{n+1} \text{ for } j = 1, \ldots, a \tag{4.10}$$

and hence

$$u \in \mathcal{E}_H^a$$

because solutions of (4.10) smoothly depend on x_0.

The argument above also shows that $\operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'_S) = 0$ when M is formally non-characteristic in the direction ν. The vice versa is not true, as the equation

$$\frac{\partial^2 u}{\partial x\, \partial t} + u = 0 \quad \text{in } \mathbb{R}^2 ,$$

which is formally characteristic in the t-direction, has no non-trivial distribution solution with support contained in $\{t = 0\}$.

The exact sequence (4.9) can be interpreted then by:

(*a*) *The condition that* ν *be non-characteristic for* M *is necessary and sufficient to have uniqueness in the Cauchy problem for distributions.*

(*b*) *The condition* $\operatorname{Ext}^1_{\mathcal{T}}(M, \mathcal{D}'_H) = 0$ *is necessary and sufficient in order that every element of* $\operatorname{Ext}^1_{\mathcal{T}}(M, \mathcal{D}'_S)$ *be the Cauchy data of an element* $u \in \operatorname{Ext}^0_{\mathcal{T}}(M, \check{\mathcal{D}}'_H)$.

5. – Hyperbolicity with respect to a half space.

We keep the notations of the preceding section.

We have the following

THEOREM 5.1. – *Let* M *be a unitary left* $\mathcal{T}$*-module of finite type and let* $\nu \in \mathbb{R}^{n-1} - \{0\}$. *Then the following conditions are equivalent:*

(i) *The natural map*

$$\operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}(\mathbb{R}^{n+1})) \to \operatorname{Ext}^0_{\mathcal{T}}(M, W_{\check{H}}) .$$

is a bijection.

(ii) *We have*

$$\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_H) = 0 \quad \text{for } j \geqq 0 .$$

(ii′) *We have*

$$\mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H) = \mathrm{Ext}^1_{\mathcal{T}}(M, \mathcal{E}_H) = 0 .$$

(iii) *For every* $\mathfrak{p} \in \mathrm{Ass}\,(M)$, *we can find a constant* $0 < c < 1$ *such that*

$$-\mathrm{Im}\,\langle\zeta, \nu\rangle \leqq c|\nu|\cdot|\mathrm{Im}\,\zeta| + c^{-1} \quad \textit{for every } \zeta \in V(\mathfrak{p}) . \tag{5.1}$$

(iii′) *For every* $\mathfrak{p} \in \mathrm{Supp}\,(M)$, *we can find a constant* $0 < c < 1$ *such that*

$$-\mathrm{Im}\,\langle\zeta, \nu\rangle \leqq c|\nu|\cdot|\mathrm{Im}\,\zeta| + \bar{c}_1 \quad \textit{for every } \zeta \in V(\mathfrak{p}) .$$

DEF. – When the equivalent conditions above are satisfied, we say that M is *hyperbolic* in the direction ν.

PROOF OF THEOREM 5.1. – From the exact sequence (3.3) we deduce that (i) and (ii′) are equivalent, while (ii′) is equivalent to (ii) because $\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{E}_H) = 0$ for every $j \geqq 2$ and every M. Clearly (iii) is equivalent to (iii′). To show that (i) implies (iii), we can assume that $M = \mathcal{T}/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$ (this is a consequence of propositions 1.1 and 1.2). Let

$$\ldots \to \mathcal{T}^a \xrightarrow{{}^tB} \mathcal{T}^b \xrightarrow{{}^tA} \mathcal{T} \to \mathcal{T}/\mathfrak{p} \to 0 \tag{5.2}$$

be a Hilbert resolution of $\mathcal{T}/\mathfrak{p}$, where

$${}^tA(\zeta) = \big(p_1(\zeta), \ldots, p_a(\zeta)\big)$$

is a set of generators of $\mathfrak{p}$.

By (ii) the sequence

$$0 \to \mathcal{E}_H \xrightarrow{A(D)} \mathcal{E}_H^a \xrightarrow{B(D)} \mathcal{E}_H^b \to \ldots \tag{5.3}$$

is exact and therefore, by duality, also the sequence

$$\ldots (\mathcal{E}'_H)^b \xrightarrow{{}^tB(D)} (\mathcal{E}'_H)^a \xrightarrow{{}^tA(D)} \mathcal{E}'_H \to 0 \tag{5.4}$$

is exact. This means in particular that for every distribution T with compact support contained in H we can find distributions $T_1, \ldots, T_a$ with compact support contained in H such that

$$T - p_1(-D)\,T_1 - \ldots - p_a(-D)\,T_a$$

has support contained in S. Assume that $\nu = (1, 0, \ldots, 0)$.

Let us take $T = \delta(x_0 - 1, x_1, \ldots, x_n)$. Then we have, for some constants A, $C > 0$ and an integer $N \geqq 0$:

$$|\exp[-i\zeta_0] - p_1(-\zeta)\hat{T}_1(\zeta) - \ldots - p_a(-\zeta)\hat{T}_a(\zeta)| \leqq C(1 + |\zeta|)^N \exp[A|\mathrm{Im}(\zeta_1, \ldots, \zeta_n)|]$$

and hence, taking $-\zeta \in V(\mathfrak{p})$, we obtain:

$$-\mathrm{Im}\,\zeta_0 \leqq c + N \log(1 + |\zeta|) + A|\mathrm{Im}(\zeta_1, \ldots, \zeta_n)| \qquad \forall \zeta \in V(\mathfrak{p})\,. \tag{5.5}$$

Let us show that, for some constant $C > 0$, we have actually:

$$-\mathrm{Im}\,\zeta_0 \leqq C(1 + |\mathrm{Im}(\zeta_1, \ldots, \zeta_n)|) \qquad \forall \zeta \in V(\mathfrak{p})\,. \tag{5.6}$$

We consider, for fixed $s \in \mathbb{R}$, the semi-algebraic function

$$f_s(t) = \sup\{-\mathrm{Im}\,\zeta_0 | \zeta \in V(\mathfrak{p}), |\mathrm{Im}(\zeta_1, \ldots, \zeta_n)| \leqq s, |\zeta| \leqq t\}\,.$$

As we have

$$f_s(t) \leqq t \quad \text{ for } t > 0\,,$$

either f_s is constantly equal to $-\infty$ or 0, or, for large t, we have the asymptotic expansion

$$f_s(t) = \alpha t^q(1 + O(1)) \quad \text{ with } \alpha \neq 0 \text{ and } q \in \mathbb{Q}\,.$$

From (5.5) it follows then that either $\alpha < 0$, or $q \leqq 0$. In both cases we conclude that f_s is bounded from above uniformly on $\mathbb{R}$ for every s. Therefore the semi-algebraic function

$$\varphi(s) = \sup_{t \in \mathbb{R}} f_s(t) = \sup\{-\mathrm{Im}\,\zeta_0 \colon \zeta \in V(\mathfrak{p}), |\mathrm{Im}(\zeta_1, \ldots, \zeta_n)| \leqq s\}$$

never takes the value $+\infty$ and therefore, for large s, it is either constantly equal to 0 or $-\infty$, or has an asymptotic expansion

$$\varphi(s) = s^q(1 + O(1)) \quad \text{ with } \alpha \neq 0 \text{ and } q \in \mathbb{Q}\,.$$

To prove (5.6) we have to show that we cannot have at the same time $A > 0$ and $q > 1$.

To this aim we consider the semi-algebraic set

$$F = \{(s, t, \zeta) \colon \zeta \in V(\mathfrak{p}), \mathrm{Im}\,\zeta_0 + \varphi(s) \leqq 1, |\mathrm{Im}(\zeta_1, \ldots, \zeta_n)| \leqq s, |\zeta| \leqq t\}\,.$$

It is not empty and its projection

$$F_1 = \{(s, t): (s, t, \zeta) \in V(\mathfrak{p}) \text{ for some } \zeta \in \mathbb{C}^{n+1}\}$$

is semi-algebraic and contains all $s > 0$ sufficiently large. Therefore

$$\psi(s) = \inf\{t: (s, t) \in F_1\}$$

is a semi-algebraic function that has an asymptotic expansion

$$\psi(s) = \beta \cdot s^{q'}(1 + O(1)) \quad \text{for } s \to \infty \text{ with } \beta \neq 0 \text{ and } q' \in \mathbb{Q}\,.$$

Substituting in (5.5) we obtain

$$\alpha \cdot s^q(1 + O(1)) \leqq c + N \log\left(1 + \beta s^{q'}(1 + O(1)) + A \cdot s\right) \quad \text{for } s \to \infty$$

and hence we must have either $\alpha < 0$ or $q \leqq 1$. The estimate (5.6) follows and it is clearly equivalent to (5.1).

Also in the proof of the implication (iii) $\Rightarrow$ (ii) we assume, as we can, that $M = \mathcal{P}/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \subset \mathcal{P}$ and that $\nu = (1, 0, \ldots, 0)$, so that (5.1) can be written in the equivalent form (5.6).

We are reduced then to the proof of the exactness of (5.4), and this also reduces to the fact that

$${}^tA(D)\colon (\check{\mathcal{E}}'_H)^a \to \check{\mathcal{E}}'_H \to 0$$

is onto. The exactness of the sequence (5.4) at the other steps follows from the theorem on division of distributions (we shall consider this point in more details later on, in the proof of proposition 7.8).

If $T \in \mathcal{E}'_H$, then the Fourier-Laplace transform of T satisfies, for suitable $A > 0$, $B > 0$, $N > 0$, $c > 0$, an estimate of the form:

$$|\hat{T}(\zeta)| \leqq C(1 + |\zeta|)^N \exp\left[A'|\mathrm{Im}\,(\zeta_1, \ldots, \zeta_n)| + B(\mathrm{Im}\,\zeta_0)^+\right]$$

where $(\mathrm{Im}\,\zeta_0)^+ = \sup(0, \mathrm{Im}\,\zeta_0)$. Then we have by (5.6)

$$|\hat{T}(-\zeta)| \leqq c(1 + |\zeta|)^N \exp[C] \exp\left[(A + C)|\mathrm{Im}\,(\zeta_1, \ldots, \zeta_n)|\right]$$

and hence, by the extension theorem (cf. [6]) we can find $T_1, \ldots, T_a \in \check{\mathcal{E}}'_H$ such that

$$T - p_1(-D)T_1 - \ldots - p_a(-D)T_a \text{ has support contained in } S\,.$$

We also have an analogous of Theorem 5.1 for distributions.

THEOREM 5.2. – *Let M be a unitary left $\mathcal{T}$-module of finite type. Then, for a fixed direction $\nu \in \mathbb{R}^{n+1} - \{0\}$, the following statements are equivalent:*

(i) *M is hyperbolic in the direction ν.*

(ii) *The natural map*

$$\mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'(\mathbb{R}^{n+1})) \to \mathrm{Ext}^0_{\mathcal{T}}(M, \check{\mathcal{D}}'_{\check{H}})$$

is a bijection.

(iii) *We have*

$$\mathrm{Ext}^j_{\mathcal{T}}(M, \mathcal{D}'_H) = 0 \quad \text{for every } j \geq 0 .$$

(iii') *We have*

$$\mathrm{Ext}^0_{\mathcal{T}}(N, \mathcal{D}'_H) = \mathrm{Ext}^1_{\mathcal{T}}(M, \mathcal{D}'_H) = 0 .$$

The theorem is still true if we substitute distributions of finite type for general distributions in all statements above.

PROOF. – From the long exact sequence for Ext deduced from the exact sequence (4.8) one easily obtains the equivalence of (ii), (iii) and (iii'). Moreover, both (i) and (iii) imply that M is non-characteristic in the direction ν. The implication (i) $\Rightarrow$ (iii) will be then a consequence of a more general one that we will prove later on (theorem 7.12). The implication (iii) $\Rightarrow$ (i) follows because of a result or Hörmander [11, vol. II, Corollary 11.3.7, p. 78], because Ann (M) contains a polynomial for which the direction ν is non-characteristic: an element $u \in \mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'(\mathbb{R}^{n+1}))$ whose restriction to H is in $\mathrm{Ext}^0_{\mathcal{T}}(M, W_H)$ propagates its regularity to all of $\mathbb{R}^{n+1}$ and hence condition (ii) above implies condition (i) of Theorem 5.1.

6. – Algebraic properties of hyperbolic $\mathcal{T}$-modules. Propagation cones.

A) Let us introduce, for any prime ideal $\mathfrak{p}$ in $\mathcal{T}$, the semialgebraic sets:

$$W^{\mathbb{R}}(\mathfrak{p}) = \{\mathrm{Im}\,\zeta : \zeta \in W(\mathfrak{p})\} \subset \mathbb{R}^{n+1}$$

$$\overline{W}^{\mathbb{R}}(\mathfrak{p}) = \text{closure of } W^{\mathbb{R}}(\mathfrak{p}) \text{ in } \mathbb{R}^{n+1}$$

$$V^{\mathbb{R}}(\mathfrak{p}) = \{\mathrm{Im}\,\zeta : \zeta \in V(\mathfrak{p})\} \in \mathbb{R}^{n+1}$$

$$\tilde{V}^{\mathbb{R}}(\mathfrak{p}) = \text{asymptotic cone of } V^{\mathbb{R}}(\mathfrak{p}) .$$

A vector $\theta \in \mathbb{R}^{n+1}$ belongs to $\tilde{V}^{\mathbb{R}}(\mathfrak{p})$ iff we can find sequences $\{\zeta^m\}$ in $V(\mathfrak{p})$, $\{\varepsilon_m\}$ in $\mathbb{R}$ such that

$$\varepsilon_m > 0 \quad \text{and } \varepsilon_m \to 0 \quad \text{for } m \to \infty$$

$$\varepsilon_m \,\mathrm{Im}\,\zeta^m \to \theta \quad \text{for } m \to \infty .$$

Then we have:

PROPOSITION 6.1. – *For every prime ideal $\mathfrak{p}$, $\tilde{V}^{\mathbb{R}}(\mathfrak{p})$ is a closed cone in $\mathbb{R}^{n+1}$ and we have inclusions*:

$$W^{\mathbb{R}}(\mathfrak{p}) \subset \overline{W}^{\mathbb{R}}(\mathfrak{p}) \subset \tilde{V}^{\mathbb{R}}(\mathfrak{p})\,.$$

A necessary and sufficient condition in order that a unitary left $\mathfrak{I}$-module M be hyperbolic in the direction $\nu \in \mathbb{R}^{n+1} - \{0\}$ is that

$$-\nu \notin \tilde{V}^{\mathbb{R}}(\mathfrak{p}) \qquad \forall \mathfrak{p} \in \operatorname{Ass}(M)\,.$$

PROOF. – If (5.1) is false for some $\mathfrak{p} \in \operatorname{Ass}(M)$, then we can find a sequence $\{\zeta^m\}$ in $V(\mathfrak{p})$ such that

$$(6.1) \qquad -\langle \operatorname{Im}\zeta^m, \nu\rangle \geqq (1 - 2^{-m})|\nu|\cdot|\operatorname{Im}\zeta^m| + 2^m \qquad \text{for every } m\,.$$

Then, with $\varepsilon_m = |\operatorname{Im}\zeta^m|^{-1}$, we have $\varepsilon_m \to 0$ and, passing to a subsequence, we can assume that

$$|\nu||\operatorname{Im}\zeta^m|^{-1}\operatorname{Im}\zeta^m \to \theta \in \tilde{V}^{\mathbb{R}}(\mathfrak{p}) \qquad \text{for } m \to \infty\,.$$

Passing to the limit in (6.1) we obtain

$$|\theta| = |\nu|\,, \qquad -\langle\theta, \nu\rangle \geqq |\nu|\cdot|\theta|$$

from which it follows that $\theta = -\nu$ and hence $-\nu \in \tilde{V}^{\mathbb{R}}(\mathfrak{p})$.

The condition is therefore sufficient. The necessity can be easily derived by passing to the limit in (5.1).

REMARK. When all ideals $\mathfrak{p}$ in $\operatorname{Ass}(M)$ are principal, then the hyperbolicity in the direction ν is equivalent to a seemingly much weaker assumption:

(6.2)

(*a*) $\nu \notin W(\mathfrak{p})$

(*b*) We can find a constant $c > 0$ such that $\langle \operatorname{Im}\zeta, \nu\rangle \geqq c$ $\forall \zeta = \xi + i\tau\nu \in V(\mathfrak{p})$ with $\xi \in \mathbb{R}^{n+1}$, $\tau \in \mathbb{R}$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$.

Indeed, when $\mathfrak{p}$ is principal, setting $\nu = (1, 0, \ldots, 0)$ and considering the plurisubharmonic function

$$\psi(\zeta_1, \ldots, \zeta_n) = \sup\{-\operatorname{Im}\zeta_0\colon (\zeta_0, \zeta_1, \ldots, \zeta_n) \in V(\mathfrak{p})\}$$

on $\mathbb{C}^n$, we deduce (5.1) from (6.2) by the Phragmèn-Lindelöf principle.

The case of principal ideals is very special, as also we have, in that case:

$$\tilde{V}^{\mathbb{R}}(\mathfrak{p}) = W^{\mathbb{R}}(\mathfrak{p});$$

all connected components of $\mathbb{R}^{n+1} - W^{\mathbb{R}}(\mathfrak{p})$ are convex;

$$\tilde{V}^{\mathbb{R}}(\mathfrak{p}) = -\,\tilde{V}^{\mathbb{R}}(\mathfrak{p}) .$$

These properties fail to hold for more general $\mathcal{F}$-modules.
Let us illustrate this fact by discussing some examples.

EXAMPLE 1. – *Let* $\mathfrak{p}$ be the ideal in $\mathcal{F} = \mathbb{C}[\zeta_0, \zeta_1, \zeta_2, \zeta_3]$ generated by

$$p(\zeta) = \zeta_0^2 + \zeta_1^2 - \zeta_2^2 - \zeta_3^2 , \qquad q(\zeta) = \zeta_3 - (1+i)\zeta_0 .$$

Then with

$$A_1 = \left\{\left(-\frac{3+\sqrt{5}}{2}\,t,\, 0,\, 0,\, t\right): t \in \mathbb{R}\right\},$$

$$A_2 = \left\{\left(t,\, 0,\, 0,\, -\frac{3+\sqrt{5}}{2}\,t\right): t \in \mathbb{R}\right\},$$

$$F = \{\theta \in \mathbb{R}^4\colon (\theta_0^2 + 3\theta_0\theta_3 + \theta_3^2)^2 \geqq (\theta_2^2 - \theta_1^2)(\theta_2^2 + \theta_3^2 - \theta_1^2 - 4\theta_0^2 - 2\theta_0\theta_3)\} ,$$

$$A_3 = F \cap \{\theta_1 \neq 0\} ,$$

$$A_4 = F \cap \{\theta_2 \neq 0\} ,$$

we have

$$W^{\mathbb{R}}(\mathfrak{p}) = A_1 \cup A_2 \cup A_3 \cup A_4 \neq \overline{W}^{\mathbb{R}}(\mathfrak{p}) = \tilde{V}^{\mathbb{R}}(\mathfrak{p}) = F .$$

This example shows that $W^{\mathbb{R}}(\mathfrak{p})$ is not closed in general and, since (6.2) holds when $\nu \notin W^{\mathbb{R}}(\mathfrak{p})$, that this condition is not sufficient to imply hyperbolicity for a general system. Notice that $\mathcal{F}/\mathfrak{p}$ is hyperbolic in many directions: for instance in the direction $\nu = (0, 0, 1, 0)$.

EXAMPLE 2. – Let $\mathfrak{p}$ be the ideal in $\mathcal{F} = \mathbb{C}[\zeta_0, \zeta_1, \zeta_2]$ generated by

$$p(\zeta) = i\zeta_0 + \zeta_2 , \qquad q(\zeta) = \zeta_2^3 - \zeta_1^2 .$$

Then $M = \mathcal{F}/\mathfrak{p}$ is hyperbolic in the direction $(1, 0, 0)$, but not in the direction $(-1, 0, 0)$.

This in particular contradicts the existence of a polynomial, hyperbolic in the direction $(1, 0, 0)$, in the ideal $\mathfrak{p}$. The existence of such a polynomial was stated in [9], p. 208, but there hyperbolicity was stated in a slightly different way, requiring that it holds both for the direction ν and for the opposite direction $-\nu$.

We note also that the system associated to the ideal $\mathfrak{p}$ above is also hypoelliptic (cf. [14]). A counterexample to the statement in [9] is contained in the next example.

EXAMPLE 3. – Let $\mathfrak{p}$ be the ideal in $\mathbb{C}[\zeta_0, \zeta_1, \zeta_2, \zeta_3]$ generated by

$$p(\zeta) = \zeta_0 + i\zeta_1 , \qquad q(\zeta) = \zeta_1^2 + \zeta_2^2 + i\zeta_3^2 .$$

Then, for $\nu = (1, 0, 0, 0)$, the module $\mathcal{T}/\mathfrak{p}$ is hyperbolic both in the direction ν and $-\nu$. But $\mathcal{T}$ does not contain any polynomial which is hyperbolic in the direction ν. Indeed, if $\mathcal{T}$ would contain such a polynomial, it would also contain a hyperbolic homogeneous polynomial P. This could be taken with real coefficients. Then such a polynomial would vanish on both $V(\mathfrak{p})$ and $\overline{V(\mathfrak{p})}$.

But $V(\mathfrak{p}) \cap \overline{V(\mathfrak{p})} = \{0\}$ and therefore they cannot be contained in any algebraic variety of dimension 3 in $\mathbb{C}^4$. Hence we can not find such a polynomial P.

EXAMPLE 4. – It is obvious on the other hand that M is hyperbolic in the direction ν if Ann (M) contains a polynomial which is hyperbolic in the direction ν. Another example is the case in which $\nu = (1, 0, \dots, 0)$ is non-characteristic for M, and $(M)_n$, i.e. M considered as a $\mathcal{T}_n = \mathbb{C}[\zeta_1, \dots, \zeta_n]$-module, is elliptic. In this case M itself is elliptic and also hyperbolic in the direction ν. This particular case could be discussed directly using Cauchy-Kowalewska theorem.

B) If the unitary left $\mathcal{T}$-module M of finite type is hyperbolic in the direction ν, then it is also non-characteristic and hence formally non-characteristic in the direction ν. Thus by the exact sequence (4.2) and the isomorphism

$$\mathrm{Ext}^0_{\mathcal{T}}(M, W_S) \cong \mathrm{Ext}^0_{\mathcal{T}}((M)_n, \mathcal{E}(S))$$

we obtain:

PROPOSITION 6.2. – *If M is hyperbolic in the direction $\nu = (1, 0, \dots, 0)$, then we have an isomorphism*:

$$\mathrm{Ext}^0_{\mathcal{T}}(M, W_H) \xrightarrow{\sim} \mathrm{Ext}^0_{\mathcal{T}}((M)_n, \mathcal{E}(S)) .$$

This isomorphism has to be read in classical terms by saying that the Cauchy Problem (4.4‴) has one and only one solution $u \in W_H^{a_0}$ for every choice of the data $f \in W_H^{a_1}$ and $u_0 \in \mathcal{E}^{b_0}(S)$ satisfying the compatibility condition (4.4$^{\mathrm{iv}}$).

To improve the statement above we can also consider propagation cones.

THEOREM 6.3. *Let Γ be an open convex cone contained in $\mathbb{R}^n - \tilde{V}^{\mathbb{R}}(\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Ass}\,(M)$. Let $\nu \in \Gamma$ and, denoting by Γ^0 the polar cone to Γ*:

$$\Gamma^0 = \{x \in \mathbb{R}^{n+1} : \langle x, \theta \rangle \geqq 0 \ \ \forall \theta \in \Gamma\} ,$$

set:

$$K = S \cap (\nu - \Gamma^0), \quad \tilde{K} = H \cap (\nu - \Gamma^0).$$

Then the natural restriction map:

$$\mathrm{Ext}^0_{\mathcal{P}}(M, W_{\tilde{K}}) \to \mathrm{Ext}^0_{\mathcal{P}}(M, W_K)$$

is an isomorphism.

PROOF. – If we show that, for

$$h_K(\xi) = \sup_K \langle x, \xi \rangle, \quad h_{\tilde{K}}(\xi) = \sup_{\tilde{K}} \langle x, \xi \rangle,$$

we have

$$h_K(\theta) = h_{\tilde{K}}(\theta) \quad \forall \theta \in \tilde{V}^{\mathbf{R}}(\mathfrak{p}), \qquad \mathfrak{p} \in \mathrm{Ass}(M), \tag{6.3}$$

then the statement will follow with an argument analogous to the implication (iii) $\Rightarrow$ (i) in Theorem 5.1.

Let us first show that

$$\langle \xi, \nu \rangle \leqq h_K(\xi) \qquad \forall \xi \in \tilde{V}^{\mathbf{R}}(\mathfrak{p}). \tag{6.4}$$

We have, indeed

$$h_K(\xi) = \sup_{\substack{\theta \in \Gamma^0 \\ |\nu|^2 = \langle \theta, \nu \rangle}} \langle \nu - \theta, \xi \rangle. \tag{6.5}$$

For $\xi \in \tilde{V}^{\mathbf{R}}(\mathfrak{p})$, since $\xi \notin \Gamma$, we can find $\theta_0 \in \Gamma^0$ with

$$\langle \theta_0, \xi \rangle \leqq 0.$$

Hence, with $\theta = |\nu|^2 \theta_0 / \langle \theta_0, \nu \rangle$ in (6.5) we obtain (6.4).

We note that K is characterized by

$$\langle x, \nu \rangle = 0, \quad \langle x, \xi \rangle \leqq \langle \nu, \xi \rangle \qquad \forall \xi \in \Gamma. \tag{6.6}$$

For $x \in \tilde{K}$, we set

$$x = y + |\nu|^{-2} \langle x, \nu \rangle \nu. \tag{6.7}$$

Because we have

$$\langle x, \xi \rangle \leqq \langle \nu, \xi \rangle \qquad \forall \xi \in \Gamma, \quad \forall x \in \tilde{K}, \tag{6.8}$$

we also have, for $\xi \in \tilde{V}^{\mathbf{R}}(\mathfrak{p})$:

$$\langle x, \xi \rangle = |\nu|^{-2} \langle x, \nu \rangle \cdot \langle \nu, \xi \rangle + \langle y, \xi \rangle \leqq |\nu|^{-2} \langle x, \nu \rangle h_K(\xi) + \langle y, \xi \rangle. \tag{6.9}$$

Let $z \in K$ and let us show that

$$w = |\nu|^{-2}\langle x, \nu\rangle z + y \in K .$$

Indeed, for $\xi \in \Gamma$ we have

$$\langle w, \xi\rangle = |\nu|^{-2}\langle x, \nu\rangle \cdot \langle z, \xi\rangle + \langle y, \xi\rangle \leqq$$
$$\leqq |\nu|^{-2}\langle x, \nu\rangle \cdot \langle \nu, \xi\rangle + \langle x, \xi\rangle - |\nu|^{-2}\langle x, \nu\rangle \cdot \langle \nu, \xi\rangle = \langle x, \xi\rangle \leqq \langle \nu, \xi\rangle .$$

Therefore, $w \in K$ by (6.6) and we have

$$|\nu|^{-2}\langle x, \nu\rangle\langle z, \xi\rangle + \langle y, \xi\rangle \leqq h_K(\xi) \qquad \forall \xi \in \tilde{V}^{\mathbf{R}}(\mathfrak{p}) , \quad \forall z \in K .$$

Taking the supremum for $z \in K$ we get from (6.9):

$$\langle x, \xi\rangle \leqq h_K(\xi) \qquad \forall x \in \tilde{K} , \quad \forall \xi \in \tilde{V}^{\mathbf{R}}(\mathfrak{p}) ,$$

proving (6.3).

Theorem 6.3 has an obvious corollary in the non-convex case. If Ω is an open subset of S, we denote by $\tilde{\Omega}_\Gamma$ the reunion of all convex cones of the form

$$(y - \Gamma^0) \cap H , \qquad \text{for } (y - \Gamma^0) \cap S \subset \Omega .$$

Then we have

THEOREM 6.4. – *With the same assumptions of* Theorem 6.3, *the natural restriction map*

$$\mathrm{Ext}^0_{\mathcal{T}}(M, W_{\tilde{\Omega}_\Gamma}) \to \mathrm{Ext}^0_{\mathcal{T}_n}\big((M)_n, \mathcal{E}(\Omega)\big)$$

is an isomorphism.

Note that Theorem 6.4 also implies that the Cauchy problem (4.4‴) has a unique solution in $W^{a_0}_{\tilde{\Omega}_\Gamma}$ for all data $f \in W^{a_1}_{\tilde{\Omega}_\Gamma}$ and $u_0 \in \mathcal{E}^{b_0}(\Omega)$ satisfying the compatibility conditions (4.4$^{\mathrm{iv}}$).

7. – Evolution modules.

A) Let M be a unitary left $\mathcal{T}$-module of finite type and let $\nu \in \mathbb{R}^{n+1} - \{0\}$ be fixed. We say that M is *of evolution in the direction* ν if, for $H = H(\nu)$, any of the following two equivalent conditions is satisfied:

(7.1) The natural restriction map

$$\mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{E}(\mathbb{R}^n)) \to \mathrm{Ext}^0_{\mathcal{T}}(M, W_{\check{H}})$$

is onto.

(7.2) $\mathrm{Ext}^1_{\mathcal{T}}(M, \mathcal{E}_H) = 0$.

We say that M is *of evolution for* $\mathcal{D}'$ *in the direction* ν if instead one of the following equivalent conditions is satisfied:

(7.1′) The natural restriction map

$$\mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'(\mathbb{R}^{n+1})) \to \mathrm{Ext}^0_{\mathcal{T}}(M, \check{\mathcal{D}}'_{\check{H}})$$

is onto.

(7.2′) $\mathrm{Ext}^1_{\mathcal{T}}(M, \mathcal{D}'_H) = 0$.

The two notions are clearly equivalent for hypoelliptic $\mathcal{T}$-modules M, i.e. when the map

$$\mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{E}(\mathbb{R}^{n+1})) \hookrightarrow \mathrm{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'(\mathbb{R}^{n+1}))$$

is an isomorphism.

A hypoelliptic $\mathcal{T}$-module M that is of evolution in the direction ν is said to be *parabolic* in the direction ν.

When M is non-characteristic in the direction ν, then is of evolution if and only if is hyperbolic in the direction ν.

Another very different example of evolution module is given by the $\mathcal{T}$-modules M obtained by syspension from a $\mathcal{T}_n$-module N. We have:

PROPOSITION 7.1. – *Let* $\nu = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ *and let* N *be a unitary left* $\mathcal{T}_n$*-module of finite type. Let us consider the* $\mathcal{T}$*-module*

$$M = N \otimes_{\mathcal{T}_n} \mathcal{T}_{n+1}.$$

Then M *is of evolution in the direction* ν.

PROOF. – Let

$$\mathcal{T}_n^{a_1} \xrightarrow{{}^tA} \mathcal{T}_n^{a_0} \to N \to 0$$

be a finite presentation of N. Then

$$\mathcal{T}^{a_1} \xrightarrow{{}^tA} \mathcal{T}^{a_0} \to M \to 0,$$

where we consider ${}^tA(\zeta)$ as a matrix of polynomials in $\mathcal{I}$ independent of ζ_0, is a finite presentation of M. If $f \in W_H^{q_0}$ is a solution of

$$A(D)f = A(D_1, \ldots, D_n)f = 0 ,$$

then all coefficients f_h of the formal power series

$$\sum_{h=0}^{\infty} f_h(x_1, \ldots, x_n) \cdot x_0^h$$

defined by f at $x_0 = 0$ satisfy

$$A(D_1, \ldots, D_n) f_h = 0 .$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be equal to 1 on a neighborhood of 0 in $\mathbb{R}$. For a sequence $\{t_h\}$ of positive real numbers with $t_h \nearrow +\infty$, the series

$$\sum_{j=0}^{\infty} f_h(x_1, \ldots, x_n) x_0^h \chi(t_h x_0)$$

converges in $\mathcal{E}(\mathbb{R}^{n+1})$ to a function g such that

$$A(D)g = 0 \qquad \text{on } \mathbb{R}^{n+1}$$

and f and g coincide with all derivatives on S. Then

$$\tilde{f} = f \quad \text{for } x_0 \leqq 0 , \qquad \tilde{f} = g \quad \text{for } x_0 \geqq 0$$

defines a solution of

$$A(D)\tilde{f} = 0 \qquad \text{on } \mathbb{R}^{n+1}$$

extending f.

Note that, by Proposition 6 and the Corollary following Proposition 8 in [18], we have:

$$\operatorname{Ext}^j_{\mathcal{I}}(M, \mathcal{E}_H) = 0 \qquad \text{for } j \geqq 2$$

$$\operatorname{Ext}^j_{\mathcal{I}}(M, \mathcal{D}'_H) = 0 \qquad \text{for } j \geqq 2$$

for every unitary left $\mathcal{I}$-module of finite type. Then, by Proposition 1.2 we have:

THEOREM 7.2. – *A necessary and sufficient condition in order that a unitary left $\mathcal{I}$-module M be of evolution (resp. of evolution for $\mathcal{D}'$) in the direction ν is that, for every $\mathfrak{p} \in \operatorname{Ass}(M)$, the module $\mathcal{I}/\mathfrak{p}$ be of evolution (resp. of evolution for $\mathcal{D}'$) in the direction ν.*

REMARK. – Proposition 7.1 shows at once that the condition that there is a subideal $\mathfrak{p}' \subset \mathfrak{p}$ such that $\mathfrak{I}/\mathfrak{p}'$ be of evolution in the direction ν does not imply that $\mathfrak{I}/\mathfrak{p}$ is of evolution in the direction ν.

B) We say that a prime ideal $\mathfrak{p}$ in $\mathfrak{I}$ is of evolution in the direction ν if the $\mathfrak{I}$-module $\mathfrak{I}/\mathfrak{p}$ is of evolution in the direction ν.

Assume that $\nu = (1, 0, ..., 0)$. We have:

LEMMA 7.3. – *If $\mathfrak{p}$ is a prime ideal in $\mathfrak{I}$, then $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{I}_n$ is a prime ideal in $\mathfrak{I}_n$.*

Indeed, $\mathfrak{I}$ is a flat ring extension of $\mathfrak{I}_n$.

Given a prime ideal $\mathfrak{p}$ in $\mathfrak{I}$, let us fix a set of generators

$$p_1, \dots, p_h, \qquad q_1, \dots, q_k$$

of $\mathfrak{p}$ with the properties:

$$p_1, \dots, p_h \text{ generate } \mathfrak{p}' = \mathfrak{p} \cap \mathfrak{I}_n \text{ in } \mathfrak{I}_n. \tag{7.3}$$

For every $j = 1, ..., k$,

$$q_j(\zeta) = \sum_{h=0}^{m_j} q_{jh}(\zeta_1, \dots, \zeta_n)\zeta_0^j \text{ is irreducible, of degree } m_j \text{ in } \zeta_0 \tag{7.4}$$

and its discriminant $\Delta_j(\zeta_1, ..., \zeta_n)$ with respect to ζ_0 does not belong to $\mathfrak{p}$.

$$m_1 \leqq m_2 \leqq \dots \leqq m_k, \tag{7.5}$$

$$m_1 = \inf\{\text{degree of } p \text{ with respect to } \zeta_0 \colon p \in \mathfrak{p} - \mathfrak{p}'\}. \tag{7.6}$$

By the choice of m_1, we obtain:

We can find polynomials

$$\alpha_2, \dots, \alpha_k \in \mathfrak{I}_n, \qquad \beta_2, \dots, \beta_k \in \mathfrak{I}$$

such that

$$\alpha_j(\zeta_1, \dots, \zeta_n) q_j(\zeta) - \beta_j(\zeta) q_1(\zeta) \in \mathfrak{p}' \quad \text{for } j = 2, \dots, k, \tag{7.7}$$

(7.8) for each $j = 2, ..., k$, the polynomials α_j and β_j have no common factor and do not belong to $\mathfrak{p}'$.

This follows by division in the Euclidean ring of polynomials in ζ_0 having coefficients that are rational functions of $\zeta_1, ..., \zeta_n$.

Let us prove now:

LEMMA 7.4. – *Let* $f \in \mathcal{E}_H$ *satisfy*

$$p_j(D_1, \ldots, D_n) f = 0 \quad \text{for } j = 1, \ldots, h. \tag{7.9}$$

Then we can find $g \in \mathcal{E}_H$ *such that*

$$\begin{cases} \alpha_2(D_1, \ldots, D_n) \ldots \alpha_k(D_1, \ldots, D_n) g = f \\ p_j(D) g = 0 \quad \text{for } j = 1, \ldots, h. \end{cases} \tag{7.10}$$

PROOF. – By proposition 7.1, $\mathcal{I}/(\alpha_2, \ldots, \alpha_k, p_1, \ldots, p_h)$ is of evolution in the direction $\nu = (1, 0, \ldots, 0)$. Therefore we only have to check that the right hand side of (7.10) satisfies the right compatibility conditions. Let $\gamma_0, \ldots, \gamma_h \in \mathcal{I}$ be such that

$$\gamma_0 \alpha_2 \ldots \alpha_k + \gamma_1 p_1 + \ldots + \gamma_h p_h = 0.$$

Because $\alpha_2 \ldots \alpha_k \notin \mathfrak{p}'$, this equality implies that γ_0 belongs to the ideal in $\mathcal{I}$ generated by $\mathfrak{p}'$ and hence

$$\gamma_0(D) f = 0.$$

The proof is complete.

From this we deduce

LEMMA 7.5. – *Let* $f \in \mathcal{E}_H$ *satisfy* (7.9) *above. Then we can find* $f_1, \ldots, f_k \in \mathcal{E}_H$ *with*

$$f_1 = f$$

such that the system

$$\begin{cases} q_1(D) u = f_1 \\ \ldots\ldots\ldots \\ q_k(D) u = f_k \\ p_1(D) u = 0 \\ \ldots\ldots\ldots \\ p_h(D) u = 0 \end{cases} \tag{7.11}$$

is solvable with $u \in \mathcal{E}(\mathbb{R}^{n+1})$, *i.e. the right hand side satisfies all compatibility conditions.*

PROOF. – By the preceding lemma, we can find $g \in \mathcal{E}_H$ solving (7.10).

We define then

$$
\begin{aligned}
& f_1 = f \\
(7.12) \qquad & f_j = \alpha_2(D_1, \ldots, D_n) \ldots \alpha_{j-1}(D_1, \ldots, D_n)\alpha_{j+1}(D_1, \ldots, D_n) \ldots \alpha_k(D_1, \ldots, D_n)\beta_j(D) g \\
& \qquad \text{for } j = 2, \ldots, k .
\end{aligned}
$$

Let us show now that (7.11) satisfies then all compatibility conditions.

Let $\gamma_1, \ldots, \gamma_h, \eta_1, \ldots, \eta_k \in \mathfrak{T}$ be such that

$$\gamma_1 p_1 + \ldots + \gamma_h p_h + \eta_1 q_1 + \ldots + \eta_k q_k = 0 .$$

Denoting by $\tilde{\mathfrak{p}}'$ the ideal in $\mathfrak{T}$ generated by $\mathfrak{p}'$, we have

$$
\begin{aligned}
& q_1(\eta_1\alpha_2 \ldots \alpha_k + \eta_2\alpha_3 \ldots \alpha_k\beta_2 + \ldots + \eta_k\alpha_2 \ldots \alpha_{k-1}\beta_{k-1}) \equiv \\
& \qquad \equiv (q_1\eta_1 + q_2\eta_2 + \ldots + q_k\eta_k)\cdot\alpha_2 \ldots \alpha_k (\operatorname{mod} \tilde{\mathfrak{p}}');
\end{aligned}
$$

but, since $q_1 \notin \mathfrak{p}'$ and $\mathfrak{p}'$ is a prime ideal, this implies that

$$\eta_1\alpha_2 \ldots \alpha_k + \eta_2\alpha_3 \ldots \alpha_k\beta_2 + \ldots + \eta_k\alpha_2 \ldots \alpha_{k-1}\beta_k \in \mathfrak{p}' .$$

Hence

$$
\begin{aligned}
\eta_1(D) f_1 + \ldots + \eta_k(D) f_k = \big(\eta_1(D)\alpha_2(D) \ldots \alpha_k(D) + \eta_2(D)\alpha_3(D) \ldots \alpha_k(D)\beta_2(D) + \\
+ \ldots + \eta_k(D)\alpha_2(D) \ldots \alpha_{k-1}(D)\beta_k(D)\big) g = 0 .
\end{aligned}
$$

The proof is complete.

We obtain then:

PROPOSITION 7.6. – *With the notations introduced above: let $\mathscr{I}$ denote the ideal of $\mathfrak{T}$ generated by $q_1, p_1, \ldots, p_h$.*

Then a necessary and sufficient condition in order that $\mathfrak{p}$ be of evolution in the direction $\nu = (1, 0, \ldots, 0)$ is that the $\mathfrak{T}$-module $\mathfrak{T}/\mathscr{I}$ be of evolution in the direction ν.

PROOF. – The condition is obviously sufficient because $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{T}/\mathscr{I})$. The necessity follows from the previous lemma, because, if the right hand side $(f, g_1, \ldots, g_h) \in \mathcal{E}_H^{h+1}$ of the system

$$
(7.12) \qquad \left\{
\begin{aligned}
& q_1(D) u = f \\
& p_1(D) u = g_1 \\
& \vdots \\
& p_h(D) u = g_h
\end{aligned}
\right.
$$

satisfies all compatibility relations, then we can find $w \in \mathcal{E}_H$ such that

$$p_j(D)w = g_j \quad \text{for } j = 1, \ldots, h$$

because $\tilde{\mathfrak{p}}'$ is of evolution in the direction ν.

We obtain a system for $v = u - w$:

$$(7.13) \qquad \left\{ \begin{array}{l} q_1(D)v = \tilde{f} \\ p_1(D)v = 0 \\ \cdots\cdots\cdots\cdots \\ p_h(D)v = 0 \end{array} \right.$$

that by the previous lemma can be lifted to a system of the form (7.11).

Therefore we have $\mathrm{Ext}^1_{\mathcal{T}}(\mathcal{T}/\mathscr{I}, \mathcal{E}_H) = 0$ when $\mathrm{Ext}^1_{\mathcal{T}}(\mathcal{T}/\mathfrak{p}, \mathcal{E}_H) = 0$ and therefore the proof is complete.

Proposition 7.6 gives a classical interpretation of the meaning of the Cauchy problem for overdetermined systems: given a scalar partial differential operator, we try to solve the usual Cauchy problem under the additional condition that the data and the solution satisfy a system of partial differential equations tangent to the initial hypersurface.

EXAMPLE. – In $\mathbb{R}^{2n+1}$ we consider the ideal $\mathfrak{p}$ generated by

$$\zeta_0 - i \sum_1^m \zeta_j^2, \zeta_1 + i\zeta_{n+1}, \ldots, \zeta_n + i\zeta_{2n} .$$

Then $\mathfrak{p}$ is of evolution both in the direction ν and in the direction $-\nu$, for $\nu = (1, 0, \ldots, 0)$.

This is a consequence of the fact that the solutions $u \in \mathcal{E}(\mathbb{R}^{n+1})$ of the heat equation

$$\frac{\partial u}{\partial x_0} = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2}$$

for each fixed x_0 extend to entire functions of $x_1, \ldots, x_n$ in $\mathbb{C}^n$, of the fact that $\mathfrak{p}$ also contains the polynomial

$$\zeta_0 + i \sum_{n+1}^{2n} \zeta_j^2$$

and that

$$\frac{\partial}{\partial x_0} - \sum_1^n \frac{\partial^2}{\partial x_j^2} \text{ is of evolution in } \mathbb{R}^{n+1} \text{ in the direction } \nu = (1, 0, \ldots, 0)$$

and

$$\frac{\partial}{\partial x_0}+\sum_{n+1}^{2n}\frac{\partial^2}{\partial x_j^2}$$

is of evolution in $\mathbb{R}^1_{x_0}\times\mathbb{R}^n_{x_{n+1},\dots,x_{2n}}$ in the direction $-\nu=(-1,0,\dots,0)$.

(For scalar evolution operators, cf. [11]).

This example has been discussed with S. SPAGNOLO, in connection with the question of existence of solutions analytic in the tangential variables for the Cauchy problem in the scalar case. The discussion above shows how the general theory of evolution $\mathfrak{T}$-modules is a natural generalization of that question.

The following result, that we quote here for completeness, is a consequence of statements that will be proved in the next section.

PROPOSITION 7.7. – *With the notations of proposition 7.6: a necessary and sufficient condition in order that $\mathfrak{T}/\mathfrak{p}$ be of evolution for $\mathfrak{D}'$ in the direction ν is that $\mathfrak{T}/\mathscr{I}$ be of evolution for $\mathfrak{D}'$ in the direction ν.*

EXAMPLE. – If an ideal $\mathscr{I}$ in $\mathfrak{T}$ contains a polynomial p that is hyperbolic in the direction ν, then the $\mathfrak{T}$-module $\mathfrak{T}/\mathscr{I}$ is also hyperbolic in the direction ν. In [11] HÖRMANDER characterizes partial differential operators with constant coefficients that are of evolution in the direction ν. However, the fact that $\mathscr{I}$ contains such an operator is not sufficient in order that $\mathfrak{T}/\mathscr{I}$ be itself of evolution in the direction ν. Indeed, the ideal $\mathscr{I}$ generated by $q(\xi)=i\zeta_0+\zeta_1^2$ and $p(\zeta)=-\zeta_2-\zeta_1^4$ in $\mathbb{C}[\zeta_0,\zeta_1,\zeta_2]$ is not hyperbolic (and hence not of evolution, being non-characteristic) in the direction $\nu=(1,0,0)\in\mathbb{R}^3-\{0\}$, while $q(D)=\partial/\partial x_0-\partial^2/\partial x_1^2$ is an evolution operator.

C) Let M be a unitary left $\mathfrak{T}$-module of finite type. Let

$$\cdots\to\mathfrak{T}^c\xrightarrow{{}^tB}\mathfrak{T}^b\xrightarrow{{}^tA}\mathfrak{T}^a\to M\to 0 \tag{7.14}$$

be a Hilbert resolution of M. Then, by using duality, we obtain the criterion:

PROPOSITION 7.8. – *A necessary and sufficient condition in order that M be of evolution in the direction ν is that the map*

$${}^tA(D)\colon(\check{\mathcal{E}}'_H)^b\to(\check{\mathcal{E}}'_H)^a$$

has a closed image.

A necessary and sufficient condition in order that M be of evolution for $\mathfrak{D}'$ in the direction ν is that the map

$${}^tA(D)\colon(W_H^{\mathrm{comp}})^b\to(W_H^{\mathrm{comp}})^a$$

has a closed image.

PROOF. - By duality, the exactness of the sequence

$$\mathcal{E}_H^a \xrightarrow{A(D)} \mathcal{E}_H^b \xrightarrow{B(D)} \mathcal{E}_H^c \tag{7.15}$$

(resp. of the sequence

$$(\mathcal{D}'_H)^a \xrightarrow{A(D)} (\mathcal{D}'_H)^b \xrightarrow{B(D)} (\mathcal{D}'_H)^c) \tag{7.16}$$

is equivalent, because the last maps of (7.15) and (7.16) have a closed image, to the fact that

$$(\check{\mathcal{E}}'_H)^c \xrightarrow{{}^tB(D)} (\check{\mathcal{E}}'_H)^b \xrightarrow{{}^tA(D)} (\check{\mathcal{E}}'_H)^a \tag{7.17}$$

(resp.

$$(W_H^{\mathrm{comp}})^c \xrightarrow{{}^tB(D)} (W_H^{\mathrm{comp}})^b \xrightarrow{{}^tA(D)} (W_H^{\mathrm{comp}})^a) \tag{7.18}$$

is exact and the last map of (7.17) (resp. (7.18)) has a closed image.

By the theorem of division of distributions we know that the maps:

$$\mathrm{Tor}_0^{\mathcal{T}}(M, \mathcal{E}'_S) \to \mathrm{Tor}_0^{\mathcal{T}}(M, \mathcal{E}'_H)$$

and

$$\mathrm{Tor}^0_{\mathcal{T}}(M, \mathcal{D}_{\check{H}}) \to \mathrm{Tor}_0^{\mathcal{T}}(M, \mathcal{D}(\mathbb{R}^n))$$

are injective (the last result is proved in [18]).

From the exact sequences

$$0 \to \mathcal{E}'_S \to \mathcal{E}'_H \to \check{\mathcal{E}}'_H \to 0$$

and

$$0 \to \mathcal{D}_{\check{H}} \to \mathcal{D}(\mathbb{R}^n) \to W_H^{\mathrm{comp}} \to 0$$

we deduce then the exact sequences of Tor:

$$\cdots \to \mathrm{Tor}_1^{\mathcal{T}}(M, \mathcal{E}'_H) \to \mathrm{Tor}_1^{\mathcal{T}}(M, \check{\mathcal{E}}'_H) \to \mathrm{Tor}_0^{\mathcal{T}}(M, \mathcal{E}'_S) \to \mathrm{Tor}_0^{\mathcal{T}}(M, \mathcal{E}'_H) \to$$
$$\to \mathrm{Tor}_0^{\mathcal{T}}(M, \check{\mathcal{E}}'_H) \to 0\,,$$

$$\cdots \to \mathrm{Tor}_1^{\mathcal{T}}((M, \mathcal{D}(\mathbb{R}^n)) \to \mathrm{Tor}_1^{\mathcal{T}}(M, W_H^{\mathrm{comp}}) \to \mathrm{Tor}_0^{\mathcal{T}}(M, \mathcal{D}_{\check{H}}) \to \mathrm{Tor}_0^{\mathcal{T}}(M, \mathcal{D}(\mathbb{R}^n)) \to$$
$$\to \mathrm{Tor}_0^{\mathcal{T}}(M, W_H^{\mathrm{comp}}) \to 0$$

and then

$$\mathrm{Tor}_1^{\mathcal{T}}(M, \check{\mathcal{E}}'_H) = 0 \qquad \mathrm{Tor}_1^{\mathcal{T}}(M, W_H^{\mathrm{comp}}) = 0$$

because $\mathcal{E}'_H$ and $\mathcal{D}(\mathbb{R}^n)$ are flat differential $\mathcal{T}$-modules. Thus the only condition to be required is that about the closedness of the image of the last maps of (7.15) and (7.16).

A trivial consequence of this proposition is:

COROLLARY 7.9. – *Every unitary left $\mathcal{T}$-module M of the form*

$$M = N \otimes_{\mathcal{T}_n} \mathcal{T}_{n+1}$$

with N a unitary left $\mathcal{T}_n$-module of finite type is of evolution for $\mathcal{D}'$ in the direction $\nu = (1, 0, \dots, 0)$.

The statement of Proposition 7.6 holds true when « of evolution for $\mathcal{D}'$ » substitutes « of evolution ».

From proposition 7.8 we also derive the following criterion:

PROPOSITION 7.10. – *Let M be a unitary left $\mathcal{T}$-module of finite type. Then a necessary and sufficient condition in order that M be of evolution in the direction $\nu \in \mathbb{R}^{n+1} - \{0\}$ is that, for every $\mathfrak{p} \in \operatorname{Ass}(M)$, the following condition $(E\mathfrak{p})$ holds:*

$(E\mathfrak{p})$ *if $T \in \mathcal{E}'(\mathbb{R}^{n+1})$ and $\langle T, u\rangle = 0$ for every $u \in \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H)$, then we can find $T_1 \in \mathcal{E}'_{\check{H}}$ such that $\hat{T} - \hat{T}_1 = 0$ on $V(\mathfrak{p})$.*

A necessary and sufficient condition in order that M be of evolution for $\mathcal{D}'$ in the directtion $\nu \in \mathbb{R}^{n+1} - \{0\}$ is that, for every $\mathfrak{p} \in \operatorname{Ass}(M)$, the following condition $(E'\mathfrak{p})$ holds:

$(E'\mathfrak{p})$ *if $\varphi \in \mathcal{D}(\mathbb{R}^{n+1})$ and $\langle u, \varphi\rangle = 0$ for every $u \in \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'_H)$, then we can find $u_1 \in \mathcal{D}_{\check{H}}$ such that $\hat{u} - \hat{u}_1 = 0$ on $V(\mathfrak{p})$.*

The statements are indeed a consequence of Theorem 7.2 and the fact that the image of a continuous linear map is dense in the annihilator of the kernel of its dual map.

To prove the next result, connecting the groups $\operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H)$ and $\operatorname{Ext}^1_{\mathcal{T}}(M, \mathcal{D}'_H)$, we need a precision of condition $(E\mathfrak{p})$ that is a simple consequence of the open mapping theorem on Fréchet spaces:

LEMMA 7.11. – *Condition $(E\mathfrak{p})$ above is equivalent to the following:*

$(\tilde{E}\mathfrak{p})$ *For every $A > 0$ and integer $N \geqq 0$, we can find a compact K in $\check{H}$, an integer $N_1 \geqq 0$ and a constant $B > 0$ such that if $T \in \mathcal{E}'(\mathbb{R}^{n+1})$ satisfies*

$$|\hat{T}(\zeta)| \leqq (1 + |\zeta|)^N \exp[A|\operatorname{Im}\zeta|] \qquad \forall \zeta \in \mathbb{C}^{n+1}$$

$$\langle T, u\rangle = 0 \qquad \forall u \in \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H)$$

then it also satisfies:

$$|\hat{T}(\zeta)| \leqq B(1+|\zeta|)^{N_1} \exp [H_K(\operatorname{Im} \zeta)] \qquad \forall \zeta \in V(\mathfrak{p}) .$$

REMARK. – Clearly condition $(\tilde{E}\mathfrak{p})$ is equivalent to an analogous condition for distributions $T \in \mathcal{E}'_H$: We shall refer to this equivalent condition as $(\tilde{\tilde{E}}\mathfrak{p})$.

From this criterion we deduce the following

THEOREM 7.12. – *If a unitary let $\mathcal{T}$-module M is of evolution in the direction v, then it is also of evolution for $\mathcal{D}'$ in the direction v.*

PROOF. – By Theorem 7.2 we can assume in the proof that $M = \mathcal{T}/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^{n+1})$ be such that

$$\langle u, \varphi \rangle = 0 \qquad \text{for every } u \in \operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{D}'_H) .$$

By Paley-Wiener theorem, the Fourier-Laplace transform $\hat{\varphi}$ of φ satisfies:

$$|\hat{\varphi}(\zeta)| \leqq C_N(1+|\zeta|)^{-N} \exp [A|\operatorname{Im} \zeta|]$$

for some constant $A \geqq 0$ and for a sequence $\{C^N\}$ of non-negative real numbers.

In particular, because $D^\alpha \varphi$ is orthogonal to $\operatorname{Ext}^0_{\mathcal{T}}(M, \mathcal{E}_H)$ for every α and

$$|\zeta^\alpha \hat{\varphi}(\zeta)| \leqq C_{|\alpha|} \exp [A|\operatorname{Im} \zeta|]$$

whe have by $(\tilde{E}\mathfrak{p})$:

$$|\zeta^\alpha \hat{\varphi}(\zeta)| \leqq c_{|\alpha|}(1+|\zeta|)^m \exp [H_K(\operatorname{Im} \zeta)] \qquad \text{on } V(\mathfrak{p}) \text{ for every } \alpha \in \mathbb{N}^{n+1}$$

where K is a fixed compact subset of H and m is a fixed integer.

Hence we have, with a new sequence $\{C'_N\}$ of non-negative real numbers:

$$|\hat{\varphi}(\zeta)| \leqq C'_N(1+|\zeta|)^{-N} \exp [H_K(\operatorname{Im} \zeta)] \qquad \forall \zeta \in V(\mathfrak{p}) \text{ and } \forall \text{integer } N \geqq 0 .$$

By Proposition 1 in [18], we can find a continuous plurisubharmonic function ψ on $\mathbb{C}^n$, with

$$|\psi(\zeta_1) - \psi(\zeta_2)| \leqq \text{const. for } |\zeta_1 - \zeta_2| \leqq 1$$

such that, for a sequence $\{c''_N\}$ of non-negative real numbers, we have

$$\exp [\psi(\zeta)] \leqq c''_N(1+|\zeta|)^{-N} \exp [H_K(\operatorname{Im} \zeta)] \qquad \forall \zeta \in \mathbb{C}^{n+1}, \quad \forall N$$

and

$$\exp[\psi(\zeta)] \geqq c'_N(1+|\zeta|)^{-N}\exp[H_K(\operatorname{Im}\zeta)] \qquad \forall\zeta\in\mathbb{C}^{n+1}, \quad \forall N .$$

Then (cf. [6]) we can find an entire function F on $\mathbb{C}^{n+1}$ such that

$$F = \hat{\varphi} \text{ on } V(\mathfrak{p})$$

and

$$|F(\zeta)| \leqq \text{const}\,(1+|\zeta|)^M \exp[\psi(\zeta)] \qquad \text{on } \mathbb{C}^{n+1} .$$

The function F is the Fourier-Laplace transform of a function $g\in\mathfrak{D}_{\check{H}}$.
The statement follows then from condition $(E'\mathfrak{p})$.

8. – A Phragmèn-Lindelöf principle for evolution modules.

A) Let us go back to the notations introduced in the previous section after Lemma 7.3. In particular we shall consider modules $M = \mathfrak{T}/\mathscr{I}$ for an ideal $\mathscr{I}$ of $\mathfrak{T}$ which is of the form described in Proposition 7.6:

$$\mathscr{I}\cap\mathfrak{T}_n = \mathfrak{p}' \quad \text{is a prime ideal in } \mathfrak{T}_n \tag{8.1}$$

and $\mathscr{I}$ is generated by $\mathfrak{p}'$ and a polynomial $q(\zeta)$ of the form

$$q(\zeta) = \sum_{h=0}^{m} q_h(\zeta_1, \dots, \zeta_n)\zeta_0^h \tag{8.2}$$

irreducible, with

$$q_m(\zeta_1, \dots, \zeta_n)\Delta(\zeta_1, \dots, \zeta_n) \notin \mathfrak{p}' , \tag{8.3}$$

$\Delta(\zeta_1, \dots, \zeta_n)$ denoting the discriminant of $q(\zeta)$ with respect to ζ_0.
We denote by $\mathcal{E}_H(\mathfrak{p}')$ (resp. $\mathfrak{D}'_H(\mathfrak{p}')$) the space of all $f\in\mathcal{E}_H$ (resp. $f\in\mathfrak{D}'_H$) such that

$$p(D_1, \dots, D_n)f = 0 \qquad \forall p\in\mathfrak{p}' . \tag{8.4}$$

It follows from the proof of Proposition 7.6 that we have:

LEMMA 8.1. – *A necessary and sufficient condition in order that $M = \mathfrak{T}/\mathscr{I}$ be of evolution in the direction $\nu = (1, 0, \dots, 0)$ is that the sequence*

$$\mathcal{E}_H(\mathfrak{p}') \xrightarrow{q(D)} \mathcal{E}_H(\mathfrak{p}') \to 0 \tag{8.5}$$

be exact. A necessary and sufficient condition in order that $M = \mathcal{P}/\mathcal{I}$ be of evolution for $\mathcal{D}'$ in the direction $\nu = (1, 0, \ldots, 0)$ is that the sequence

$$\mathcal{D}'_H(\mathfrak{p}') \xrightarrow{q(D)} \mathcal{D}'_H(\mathfrak{p}') \to 0 \tag{8.6}$$

be exact.

As we want to exploit Lemma 8.1 only in some particular cases, in the following lemma, where essentially we apply duality to (8.5) and (8.6), we make additional assumptions on the ideal $\mathcal{I}$.

LEMMA 8.2. – *Assume that $\mathfrak{p}'$ is elliptic, i.e. that*

$$|(\zeta_1, \ldots, \zeta_n)| \leqq c(1 + |\mathrm{Re}\,(\zeta_1, \ldots, \zeta_n)|)$$

on $V(\mathfrak{p}')$. Then a necessary and sufficient condition in order that $\mathcal{P}/\mathcal{I}$ be of evolution in the direction $\nu = (1, 0, \ldots, 0)$ is that, given $R > 0$, we can find constants $C > 0$, $N > 0$, $\varepsilon > 0$ such that

$$\sup_{\theta \in V(\mathfrak{p}')} \int_0^\varepsilon x_0^N |\tilde{\varphi}(x_0, \theta)| (1 + |\theta|)^{-N} \exp\left[-N|\mathrm{Im}\,\theta|\right] dx_0 \leqq \tag{8.7}$$

$$\leqq C \sup_{\theta \in V(\mathfrak{p}')} \int_0^\varepsilon |q(-D_0, -\theta)\tilde{\varphi}(x_0, \theta)| \exp\left[-R|\mathrm{Im}\,\theta|\right] dx_0$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^{n+1})$ with $\operatorname{supp}\varphi \subset \{x_0 < \varepsilon\}$. Here

$$\tilde{\varphi}(x_0, \theta) = \int_{\mathbb{R}^n} \varphi(x_0, x_1, \ldots, x_n) \exp\left[-i(x_1\theta_1 + \ldots + x_n\theta_n)\right] dx_1, \ldots, dx_n$$

denotes the partial Fourier-Laplace transform of φ with respect to the variables $x_1, \ldots, x_n$.

PROOF. – The statement is a consequence of the open mapping theorem and of the fact that the two sides of (8.7) are continuous seminorms in $\mathcal{D}_{K_j}/\mathcal{D}_{K_j} \cap \mathcal{D}_{\hat{H}}$ where

$$K_1 = [-\varepsilon, \varepsilon] \times \left\{(x_1, \ldots, x_n) | \sum_1^n x_j^2 \leqq N^2\right\} \quad \text{and} \quad K_2 = [-\varepsilon, \varepsilon] \times \left\{(x_1, \ldots, x_n) | \sum_1^n x_j^2 \leqq R^2\right\}$$

respectively, if the suprema are taken over all $\theta \in \mathbb{C}^n$. Then we need to apply the fundamental principle of Ehrenpreis to show that (8.7) implies that the equation

$$q(D)u = f$$

has a solution $u \in \mathcal{D}'_H(\mathfrak{p}')$ for every given $f \in \mathcal{E}_H(\mathfrak{p}')$.

LEMMA 8.3. – *Let* $c(a) = \big(1 - \exp(-\operatorname{Re} a)\big)/\operatorname{Re} a$ *for* $a \in \mathbb{C}$, $\operatorname{Re} a \neq 0$, $c(a) = 1$ *if* $\operatorname{Re} a = 0$.

Then we have:

$$\int_0^1 |u(t)|\,dt \leqq c(a)\int_0^1 |u'(t) - a\cdot u(t)|\,dt$$

for every $a \in \mathbb{C}$ *and* $u \in \mathcal{E}(\mathbb{R})$ *with* $u = 0$ *for* $t > 1$.

Let $q \in \mathcal{P}$ be a polynomial monic and of degree m with respect to ζ_0. We assume that q is irreducible and of positive degree with respect to $\zeta_1, \ldots, \zeta_n$:

$$q(\zeta_0, \zeta_1, \ldots, \zeta_0) = \zeta_0^m + \sum_{j<m} q_j(\zeta_1, \ldots, \zeta_n)\zeta_0^j$$

and set

$$r = \sup\{(\text{degree of } q_j)/(m-j)\colon j = 0, \ldots, m-1\}\,.$$

The equation $q = 0$ defines an algebraic Riemann domain Σ over $\mathbb{C}^n$

$$\Sigma \xrightarrow{\pi} \mathbb{C}^n$$

on which holomorphic functions $\tau_1, \ldots, \tau_m$ are defined in such a way that

$$q\big(\tau_j(\sigma), \pi(\sigma)\big) = 0 \qquad \text{on } \Sigma \text{ for } j = 1, \ldots, m$$

$$q\big(\zeta_0, \pi(\sigma)\big) = \big(\zeta_0 - \tau_1(\sigma)\big) \ldots \big(\zeta_0 - \tau_m(\sigma)\big) \qquad \text{for } \zeta_0 \in \mathbb{C}\,,\ \sigma \in \Sigma$$

$$\sup\{|\tau_j(\sigma)|\colon |\pi(\sigma)| \leqq t\} = O(t^r) \text{ for } t \to \infty\,.$$

The projection π is holomorphic.

Let $\Delta = \Delta(\zeta_1, \ldots, \zeta_n) \in \mathcal{P}_n$ be the discriminant of q with respect to ζ_0; the restriction of π defines then an m-fold covering map

$$\Sigma - \pi^{-1}(\Gamma) \to \mathbb{C}^n - \Gamma\,,$$

where $\Gamma = \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n\colon \Delta(\zeta_1, \ldots, \zeta_n) = 0\}$ and both spaces are connected. (Σ can be defined as the normalization of the analytic subspace $q = 0$ of $\mathbb{C}^{n+1}$).

LEMMA 8.4. – *With the notations and assumptions introduced above:*

We can find positive constants c_1, c_2 *such that, setting, for* $1 \leqq j \leqq m$,

$$E_j = \{\sigma \in \Sigma\colon \operatorname{Re} \tau_j(\sigma) \leqslant c_1(1 + |\pi(\sigma)|^2)^{\frac{1}{2}}\}\,, \tag{8.8}$$

for every plurisubharmonic function ψ *on* Σ *such that, for some* $R > 0$,

$$\sup_{\Sigma} \psi(\sigma) - R|\pi(\sigma)| < +\infty \tag{8.9}$$

we have also

$$\sup_{\Sigma} \left(\psi(\sigma) - (1 + c_2)R|\pi(\sigma)|\right) \leqq \sup_{E_1} \left(\psi(\sigma) - R|\pi(\sigma)|\right). \tag{8.10}$$

PROOF. – Note that, for $r \leqq 1$, and c_1 sufficiently large, we have $E_1 = \Sigma$, so that the statement of the lemma is trivial. Assume therefore that $r > 1$.

A) We consider first the case $n = 1$.

If c_1 is large, then E_1 contains $\pi^{-1}(D)$ for a disk D in $\mathbb{C}$ containing Γ in its interior part. One can show then that E_1 also contains curves $\gamma_1, \ldots, \gamma_s$ such that $\pi(\gamma_1), \ldots, \pi(\gamma_s)$ are rays from the origin in $\mathbb{C}$ and the angles

$$\widehat{\pi(\gamma_1)\pi(\gamma_2)}, \ldots, \widehat{\pi(\gamma_j)\pi(\gamma_{j+1})}, \ldots, \widehat{\pi(\gamma_s)\pi(\gamma_1)}$$

are all acute, while the complement of E_1 in Σ is covered by the connected, simply connected open sets $\Omega_1, \ldots, \Omega_s$, with

$$\partial\Omega_1 = \gamma_s' \cup \delta_1 \cup \gamma_2', \ldots, \partial\Omega_s' = \gamma_s \cup \delta_s \cup \gamma_1'$$

for $\delta_1, \ldots, \delta_s$ connected arcs in $\pi^{-1}(\partial D)$ and $\gamma_h' = \gamma_h - \pi^{-1}(D)$ for $h = 1, \ldots, s$. On each Ω_h, the function $|\pi(\sigma)|$ is bounded by an $\mathbb{R}$-linear function f_h of $\pi(\sigma)$. Then we obtain the thesis by applying the Phragmèn-Lindelöff maximum principle to the subharmonic function

$$\psi(\sigma) - Rf_h(\pi(\sigma)) \quad \text{on } \bar{\Omega}_h:$$

if $\lambda = \sup_{E_1} \left(\psi(\sigma) - R|\pi(\sigma)|\right)$, then we have

$$\psi(\sigma) - Rf_h(\pi(\sigma)) \leqq \lambda \quad \text{on } \partial\Omega_h$$

and therefore

$$\psi(\sigma) - Rf_h(\pi(\sigma)) \leqq \lambda \quad \text{on } \Omega_h.$$

But

$$f_h(\pi(\sigma)) \leqq (1 + c_2)|\pi(\sigma)| \quad \text{on } \Omega_h$$

for some constant $c_2 > 0$ and then

$$\psi(\sigma) - R(1 + c_2)|\pi(\sigma)| \leqq \lambda \quad \text{on } \Omega_h.$$

Repeating the argument for $h = 1, \ldots, s$, we obtain the thesis.

B) Before discussing the general case, let us consider first the situation in which $n > 1$, but $m = 1$. We have $q = \zeta_0 - p(\zeta_1, \dots, \zeta_n)$ with $p \in \mathcal{T}_n$ of degree $d > 1$, so that $\tau_1 = \tau = p$. We denote by p_0 the principal part, homogeneous of degree d, of p and let $W = \{\xi \in \mathbb{C}^n : p_0(\xi) = 0\}$. With $S_1 = \{\xi \in \mathbb{C}^n : |\xi| = 1\}$, we note that, for every $\theta \in W \cap S_1$, we can find a vector $w = w(\theta) \in S_1$ and a real $\eta = \eta(\theta)$ with $0 < \eta < 1$ such that

$$\{\theta + zw : z \in \mathbb{C},\ |z| = \eta\} \cap W = \emptyset .$$

Then, for each $\theta \in W \cap S_1$, we can find an open neighborhood U_θ of θ in $\mathbb{C}^n$ such that

$$\{\xi + zw : z \in \mathbb{C},\ |z| = \eta\} \cap W = \emptyset$$

for every ξ in U_θ. By Borel-Lebesgue's lemma, we can find $\theta_1, \dots, \theta_N$ in $W \cap S_1$ such that

$$W \cap S_1 \subset U_{\theta_1} \cup \dots \cup U_{\theta_N} .$$

It follows that it is possible to choose $\varepsilon > 0$ with the property:

if $\xi \in S_1$ *and* $|p_0(\xi)| < \varepsilon$, *then there are* $w = w(\xi) \in S_1$ *and* $\eta = \eta(\xi)$ *real with* $0 < \eta < 1$, *such that*

$$|p_0(\xi + zw)| \geqq \varepsilon |\xi + zw|^d \quad \textit{for } |z| = \eta .$$

On each complex line $L_\theta = \{z\theta : z \in \mathbb{C}\}$ with $\theta \in S_1$ and $|p_0(\theta)| \geqq \varepsilon$, we can argue as in the point A) of the proof, with estimates that come out uniform with respect to θ. Hence we obtain: there are constants c_1 and c_2' such that, if ψ is plurisubharmonic on $\mathbb{C}^n$ and satisfies (8.9) with $\Sigma = \mathbb{C}^n$, we have

$$\psi(z\theta) - (1 + c_2')|z| \leqq \sup_E (\psi(\xi) - R|\xi|)$$

where

$$E = \{\xi \in \mathbb{C}^n : \operatorname{Re} p(\xi) \leqq c_1(1 + |\xi|^2)^{\frac{1}{2}}\}$$

provided that $|\theta| = 1$ and $|p_0(\theta)| \geqq \varepsilon$.

Therefore we have

$$\psi(\theta) - (1 + c_2)R|\theta| \leqq \sup_E (\psi(\xi) - R|\xi|)$$

if $|p_0(\theta)| \geqq \varepsilon |\theta|^d$.

If $\theta \in \mathbb{C}^n$ is such that $|p_0(\theta)| < \varepsilon |\theta|^d$, then we can find $w \in S_1$ and $0 < \eta < 1$ such that

$$|p_0(\theta + z\eta w)| \geqq \varepsilon |\theta + z\eta w|^d \quad \text{for } |z| = |\theta| .$$

By the maximum principle we have

$$\psi(\theta) \leqq \max_{|z|=|\theta|} \psi(\theta + z\eta w) .$$

But, for $|z| = |\theta|$, we have also

$$\psi(\theta + z\eta w) - (1 + c_2')R|\theta + z\eta w| \leqq \sup_E (\psi(\xi) - R|\xi|) .$$

Because

$$|\theta + z\eta w| \leqq 2|\theta| \quad \text{for } |z| = |\theta| ,$$

we deduce that

$$\psi(\theta) - (1 + c_2)R|\theta| \leqq \sup_E (\psi(\xi) - R|\xi|)$$

with $c_2 = 1 + 2c_2'$, and this inequality now holds for every θ in $\mathbb{C}^n$.

C) Let us turn now to the general case. Only slight modifications are needed of the arguments of point *B*).

For $\sigma \in \Sigma - \pi^{-1}(\Gamma \cup \{0\})$, there is a unique irreducible curve $\tilde{L}_\sigma$ in Σ containing σ and contained in $\pi^{-1}(\{z\pi(\sigma): z \in \mathbb{C}\})$. Then we define (³)

$$h_c(\sigma) = |\pi(\sigma)|^r \cdot \limsup_{\substack{\sigma' \in \tilde{L}_\sigma \\ |\pi(\sigma')| \to \infty}} \frac{|\tau_j(\sigma')|}{|\pi(\sigma')|^r}$$

and denote by $h_c^*(\sigma)$ the upper semicontinuous majorant of h_c. Because Σ is normal, we obtain in this way a plurisubharmonic function defined on Σ, whose restrictions to the lines $\tilde{L}_\sigma$ is complex homogeneous of degree r. Due to the continuous dependence of the roots of a polynomial on its coefficients, the function h_c^* is continuous and moreover, if we fix $\varepsilon > 0$, the restrictions of plurisubharmonic functions ψ satisfying (8.9) satisfy (8.10) on $\tilde{L}_\sigma$ with uniform constants c_1 and c_2' (only depending on ε), provided that $h_c^*(\sigma) \geqq \varepsilon|\pi(\sigma)|^r > 0$. As in point *B*) the conclusion comes from the fact that every point $\sigma_0 \in \Sigma$ is the center on an analytic disc, with radius less or equal to $|\pi(\sigma)|$, with the boundary contained in the region where $h_c^*(\sigma) \geqq \varepsilon|\pi(\sigma)|^r$. As a streightforward consequence of lemmata 8.2, 8.3 and 8.4 we have:

THEOREM 8.5. – *Let* $p = p(\zeta_0, \zeta_1, \ldots, \zeta_n) \in \mathfrak{P}$ *be monic with respect to* ζ_0. *Let* $\mathscr{I} \subset \mathbb{C}[\zeta_0, \zeta_1, \ldots, \zeta_n, \zeta_{n+1}, \ldots, \zeta_n]$ *be generated by* p *and by* $\zeta_1 + i\zeta_{n+1}, \ldots, \zeta_n + i\zeta_{2n}$. *Then* $\mathbb{C}[\zeta_0, \ldots, \zeta_{2n}]/\mathscr{I}$ *is of evolution in the direction* $\nu = (1, 0, \ldots, 0, 0, \ldots, 0) \in \mathbb{R}^{2n+1}$.

From theorem 8.5 we deduce now a statement for general $\mathfrak{P}$-modules.

(³) We introduce here the « circled indicator of growth » function of Lelong (cf. P. LELONG - L. GRUMAN, *Entire Functions of Several Complex Variables*, Springer, Berlin, 1986).

First of all let us introduce the notion of holomorphic suspension in the tangential directions.

Given a unitary left $\mathcal{T}$-module M of finite type, with $\mathcal{T} = \mathcal{T}_{n+1} \equiv \mathbb{C}[\zeta_0, \ldots, \zeta_n]$, let $\mathfrak{a}$ be the ideal of $\mathcal{T}_{2n+1} = \mathbb{C}[\zeta_0, \ldots, \zeta_n, \zeta_{n+1}, \ldots, \zeta_{2n}]$ generated by

$$\zeta_1 + i\zeta_{n+1}, \ldots, \zeta_n + i\zeta_{2n} .$$

Then we set

$$\tilde{M} = M \otimes_{\mathcal{T}} \mathcal{T}_{2n+1}/\mathfrak{a} .$$

The module $\tilde{M}$ is called the holomorphic suspension of M tangential to the hyperplane $S = \{x_0 = 0\} \subset \mathbb{R}^{n+1}$.

We have:

THEOREM 8.6. – *The holomorphic suspension $\tilde{M}$ of a unitary left $\mathcal{T}$-module M of finite type and formally non-characteristic in the direction $\nu = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$, tangential to the hyperplane $S = \{x_0 = 0\} \subset \mathbb{R}^{n+1}$ is an evolution module in the direction $\tilde{\nu} = (1, 0, \ldots, 0, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.*

PROOF. – It is sufficient to consider modules M of the form $M = \mathcal{T}/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$, containing a polynomial that is monic with respect to ζ_0.

Note that, if the module $\mathcal{T}/\mathfrak{p}$ is non-characteristic in the direction ν, then the statement reduces to the theorem of Cauchy-Kowalewska. So we can assume that $\mathcal{T}/\mathfrak{p}$ is characteristic, but formally non-characteristic, in the direction ν.

Let $\mathfrak{p}' = \mathfrak{p} \cap \mathcal{T}_n$. By the preparation theorem (cf. [3]) we can choose coordinates in $\mathbb{R}^n = S$ in such a way that, if $d = \dim_{\mathbb{C}} V(\mathfrak{p}')$ (by $V(\mathfrak{p}')$ we denote the affine variety of common zeros of polynomials of $\mathfrak{p}'$ in $\mathbb{C}^n$), then we have

$$(8.11) \qquad |(\zeta_1, \ldots, \zeta_n)| \leq C(1 + |(\zeta_1, \ldots, \zeta_d)|) \quad \forall (\zeta_1, \ldots, \zeta_d) \in V(\mathfrak{p}') .$$

Then there is in $\mathfrak{p}$ a monic polynomial q in ζ_0 of the form

$$(8.12) \qquad q(\zeta) = \zeta_0^m + \sum_{j=0}^{m-1} q_j(\zeta_1, \ldots, \zeta_d)\, \zeta_0^j$$

such that $\mathfrak{p}$ is generated by q and $\mathfrak{p}'$.

Let now $\varphi_0, \ldots, \varphi_{n-1}$ be entire functions on $\mathbb{C}^n$ satisfying

$$(8.13) \qquad p(D)\varphi_h = 0 \qquad \forall p \in \mathfrak{p}' .$$

We note that $(\mathcal{T}_n/\mathfrak{p}')_d$ is a free $\mathcal{T}_d$-module and hence we can find differential operators $B_1, \ldots, B_h$ on $\mathbb{R}^n$ such that for every given h-uple $g_1, \ldots, g_h$ of entire functions on $\mathbb{C}^d$ there is a unique g entire on $\mathbb{C}^n$ such that

$$\left\{\begin{array}{ll} p(D)g = 0 & \text{for every } p \in \mathfrak{p}' \\ B_j(D)g|_{x_{d+1}=\ldots=x_n=0} = g_j & \text{for } j = 1, \ldots, h\,. \end{array}\right. \tag{8.14}$$

By Theorem 8.5 we can find functions $f_1, \ldots, f_h$ in $\mathcal{E}(\mathbb{R}\times\mathbb{C}^d)$ that satisfy

$$\left\{\begin{array}{l} q(D)f_j = 0\,, \\ \left(\dfrac{\partial}{\partial x_k} + i\dfrac{\partial}{x_{n+k}}\right)f_j = 0 \qquad \text{for } k = 1, \ldots, n\,, \\ \left.\dfrac{\partial f_j}{\partial x_0^h}\right|_{x_0=0,x_{d+1}=\ldots=x_n=0} = B_j(D)\varphi_n|_{x_{d+1}=\ldots=x_n=0} \qquad \text{for } h = 0, \ldots, m-1\,. \end{array}\right. \tag{8.15}$$

Then we denote by f the unique function in $\mathcal{E}(\mathbb{R}\times\mathbb{C}^n)$ such that, for each fixed $x_0 \in \mathbb{R}$, $g(x_1, \ldots, x_n) = f(x_0, x_1, \ldots, x_n)$ is the solution of (8.14) for $g_j(x_1, \ldots, x_d) = f_j(x_0, x_1, \ldots, x_d)$. We obtain in this way a solution $f \in \mathcal{E}(\mathbb{R}\times\mathbb{C}^n)$ of the Cauchy problem

$$\left\{\begin{array}{l} f \in \operatorname{Ext}^0_{\mathcal{P}_{2n+1}}(\tilde{M}, \mathcal{E}(\mathbb{R}\times\mathbb{C}^n))\,, \\ \dfrac{\partial^h f}{\partial x_0^h}(0, x_1, \ldots, x_n) = \varphi_h(x_1, \ldots, x_n)\,, \qquad h = 0, \ldots, m-1\,. \end{array}\right. \tag{8.16}$$

The proof is complete.

B) Let $\mathfrak{p}$ be a prime ideal in $\mathcal{P} = \mathbb{C}[\zeta_0, \zeta_1, \ldots, \zeta_n]$. We say that $V(\mathfrak{p})$ satisfies a *Phragmèn-Lindelöf principle in the direction* $\nu = (1, 0, \ldots, 0)$ if *we can find* $\varepsilon > 0$ *such that every plurisubharmonic function* φ *on* $\mathbb{C}^{n+1}$ *that satisfies*

$$\varphi(\zeta) \leqq A|\operatorname{Im}\zeta'| + \varepsilon \sup(0, -\operatorname{Im}\zeta_0) \quad \textit{on } V(\mathfrak{p}) \textit{ for some } A \geqq 0 \tag{8.17}$$

$$\varphi(\zeta) \leqq A|\zeta'| \quad \textit{on } V(\mathfrak{p}) \textit{ for some } A \geqq 0 \tag{8.18}$$

also satisfies

$$\varphi(\zeta) \leqq B(|\operatorname{Im}\zeta'| + 1) \quad \textit{on } V(\mathfrak{p}) \textit{ for some } B \geqq 0\ (^4)\,. \tag{8.19}$$

REMARK. – When $\mathcal{P}/\mathfrak{p}$ is formally non-characteristic in the direction ν the implication (8.17), (8.18) $\Rightarrow$ (8.19) is equivalent to the implication $(8.17')_N$,

(4) We have set here $\zeta' = (\zeta_1, \ldots, \zeta_n)$.

(8.18) $\Rightarrow$ (8.19′)$_N$ where

$$(8.17')_N \quad \varphi(\zeta) \leqq A|\operatorname{Im}\zeta'| + \varepsilon \sup(0, -\operatorname{Im}\zeta_0) + \\ + N\log(1+|\zeta|) \quad \textit{on } V(\mathfrak{p}) \textit{ for some } A \geqq 0$$

and

$$(8.19')_N \quad \varphi(\zeta) \leqq B(|\operatorname{Im}\zeta'|+1) + N\log(1+|\zeta|) \quad \textit{on } V(\mathfrak{p}) \textit{ for some } B \geqq 0\,.$$

Indeed, arguing as in [18] (Proposition 1, p. 217), we can show that one can construct, for $\delta > 0$ and any integer $M > 0$ a plurisubharmonic function $\varphi_{\delta,M}$ on $\mathbb{C}^n$ with

$$-C_M - M\log(1+|(\zeta_1, \ldots, \zeta_n)|) + \delta|\operatorname{Im}(\zeta_1, \ldots, \zeta_n)| \leqq \\ \leqq \varphi_{\delta,M}(\zeta_1, \ldots, \zeta_n) \leqq C_M - M\log(1+|(\zeta_1, \ldots, \zeta_n)|) + \delta|\operatorname{Im}(\zeta_1, \ldots, \zeta_n)|$$

for some constant $C_M > 0$. Then (8.17′) implies, because $\mathcal{T}/\mathfrak{p}$ is formally non-characteristic in the direction ν, that

$$(8.17'') \quad \varphi(\zeta) \leqq A|\operatorname{Im}\zeta'| + \varepsilon \sup(0, -\operatorname{Im}\zeta_0) + \\ + M\log(1+|(\zeta_1, \ldots, \zeta_n)|) + C'_M \quad \text{on } V(\mathfrak{p})\,,$$

for an integer $M > 0$ and a constant $C'_M > 0$, both depending only on N and $\mathfrak{p}$.

Then $\tilde{\varphi}(\zeta) = \varphi(\zeta) + \varphi_{M,\delta}(\zeta_1, \ldots, \zeta_n) - C_M - C'_N$ satisfies (8.17) and (8.18) with a new constant $A' > A$. Then (8.19′) is a consequence of (8.19) for $\tilde{\varphi}$ and of (8.20). The viceversa is obvious as (8.17) is the same as $(8.17')_0$ and (8.19) the same as $(8.19')_0$.

Then we obtain:

THEOREM 8.7. – *Let M be a unitary left $\mathcal{T}$-module of finite type. We assume that for $\mathfrak{p} \in \operatorname{Ass}(M)$ either $\mathfrak{p}$ is generated by $\mathfrak{p}' = \mathfrak{p} \cap \mathcal{T}_n$, or that $\mathfrak{p}$ satisfies a Phragmèn-Lindelöf principle in the direction $\nu = (1, 0, \ldots, 0)$ and $\mathcal{T}/\mathfrak{p}$ is formally non-characteristic in the direction ν. Then M is of evolution in the direction ν.*

PROOF. – Assume that $\mathcal{T}/\mathfrak{p}$ is formally non-characteristic in the direction ν. Then, by Theorem 8.6 and the open mapping theorem, if T is a distribution with compact support in $H \cap \{x_0 < \varepsilon\}$ and $\hat{T}$ its Fourier-Laplace transform, then, for some constant $C \geqq 0$, the plurisubhamonic function

$$\varphi(\zeta) = \log|\hat{T}(-\zeta)| - C$$

satisfies $(8.17')_N$ and (8.18) for some integer $N \geqq 0$ and a suitable constant $A \geqq 0$. The implication $(8.17')_N$ and $(8.18) \Rightarrow (8.19')_N$ tells us then that condition $(\approx{E}\mathfrak{p})$ is satisfied (cf. § 7) and therefore that $\mathfrak{I}/\mathfrak{p}$ is of evolution in the direction ν.

The statement follows by Proposition 7.1 and Theorem 7.1.

REMARK. – When $\nu = (1, 0, \ldots, 0)$ is non-characteristic for $\mathfrak{I}/\mathfrak{p}$, then we can choose $\varphi(\zeta) = \varepsilon \sup (0, -\operatorname{Im} \zeta_0)$ as a plurisubharmonic function on $V(\mathfrak{p})$ satisfying (8.17) and (8.18) and then (8.19) is the condition of hyperbolicity.

For a principal ideal $\mathfrak{p}$, the fact that the Petrowski condition

$$(8.21) \qquad -\operatorname{Im} \zeta_0 \leqq C \quad \text{for } (\zeta_0, \zeta_1, \ldots, \zeta_n) \in V(\mathfrak{p}), \quad (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$$

implies that $\mathfrak{I}/\mathfrak{p}$ is of evolution in the direction $\nu = (1, 0, \ldots, 0)$ reduces to the classical Phragmèn-Lindelöff inequality for plurisubharmonic functions on $\mathbb{C}^n$. Indeed, let φ be a plurisubharmonic function on $\mathbb{C}^{n+1}$ satisfying (8.17) and (8.18).

Set

$$\psi(\zeta_1, \ldots, \zeta_n) = \sup \{\varphi(\zeta_0, \zeta_1, \ldots, \zeta_n) : (\zeta_0, \zeta_1, \ldots, \zeta_n) \in V(\mathfrak{p})\} .$$

From (8.17), (8.18) we have

$$\begin{aligned} &\psi(\xi_1, \ldots, \xi_n) \leqq \varepsilon C && \text{if } (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \\ &\psi(\zeta_1, \ldots, \zeta_n) \leqq A|(\zeta_1, \ldots, \zeta_n)| && \text{if } (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n \end{aligned}$$

and these inequalities imply that

$$\begin{aligned} &\psi(\zeta_1, \ldots, \zeta_n) \leqq \varepsilon C + A|\operatorname{Im} (\zeta_1, \ldots, \zeta_n)| && \text{on } \mathbb{C}^n, \text{ i.e.} \\ &\varphi(\zeta_0, \zeta_1, \ldots, \zeta_n) \leqq \varepsilon C + A|\operatorname{Im} (\zeta_1, \ldots, \zeta_n)| && \text{on } V(\mathfrak{p}) . \end{aligned}$$

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