ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2014, Vol. 58, No. 2, pp. 39–44. © Allerton Press, Inc., 2014.
Original Russian Text © A.G. Pinus, 2014, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2014, No. 2, pp. 38–46.

The Classical Galois Closure for Universal Algebras

A. G. Pinus[1*]

[1] *Novosibirsk State Technical University, pr. K. Marksa 20, Novosibirsk, 630092 Russia*

Received October 9, 2012

Abstract—We study the Galois correspondence between subgroups of groups of universal algebras automorphisms and subalgebras of fixed points of these automorphisms.

DOI: 10.3103/S1066369X14020066

Keywords and phrases: *automorphisms, subalgebras, Galois-correspondence.*

The algebraic structures automorphisms relationship with the sets of their fixed points is related to a traditional algebraic subjects whose origin can be traced to the classical Galois theory which establishes the correspondence between subgroups of the Galois fields automorphisms groups and fixed points subfields of these automorphisms. In somewhat more general setting of the universal algebras the similar study was presented in the B. I. Plotkin monograph [1]. Nevertheless, later the universal algebras specialists concentrated more on the Galois-correspondence between clones of the relations on the fixed set and functions clones which do not change these relations (see, e.g., [2]). Here we consider the Galois-correspondence between the universal algebra $\mathfrak{A}$ subgroups of the automorphism group $\operatorname{Aut}\mathfrak{A}$ and the fixed points automorphism subalgebras from these groups in the classical Galois theory setting.

Let us denote by $\operatorname{Sub}\mathfrak{A}$ a lattice of subalgebras of some universal algebra $\mathfrak{A} = \langle A; \sigma\rangle$. For any $f \in \operatorname{Aut}\mathfrak{A}$ we say that $\operatorname{Fix} f$ is the set $\{a \in A \mid f(a) = a\}$ of all fixed points for the automorphism f, and $\operatorname{Fix} G = \bigcap_{f\in G} \operatorname{Fix} f$ is the set of all fixed points for all automorphisms from the subgroup G of the group $\operatorname{Aut}\mathfrak{A}$. Clearly, for any $f \in \operatorname{Aut}\mathfrak{A}$ and any subgroup G of the group $\operatorname{Aut}\mathfrak{A}$ the sets $\operatorname{Fix} f$ and $\operatorname{Fix} G$ are subalgebras of the algebra $\mathfrak{A}$. We denote by $\operatorname{Stab}\mathfrak{B}$ the set of functions $f \in \operatorname{Aut}\mathfrak{A}$ such that $\mathfrak{B} \subseteq \operatorname{Fix} f$ for any subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}$. Again it seems clear that $\operatorname{Stab}\mathfrak{B}$ is the subgroup of the group $\operatorname{Aut}\mathfrak{A}$. Thus we have the Galois mappings (analogs of the relative mapping from the classical Galois theory for the fields):

$$\operatorname{Stab} : \operatorname{Sub}\mathfrak{A} \to \operatorname{Sub}\operatorname{Aut}\mathfrak{A},$$
$$\operatorname{Fix} : \operatorname{Sub}\operatorname{Aut}\mathfrak{A} \to \operatorname{Sub}\mathfrak{A}.$$

Let us consider the operation of *Galois-closure* corresponding to these mappings on the lattice $\operatorname{Sub}\mathfrak{A}$ of the algebra $\mathfrak{A}$ subalgebras: We put in correspondence to any subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}$ the closure of this subalgebra $\overline{\mathfrak{B}} = \operatorname{Fix}\operatorname{Stab}\mathfrak{B}$. The definition of subalgebra $\overline{\mathfrak{B}}$ implies that for any automorphisms f and g of the algebra $\mathfrak{A}$ the equality $f\restriction\mathfrak{B}=g\restriction\mathfrak{B}$ yields the equality $f\restriction\overline{\mathfrak{B}}=g\restriction\overline{\mathfrak{B}}$. Here $f \restriction \mathfrak{B}$ is the restriction of the function f to subalgebra $\mathfrak{B}$. Particularly, the identity of automorphism f for $\mathfrak{A}$ on its subalgebra $\mathfrak{B}$ implies this automorphism identity also on $\overline{\mathfrak{B}}$.

Note the principal properties of the Galois-closure operation (these properties are implicit consequences of definition of the operation):

1) $\mathfrak{B} \subseteq \overline{\mathfrak{B}}$,

2) $\overline{\overline{\mathfrak{B}}} = \overline{\mathfrak{B}}$,

3) $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \to \overline{\mathfrak{B}}_1 \subseteq \overline{\mathfrak{B}}_2$.

[*] E-mail: ag.pinus@gmail.com.