ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2011, Vol. 55, No. 8, pp. 33–37. © Allerton Press, Inc., 2011.
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On ∞-Quasivarieties

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Received May 7, 2010

Abstract—We introduce the notion of an ∞-quasivariety and characterize ∞-quasivarieties as classes closed with respect to certain operators.

DOI: 10.3103/S1066369X11080068

Keywords and phrases: *universal algebras, quasiidentities, direct limits, direct products, subalgebras.*

In the algebraic geometry of universal algebras the so-called ∞-quasiidentities (for example, [1]) play an important role. We understand ∞-quasiidentities as formulas in the form

$$\forall \overline{x}\ (\Phi(\overline{x}) \to s(\overline{x}) = t(\overline{x}))\,,$$

where $\overline{x}$ is a finite collection of variables, $\Phi(\overline{x})$ is a conjunction (possibly, infinite) of equalities between terms of some fixed signature σ of variables $\overline{x}$, while $s(\overline{x})$ and $t(\overline{x})$ are terms of the signature σ of variables $\overline{x}$.

For very important in the algebraic geometry correlations $\leqslant^{\Delta}$ and $\overset{\Delta}{\sim}$ (see the definition in [1]) between algebras the following theorem (proved in [1]) is valid: Algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ satisfy the correlation $\mathfrak{A}_1 \leqslant^{\Delta} \mathfrak{A}_2$ ($\mathfrak{A}_1 \overset{\Delta}{\sim} \mathfrak{A}_2$) if and only if all ∞-quasiidentities that are valid on the algebra $\mathfrak{A}_1$ are also valid on $\mathfrak{A}_2$ (and vice versa).

We understand an ∞-quasivariety as any class $\mathcal{K}$ of algebras of a fixed signature σ, on which ∞-quasiidentities from some fixed collection of those (the ∞-q-theory of the class $\mathcal{K}$) are valid. The goal of this paper is to study the main properties of ∞-quasivarieties. See [2] for some information on analogs of quasivarieties in infinite languages.

First of all, let us characterize ∞-quasivarieties as classes closed with respect to certain operators on classes of algebras. Evidently, any ∞-quasivariety is closed with respect to the passage to subalgebras and with respect to direct products, i.e., $S\mathcal{K} = \mathcal{K}$ and $P\mathcal{K} = \mathcal{K}$. Here, as usual, $S\mathcal{K}$ is the class of all subalgebras of $\mathcal{K}$-algebras, and $P\mathcal{K}$ is the class of all directs products of $\mathcal{K}$-algebras.

Let the symbol $I\mathcal{K}$ denote the class of all algebras isomorphic to some $\mathcal{K}$-algebra.

Recall that the direct spectrum $\langle \mathfrak{A}_i, \varphi_{ij}, \langle I; \leqslant\rangle\rangle$ of algebras $\mathfrak{A}_i$ is a set of algebras $\mathfrak{A}_i$ ($i \in I$), where $\langle I; \leqslant\rangle$ is some directed upwards partially ordered set, for i ($i \leqslant j$) elements of I the symbol φ_{ij} denotes a homomorphism of the algebra $\mathfrak{A}_i$ to that $\mathfrak{A}_j$, and $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ for $i \leqslant j \leqslant k$. The direct limit $\varinjlim\langle \mathfrak{A}_i, \varphi_{ij}, \langle I; \leqslant\rangle\rangle$ of the direct spectrum $\langle \mathfrak{A}_i, \varphi_{ij}, \langle I; \leqslant\rangle\rangle$ is defined in the usual way (for example, [2]). If all algebras $\mathfrak{A}_i$ belong to some class $\mathcal{K}$, then we speak of the direct $\mathcal{K}$-spectrum of algebras. If, in addition, all homomorphisms φ_{ij} are isomorphic embeddings (of the algebra $\mathfrak{A}_i$ in that $\mathfrak{A}_j$, correspondingly), then we speak of the direct $\mathcal{K}$-spectrum of embeddability. We naturally identify algebras $\mathfrak{A}_i$ from the direct spectrum of embeddability $\langle \mathfrak{A}_i, \varphi_{ij}, \langle I; \leqslant\rangle\rangle$ with the corresponding subalgebras of the direct limit $\varinjlim\langle \mathfrak{A}_i, \varphi_{ij}, \langle I; \leqslant\rangle\rangle$. Recall also that a set of subalgebras $\mathfrak{A}_i$ ($i \in I$) of an algebra $\mathfrak{A}$ is called a local covering of the algebra $\mathfrak{A}$, if the set $\{\mathfrak{A}_i \mid i \in I\}$ partially ordered by the relation "to be a subalgebra" is directed upwards, and the union of basic sets of algebras $\mathfrak{A}_i$ ($i \in I$) coincides with the basic set of the algebra $\mathfrak{A}$. If, in addition, $\mathfrak{A}_i \in \mathcal{K}$, then we speak of the local $\mathcal{K}$-covering of the algebra $\mathfrak{A}$.

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